Supplementary Remarks on Frobeniusean Algebras II

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An algebra $A$ over a field $F$ is quasi-Frobeniusean $^1$ if and only if the dualities

$$\alpha_1) \quad l(r(I)) = I, \quad \alpha_2) \quad r(l(x)) = x$$

between left and right ideals with respect to annihilation hold, where $I$ and $x$ are left and right ideals respectively, and $l(\#)$ and $r(\#)$ denote left and right annihilators in $A$ respectively. Further $A$ is Frobeniusean if and only if besides the annihilation dualities $\alpha_1)$ and $\alpha_2)$ also the rank relations

$$\beta_1) \quad (l:F) + (r(F):F) = (A:F), \quad \beta_2) \quad (r:F) + (l(F):F) = (A:F)$$

are valid. $^2$

The notion of Frobeniusean and quasi-Frobeniusean algebras as well as these duality criteria have been extended to general rings with minimum condition. In this note we shall study some properties of a ring with the duality $\alpha_1)$ for left ideals, and show, among others, that only the duality relations $\alpha_1)$ and $\beta_1)$ or $\alpha_2)$ are sufficient for an algebra to be Frobeniusean or quasi-Frobeniusean respectively.

Let $A$ be a ring with minimum condition for left and right ideals. (We shall understand by a ring always such a ring.) Let $N$ be the radical of $A$, $A/N = \bar{A} = \bar{A}_1 + \bar{A}_2 + \ldots + \bar{A}_k$ be the direct decomposition of $\bar{A}$ into simple two-sided ideals $\bar{A}_k$ and let $f_{K_i}$, $e_{K_i}$, $e_k = e_{K_1}$, $e_{K_{i,j}}$ and $E_k = \sum_{i=1}^{K_k} e_{K_{i,j}}$ ($k = 1, \ldots, k$) have the same meaning as in S.I or Fr. I §1; namely $e_{K_i}$ ($k = 1, \ldots, k$; $i = 1, \ldots, f_{K_i}$) are mutually


$^2$) Of course $\beta_1)$, $\beta_2)$ follow respectively from $\beta_1)$, $\beta_2)$, since $l(r(I)) \supseteq I$, $r(l(x)) \supseteq x$, $r(l(r(I))) = r(I)$ and $l(r(l(x))) = l(x)$ always. The same is the case with the modified rank relation $\beta')$ in Fr II, Theorem 7; namely $\beta')$ implies $\alpha_1)$ and $\alpha_2)$, and is sufficient, by itself, to secure that a ring $A$ is Frobeniusean, provided $A$ has composition series for left and right ideals.

$$C_l(I) = C_r(A/r(I)) \quad C_r(\tau) = C_l(A/l(\tau))$$

with $C_l$ and $C_r$ denoting left and right composition lengths, are sufficient (and necessary) in order that an algebra or a ring be quasi-Frobeniusean.
orthogonal primitive idempotent elements whose sum is a principal idempotent element $E$ of $A$, whence $A = \sum_{k=1}^{K} \sum_{i=1}^{r(E)} A_{k,i} + l(E)$ (where $e_{k,i} A + r(E)$) is the direct decomposition of $A$ into directly indecomposable left (right) ideals $A_{k,i}$ and $l(E)$ ($r(E)$); here $A_{k,i}$ $(e_{k,i} A)$ and $A_{j,i}$ $(e_{j,i} A)$ are operator isomorphic if and only if $\kappa = \lambda$. Then $A = \sum_{k=1}^{K} \sum_{i=1}^{r(E)} A_{k,i}$ equals $\sum_{k=1}^{K} \sum_{i=1}^{r(E)} A_{k,i}$ for each $\kappa, i$, and $C_{k,i} = \delta_{k,i} e_{k,i} C_{k,i}$ mod. $N$ is, for each $\kappa$, the unit element of simple ring $A_{\kappa}$.

Theorem 1. Let $A$ be a ring with the dualily $\alpha_i$. Then $A$ possesses a unit element, and there exists a permutation $\pi$ of $(1, 2, \ldots, k)$ such that for each $\kappa$,

(i) The largest completely reducible right subideal $r_{\kappa}$ of $e_{\kappa} A$ is a direct sum of simple right ideals isomorphic to $e_{\pi(\kappa)} A/e_{\pi(\kappa)} N$, and

(ii) $A e_{\pi(\kappa)}$ has a unique simple left subideal $l_{\pi(\kappa)}$ and $l_{\pi(\kappa)}$ is isomorphic to $A e_{\pi(\kappa)} / N e_{\pi(\kappa)}$.

Proof.

We proceed stepwise:

1) By the duality $\alpha_i$ we have $l(r(0)) = 0 = l(A)$.

2) We denote $l(N)$ by $R$. Then $r(Re_{\pi(\kappa)}) \supseteq N \cup (E - e_{\pi(\kappa)}) A$. Here the right hand side is a maximal right ideal of $A$, whence $r(Re_{\pi(\kappa)})$ coincides with either $N \cup (E - e_{\pi(\kappa)}) A$ or $A$. If $r(Re_{\pi(\kappa)}) = A$, then, $Re_{\pi(\kappa)} = l(A) = 0$, by the duality. If $r(Re_{\pi(\kappa)}) = N \cup (E - e_{\pi(\kappa)}) A$ and $Re_{\pi(\kappa)}$ contains a proper left subideal $l$ then $r(l) \supseteq r(Re_{\pi(\kappa)})$ by the duality, and so $r(l) = A$, $l = l(A)$ = 0. Hence $Re_{\pi(\kappa)}$ is either a simple left ideal or zero.

3) $R = Re_{\pi(\kappa)} + (R \cap l(E))$ and $A = EA + r(E)$. Here $l(E)$ and $r(E)$ are contained in $N$; hence $(R \cap l(E)) A = (R \cap l(E)) (EA + r(E)) = 0$. Therefore, by 1) $R \cap l(E) = 0$, whence $R = Re_{\pi(\kappa)}$.

4) $R = Re_{\pi(\kappa)} = \sum_{\kappa=1}^{K} Re_{\kappa,i}$, and $Re_{\kappa,i}$ is either a simple left ideal or zero. So $NR = 0$, namely $R \subseteq r(N)$. We denote $r(N)$ by $L$.

5) $RE_{\kappa}$ is a two-sided ideal. For, $A = \sum_{\lambda} E_{\lambda} A E_{\lambda} \cup N$, whence $RE_{\kappa} A = RE_{\kappa} (\sum_{\lambda} E_{\lambda} A E_{\lambda} \cup N) \subseteq RE_{\kappa}$. Moreover $r(RE_{\kappa}) \supseteq N \cup (E - E_{\kappa}) A$, and here the right hand side is a maximal two-sided ideal. Hence $RE_{\kappa}$ is either a simple two-sided ideal or zero, as we see in a similar way as in 2).

6) $E_{\kappa} R$ is a (not only right, but) two-sided ideal. For, $A = \sum_{\lambda} E_{\lambda} A E_{\lambda} \cup N$, $R \subseteq L$, and so $AE_{\kappa} R = (\sum_{\lambda} E_{\lambda} A E_{\lambda} \cup N) E_{\kappa} R \subseteq E_{\kappa} R$. $E_{\kappa} R$ is a non-zero ideal for each $\kappa$, since $e_{\kappa,i} R$ is the largest com-
pletely reducible right subideal of \( e_{\kappa, i} A \).

7) \( R = R E = \sum \kappa RE_\kappa = ER + (R \cap r(E)) = \sum \kappa E_\kappa R + (R \cap r(E)). \)

\( R \cap r(E) \) is a (not only right, but) two-sided ideal, because \( A = EAE \cup N, R \subseteq L \) and so \( A (R \cap r(E)) = (EAE \cup N) (R \cap r(E)) = 0. \) Hence the extreme right member of the above equation is a direct decomposition into \( k \) non-zero two-sided ideals \( E_\kappa R \) and \( R \cap r(E) \), while the third is a direct decomposition into at most \( k \) simple two-sided ideals \( RE_\kappa \). This shows that \( E_\kappa R \) is a simple two-sided ideal, \( R \cap r(E) = 0 \) and \( RE_\kappa \) is a non-zero two-sided ideal, for each \( \kappa \).

8) \( R \cap r(E) = 0 \) implies \( r(E) = 0. \) Hence \( A = EA + r(E) = EA \) and \( l(E) = l(EA) = l(A) = 0 \) according to 1). So \( A = AE + l(E) = AE \), and \( E \) is a unit element of \( A \).

9) \( E_\kappa L \) is a (not only right, but) two-sided ideal. For, \( A = \sum \kappa E_\kappa AE_\kappa \cup N \) whence \( AE_\kappa L = (\sum \kappa E_\kappa AE_\kappa \cup N) E_\kappa L \subseteq E_\kappa L \). \( l(E_\kappa L) \supseteq N \cup A (1-E_\kappa) \). Since the right hand side is a maximal two-sided ideal and \( l(E_\kappa L) \) is also a two-sided ideal, \( l(E_\kappa L) \) is equal either to \( N \cup A (1-E_\kappa) \) or to \( A \). If \( l(E_\kappa L) = A \), then \( r(A) = r(l(E_\kappa L)) \supseteq E_\kappa L \). But the left hand side is zero, since \( A \) possesses unit element, and so \( E_\kappa L \) must vanish too. If \( l(E_\kappa L) = N \cup A (1-E_\kappa) \), however, then \( r(l(E_\kappa L)) = r(N \cup A (1-E_\kappa)) = E_\kappa L. \) Now we consider a composition series

\[ \begin{align*}
A \supseteq N \cup A (1-E_\kappa) \supsetneq 3_1 \supsetneq 3_2 \supsetneq \ldots \supseteq 0
\end{align*} \]

of two-sided ideals of \( A \). If a left ideal \( I' \) is a proper subideal of a left ideal \( I \), then \( r(I) \) is a proper subideal of \( r(I') \) by the duality for left ideals. Hence the series

\[ r(A) = 0 \subsetneq r(N \cup A (1-E_\kappa)) \subsetneq r(3_1) \subsetneq r(3_2) \subsetneq \ldots \subsetneq r(0) = A \]

has the same length as the above series, and, therefore this is also a composition series of two-sided ideals of \( A \), and \( r(N \cup A (1-E_\kappa)) = E_\kappa L \) is a simple two-sided ideal. Thus \( E_\kappa L \) is either a simple two-sided ideal or zero.

Therefore \( E_\kappa L \cdot N = 0 \) for every \( \kappa \), and \( L \) is contained in \( R \), which gives, when combined with 4), \( L = R \). We denote this by \( M \).

10) According to 7) we have two direct decompositions of \( M \) into simple two-sided ideals:

\[ M = \sum \kappa ME_\kappa = \sum \kappa E_\kappa M \]

There exists then a permutation \( \pi \) of \( (1, 2, \ldots, k) \) such that \( ME_\pi(\kappa) = E_\kappa M \) for each \( \kappa \). This shows that \( ME_{\pi(\kappa)} \), which is by 2) and \( R = \)
\[ L = M \] the unique simple left subideal of \( Ae_{\pi(Ki)} \), is isomorphic to \( Ae_{\pi(Ki)} Ne_{K} \), while the largest completely reducible right subideal \( e_{K} M \) of \( e_{K} A \) is a direct sum of simple subideals isomorphic to \( e_{\pi(K)} A / e_{\pi(K)} N \). This completes our proof.

In the case of algebras, we have, on combining this theorem and Theorem 1 of SI, the following

**Theorem 2.** An algebra \( A \), with a finite rank over a field \( F \), is quasi-Frobeniusan if (and only if) \( l(r(I)) = 1 \) for every left ideal \( I \) in \( A \). A is further Frobeniusan if (and only if) besides \( \alpha_{i} \)

\[ \beta_{i} (I : F) + (r(I) : F) = (A : F) \]

is valid for every left ideal \( I \) in \( A \). \(^3\)

The assertion of Theorem 2 is not valid for rings (with minimum condition) in general. \(^4\) But a ring \( A \) with the duality \( \alpha_{i} \) possesses a unit element, and satisfies so the maximum condition also; hence \( A \) possesses composition series of either right ideals and left ideals.

**Theorem 3.** Let \( A \) be a ring (with minimum condition) with the duality \( \alpha_{i} \) for left ideals. \( A \) is quasi-Frobeniusan if (and only if) \( A \) has the same composition lengths for right and left ideals. Further \( A \) is Frobeniusan if (and only if)

\[ f_{(K)} = f_{(\pi(Ki))} \]

for each \( \kappa \) besides the above condition.

**Proof.** Assume that \( A \) has the same composition lengths for right ideals and left ideals. By Theorem 1 \( Me_{K} \) is a simple left ideal and \( e_{K} M \) is a direct sum of simple subideal isomorphic to \( e_{\pi(K)} A / e_{\pi(K)} N \). The left annihilator of \( e_{K} M \) contains a maximal left ideal \( N \cup A (1 - e_{K}) \), and is so equal either to \( A \) or to \( N \cup A (1 - e_{K}) \). But if \( l(e_{K} M) = A \) then \( r(A) = r(l(e_{K} M)) \supseteq e_{K} M \), which gives a contradiction, since \( A \) has unit element, \( r(A) = 0 \), while \( e_{K} M = 0 \). Therefore \( l(e_{K} M) = N \cup A (1 - e_{K}) \) and \( r(l(e_{K} M)) = r(N \cup A (1 - e_{K})) = e_{K} M \).

Consider now a composition series:

\[ A \supset N \cup A (1 - e_{K}) \supset I_{1} \supset I_{2} \supset \ldots \supset 0 \]

of left ideals. By the duality for left ideals, the series

\[ r(A) = 0 \subset r(N \cup A (1 - e_{K})) = e_{K} M \subset r(I_{1}) \subset r(I_{2}) \subset \ldots \subset r(0) = A \]

has the same length as the above series. Then by our assumption this series must be a composition series of right ideals of \( A \). Therefore \( e_{K} M \) is a simple right ideal. This shows that \( A \) is quasi-Frobeniusan.

\(^3\) The last assertion is a rather immediate consequence of the first. For \( \alpha_{1}, \alpha_{2} \) and \( \beta_{1} \) together imply \( \beta_{2} \) as we see readily.

\(^4\) See SI.
In our above proof of Theorem 1 we used only \( l(A) = l(r(0)) = 0 \) and that \( l \supseteq l_0 \), with left ideals \( l \) and \( l_0 \), implies \( r(l)(r(l_0)) \). So we may restate our theorems in the following forms refined in this sense:

**Theorem 4.** If \( A \) is a ring in which \( l \supseteq l_0 \), with left ideals \( l \) and \( l_0 \), then the duality \( \alpha_l \) holds for every left ideal \( l \) in \( A \), and conversely.

**Proof.** Let \( l \) be a left ideal of \( A \). If the above condition is satisfied and \( r(l(r(l))) = r(l) \), then \( r(l)(r(l_0)) = r(l_0)(r(l)) \), which is absurd.

**Corollary.** Let \( A \) be a ring in which \( l \supseteq l_0 \), \( r \supseteq r_0 \) imply \( r(l) \subseteq r(l_0) \), \( l(r) \subseteq l(r_0) \) respectively, with left ideals \( l, l_0 \) and right ideals \( r, r_0 \).

Then \( A \) is quasi-Frobeniusean. 5)

We have further

**Theorem 5.** A ring \( A \) satisfies the duality \( \alpha_l \) for left ideals if (and only if) the duality \( \alpha_l \) holds for every nilpotent left ideal. 6)

**Proof.** Assume that the duality \( \alpha_l \) holds for every nilpotent left ideal in \( A \). Let \( l \) be a left ideal generated by an idempotent \( e \); \( l = Ae \). Then \( r(l) = r(e) \), and \( r(e) \) is the set of elements \( a - ea(a \in A) \). If \( c \) is an element of \( l(r(l)) \), then \( c(a - ea) = (c - ce)a = 0 \) for all elements \( a \) of \( A \). But \( l(A) = l(r(0)) = 0 \) by our assumption. Hence \( e - ce = 0 \) and, \( c \in Ae = l. \) This shows \( l(r(l)) \), and thus the duality \( \alpha_l \) holds for \( l \).

As in our proof of Theorem 1, \( r(Re_{K,l}) \) is equal either to \( A \) or to \( N \bigcup (E-e_{K,l}) A \). If \( r(Re_{K,l}) = A \), then \( l(A) = 0 = l(r(Re_{K,l})) \supseteq Re_{K,l} \), hence \( Re_{K,l} \) is zero. If \( r(Re_{K,l}) = N \bigcup (E-e_{K,l}) A \) and \( Re_{K,l} \) contains a simple left ideal \( l_1 \), then \( r(l_1) \supseteq r(Re_{K,l}) \), \( N \bigcup (E-e_{K,l}) A \) and, therefore, \( r(l_1) = A \), \( l_1 = 0 \). This shows that \( Re_{K,l} \) is either a simple left ideal or zero. And 3), 4), 5), 6), 7), 8), all remain valid under the present weaker assumption. Thus \( A \) possesses, in particular, a unit element. Let \( I_2 \) be a left ideal. We may express it as \( I_2 = Ae_2 + I_2' \), where \( e_2 \) is an idempotent element and \( I_2' \) is contained in \( N \). \( r(I_2) = r(Ae_2) \bigcap r(I_2') = (1-e_2) A \bigcap r(I_2') = (1-e_2) r(I_2') \), and \( l(r(I_2)) \supseteq l_2 = Ae_2 + I_2' \). Let \( z \) be an element of \( l(r(I_2)) \). Then \( z = ze_2 + z(1-e_2) \) and \( z \cdot r(I_2) = z(1-e_2) r(I_2') = 0 \), whence \( z(1-e_2) \in l(r(I_2')) \). This shows \( l(r(I_2)) = Ae_2 \bigcup l(r(I_2')) \). Since \( I_2' \) is a nilpotent left ideal, \( l(r(I_2')) = I_2' \). Thus \( l(r(I_2)) = Ae_2 + I_2' = I_2 \).

5) Also in our previous criteria of Frob. and quasi-Frob. rings we did not assume the annihilation dualities \( \alpha_{l_1} \) and \( \alpha_{e_2} \) fully. See Fr. I, Theorem 3 (and Theorem 7) and Fr II, Theorem 6 (and Theorems 7, 10).

6) See foot note 5).

7) The duality \( \alpha_l \) holds for our \( l_1 \). For, \( l_1 \) is either nilpotent or idempotent. And, if \( l_1 \) is nilpotent, then the duality for \( l_1 \) is valid by our assumption. If \( l_1 \) is idempotent, then it is generated by an idempotent element and the duality holds for \( l_1 \).
Next we have

**Lemma.** A ring \( A \) is directly decomposable into a bound ring and a semi-simple ring; here a ring is called a bound ring if \( R \cap L \subseteq N \).\(^8\)

Now we can refine Theorem 2 as follows:

**Theorem 6.** An algebra \( A \), with a finite rank over a field \( F \), is quasi-Frobeniusean if (and only if) the duality \( \alpha_i \) holds for every nilpotent left ideal. \( A \) is further Frobeniusean if (and only if) besides \( \alpha_i \) the rank relation \( \beta_i \) is valid for every nilpotent simple left ideal.\(^9\)

**Proof.** The first half is an immediate consequence of Theorems 2 and 5. To prove the second, we observe first that \( A = A_1 + A_2 \) is Frobeniusean if and only if subrings \( A_1 \) and \( A_2 \) are so, and that if the conditions of the second part are satisfied in \( A \), then they are also satisfied in \( A_1 \) and \( A_2 \). So we may, by virtue of the above lemma, restrict ourselves to the case of a bound algebra \( A \), satisfying our conditions.

\( A \) is quasi-Frobeniusean, any how, and so \( Me_{\pi(K)} \) is a simple left ideal isomorphic to \( Ae_K/Ne_{\pi(K)} \). Hence \( (Me_{\pi(K)} : F) = (Ae_K/Ne_{\pi(K)} : F) = f_{(\pi(K))}(e_K Ae_{\pi(K)}/e_K Ne_{\pi(K)} : F) \). Since \( A \) is a bound algebra, \( Me_{\pi(K)} \) is a nilpotent simple left ideal. So

\[
(Me_{\pi(K)} : F) = (A : F) - r(Me_{\pi(K)} : F) = (A : F) - (N \cap (1 - e_{\pi(K)}) A : F)
\]

\[
= (e_{\pi(K)} A/ e_{\pi(K)} N : F) = f_{(\pi(K))}((e_{\pi(K)} A e_{\pi(K)}/ e_{\pi(K)} Ne_{\pi(K)} : F))
\]

But \( e_K Ae_{\pi(K)}/e_K Ne_{\pi(K)} \approx e_{\pi(K)} A e_{\pi(K)}/e_{\pi(K)} Ne_{\pi(K)} \) as was shown in the proof of Theorem 3 in Fr. I, hence \( f_{(\pi(K))} = f_{(\pi(K))} \) is valid, for each \( \kappa \). This shows that \( A \) is Frobeniusean.

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9) See footnote 5,