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CLOSED HYPERSURFACES
WITH CONSTANT MEAN CURVATURE
IN A SYMMETRIC MANIFOLD

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Abstract
We prove a rigidity theorem for closed hypersurfaces with constant mean curvature in a symmetric Riemannian manifold, which is a generalization of main results in [3] and [15].

1. Introduction
It seems interesting to generalize the famous optimal rigidity theorem for minimal hypersurfaces in a sphere due to J. Simons, H.B. Lawson Jr., and S.S. Chern, M. do Carmo and S. Kobayashi to general cases (see [4], [8], [12]). Q.M. Cheng and H. Nakagawa [3], and H.W. Xu [15] proved the following optimal rigidity theorem for hypersurfaces of constant mean curvature in a sphere independently.

Theorem A ([3], [15]). Let $M^n$ be an $n$-dimensional closed hypersurface with constant mean curvature $H$ in a unit sphere $S^{n+1}$. If the squared norm of the second fundamental form $S$ satisfies

$$S \leq \alpha(n, H),$$

then $M$ is congruent to one of the following

1. totally umbilic sphere $S^n(1/\sqrt{1 + H^2})$;
2. one of the Clifford minimal hypersurface $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$ in $S^{n+1}(1)$, for $k = 1, 2, \ldots, n-1$;
3. the isoparametric hypersurface $S^{n-1}(1/\sqrt{1 + \lambda^2}) \times S^1(\lambda/\sqrt{1 + \lambda^2})$ in $S^{n+1}(1)$.

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Here $\lambda$ and $\alpha(n, H)$ are given by

$$\lambda = \frac{nH + \sqrt{n^2H^2 + 4(n - 1)}}{2(n - 1)}$$

and

$$\alpha(n, H) = n + \frac{n^3}{2(n - 1)}H^2 - \frac{n(n - 2)}{2(n - 1)}\sqrt{n^2H^2 + 4(n - 1)H^2}.$$

Motivated by Theorem A and a theorem due to G. Huisken [7], B. Andrews [2] proposed a following conjecture on mean curvature flow for closed hypersurfaces in a unit sphere.

**Conjecture.** Let $M_0 = F_0(M)$ be a closed hypersurface in $S^{n+1}$ which satisfies

\[ S < \alpha(n, H). \]

Then there exists a smooth family of hypersurfaces $\{M_t = F_t(M)\}_{0 \leq t < T}$ which satisfy (1.1) and move by mean curvature flow with initial data $M_0$. Either $T < \infty$ and $M_t$ is asymptotic to a family of geodesic spheres shrinking to their common centre, or $T = \infty$ and $M_t$ approaches to a great sphere.

The topological sphere theorem due to K. Shiohama and H.W. Xu [11] says that any closed hypersurface in $S^{n+1}$ which satisfies $S < \alpha(n, H)$ must be a topological sphere, which provides an positive evidence to the conjecture above. In this paper, we generalize Theorem A as follows.

**Main Theorem.** Let $N^{n+1}$ be an $(n + 1)$-dimensional simply connected symmetric Riemannian manifold with $\delta$ pinched curvature, i.e., $\delta \leq K_N \leq 1$, and $M^n$ be a closed hypersurface with constant mean curvature $H$ in $N^{n+1}$. If

\[ (S - nH^2)[\alpha(n, H) - S - 2n(1 - \delta)] - \frac{1}{2}(1 - \delta)n^{3/2}H\sqrt{S - nH^2} \geq 0, \]

then $M$ is congruent to one of the following

1. totally umbilical hypersurface;
2. one of the Clifford minimal hypersurface $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n - k)/n})$ in $S^{n+1}(1)$, for $k = 1, 2, \ldots, n - 1$;
3. the isoparametric hypersurface $S^{n-1}(1/\sqrt{1 + \lambda^2}) \times S^1(\lambda/\sqrt{1 + \lambda^2})$ in $S^{n+1}(1)$.

Here $\alpha(n, H), \lambda$ are defined as in Theorem A.
Consequently we have

**Corollary.** Let $M^n$ be an $n$-dimensional closed minimal hypersurface in $N^{n+1}$ with curvature $K_N$ satisfying $\delta \leq K_N \leq 1$. If the squared norm of the second fundamental form $S$ satisfies

$$S \leq (2\delta - 1)n,$$

then $M$ is congruent to one of the following

1. totally geodesic submanifold;
2. one of the Clifford minimal hypersurface $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$ in $S^{n+1}(1)$, for $k = 1, 2, \ldots, n - 1$.

It should be mentioned that when $M^n$ is a minimal hypersurface in $N^{n+1}$, then our pinching condition reduces to $S \leq (2\delta - 1)n$, which is weaker than the one in [5], [10] and [14].

Motivated by the main theorem, one can propose an analogue of the conjecture above for closed hypersurfaces in a symmetric Riemannian manifold with $\delta$ pinched curvature.

## 2. Preliminaries

Throughout this paper, let $M^n$ be an $n$-dimensional closed hypersurface isometrically immersed in an $(n+1)$-dimensional simply connected symmetric Riemannian manifold $N^{n+1}$. The following convention of indices are used throughout.

$$1 \leq i, j, k, \ldots, \leq n,$$

$$1 \leq A, B, C, \ldots, \leq n + 1.$$

Choose an orthonormal frame field $\{e_A\}$ in a neighborhood of $p \in M$ such that the $\{e_i\}$ span the tangent space $T_pM$ to $M$ at $p$. Let $\{\omega_A\}$ be the dual frame fields of $\{e_A\}$ and $\{\omega_{AB}\}$ be the connection 1-forms of $N$. Restricting these forms to $M$, we have

$$\omega_{n+1i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$

The curvature tensors of $N, M$ are denoted by $K_{ABCD}, R_{ijkl}$ respectively. The second fundamental form of $M$ is denoted by $h$ and the mean curvature normal field by $\xi$. Denote the mean curvature of $M$ and squared norm of $h$ by $H = ||\xi||$ and $S$ respectively. We have then

$$h = \sum_{i,j} h_{ij} \omega^i \otimes \omega^j \otimes e_{n+1},$$

(2.1)
\[
\xi = \frac{1}{n} \sum_{i=1}^{n} h_{ij} e_{n+1},
\]

(2.2)

\[
R_{ijkl} = K_{ijkl} + h_{ik} h_{jl} - h_{il} h_{jk},
\]

(2.3)

\[
S = \sum_{i,j} (h_{ij})^2.
\]

(2.4)

**Definition 2.1.** \(M\) is called a hypersurface with constant mean curvature if \(H\) is constant. In particular, \(M\) is called minimal hypersurface if \(H = 0\).

We denote the first and second covariant derivatives of \(h_{ij}\) by \(h_{ijk}\) and \(h_{ijkl}\) respectively, which are defined as in [4]. Following to [4] and [16], we have

\[
h_{ijk} - h_{ikj} = -K_{n+1ijk},
\]

(2.5)

and the Ricci formula

\[
h_{ijkl} - h_{ijlk} = \sum_s h_{ij} R_{skkl} + \sum_s h_{ik} R_{sjkl}.
\]

(2.6)

Let \(K_{n+1ijk}\) be the covariant derivative of \(K_{n+1ijk}\) as the section of \(T^\perp M \otimes T^* M \otimes T^* M \otimes T^* M\) and \(K_{ABCD;E}\) the covariant derivative of \(K_{ABCD}\) as curvature tensor of \(N\). Restricted to \(M\) we have

\[
\sum_l K_{n+1ijk;ol} = dK_{n+1ijk} + \sum_s K_{n+1lsjk;ol} + \sum_s K_{n+1lsjk;olks},
\]

(2.7)

and

\[
K_{n+1ijk,l} = K_{n+1ijk} - K_{n+1imn+1k} h_{jl} - K_{n+1inj+1} h_{kl} + \sum_m K_{mihk} h_{ml}.
\]

(2.8)

**Definition 2.2.** \(N\) is called a symmetric Riemannian manifold if for every \(p \in N\) there exists an isometric \(\sigma_p: N \to N\) such that \(\sigma_p(p) = p\), and the differential of \(\sigma_p\) at \(p\) is equal to \(-I_p\), where \(I_p\) is the identity transformation of \(T_p N\). The Laplacian of the second fundamental form is defined by \(\Delta h_{ij} = \sum_k h_{ikkk}\).

The following propositions will be used in the proof of Main Theorem.

**Proposition 2.3** ([3], [15]). If \(a_1, \ldots, a_n\) are \(n\) real numbers with \(\sum_{i=1}^{n} a_i = 0\), then

\[
\left| \sum_{i=1}^{n} a_i^3 \right| \leq (n - 2)(n(n - 1))^{-1/2} \left( \sum_{i=1}^{n} a_i^2 \right)^{3/2}.
\]

Moreover, the equality holds if and only if at least \(n - 1\) numbers of \(a_i\)'s are equal.
Proposition 2.4. If the function \( f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i \) satisfies
\[
\sum_{i=1}^{n} x_i = 0, \quad \sum_{i=1}^{n} x_i^2 = \Lambda, \quad \delta \leq y_i \leq 1.
\]
Then
\[
f(x_1, \ldots, x_n, y_1, \ldots, y_n) \geq \frac{1}{2}(\delta - 1)(n\Lambda)^{1/2}.
\]

Proof. We assume
\[
x_1 \leq x_2 \leq \cdots \leq x_k \leq 0 \leq x_{k+1} \leq \cdots \leq x_n.
\]
Thus
\[
f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{k} x_i y_i + \sum_{i=k+1}^{n} x_i y_i
\]
\[
\geq \sum_{i=1}^{k} x_i + \delta \sum_{i=k+1}^{n} x_i
\]
\[
= (\delta - 1) \sum_{i=k+1}^{n} x_i.
\]
By (2.9) we have
\[
k\Lambda = k \sum_{i=1}^{k} x_i^2 + k \sum_{i=k+1}^{n} x_i^2
\]
\[
\geq \left( \sum_{i=1}^{k} x_i \right)^2 + k \left( \sum_{i=k+1}^{n} x_i \right)^2
\]
\[
= \frac{n}{n-k} \left( \sum_{i=k+1}^{n} x_i \right)^2.
\]
So by (2.11) we have
\[
\left( \sum_{i=k+1}^{n} x_i \right)^2 \leq \frac{k(n-k)}{n-\Lambda} \leq \frac{n}{4} \Lambda.
\]
Thus
\[
f(x_1, \ldots, x_n, y_1, \ldots, y_n) \geq (\delta - 1) \sum_{i=k+1}^{n} x_i \geq \frac{1}{2}(\delta - 1)(n\Lambda)^{1/2}.
\]
This proves Proposition 2.4.

From (2.5) and (2.6),

\[
\Delta h_{ij} = - \sum_{k} (K_{n+1}^{klik} + K_{n+1}^{lijkkk}) + \sum_{k,m} (h_{mk} R_{mijk} + h_{im} R_{mkjk}).
\]

Since $N$ is a symmetric manifold, $N$ is complete and locally symmetric. Thus

\[
K_{ABCD,E} = 0
\]

for all $A, B, C, D, E$. This together with (2.3), (2.8) and (2.12) implies

\[
\frac{1}{2} \Delta S = \sum_{i,j,k} (h_{ijk})^2 + \sum_{i,j} h_{ij} \Delta h_{ij} = X + Y + Z,
\]

where

\[
X = n H \text{tr} H_{n+1}^3 - (\text{tr} H_{n+1}^2)^2,
\]

\[
Y = 2 \sum_{m,k,i,j} (h_{mj} h_{ij} K_{mkik} + h_{mk} h_{ij} K_{miij}) + \sum_{i,j,k} (h_{ijk})^2,
\]

\[
Z = - \sum_{i,j,k} (K_{n+1}^{k+1n+1k} h_{ij} h_{ij} + K_{n+1}^{k+1n+1k} h_{ij} h_{ij} + K_{n+1}^{k+1n+1k} h_{ij} h_{ij} + K_{n+1}^{k+1n+1k} h_{ij} h_{ij}).
\]

3. Proof of Main Theorem

The following lemmas are useful in the proof of the main theorem.

**Lemma 3.1.** $X \geq (S - n H^2)[2n H^2 - S - (n(n - 2)/\sqrt{n(n - 1)}) H(S - n H^2)^{1/2}]$.

Proof. Let $\{e_i\}$ be an orthonormal frame at a point on $M$ such that the matrix $H_{n+1} = (h_{ij})_{n \times n}$ takes the diagonal form and such that $h_{ij} = \lambda_i \delta_{ij}$ for all $i, j$. Set

\[
f_k = \sum_{i=1}^{n} (\lambda_i)^k,
\]

\[
B_k = \sum_{i=1}^{n} (\mu_i)^k,
\]

\[
\mu_i = H - \lambda_i.
\]

Then we have

\[
B_1 = 0, \quad B_2 = S - n H^2,
\]

and

\[
B_3 = 3 HS - 2n H^3 - f_3.
\]
From (3.1), (3.2) and Proposition 2.3, we get

\[
X = nHf_1 - S^2
\geq nH \left[ 3HS - 2nH^3 - \frac{n - 2}{\sqrt{n(n-1)}} B_2^{3/2} \right] - S^2
\geq (S - nH^2) \left[ 2nH^2 - S - \frac{n(n - 2)}{\sqrt{n(n-1)}} H(S - nH^2)^{1/2} \right].
\]

This proves Lemma 3.1.

**Lemma 3.2.** \( Y \geq 2\delta n(S - nH^2). \)

**Proof.** It follows that

\[
Y = 2 \sum_{i,k} \left( K_{i\bar{k}k}(h_{ii})^2 + K_{i\bar{k}k}h_{kk}h_{ii} \right)
= \sum_{i,k} K_{i\bar{k}k}(\lambda_i - \lambda_k)^2
\geq \delta \sum_{i,k} (\lambda_i - \lambda_k)^2
= 2\delta (S - nH^2).
\]

This proves Lemma 3.2.

**Lemma 3.3.** \( Z \geq -n(S - nH^2) - (1/2)(1 - \delta)n^{3/2}H \sqrt{S - nH^2}. \)

**Proof.**

\[
Z = -\sum_{k,i} K_{n+1\bar{k}n+1k}(\lambda_i)^2 + \sum_{k,i} K_{n+1\bar{k}n+1k}\lambda_k \lambda_i
\geq -nS + nH \sum_k K_{n+1\bar{k}n+1k}\lambda_k
= -n(S - nH^2) + nH \sum_k K_{n+1\bar{k}n+1k}\mu_k,
\]

where we set \( \mu_k = \lambda_k - H \). Since \( \sum_k \mu_k = 0 \), \( \sum_k \mu_k^2 = S - nH^2 \) and \( \delta \leq K_{n+1\bar{k}n+1k} \leq 1 \), by Proposition 2.4, we have

\[
Z \geq -n(S - nH^2) - \frac{1}{2}(1 - \delta)n^{3/2}H \sqrt{S - nH^2}.
\]

This proves Lemma 3.3.
Proof of Main Theorem. If $S = nH^2$, then $\lambda_i = H$ for $i = 1, 2, \ldots, n$, which means that $M$ is a totally umbilic submanifold.

If $S \neq nH^2$, then $S > nH^2$. By the assumption that

$$(S - nH^2)[\alpha(n, H) - S - 2n(1 - \delta)] - \frac{1}{2}(1 - \delta)n^{3/2}H\sqrt{S - nH^2} \geq 0,$$

we get

$$S \leq \alpha(n, H) - 2n(1 - \delta) \leq \alpha(n, H).$$

Combining this with Lemmas 3.1, 3.2 and 3.3, we have

$$\frac{1}{2} \Delta S \geq (S - nH^2) \left[ -n + 2n\delta + 2nH^2 - S - \frac{n(n - 2)}{\sqrt{n(n - 1)}}H(S - nH^2)^{1/2} \right]$$

$$- \frac{1}{2}(1 - \delta)n^{3/2}H\sqrt{S - nH^2}$$

$$\geq (S - nH^2) \left[ n + 2nH^2 - \alpha(n, H) - \frac{n(n - 2)}{\sqrt{n(n - 1)}}H(\alpha(n, H) - nH^2)^{1/2} \right]$$

$$+ \alpha(n, H) - S - 2n(1 - \delta) - \frac{1}{2}(1 - \delta)n^{3/2}H\sqrt{S - nH^2}$$

$$= (S - nH^2)[\alpha(n, H) - S - 2n(1 - \delta)] - \frac{1}{2}(1 - \delta)n^{3/2}H\sqrt{S - nH^2}.$$

By the assumption and Hopf’s maximum principle, we see that $S$ must be a constant. This implies that the inequalities in (3.3) and (3.4) become equalities. Since $S > nH^2 \geq 0$, it follows from (3.3) that $K_{n+1|n+1k} = 1$ for all $k = 1, \ldots, n$. On the other hand, it follows from $S > nH^2$ and $\sum_{k} \mu_k = 0$ that there exist $k$ and $l$ such that $\mu_k < 0$ and $\mu_l > 0$, where $1 \leq k < l \leq n$. By (3.4) and Proposition 2.4, we have $K_{n+1|n+1l} = \delta$ for some $l$. Therefore

$$\delta = 1 \quad \text{and} \quad (S - nH^2)[S - \alpha(n, H)] = 0,$$

which implies that $S = \alpha(n, H)$ and $N$ is isometric to a unit sphere. It follows from Theorem A that $M$ must be congruent to either

(i) one of the Clifford minimal hypersurfaces $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$ in $S^{n+1}(1)$, for $k = 1, 2, \ldots, n - 1$; or

(ii) the isoparametric hypersurface $S^{n-1}(1/\sqrt{1+\lambda^2}) \times S^1(\lambda/\sqrt{1+\lambda^2})$ in $S^{n+1}(1)$, where $\lambda$ is given by

$$\lambda = \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2(n-1)}.$$

This proves the theorem.
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