Title
CAUCHY PROBLEM FOR SOME HYPERBOLIC SYSTEMS OF NONLINEAR FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Citation
Annual Report of FY 2004, The Core University Program between Japan Society for the Promotion of Science (JSPS) and Vietnamese Academy of Science and Technology (VAST) P.305-P.317

Issue Date
2005

Text Version
publisher

URL
http://hdl.handle.net/11094/13244

DOI

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CAUCHY PROBLEM FOR SOME
HYPERBOLIC SYSTEMS OF NONLINEAR
FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The Cauchy problem for a normal weakly hyperbolic system of first-order nonlinear partial differential equations in two variables is considered. Sufficient conditions for its diagonalisation are given. The local solvability of the noncharacteristic Cauchy problem for classical weakly hyperbolic Monge-Ampère equation is proved.

1. Cauchy problem

We consider the following normal quasilinear first-order system of

1991 Mathematics Subject Classification. Primary 35L70; Secondary 53C21, 58C27, 58G17.

Keywords: Hyperbolic system, Diagonalised system, Weakly hyperbolic Monge-Ampère equation, Cauchy problem.

* Supported in part by the National Basic Research Program in Natural Sciences (Vietnam).

Typeset by AMS-TEX
equations in two variables

\[
\begin{cases}
\frac{\partial X_1}{\partial \alpha_2} = (a_{12} - 1) \frac{\partial X_1}{\partial \alpha_1} + a_{22} \frac{\partial X_2}{\partial \alpha_1} + \frac{\partial P_2}{\partial \alpha_1} \\
\frac{\partial X_2}{\partial \alpha_2} = -a_{11} \frac{\partial X_2}{\partial \alpha_1} - (a_{21} + 1) \frac{\partial X_2}{\partial \alpha_1} - \frac{\partial P_1}{\partial \alpha_1} \\
\frac{\partial Z}{\partial \alpha_2} = (a_{12}P_1 - a_{11}P_2) \frac{\partial X_1}{\partial \alpha_1} + (a_{22}P_1 - a_{21}P_2) \frac{\partial X_2}{\partial \alpha_1} - \frac{\partial Z}{\partial \alpha_1} - P_2 \frac{\partial P_1}{\partial \alpha_1} + P_1 \frac{\partial P_2}{\partial \alpha_1} \\
\frac{\partial P_1}{\partial \alpha_2} = (-a_{11}a_{22} + a_{12}a_{21}) \frac{\partial X_2}{\partial \alpha_1} + (a_{12} - 1) \frac{\partial P_1}{\partial \alpha_1} - a_{11} \frac{\partial P_2}{\partial \alpha_1} \\
\frac{\partial P_2}{\partial \alpha_2} = (a_{11}a_{22} - a_{12}a_{21}) \frac{\partial X_1}{\partial \alpha_1} + a_{22} \frac{\partial P_1}{\partial \alpha_1} - (a_{21} + 1) \frac{\partial P_2}{\partial \alpha_1}
\end{cases}
\]

(1)

Here \((X_1, X_2, Z, P_1, P_2)\) are unknown functions of the variables \(\alpha_1, \alpha_2; a_{ij}\) are functions of \((X_1, X_2, Z, P_1, P_2)\).

Suppose that in \(R^2_\alpha\) there is a curve \(\Gamma\) that is given by equations:

\[
\begin{cases}
X_1 = X_1^0(\alpha_1), \\
X_2 = X_2^0(\alpha_1).
\end{cases}
\]

Suppose that we are given also 3 functions \(Z^0(\alpha_1), P_1^0(\alpha_1), P_2^0(\alpha_1)\).

The Cauchy problem for the system (1) consists in looking for \((X(\alpha_1, \alpha_2), Z(\alpha_1, \alpha_2), P(\alpha_1, \alpha_2))\) such that

\[
\begin{cases}
Z(\alpha_1, \alpha_2), P(\alpha_1, \alpha_2)) \equiv \\
(X_1(\alpha_1, \alpha_2), X_2(\alpha_1, \alpha_2), Z(\alpha_1, \alpha_2), P_1(\alpha_1, \alpha_2), P_2(\alpha_1, \alpha_2)).
\end{cases}
\]

\(\in C^2\) that is a solution of (1) such that

\[
\begin{cases}
X(\alpha_1, \alpha_2)|_{\alpha_2=0} = X^0(\alpha_1), \\
Z(\alpha_1, \alpha_2)|_{\alpha_2=0} = Z^0(\alpha_1), \\
P(\alpha_1, \alpha_2)|_{\alpha_2=0} = P^0(\alpha_1),
\end{cases}
\]

(2)

where \(X^0(\alpha_1) \equiv (X_1^0(\alpha_1), X_2^0(\alpha_1)), P^0(\alpha_1) \equiv (P_1^0(\alpha_1), P_2^0(\alpha_1)).\)
From (3) we have the following necessary condition for the initial Cauchy data

$$\frac{\partial Z^0(\alpha_1)}{\partial \alpha_1} = P_1^0(\alpha_1) \frac{\partial X_1^0(\alpha_1)}{\partial \alpha_1} + P_2^0(\alpha_1) \frac{\partial X_2^0(\alpha_1)}{\partial \alpha_1}, \quad (3)$$

which is assumed to be fulfilled.

We introduce now the following condition for the system (1)

$$ (C_1): \quad a_{ij}(X, Z, P)$$

$$X_1^0(\alpha_1), X_2^0(\alpha_1),$$

$$Z^0(\alpha_1), P_1^0(\alpha_1), P_2^0(\alpha_1) \quad (\quad)$$

$$\left| \frac{D(a_{11}, a_{12})}{D(P_1, P_2)} \right| (X_1^0(\alpha_1))^2 + \left| \frac{D(a_{21}, a_{22})}{D(P_1, P_2)} \right| (X_2^0(\alpha_1))^2 +$$

$$\left[ \frac{D(a_{11}, a_{22})}{D(P_1, P_2)} \right] X_1^0(\alpha_1) X_2^0(\alpha_1) +$$

$$+ \left( \frac{\partial a_{11}}{\partial P_1} + \frac{\partial a_{12}}{\partial P_2} \right) X_1^0(\alpha_1) + \left( \frac{\partial a_{21}}{\partial P_1} + \frac{\partial a_{22}}{\partial P_2} \right) X_2^0(\alpha_1) + 1 \neq 0, \quad (4)$$

$$a_{ij}(X_1^0(\alpha_1), X_2^0(\alpha_1),$${

$$Z^0(\alpha_1), P_1^0(\alpha_1), P_2^0(\alpha_1)).$$

2. Hyperbolicity

We set

$$U = (X_1, X_2, Z, P_1, P_2)^T,$$

$$A(U)=$$

$$\begin{bmatrix}
  a_{12} - 1 & a_{22} & 0 & 0 & 1 \\
  -a_{11} & -a_{21} - 1 & 0 & -1 & 0 \\
  a_{12} P_1 - a_{11} P_2 & a_{22} P_1 - a_{21} P_2 & -1 & -P_2 & P_1 \\
  0 & -a_{11} a_{22} + a_{12} a_{21} & 0 & a_{12} - 1 & -a_{11} \\
  a_{11} a_{22} - a_{12} a_{21} & 0 & 0 & a_{22} & -a_{21} - 1
\end{bmatrix}.$$
We write the system (1) in the matrix form

$$\frac{\partial U}{\partial \alpha_2} = A(U) \frac{\partial U}{\partial \alpha_1}, \quad (6)$$

Now we recall some definitions and results on hyperbolic systems. To do this we may consider following more general normal system in two variables

$$\frac{\partial V}{\partial \alpha_2} = H(V) \frac{\partial V}{\partial \alpha_1} + G(V), \quad (7)$$

where $V, G(V)$ are column-vectors of size $m \times 1$ and $H(V)$ is matrix of size $m \times m$.

The Cauchy problem for system (7) consists in looking for $V(\alpha_1, \alpha_2) \in C^1$ such that

$$V(\alpha_1, \alpha_2)\big|_{\alpha_2=0} = V^0(\alpha_1), \quad (8)$$

where $V^0(\alpha_1)$ is a given vector function.

**Definition 1.** ([9])

$$V \in R^m$$

$$H(V) \in R^m,$$

**Theorem 1.** ([9])

$$H(V) \in R^m, \quad (9)$$

**Theorem 2.** ([9])

$$2m$$

**Theorem 3.**

$$a_{12} \neq a_{21}, \quad (X_1, X_2, Z, P_1, P_2) \in R^5,$$

For the system (6) we do not assume that $a_{12} \neq a_{21}$. In this case only condition 1) in Definition 1 is valid. We show below (Theorem 5) that under some restrictions on coefficients $a_{ij}(X, Z, P)$,
the system (6) can be reduced to a diagonal one of 7 quasilinear equations with respect to 7 unknowns. From the Theorem 1 it follows that there exists locally unique smooth solution for the Cauchy problem (6), (2). In this case the system (6) could be said to be weakly hyperbolic.

3. Reduced system

Set

$$C(X_1, X_2, Z, P_1, P_2) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
-a_{11} & -a_{21} & 0 & 1 & 0 \\
-a_{12} & -a_{22} & 0 & 0 & 1
\end{bmatrix}.$$  \hspace{1cm} (9)

Then

$$C^{-1}(X_1, X_2, Z, P_1, P_2) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
a_{11} & a_{21} & 0 & 1 & 0 \\
a_{12} & a_{22} & 0 & 0 & 1
\end{bmatrix}.$$  \hspace{1cm} (10)

Set

$$V \equiv (\tilde{X}_1, \tilde{X}_2, \tilde{Z}, Q_1, Q_2)^T = C^{-1}U.$$  \hspace{1cm} (11)

That means

$$\begin{cases}
\tilde{X}_1 = X_1, \\
\tilde{X}_2 = X_2, \\
\tilde{Z} = Z, \\
Q_1 = P_1 + a_{11}(X, Z, P)X_1 + a_{21}(X, Z, P)X_2,
\end{cases} \hspace{1cm} (C_1)$$

$$\begin{cases}
Q_2 = P_2 + a_{12}(X, Z, P)X_1 + a_{22}(X, Z, P)X_2.
\end{cases}$$

Proposition 1.
\[
\begin{align*}
\in R^5, \quad \ffi & \quad (X^0(\alpha_1), Z^0(\alpha_1), Q^0(\alpha_1)), \\
P_1, P_2 & \\
\begin{cases}
P_1 + a_{11}(X, Z, P)X_1 + a_{21}(X, Z, P)X_2 = Q_1, \\
P_2 + a_{12}(X, Z, P)X_1 + a_{22}(X, Z, P)X_2 = Q_2
\end{cases} \tag{11}
\end{align*}
\]

\[
P_1 = f(X_1, X_2, Z, Q_1, Q_2), P_2 = g(X_1, X_2, Z, Q_1, Q_2), \tag{12}
\]

\[
Q^0(\alpha_1) = P^0(\alpha_1) + X^0(\alpha_1)H(X^0(\alpha_1), Z^0(\alpha_1), P^0(\alpha_1))
\]

\[
H(X, Z, P) = \begin{bmatrix}
a_{11}(X, Z, P) & a_{12}(X, Z, P) \\
a_{21}(X, Z, P) & a_{22}(X, Z, P)
\end{bmatrix}. \tag{13}
\]

**Theorem 4.** \[ \frac{\partial V}{\partial \alpha_2} = A(V) \frac{\partial V}{\partial \alpha_1} + B(V)V, \tag{14} \]

\[
A = \begin{bmatrix}
-1 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & -P_2 & P_1 \\
0 & 0 & 0 & a_{12} - a_{21} - 1 & 0 \\
0 & 0 & 0 & 0 & a_{12} - a_{21} - 1
\end{bmatrix}, \tag{15}
\]

\[
B = C^{-1}(A \frac{\partial C}{\partial \alpha_1} - \frac{\partial C}{\partial \alpha_2}) =
\]

\[
\begin{bmatrix}
\frac{\partial a_{12}}{\partial \alpha_1} & -\frac{\partial a_{11}}{\partial \alpha_1} - \frac{\partial a_{22}}{\partial \alpha_1} & 0 & 0 & 0 \\
-\frac{\partial a_{11}}{\partial \alpha_1} P_2 + \frac{\partial a_{12}}{\partial \alpha_1} P_1 & -\frac{\partial a_{21}}{\partial \alpha_1} P_2 + \frac{\partial a_{22}}{\partial \alpha_1} P_1 & 0 & 0 & 0 \\
(a_{12} - a_{21} - 1) \frac{\partial a_{11}}{\partial \alpha_1} - \frac{\partial a_{11}}{\partial \alpha_2} & (a_{12} - a_{21} - 1) \frac{\partial a_{21}}{\partial \alpha_1} - \frac{\partial a_{21}}{\partial \alpha_2} & 0 & 0 & 0 \\
(a_{12} - a_{21} - 1) \frac{\partial a_{12}}{\partial \alpha_1} - \frac{\partial a_{12}}{\partial \alpha_2} & (a_{12} - a_{21} - 1) \frac{\partial a_{22}}{\partial \alpha_1} - \frac{\partial a_{22}}{\partial \alpha_2} & 0 & 0 & 0
\end{bmatrix}. \tag{16}
\]
\[
\begin{align*}
   a_{ij}(X_1, X_2, Z, P_1, P_2) & \quad P_1, P_2 \\
   P_1 = f(X_1, X_2, Z, Q_1, Q_2), P_2 = g(X_1, X_2, Z, Q_1, Q_2)
\end{align*}
\]

4. Diagonalization

It is clear from (15) that the system (14) is not diagonal. We give now some sufficient conditions under which the system (14) can be reduced to a diagonal quasilinear one.

We introduce now other condition for the system (1)

\((C_2)\) :

\[
\begin{align*}
   & a_{ij}(X, Z, P) \quad fi \\
   \begin{cases}
   \frac{\partial a_{ij}}{\partial Z} = 0, \\
   \frac{\partial a_{ij}}{\partial X_1} - a_{11} \frac{\partial a_{ij}}{\partial P_1} - a_{12} \frac{\partial a_{ij}}{\partial P_2} = 0, \\
   \frac{\partial a_{ij}}{\partial X_2} - a_{21} \frac{\partial a_{ij}}{\partial P_1} - a_{22} \frac{\partial a_{ij}}{\partial P_2} = 0.
   \end{cases}
\end{align*}
\]

We set

\[
S_1 = \frac{\partial Q_1}{\partial \alpha_1}, S_2 = \frac{\partial Q_2}{\partial \alpha_1}.
\]

Proposition 2.  

\((C_1) \quad (C_2)\)

\[
(a_{12} - a_{21} - 1) \frac{\partial a_{ij}}{\partial \alpha_1} - \frac{\partial a_{ij}}{\partial \alpha_2} = 0, \quad \text{(18)}
\]

\[
n_{ij}(X, Q) \quad c_{ij}(X, Q)
\]

\[
\frac{\partial a_{ij}}{\partial \alpha_1} = n_{ij}(X, Q)S_1 + c_{ij}(X, Q)S_2, \forall i, j = 1, 2. \quad \text{(19)}
\]

We introduce new dependent variables

\[
W = (X_1, X_2, Z, Q_1, Q_2, S_1, S_2)^T.
\]
From Proposition 2 it follows

**Theorem 5.** Assume the conditions \((C_1)\) and \((C_2)\). Then the system (14) can be diagonalized, i.e. it may be reduced to following diagonal one:

\[
\frac{\partial W}{\partial \alpha_2} = \tilde{A}(W) \frac{\partial W}{\partial \alpha_1} + F(W),
\]

where

\[
\tilde{A} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & t & 0 & 0 \\
0 & 0 & 0 & 0 & t & 0 \\
0 & 0 & 0 & 0 & 0 & t
\end{bmatrix},
\]

where \(t = a_{12} - a_{21} - 1\) and

\[
F(W) = F_1(W) + F_2(W),
\]

where

\[
F_1(W) = \begin{bmatrix}
- S_2 + (b_{12}S_1 + c_{12}S_2)P_1 + (b_{22}S_1 + c_{22}S_2)P_2 \\
- S_1 + (b_{11}S_1 + c_{11}S_2)P_1 - (b_{21}S_1 + c_{21}S_2)P_2 \\
- P_2S_1 + P_1S_2 + [(b_{11}S_1 + c_{11}S_2)P_2 + (b_{12}S_1 + c_{12}S_2)P_1]X_1 \\
0 \\
0 \\
[(b_{12} - b_{21})S_1 + (c_{12} - c_{21})S_2]S_1 \\
[(b_{12} - b_{21})S_1 + (c_{12} - c_{21})S_2]S_2
\end{bmatrix},
\]

\[
F_2(W) = \begin{bmatrix}
0 \\
0 \\
- (b_{21}S_1 + c_{21}S_2)P_2 + (b_{22}S_1 + c_{22}S_2)P_1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix},
\]
where in $F_1(W), F_2(W)$ the variables $P_1, P_2$ must be replaced by $f(X_1, X_2, Z, Q_1, Q_2)$ and $g(X_1, X_2, Z, Q_1, Q_2)$ respectively.

**Theorem 6.**

\begin{align*}
\text{(C)}_1 \quad \text{(C)}_2.
\end{align*}

**5. Application to the classical weakly hyperbolic Monge-Ampère equation**

The classical hyperbolic Monge-Ampère equation with two variables is that of the form

\[
F(x_1, x_2, z, p_1, p_2, r, s, t) = Ar + Bs + Ct + (rt - s^2) - E = 0, \quad (22)
\]

where $z = z(x_1, x_2)$ is an unknown function defined for $(x_1, x_2) \in \mathbb{R}^2$, $p_1 = \frac{\partial z}{\partial x_1}, p_2 = \frac{\partial z}{\partial x_2}, r = \frac{\partial^2 z}{\partial x_1^2}, s = \frac{\partial^2 z}{\partial x_1 \partial x_2}$ and $t = \frac{\partial^2 z}{\partial x_2^2}$. The coefficients $A, B, C$ and $E$ are real smooth functions of $(x_1, x_2, z, p_1, p_2)$ and satisfy the condition of hyperbolicity:

\[
\Delta := B^2 - 4(AC + E) > 0.
\]

In this case the characteristic equation

\[
\lambda^2 + B\lambda + (AC + E) = 0 \quad (23)
\]

has two different real roots $\lambda_1 = \lambda_1(x_1, x_2, z, p_1, p_2), \lambda_2 = \lambda_2(x_1, x_2, z, p_1, p_2)$. In the case, where the equation (1) is hyperbolic, it can be written in the following equivalent form

\[
\begin{vmatrix}
z_{x_1x_1} + C & z_{x_1x_2} + \lambda_1 \\
z_{x_2x_1} + \lambda_2 & z_{x_2x_2} + A
\end{vmatrix} = 0. \quad (24)
\]

Equation (22) was investigated in [1], [2] by G. Darboux and E. Goursat under the assumptions that $\Delta > 0$ and there are two independent first integrals for the equation (22). In this case the equation (22) had been also considered in [3], [4], [6], [7] by
reducing it to a hyperbolic quasilinear system of first-order partial differential equations with two variables. For the case $\Delta \geq 0$ in [5] M. Tsuji proved local solvability of Cauchy problem (22), (25) provided that there exist two independent first integrals. In [10] D. V. Tuniski considered the case $\Delta \geq 0$ and proved solvability of the Cauchy problem in class of multivalue functions, but under rather strong assumptions on coefficients $A, B, C, E$.

In [8] we have proposed a solving method for the equation (24) that reduces it to the system (1) with $a_{11} = C, a_{12} = \lambda_1, a_{21} = \lambda_2, a_{22} = A$. Applying Theorem 6 stated above to the last system we can consider the case $\Delta \geq 0$ and we do not assume existence of two independent first integrals.

Suppose the functions $X^0_1(\alpha_1), X^0_2(\alpha_1), Z^0(\alpha_1), P^0_1(\alpha_1), P^0_2(\alpha_1)$ are given as in 1, that satisfy the condition (3).

**Cauchy problem:** The Cauchy problem for the equation (22) consists in looking for $z(x) \in C^2$ that is a solution of (22) such that

\[
\begin{cases}
  z(x)|_{x=x_0(\alpha_1)} = Z^0(\alpha_1), \\
  z_{x^j}(x)|_{x=x_0(\alpha_1)} = P^0_j(\alpha_1), j = 1, 2,
\end{cases}
\]

(25)

where $X^0(\alpha_1) \equiv (X^0_1(\alpha_1), X^0_2(\alpha_1))$.

We assume that the Cauchy problem (22), (25) is not characteristic, i.e.

\[
C(X^0_1'(\alpha_1))^2 + A(X^0_2'(\alpha_1))^2 - BX^0_1'(\alpha_1)X^0_2'(\alpha_1) + \\
(X^0_1'(\alpha_1)P^0_1'(\alpha_1) + X^0_2'(\alpha_1)P^0_2'(\alpha_1)) \neq 0,
\]

(26)

where the coefficients $A, B, C$ are calculated at $(X^0_1(\alpha_1), X^0_2(\alpha_1), Z^0(\alpha_1), P^0_1(\alpha_1), P^0_2(\alpha_1))$.

**Theorem 7.**
\[ \Delta \geq 0; \]

\[
\begin{align*}
\left| \frac{D(C, \lambda_1)}{D(p_1, p_2)} \right| (X_1^0(\alpha_1))^2 + \left| \frac{D(\lambda_2, A)}{D(p_1, p_2)} \right| (X_2^0(\alpha_1))^2 + \\
\left[ \left| \frac{D(C, A)}{D(p_1, p_2)} \right| + \left| \frac{D(\lambda_2, \lambda_1)}{D(p_1, p_2)} \right| \right] X_1^0(\alpha_1) X_2^0(\alpha_1) + \\
+ \left( \frac{\partial C}{\partial p_1} + \frac{\partial \lambda_1}{\partial p_2} \right) X_1^0(\alpha_1) + \left( \frac{\partial \lambda_2}{\partial p_1} + \frac{\partial A}{\partial p_2} \right) X_2^0(\alpha_1) + 1 \neq 0, \quad (27)
\end{align*}
\]

\( A, C, \lambda_1, \lambda_2, X_1^0(\alpha_1), X_2^0(\alpha_1), Z^0(\alpha_1), P_1^0(\alpha_1), P_2^0(\alpha_1) \);

\( A(x, z, p), C(x, z, p), \lambda_1(x, z, p), \lambda_2(x, z, p) \)

\[
\begin{align*}
\frac{\partial \varphi}{\partial x_1} - C \frac{\partial \varphi}{\partial p_1} - \lambda_1 \frac{\partial \varphi}{\partial p_2} &= 0, \\
\frac{\partial \varphi}{\partial x_2} - \lambda_2 \frac{\partial \varphi}{\partial p_1} - A \frac{\partial \varphi}{\partial p_2} &= 0. \quad (28)
\end{align*}
\]

The Monge-Ampère equations, satisfying conditions 1) - 4) of Theorem 7 are said to be \textbf{weakly hyperbolic} ones.
Exemples.

1) ([5], [10]) The coefficients $A, B, C, E$ are constants with $\Delta \geq 0$. It is easy to see that all the assumptions of the Theorem 7 are satisfied.

2) Suppose $v(y, t)$ is a solution of the Burger equation

\[ v_t + vv_y = 0, \quad (29) \]

which satisfies the following condition

\[ v_y(P_1^0(\alpha_1) - P_2^0(\alpha_1), X_1^0(\alpha_1) + X_2^0(\alpha_1))(X_1^0(\alpha_1) + X_2^0(\alpha_1)) + 1 \neq 0. \quad (27') \]

Then the Monge-Ampère equation

\[ rt - s^2 + v^2(z_{x_1} - z_{x_2}, x_1 + x_2) = 0 \]

with $A = B = C = 0, E = -v^2(z_{x_1} - z_{x_2}, x_1 + x_2), \Delta = 4v^2(p_1 - p_2, x_1 + x_2), \lambda_1 = -\lambda_2 = v(p_1 - p_2, x_1 + x_2)$ satisfies all conditions of the Theorem 7.

3) Suppose $v(y, t), w(y, t)$ are some solutions of the equation (29) that satisfy the condition:

\[ v_y(-P_1^0(\alpha_1), X_1^0(\alpha_1))w_y(-P_2^0(\alpha_1), X_2^0(\alpha_1))X_1^0(\alpha_1)X_2^0(\alpha_1) - \]

\[ v_y(-P_1^0(\alpha_1), X_1^0(\alpha_1))X_1^0(\alpha_1) - w_y(-P_2^0(\alpha_1), X_2^0(\alpha_1))X_2^0(\alpha_1) + 1 \neq 0. \quad (27'') \]

Then the equation

\[ w(-z_{x_2}, x_2)r + v(-z_{x_1}, x_1)t + (rt - s^2) - v(-z_{x_1}, x_1)w(-z_{x_2}, x_2) = 0 \]

with $A = w(-p_2, x_2), B = 0, C = v(-p_1, x_1), E = v(-p_1, x_1)w(-p_2, x_2), \Delta = 0, \lambda_1 = \lambda_2 = 0$ satisfies all conditions of the Theorem 7.
REFERENCES


