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On the topologies of homeomorphism groups of topological spaces ¹)

By Hirosi NAGAO

1. Let $R$ be a topological space. Then, by the usual definition of the multiplication, all the homeomorphic mappings of $R$ onto itself form an abstract group $A_R$. The present note is devoted to the question of in respect to what topology $A_R$ may become a topological group, provided that $R$ is regular and locally bicomplete. ²)

In section 2, under the condition that $R$ is regular and locally bicomplete, we obtain the weakest one of topologies of $A_R$ in respect to which $A_R$ becomes a topological group and the mapping $(f, a) \mapsto f(a)$ from the topological product of $A_R$ and $R$ to $R$ is continuous for both $f$ and $a$.

Furthermore, in section 3, supposing that $R$ is a uniform space which is locally bicomplete, we shall show that the topology of $A_R$ introduced in section 2 coincides with the topology introduced analogously to that of character groups.

2. Let $R$ be a regular and locally bicomplete topological space, and let $A_R$ have the same significance as section 1. Let further $\{U_\alpha\}$ be a basis of $R$ (that is, a system of open sets such that every open set of $R$ can be obtained as a sum of open sets belonging to it) such that the closure $\overline{U}_\alpha$ of $U_\alpha$ is bicomplete. If we denote by $V_{x_1} \ldots x_r : \beta_1 \ldots \beta_r$ the set of homeomorphisms which transform $\overline{U}_{\alpha i}$ into $U_{\beta k}$ ($i = 1, \ldots, r$), then we have the following theorem.

Theorem 1. Denote $V_{x_1} \ldots x_r : \beta_1 \ldots \beta_r \setminus V_{x_1}^{-1} \ldots x_r : \beta_1 \ldots \beta_r$ by $W_{x_1}^{-1} \ldots x_r : \beta_1 \ldots \beta_r$.

(where $S^{-1}$ means the set of inverses of elements belonging to a subset $S$ of $A_R$), and let $\Sigma$ be the system consisting of all $W_{x_1}^{1} \ldots x_r : \beta_1 \ldots \beta_r$.

¹) The writer is grateful to Prof. K. Shoda, who gave an impulse to the present paper.
²) After having written this paper, the writer became aware of the fact that J. Dieudonné, R. Arens, J. Braconnier and J. Cholmez have already investigated this problem, but in the present situation, the writer cannot see their papers except the paper of J. Dieudonné, which appeared in Amer. Journ. Vo 1. 70 No. 3 (1948).
(occasionally abbreviated \( W^{(\alpha')};(\beta') \) which are non-empty. Then, taking \( \Sigma \) as a basis of \( A_R \), we obtain the weakest topology of \( A_R \) in respect to which \( A_R \) becomes a topological group and the mapping \((f, a) \rightarrow f(a)\) from the topological product of \( A_R \) and \( R \) to \( R \) is continuous for both \( f \) and \( a \).

**Proof.** In order to prove that \( A_R \) becomes a topological group regarding \( \Sigma \), it will be sufficient to show that if we take the system \( \Sigma' \) of all \( V_{\alpha_1 \ldots \alpha_r};\beta_1 \ldots \beta_r \) (occasionally abbreviated \( V_{(\alpha)};(\beta) \)) which are non-empty, then \( A_R \) becomes a topological space and the topology is continuous respecting the product.

Let \( f \) and \( g \) be any two different elements of \( A_R \). Then there exists \( a \in R \) such that \( f(a) \neq g(a) \), and an open set \( U_a \) belonging to \( \{U_a\} \) such that \( f(a) \in U_a \), \( g(a) \in U_a \). If we take an open set \( U_\beta \) from \( \{U_a\} \) such that \( a \in U_\beta \) and \( f(\overline{U_\beta}) \subseteq U_a \), then \( f \in V_\beta \) and \( g \in V_\beta \). Furthermore, the intersection of any two sets which belong to \( \Sigma' \) and contain some element of \( A_R \) belongs also to \( \Sigma' \). Hence \( A_R \) is a topological space regarding \( \Sigma' \).

Let \( f, g \in V_\alpha ;\beta \). Then for any \( a \in \overline{U}_a \), there exist \( U_{\rho(a)} \) and \( U_\rho(a) \) such that \( a \in U_\rho(a) \), \( g(a) \in U_\rho(a) \), \( g(\overline{U_\rho(a)}) \subseteq U_{\rho(a)} \), and \( f(U_{\rho(a)}) \subseteq U_\beta \). Since \( \overline{U}_a \) is bicom pact, there exists a finite set \( \{a_1, a_2, \ldots, a_n\} \) of elements belonging to \( \overline{U}_a \) such that \( \overline{U}_a \subseteq U_{\rho(a_1)} \cup U_{\rho(a_2)} \cup \cdots \cup U_{\rho(a_n)} \). For brevity, let us denote \( \rho(a_i) \) and \( \sigma(a_i) \) by \( \rho_i \) and \( \sigma_i \) respectively. Then \( g \in V_{p_1 \ldots p_n};\beta_1 \ldots \beta_n \), \( f \in V_{\sigma_1 \ldots \sigma_n};\beta_1 \ldots \beta_n \) and \( V_{\sigma_1 \ldots \sigma_n};\beta_1 \ldots \beta_n \subseteq V_\alpha ;\beta \).

From this fact, it can readily be seen that for any \( V_{(\alpha')};(\beta') \) containing \( f \) and \( g \) there exist \( V_{(\alpha'')};(\beta'') \) containing \( f \) and \( g \) respectively and satisfying \( V_{(\alpha'')};(\beta'') \subseteq V_{(\alpha')};(\beta') \). That is, the topology of \( A_R \) introduced by \( \Sigma' \) is continuous respecting the product.

Now we shall prove the remaining part of the theorem. The mapping \((f, a) \rightarrow f(a)\) is clearly continuous for \( f \) and \( a \) in respect to the topology of \( A_R \) introduced by \( \Sigma \). Let \( \Sigma^* = \{W^*\} \) be a system of subsets of \( A_R \) such that, when we take it as a basis, \( A_R \) becomes a topological group.

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3) The author is grateful to Prof. T. Tannaka, who suggested that this topology is the weakest one of such topologies.
and the mapping \((f, a) \mapsto f(a)\) is continuous for \(f\) and \(a\). Then, in order to prove our proposition, it is sufficient to show that for any \(V_\alpha; \beta\) containing \(f\) there exists \(W^* \in \Sigma^*\) such that \(f \in W^* \subseteq V_\alpha; \beta\). For any element \(a\) of \(U_\alpha\) there exist \(W^*_a \in \Sigma^*\) and \(U_\rho(a) \in \{U_\alpha\}\) such that \(f \in W^*_a\), \(a \in U_\rho(a)\), and \(W^*_a U_\rho(a) \subseteq U_\beta\). Since \(U_\alpha\) is bicompat, there exists a finite system \([a_1, a_2, \ldots, a_n]\) such that \(U_\alpha \subseteq U_\rho(a_1) \cup U_\rho(a_2) \cup \ldots \cup U_\rho(a_n)\). Let \(W^*\) be a subset belonging to \(\Sigma^*\) and having the property \(W^* \subseteq \bigcap_{i=1}^n W^*_a\), then obviously \(f \in W^* \subseteq V_\alpha; \beta\), q.e.d.

In theorem 1, starting from a definite basis of \(R\), we have defined one topology of \(A_R\), but now it is shown that this topology does not depend on the choice of the bases, namely

Theorem 2. Let \(\{U_\alpha\}\) and \(\{U_\rho^*\}\) be any two bases. Then the two topologies of \(A_R\) introduced in the same way as above coincide with each other.

Proof. Let \(V(\alpha); (\beta)\) and \(V(\rho); (\omega)\) have the similar meaning to the previous case according to \(\{U_\alpha\}\) and \(\{U_\rho^*\}\) respectively. In order to prove the proposition it is sufficient to show that for any \(f \in A_R\) and any \(V_\alpha; \beta\) containing \(f\) there exists \(V_{(\rho)}; (\omega)\) such that \(f \in V_{(\rho)}; (\omega) \cup V_\alpha; \beta\), and conversely. If \(f \in V_\alpha; \beta\), then for any \(a \in U_\alpha\) there exists \(U^*_\rho(a)\) such that \(f(a) \in U^*_\rho(a) \subseteq U_\beta\), and for such \(U^*_\rho\) there exists \(V^*_{(\rho)}\) such that \(a \in U^*_{(\rho)}(a)\) and \(f(U^*_{(\rho)}(a)) \subseteq U^*_{(\rho)}(\omega)\). Since \(U_\alpha\) is bicompat, we can select certain finite elements \([a_1, a_2, \ldots, a_n]\) from \(U_\rho\) such that \(U_\alpha \subseteq U^*_{(\rho)}(a_1) \cup \ldots \cup U^*_{(\rho)}(a_n)\). Then obviously \(f \in V^*_{(\rho)}(a_1) \ldots \gamma(a_n) \subseteq U_\alpha; \beta\), q.e.d.

The following proposition is almost evident.

Theorem 3. If \(R\) satisfies the second axiom of countability, then the topological group \(A_R\) defined as above satisfies the same axiom.

3. In this section, we shall assume that \(R\) is a uniform space which is locally bicompat. Let \(\{U_\alpha(a)\mid a \in \Gamma, a \in R\}\) be a uniform system of neighborhoods of \(R\) satifying the following conditions:

\(n_1\) \quad a \in U_\alpha(a)
\(n_2\) \quad \bigcap_{a \in \Gamma} U_\alpha(a) = a
\(n_3\) \quad For any \(\alpha, \beta \in \Gamma\), there exists \(\gamma \in \Gamma\) such that \(U_\gamma(a) \subseteq U_\alpha(a)\)
For any \( \alpha \in \Gamma \), there exists \( \beta \in \Gamma \) such that \( a \in U_\beta(b) \) implies \( U_\beta(b) \subseteq U_\alpha(a) \).

For any \( \alpha \in \Gamma \), there exists \( \beta \in \Gamma \) such that \( b \in U_\beta(a) \) implies \( U_\beta(b) \subseteq U_\alpha(a) \).

For any \( U_\gamma(a) \) and \( b \in U_\alpha(a) \), there exists \( \beta \in \Gamma \) (which depends on \( a \) and \( b \)) such that \( c \in U_\beta(a) \) implies \( U_\beta(b) \subseteq U_\alpha(c) \).

For each bicom pact subset \( F \) of \( R \), \( \alpha \in \Gamma \), and \( f \in A_R \), we denote by \( V_{\gamma,\alpha}(f) \) the set of homeomorphisms \( g \) of \( R \) such that \( a \in F \) implies \( g(a) \in U_\alpha(f(a)) \). Set \( W_{\gamma,\alpha}(f) = V_{\gamma,\alpha}(f) \cap V_{\gamma,\alpha}(f^{-1})^{-1} \). Then we have

**Theorem 4.** If we take \( \{W_{\gamma,\alpha}(f)\} \) as a complete system of neighborhoods of \( f \), then \( A_R \) is a topological group.

**Proof.** In order to prove the proposition it is sufficient to show that if we take \( \{V_{\gamma,\alpha}(f)\} \) as a complete system of neighborhoods of \( f \) then \( A_R \) becomes a topological space and the topology of \( A_R \) is continuous respecting the product.

a) Let \( f \) and \( g \) be any two different elements of \( A_R \), then there exists \( a \in R \) such that \( f(a) \neq g(a) \). Let \( g(a) \in U_\alpha(f(a)) \). Then, if we denote by \( F \) the set consisting of the single element \( a \) we have \( g \in V_{\gamma,\alpha}(f) \).

b) For any two neighborhoods \( V_{\gamma,\alpha}(f), V_{\gamma,\beta}(f) \) of \( f \), if we take \( \gamma \) satisfying the condition \( n_3 \) for \( \alpha \) and \( \beta \), then \( V_{\gamma,\alpha}(f) \subseteq V_{\gamma,\alpha}(f) \cap V_{\gamma,\beta}(f) \).

c) Let \( g \in V_{\gamma,\alpha}(f) \). Then \( a \in F \) implies \( g(a) \in U_\alpha(f(a)) \). From the condition \( n_3 \), there exists \( \beta \in \Gamma \) such that \( g(a) \in U_\beta(f(a)) \) implies \( U_\beta(f(a)) \subseteq U_\alpha(g(a)) \). If we take \( \gamma \) satisfying \( n_3 \) for \( \beta \), then from \( c \in U_\gamma(f(a)) \) and \( d \in U_\gamma(g(a)) \) it follows that \( U_\gamma(f(a)) \subseteq U_\alpha(c) \).

Let \( U(a) \) be an open set containing \( a \) such that its closure is bicom pact and \( f(U(a)) \subseteq U_\gamma(f(a)) \), \( g(U(a)) \subseteq U_\gamma(g(a)) \). Since \( F \) is bicom pact, \( F \) may be covered by a certain finite system of \( U(a) \):

\[ F \subseteq U(a_1) \cup U(a_2) \cup \ldots \cup U(a_n) \]

Let \( \gamma \) be an element from \( \Gamma \) such that for any \( x \in R \) \( U_\gamma(x) \subseteq U_\gamma(a_i) \) (\( i = 1, 2, \ldots, n \)), and set \( F' = U(a_1) \cup \ldots \cup U(a_n) \). Then if \( h \) and \( a \) belong to \( V_{\gamma,\alpha}(f) \) and \( F \)
respectively, there exists \(U(\alpha)\) such that \(\alpha \in U(\alpha)\), and hence \(g(\alpha) \in U_7(\alpha, g'\alpha)\) and \(f(\alpha) \in U_7(\alpha, f(\alpha))\). Accordingly \(U_7(\alpha, g(\alpha)) \subset U_7(f(\alpha))\). Since \(h(\alpha) \in U_7(g(\alpha)) \subset U_7(\alpha, g(\alpha))\) we have \(h(\alpha) \in U_7(f(\alpha))\), that is, \(V_7(\gamma, g(\alpha)) \subset V_7(\alpha, f(\alpha))\).

From a), b), c), we can conclude that \(A_\mathbb{R}\) is a topological space regarding \(\{V_7, \alpha(f)\}\).

Now we shall show that for any two elements from \(A_\mathbb{R}\) and any neighborhood \(V_7(\alpha, f(g))\) of \(f(g)\) there exist \(V_7(f_1, \alpha(f))\) and \(V_7(f_2, \alpha(g))\) such that \(V_7(f_1, \alpha(f)) \subset V_7(f_2, \alpha(g))\).

Generally, it is easily verified that if \(F\) is a bicompact subset of \(R\) and \(f\) is an element of \(A_\mathbb{R}\) then for any \(\alpha\) there exists \(\beta\) such that \(f(U_7(\alpha)) \subset V_7(f(\alpha))\) for any \(\alpha \in F\).

Let \(V_7(\alpha(\alpha, f))\) be an arbitrary neighborhood of \(f\), and let \(\beta\) satisfy the condition \(n_0\) for \(\alpha\). Since \(g(F)\) is bicompact, there exists \(\gamma\) such that \(f(U_7(g(\alpha))) \subset U_7(f(\alpha))\) for any \(\alpha \in F\), and moreover \(U_7(\alpha)\) is bicompact. \(g(F)\) may be covered by a certain finite system of \(U_7(\alpha)\): \(g(F) \subset U_7(\alpha_1) \cup \ldots \cup U_7(\alpha_n)\). Denote \(U_7(\alpha_1) \cup \ldots \cup U_7(\alpha_n)\) by \(O\). Then, since \(F' = O\) is bicompact, there exists \(\gamma'\) such that for any \(x \in R\) \(U_7'(x) \subset U_7(x)\), and also \(U_7'(g(\alpha)) \subset O\) holds for any \(\alpha \in F\). Let \(f' \in V_7(\alpha, \gamma(f))\) and \(g' \in V_7(\alpha, \gamma'(g))\). Then, since for any \(\alpha \in F\) \(g'(\alpha) \in U_7(g(\alpha))\), \(g'(\alpha) \in U_7(f(\alpha))\) and \(f'(\alpha) \in U_7(f(g(\alpha)))\), hence \(f' \in \alpha(\alpha, f) \in U_7(f(\alpha))\). That is \(V_7(\alpha, \gamma(f)) \subset V_7(\alpha, f(\alpha))\), q.e.d.

For a uniform space \(R\) which is locally bicompact, two topologies may be introduced in \(A_\mathbb{R}\) from theorem I and theorem 4. But, these two topologies coincide with each other.

To prove this, let \(\{U_\alpha(\alpha)\}\) be a uniform system of neighborhoods of \(R\) and \(\{U_\alpha(\alpha)\}\) a basis of \(R\). Let further \(V_7(\alpha, f)\), \(V_7, \alpha(f)\), \(V_7(\alpha)\), \(\langle\alpha\rangle\) have the same significance as above.

Let \(f \in V_7(\alpha, \sigma)\). Then, since \(f(U_7(\alpha)) \subset U_7(\alpha)\) and \(f(U_7(\alpha))\) is bicompact, there exists \(\alpha\) such that \(\alpha \in U_7(\alpha)\) implies \(U_7(f(\alpha)) \subset U_7(\alpha)\). Hence, if we put \(F = U_7(\alpha)\), then \(V_7(\alpha, f) \subset V_7(\alpha)\). From this fact, now it is easily seen that for any \(W(\alpha)\) containing \(f\) there exists \(W(\alpha, f)\) which is contained in \(W(\alpha)\).
Conversely, let us suppose that $V_F, \varepsilon(f)$ is given arbitrarily. For $\alpha$, there exists $\beta$ satisfying the condition $\kappa \leq \alpha$, and for any $\alpha \in F$, there exists $U^*_{\alpha}(a)$ such that $f(a) \in U^*_{\alpha}(a)$ and $U^*_{\alpha}(a) \subseteq U_\beta(f(a))$, and further for such $U^*_{\alpha}(a)$ there exists $U^*_{\beta}(a)$ such that $a \in U^*_{\beta}(a)$ and $f(U^*_{\beta}(a)) \subseteq U^*_{\alpha}(a)$. Since $F$ is bicom pact, $F$ is covered by a finite set of $U^*_{\beta}(a) : F \subseteq U^*_{\beta}(a_1) \cup \ldots \cup U^*_{\beta}(a_n)$. Let $g \in V_{\beta}(a_1) \ldots \beta(a_n) ; \varepsilon(a_1) \ldots \varepsilon(a_n)$ and $a \in F$. Then $a$ is in some $U^*_{\beta}(a_i)$. Since $g(a) \in U^*_{\beta}(a_i) \subseteq U_\beta(f(a_i))$ and $f(a) \in U_\beta(f(a_i))$, we have $g(a) \in U_\beta(f(a))$, that is $g \in V_F, \varepsilon(f)$. From this fact, it can be seen that for any $W_F, \varepsilon(f)$ there exists $W_{\beta} ; \varepsilon$ such that $f \in W_{\beta} ; \varepsilon$ and $W_{\beta} ; \varepsilon \subseteq W_F, \varepsilon(f)$.

Therefore, our proposition is proved, and now we have the following

Theorem 5. Let $R$ be a uniform space which is locally bicom pact. Then the topology of $A_R$ defined in theorem 4 of course depends on the topology, but it is independent of the uniform structure of $R$. Furthermore, this topology coincides with the topology defined in theorem 1.

Finally, we shall note that, in theorem 4, if $R$ is bicom pact then by taking $\{V_F, \varepsilon(f)\}$ as a complete system of neighborhoods of $f \in A_R$ becomes a topological group and this topology coincides with that defined by $\{W_F, \varepsilon(f)\}$.

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