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Rights
ON FINITE POINT TRANSITIVE AFFINE PLANES
WITH TWO ORBITS ON $l_{\infty}$

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1. Introduction

Kallaher [3] proposed the following conjecture.

Conjecture. Let $\pi$ be a finite affine plane of order $n$ with a collineation group $G$ which is transitive on the affine points of $\pi$. If $G$ has two orbits on the line at infinity, then one of the following statements holds:

(i) The plane $\pi$ is a translation plane, and the group $G$ contains the group of translations of $\pi$.

(ii) The plane $\pi$ is a dual translation plane, and the group $G$ contains the group of dual translations of $\pi$.

The purpose of this paper is to study this conjecture. When $G_A$ has two orbits of length 1 and $n$ on the line at infinity, where $A$ is an affine point of $\pi$, some work has been done on this conjecture. (See Johnson and Kallaher [2].)

Our notation is largely standard and taken from [3]. Let $\pi = \pi(\ell)$ be the projective extension of an affine plane $\pi$, and $G$ a collineation group of $\pi$. If $P$ is a point of $\pi$ and $\ell$ is a line of $\pi$, then $G(P, \ell)$ is the subgroup of $G$ consisting of all perspectivities in $G$ with center $P$ and axis $\ell$. If $m$ is a line of $\pi$, then $G(m, m)$ is the subgroup consisting of all elations in $G$ with axis $m$.

In § 2 we prove the following theorem.

Theorem 1. Let $\pi$ be a finite affine plane of order $n$ with a collineation group $G$ and let $\Delta$ be a subset of $\ell_\infty$ such that $|\Delta| = t \geq 2$, $(n, t) = 1$ and $(n, t - 1) = 1$. If there is an integer $k_1 > 1$ such that $|G(P, \ell_\infty)| = k_1$ for all $P \in \Delta$ and there is an integer $k_2 > 1$ such that $|G(Q, \ell_\infty)| = k_2$ for all $Q \in \ell_\infty - \Delta$, then $\pi$ is a translation plane, and $G$ contains the group $T$ of translations of $\pi$.

In § 3 and § 4, we prove the following theorem by using Theorem 1.

Theorem 2. Let $\pi$ be a finite affine plane of order $n$ with a collineation group $G$ which is transitive on the affine points of $\pi$. If $G$ has two orbits of length 2 and $n - 1$ on $\ell_\infty$, then one of the following statements holds:
(i) *The plane π is a translation plane, and the group G contains the group T of translations of π.*

(ii) $|G(\ell_0, \ell_0)| = n = 2^m$ for some $m \geq 1$, $G(P_1, \ell_0) = G(P_2, \ell_0) = 1$ and $|G(P, \ell_0)| = 2$ for all $P \in \ell_0 - \{P_1, P_2\}$.

The planes which are not André planes, satisfying the hypothesis of Theorem 2, include a class of translation planes of order $q^3$, where $q$ is an odd prime power. (See Suetake [4] and Hiramine [1].)

2. The proof of Theorem 1

In this section, we prove Theorem 1.

Let $\pi$ be a finite affine plane of order $n$ with a collineation group $G$, satisfying the hypothesis of Theorem 1. By Theorem 4.5 of [3], $G(\ell_0, \ell_0)$ is an elementary abelian $r$-group for some prime $r$ dividing $n$. Hence there exist positive integers $m$ and $s$ such that $k_1 = r^m$ and $k_2 = r^s$. Let $P$ be a point of $\pi$ such that $P \in \Delta$. Let $\ell$ be an affine line of $\pi$ such that $\ell \ni P$. Since $G(P, \ell_0)$ is semiregular on $\ell - \{P\}$, $r^m | n$. Similarly, $r^s | n$. By definition, $G(\ell_0, \ell_0) = \bigcup_{P \in \ell_0} G(P, \ell_0)$ and $G(P, \ell_0) \cap G(Q, \ell_0) = 1$ for distinct points $P, Q \in \ell_0$. Thus

$$|G(\ell_0, \ell_0)| = 1 + \sum_{P \in \Delta} (|G(P, \ell_0)| - 1) + \sum_{Q \in \ell_0 - \Delta} (|G(Q, \ell_0)| - 1)$$

$$= 1 + t(r^m - 1) + (n + 1 - t)(r^s - 1).$$

Since $r^m | |G(\ell_0, \ell_0)|$, it follows $0 \equiv 1 - t + (1 - t)r^m - 1 + t \pmod{r^m}$. Therefore $(t - 1)r^m \equiv 0 \pmod{r^m}$. Since $(t - 1, r) = 1$, this implies $r^m | r^s$. Thus $m \leq s$. On the other hand, since $r^s | |G(\ell_0, \ell_0)|$, it follows $0 \equiv 1 + t(r^s - 1) - 1 + t \pmod{r^s}$. Therefore $t(r^s - 1) \equiv 0 \pmod{r^s}$. Since $(t, r) = 1$, this implies $r^s | r^m$. Thus $m \geq s$. Therefore $m = s$ and $k_1 = k_2$. By a result of Gleason (See Theorem 5.2 of [3].), the theorem holds.

3. The proof of Theorem 2 when $n$ is odd

In this section, we prove Theorem 2 when $n$ is odd.

Let $\pi$ be a finite affine plane of odd order $n$ with a collineation group $G$ which is transitive on the affine points of $\pi$, satisfying the hypothesis of Theorem 2. Then $G$ has an orbit $\Delta = \{P_1, P_2\}$ of length 2 on $\ell_0$. Let $A$ be an affine point of $\pi$. Let $\Phi$ be the set of the affine points of $\pi$, and let $\Omega = \Phi \cup \ell_0$. Then $G$ induces a permutation group on $\Omega$. $\Phi, \Delta$ and $\ell_0 - \Delta$ are orbits of $G$. Since $(|\Phi|, |\Delta|) = (n^2, 2) = 1$ and $(|\Phi|, |\ell_0 - \Delta|) = (n^2, n - 1) = 1$, by Theorem 3.3 of [3] $\Delta$ and $\ell_0 - \Delta$ are orbits of $G_A$.

**Lemma 3.1.** $G_A$ includes an involutory homology of $\pi$. 
Proof. $G_A$ induces a permutation group on $\ell_n - \{P_1, P_2\}$. Since $n$ is odd, $|\ell_n - \{P_1, P_2\}| = n - 1$ is even. Let $S$ be a Sylow 2-subgroup of $G_A$. As $G_A$ is transitive on $\ell_n - \{P_1, P_2\}$, $n - 1 | |G_A|$. Hence $S \neq 1$. There exists an involution $\sigma$ in the center of $S$. Suppose that $\sigma$ is a Baer involution. If $P_1\sigma = P_1$, then $P_2\sigma = P_2$ and so $|\ell_n - \Delta |P\sigma = P| = \sqrt{n} - 1$. This contradicts a result of Lüneburg. (See Corollary 3.6.1 of [3].) If $P_1\sigma = P_2$, then $P_2\sigma = P_2$ and so $|\ell_n - \Delta |P\sigma = P| = \sqrt{n} + 1$. This is again a contradiction by Corollary 3.6.1 of [3]. Therefore $\sigma$ is an involutory homology.

**Lemma 3.2.** Let $\sigma$ be an involutory homology of $\pi$ such that $\sigma \in G_A$. If $P_1\sigma = P_1$, then $\pi$ is a translation plane, and $G$ contains the group $T$ of translations of $\pi$.

Proof. Since $P_1\sigma = P_1$, $P_2\sigma = P_2$. Assume that $\ell_n$ is the axis of $\sigma$. Then $\sigma \in G(A, \ell_n)$. By a result of André (See Corollary 10.1.3 of [3]), the lemma holds. Assume that $\ell_n$ is not the axis of $\sigma$. We may assume that $AP_1$ is the axis of $\sigma$. Then $\sigma \in G(P_2, AP_1)$. There exists $\tau \in G_A$ such that $P_1\tau = P_2$. Clearly $P_2\tau = P_1$. Since $P_2\tau = P_1$ and $(AP_1)\tau = AP_2$, $\tau^{-1}\sigma\tau \in G(P_1, AP_2)$. Therefore $\sigma(\tau^{-1}\sigma\tau) \in G(A, \ell_n) - \{1\}$, by a result of Ostrom. (See Lemma 4.13 of [3].) Thus the lemma holds by Corollary 10.1.3 of [3].

**Lemma 3.3.** If $G_A$ includes an involutory homology of $\pi$ which does not fix $P_1$, then the following statements hold:

(i) If $P \in \ell_n - \{P_1, P_2\}$, then there exist $Q \in \ell_n - \{P_1, P_2, P\}$ and $\sigma \in G(Q, AP)$ such that $|\sigma| = 2$.

(ii) If $Q \in \ell_n - \{P_1, P_2\}$, then there exist $P \in \ell_n - \{P_1, P_2, Q\}$ and $\tau \in G(Q, AP)$ such that $|\tau| = 2$.

Proof. By assumption, there exists an involutory homology $\sigma$ of $\pi$ such that $\sigma \in G_A$ and $P_1\sigma \neq P_1$. Clearly $P_2\sigma \neq P_1$. There exists $P_0 \in \ell_n - \{P_1, P_2\}$ such that $AP_0$ is the axis of $\sigma$. Let $Q_0$ be the center of $\sigma$. Then $Q_0 \in \ell_n - \{P_1, P_2, P_0\}$. Let $P \in \ell_n - \{P_1, P_2\}$. Then there exists $\varphi \in G_A$ such that $P = P_0\varphi$. Set $Q = Q_0\varphi$. Clearly $Q \in \{P_1, P_2\}$. Since $\sigma \in G(Q_0, AP_0)$ and $(AP_0)\varphi = AP$, $\varphi^{-1}\sigma\varphi \in G(Q, AP)$. This yields the statement (i). Similarly, we have the statement (ii).

**Lemma 3.4.** If $G_A$ includes an involutory homology of $\pi$ which does not fix $P_1$, then one of the following statements holds:

(i) The plane $\pi$ is a translation plane and $G$ contains the group $T$ of translations of $\pi$.

(ii) If $P \in \ell_n - \{P_1, P_2\}$, then $G(P, AP) \neq 1$. 
Proof. Let \( P \in \omega - \{P_1, P_2\} \). By Lemma 3.3 (i), there exist \( Q \in \omega - \{P_1, P_2, P\} \) and \( \sigma \in G(Q, AP) \) such that \( |\sigma| = 2 \). On the other hand, by Lemma 3.3 (ii) there exist \( R \in \omega - \{P_1, P_2, Q\} \) and \( \tau \in G(R, AQ) \) such that \( |\tau| = 2 \). Assume that \( R = P \). Then \( \sigma \in G(Q, AP) \) and \( \tau \in G(P, AQ) \). By Lemma 4.13 of [3], \( \sigma \tau \in G(A, \omega) - \{1\} \). Thus the statement (i) holds by Corollary 10.1.3 of [3]. Assume that \( R \neq P \). Then since \( \tau \in G(R, AQ) \) and \( (AQ) \sigma = AQ, \sigma^{-1} \tau \sigma \in G(R \sigma, AQ) \). As \( R \neq R \sigma, (\sigma^{-1} \tau \sigma) \in G(Q, AQ) - \{1\} \) by a result of Baer. (See Lemma 4.12 of [3].) Thus \( G(Q, AQ) \neq 1 \). On the other hand, since \( G_A \) acts transitively on \( \omega - \{P_1, P_2\} \), the statement (ii) holds.

**Lemma 3.5.** If \( G(P, AP) \neq 1 \) for all \( P \in \omega - \{P_1, P_2\} \), then there is an integer \( k > 1 \) such that \( |G(P, \omega)| = k \) for all \( P \in \omega - \{P_1, P_2\} \).

Proof. Let \( P \in \omega - \{P_1, P_2\} \). Let \( \ell \) be an affine line of \( \pi \) such that \( \ell \equiv P \). By a result of Ostrom and Wagner (See Theorem 4.3 of [3]), there exists \( \tau \in G_\pi \) such that \( (AP) \tau = \ell \). Since \( G(P, AP) \neq 1 \), \( \tau^{-1} G(P, AP) \tau = G(P \tau, (AP) \tau) = G(P, \ell) \neq 1 \). Therefore by the dual of Corollary 4.6.1 of [3], \( G(P, \omega) \neq 1 \). On the other hand, since \( G_A \) acts transitively on \( \omega - \{P_1, P_2\} \), the lemma holds.

**Lemma 3.6.** If \( G(P, AP) \neq 1 \) for all \( P \in \omega - \{P_1, P_2\} \), then \( |G(P, \omega)| = |G(P \tau, \omega)| > 1 \).

Proof. Since the order \( n \) of \( \pi \) is odd, by Lemma 3.5 \( |G(P, \omega)| \geq 3 \) for all \( P \in \omega - \{P_1, P_2\} \). Therefore

\[
\begin{align*}
|G(P, \omega)| &= \bigcup_{P \in \omega - \{P_1, P_2\}} |G(P, \omega)| \\
&= 1 + \sum_{P \in \omega - \{P_1, P_2\}} (|G(P, \omega)| - 1) \\
&\geq 1 + 2(n-1) \\
&= 2n - 1 \\
&> n.
\end{align*}
\]

Thus \( |G(\omega, \omega)| > n \). Hence by a result of Ostrom (See Theorem 4.6 of [3]), \( G(P, \omega) \neq 1 \) for all \( P \in \omega \). In particular \( G(P, \omega) \neq 1 \). There exists \( \tau \in G_A \) such that \( P_2 \tau = P_1 \). Thus \( |G(P_2, \omega)| = |\tau^{-1} G(P_2, \omega) \tau| = |G(P_1, \omega)| > 1 \). Hence the lemma holds.

Proof of Theorem 2 when \( n \) is odd: By Lemmas 3.2, 3.4, 3.5, 3.6 and Theorem 1, the theorem holds.

4. The proof of Theorem 2 when \( n \) is even

In this section, we prove Theorem 2 when \( n \) is even. Let \( \pi \) be a finite affine plane of even order \( n \) with a collineation group \( G \)
which is transitive on the affine points of \( \pi \) satisfying the hypothesis of Theorem 2. Then \( G \) has an orbit \( \Delta = \{ P_1, P_2 \} \) of length 2 on \( \ell_\infty \).

**Lemma 4.1.** \( G \) includes a translation of order 2 of \( \pi \).

**Proof.** Since \( n^2 | | G |, 2 | | G | \). Let \( S \) be a Sylow 2-subgroup of \( G \). Then there exists an involution \( \sigma \) in the center of \( S \). By Corollary 3.6.1 of [3] the involution \( \sigma \) is neither a Baer involution, nor an affine elation. It follows that \( \sigma \) is a translation of \( \pi \).

**Lemma 4.2.** \( G(\ell_\infty, \ell_\infty) \) is an elementary abelian 2-group and \( | G(\ell_\infty, \ell_\infty) | \geq 2 \).

**Proof.** If \( n = 2 \), then the lemma holds. Let \( n \neq 2 \). Considering the action of \( G \) on \( \ell_\infty \), by Lemma 4.1 there exist distinct points \( Q_1, Q_2 \in \ell_\infty \) such that \( G(Q_1, \ell_\infty) \neq 1 \) and \( G(Q_2, \ell_\infty) \neq 1 \). By Theorem 4.5 of [3], the lemma holds.

**Lemma 4.3.** If \( G(P_1, \ell_\infty) \neq 1 \), then the plane \( \pi \) is a translation plane, and the group \( G \) contains the group \( T \) of translations of \( \pi \).

**Proof.** There exists an involution \( \sigma \), such that \( \sigma \in G(P_i, \ell_\infty) \) for \( i \in \{ 1, 2 \} \). Then \( \sigma_1 \sigma_2 \in G(\ell_\infty, \ell_\infty) \) and \( | \sigma_1 \sigma_2 | = 2 \). Let \( Q \) be the center of \( \sigma_1 \sigma_2 \). Then \( Q \in \ell_\infty - \{ P_1, P_2 \} \). Since \( G \) acts transitively on \( \ell_\infty - \{ P_1, P_2 \} \), there exists \( r \geq 1 \) such that \( | G(P, \ell_\infty) | = 2^r \) for all \( P \in \ell_\infty - \{ P_1, P_2 \} \). There exists \( s \geq 1 \) such that \( | G(P_1, \ell_\infty) | = | G(P_2, \ell_\infty) | = 2^s \). Let \( | G(\ell_\infty, \ell_\infty) | = 2^t \). Then \( t \geq r + s \). Since

\[
| G(\ell_\infty, \ell_\infty) | = 1 + \sum_{P \in \ell_\infty - \{ P_1, P_2 \}} (| G(P, \ell_\infty) | - 1) + \sum_{Q \in \{ P_1, P_2 \}} (| G(Q, \ell_\infty) | - 1),
\]

\( 2^t = 1 + (n-1)(2^r-1) + 2(2^s-1) \) (\(*\))

By the same argument as in the proof of Theorem 1, \( 2^r \equiv 0 \) (mod 2^s) and \( 2^{s+1} \equiv 0 \) (mod 2^r). Thus \( s \leq r \leq s + 1 \).

Suppose that \( r = s + 1 \). From (\(*\)), \( 2^t = 1 + (n-1)(2^{s+1}-1) + 2(2^s-1) \) follows. Therefore \( n = 2^t (2^{s+1}-1) \). As \( n \) is an integer, this is a contradiction. Hence \( r = s \). By Theorem 5.2 of [3], the lemma holds.

**Lemma 4.4.** If \( G(P_1, \ell_\infty) = 1 \), then \( | G(\ell_\infty, \ell_\infty) | = n = 2^m \) for some \( m \geq 1 \), \( G(P_1, \ell_\infty) = 1 \) and \( | G(P, \ell_\infty) | = 2 \) for all \( P \in \ell_\infty - \{ P_1, P_2 \} \).

**Proof.** By assumption, \( G(P_2, \ell_\infty) = 1 \) follows. If \( P \in \ell_\infty - \{ P_1, P_2 \} \), then \( G(P, \ell_\infty) \neq 1 \). Therefore there exists an integer \( r \geq 1 \) such that \( | G(Q, \ell_\infty) | = 2^r \) for all \( Q \in \ell_\infty - \{ P_1, P_2 \} \). Suppose that \( r \geq 2 \). Then

\[
| G(\ell_\infty, \ell_\infty) |
\]

\[
= \sum_{Q \in \ell_\infty - \{ P_1, P_2 \}} (| G(Q, \ell_\infty) | - 1) + 1
\]

\[
= (2^r - 1)(n-1) + 1
\]
\[ \geq 3(n-1) + 1 \]
\[ = 3n - 2 \]
\[ > n. \]

By Theorem 4.6 of [3], it follows that \( G(Q, \ell_w) \neq 1 \) for all \( Q \in \ell_w \). In particular \( G(P, \ell_w) \neq 1 \), a contradiction. Hence \( r = 1 \). Therefore \( |G(\ell_w, \ell_w)| = (2-1) \cdot (n-1) + 1 = n \). Therefore there exists an integer \( m \geq 1 \) such that \( n = 2^m \). Thus the lemma holds.

Proof of Theorem 2 when \( n \) is even: By Lemmas 4.3 and 4.4, the theorem holds.

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References

[1] Y. Hiramine: On translation planes of order \( q^3 \) with an orbit of length \( q^3 - 1 \) on \( \ell_w \), Osaka J. Math. 23 (1986), 563–575.