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ON $\pi$-SOLVABLE GROUPS WHOSE CHARACTER DEGREES ARE $\pi$-NUMBERS

TAKASHI YAO

(Received June 8, 1976)

1. Introduction. Let $\pi$ be a set of primes, and let $n=p_1 \cdot p_2 \cdots p_t$ be a positive integer, where the $p_i$ are (not necessarily distinct) primes. Then we say that the total exponent (shortly T-exponent) of $n$ is $t$ and write $e(n)=t$. If $p_i \in \pi$ for $i=1, 2, \cdots, t$ with the above notation, then $n$ is said to be a $\pi$-number.

Let $Irr(G)$ be the set of irreducible complex characters of a group $G$. We say that a group $G$ has c.d.$\pi$ (character degrees $\pi$) if $\chi(1)$ is a $\pi$-number for any $\chi \in Irr(G)$, a group $G$ has r.x.e (representation exponent $e$) if $e(\chi(1)) \leq e$ for any $\chi \in Irr(G)$, and $G$ has r.x.e for $\pi$ (representation exponent $e$ for $\pi$) if $G$ has c.d.$\pi$ and r.x.e.

In this paper we shall prove the following theorems.

Theorem I. Let $G$ have r.x.e for $\pi$. Suppose $G$ is $\pi$-solvable when $|\pi| \geq 3$. Then $G$ has a normal series

$$ G = A_t \triangleright B_{t-1} \triangleright A_{t-1} \triangleright \cdots \triangleright B_0 \triangleright A_0 $$

and there exists some prime $p_i \in \pi$ for any $i$ such that

1. $A_i$ has r.x.i for $\pi$,
2. $A_i/B_{i-1}$ is a cyclic $\pi_i$-group, where $\pi_i = \pi - \{p_i\}$,
3. $B_{i-1}/A_{i-1}$ is an elementary abelian $p_i$-group, and
4. $|A_i/A_{i-1}|$ is a $\pi$-number with $e(|A_i/A_{i-1}|) \leq 2i+1$.

In particular $G$ has a subnormal abelian subgroup $A_0$ whose index is a $\pi$-number with $e(|G:A_0|) \leq e(e+2)$.

This theorem generalizes the result of I.M. Isaacs and D.S. Passman [5] in the case $\pi = \{p\}$. In the case $\pi = \{p\}$, indeed, $p_1 = p_2 = \cdots = p_t = p$ and the $\pi_i$ are empty with the above notation. Thus $A_i = B_{i-1}$, that is, the normal series in Theorem I has elementary abelian factor groups.

In Theorem I $G$ may have, however, larger subnormal abelian subgroups. We shall show the existence of such subgroups. First we make the following definition.

Let $f_s$ (resp. $f_n$) be a function with the following property. If $G$ is a sol-
vable (resp. nilpotent) group with \( r.x.e \), then \( G \) has a subnormal abelian subgroup \( A \) with \( e(\frac{|G|}{|A|}) \leq f_s(e) \) (resp. \( f_n(e) \)). Moreover we assume that \( f_s \) (resp. \( f_n \)) is the smallest such function. Let \( f_{(p)} \) be the corresponding function for the class of groups with \( r.x.e \) for a prime \( p \).

In what follows, we denote the largest integer \( \leq x \) by \( [x] \).

In [6] we know the existence of \( f_{(p)} \) for any prime \( p \). Actually \( f_{(p)}(0)=0 \) and

\[
2e \leq f_{(p)}(e) \leq [4e - \log_2 4e] \quad \text{when } e \geq 1.
\]

In this paper we have:

**Theorem II.** The functions \( f_s \) and \( f_n \) exist and satisfy

1. \( f_s(0)=0, f_s(1)=2 \) and
   \[
   2e \leq f_s(e) \leq [4e - \log_2 8e] \quad \text{when } e \geq 2.
   \]
2. \( f_n(e) \leq f_s(e) \leq e(e+3)/2. \)

This yields in particular

\[
f_n(0)=f_n(1)=0, f_n(1)=f_s(1)=2, f_n(2)=4 \quad \text{and } f_s(2)=4 \text{ or } 5.
\]

All groups in this paper are assumed to be finite unless otherwise stated. Let \( N \triangleleft G \). If \( \chi \in \text{Irr}(G/N) \), then \( \chi \) may be viewed as a character of \( G \). For example \( \hat{G} = \text{Irr}(G/G') \), where \( G' \) is the commutator subgroup of \( G \), is the set of linear characters of \( G \). In what follows an irreducible character means an irreducible complex character. If \( G \) is a group, then \( Z(G) \) and \( \Phi(G) \) denote the center and Frattini subgroup of \( G \) respectively. If \( S \) is a set, then \( |S| \) denotes the cardinality of \( S \). We write

\[
\pi(G) = \{ \text{primes } p \mid p \text{ divides } |G| \},
\]

\[
\pi' = \{ \text{primes } p \mid p \notin \pi \}, \text{ and}
\]

\[
p' = \{ p \}'.
\]

Let \( \chi \) be a character. We denote simply \( e(\chi(1)) \) by \( e(\chi) \). If \( e(\chi)=e \), then we say that \( \chi \) is a character with total exponent \( e \) (shortly \( T \)-exponent \( e \)). All the other notation can be seen in [3] or [6].

The author would like to express his hearty thanks to Professor H. Nagao who encouraged him in whole study.

2. **Groups with \( c.d.\pi \).** The following theorem is a slight extension of the Burnside’s \( p^aq^b \)-Theorem, (see [3] 4.3.3).

**Theorem 2.1.** Let \( G \) have \( c.d.\pi \). If \( |\pi| \leq 2 \), then \( G \) is solvable.

Proof. Since any normal subgroup or factorgroup of \( G \) satisfies the same assumption, the theorem follows at once by induction on \( |G| \) if \( G \) is not simple. So we may assume \( G \) is simple. Therefore we may also assume \( p \in \pi \subseteq \{ p, q \} \) and \( G \) has a nontrivial Sylow \( p \)-subgroup \( P \). Choose \( 1 \neq x \in Z(P) \). Let \( 1_G \neq x \in \text{Irr}(G) \). If \( \chi(1) \) is a power of \( p \), then the simplicity of \( G \) and Burnside’s
lemma (see [3] 4.3.1) imply $\chi(x) = 0$. Thus by orthogonality relations,

$$0 = \sum_{x \in \text{Irr}(G)} \chi(1)\chi(x) = 1 + q\alpha$$

where $\alpha$ is an algebraic integer. So $\alpha = -1/q$, which is clearly impossible.

There exists no extension of Theorem 2.1 to the case $|\pi| \geq 3$ as $SL(2,5)$ shows.

The following results on groups with $c.d.\, p'$ for a prime $p$ are shown in [8] and [1].

**Proposition A** (N. Ito). *If $G$ is a solvable group with $c.d.\, p'$, then $G$ has a normal abelian Sylow $p$-subgroup.*

**Proposition B** (P. Fong). *If $G$ is a $p$-solvable group with $c.d.\, p'$, then $G$ has a normal abelian Sylow $p$-subgroup.*

The latter includes the former. We shall extend these propositions in Theorem 2.5. We start with some lemmas.

If a $\tau$-number $n$ is also a $\tau'$-number, then $n=1$. Therefore the following lemma is immediate.

**Lemma 2.2.** *If $G$ is a $\tau$-group with $c.d.\, \tau$, then $G$ is abelian.*

**Lemma 2.3** (P. X. Gallagher [2], Theorem 8). *Suppose $G$ is a $\tau$-separable group with a Hall $\tau'$-subgroup $H$. If the degree of any irreducible constituent of $(1_H)^G$ is a $\tau$-number, then $H \triangleleft G$.*

**Remark.** In [2] the term "$\tau$-solvable" seems to be used in the sense of "$\tau$-separable".

The following lemma is proved by using the Schur-Zassenhaus Theorem, (see [3] 6.3.5).

**Lemma 2.4.** *If $G$ is $\tau$-separable, then $G$ possesses a Hall $\tau'$-subgroup.*

We are now ready to extend Proposition B. If $G$ is a $\tau$-separable group with $c.d.\, \tau$, then $G$ has a Hall $\tau'$-subgroup $H$ by Lemma 2.4 and hence Lemma 2.3 is applicable. Therefore $H \triangleleft G$ and $H$ is a $\tau'$-group with $c.d.\, \tau$. So $H$ is abelian by Lemma 2.2. By combining Theorem 2.1 and Ito's Theorem we have:

**Theorem 2.5.** *Suppose $G$ is $\tau$-separable when $|\pi| \geq 3$. Then $G$ has a normal abelian Hall $\tau'$-subgroup if and only if $G$ has $c.d.\, \tau$.*

The following corollary is useful in the proof of Theorem I in section 3.

**Corollary 2.6.** *Let $G$ have $c.d.\, \tau$. Suppose $G$ is $\tau$-solvable when $|\pi| \geq 3$.*
Then $G$ is solvable.

Proof. By the theorem $G$ has a normal abelian Hall $\pi'$-subgroup $H$. Then $G/H$ is a $\pi$-solvable $\pi$-group, and hence $G/H$ is solvable. Therefore $G$ is also solvable.

Now it is clear the following corollary holds for subnormal subgroups of arbitrary groups.

**Corollary 2.7.** Let $G$ have c.d.$\pi$. Suppose $G$ is $\pi$-separable when $|\pi| \geq 3$. Then every subgroup of $G$ has also c.d.$\pi$.

Proof. Let $G$ be as above. By the theorem $G$ has a normal abelian Hall $\pi'$-subgroup $H$. Let $K$ be a subgroup of $G$. Then $H \cap K$ is a normal abelian Hall $\pi'$-subgroup of $K$ and hence the theorem implies the corollary.

3. Groups with $r.x.e$ for $\pi$. In this section we shall prove Theorem I.

The following properties of the total exponent immediately follow from our definition.

**Lemma 3.1.**
1. $e(m) \geq 0$, and $e(m) = 0$ if and only if $m = 1$.
2. $e(mn) = e(m) + e(n)$.

In particular these yield:
3. When $s$ divides $t$, $e(s) \leq e(t)$, and the equality holds if and only if $s = t$.

If $G$ has $r.x.0$, then $G$ has no nonlinear irreducible characters and hence $G$ is abelian. We know that groups with $r.x.1$ are solvable ([7] Theorem 6.1), but groups with $r.x.2$ are not necessarily solvable. Indeed the simple group $A_5$, the alternating group on 5 letters, has character degrees $1, 3, 2^2, 5$.

By using Frobenius Reciprocity Theorem, Clifford’s Theorem and our definition, we have the following immediately.

**Lemma 3.2.** Let $N$ be subnormal in $G$ where $G$ has $r.x.e$ for $\pi$. Then $N$ has $r.x.e$ for $\pi$.

The following lemma will be useful in applying induction on the total exponent.

**Lemma 3.3.** Let $N \triangleleft G$ where $G$ has $r.x.e$ for $\pi$. If $G/N$ is nonabelian, then $N$ has $r.x.(e-1)$ for $\pi$.

Proof. By Lemma 3.2, it will be sufficient to show that $N$ has no irreducible characters with $T$-exponent $e$. Assume that $N$ has an irreducible character $\theta$ with $e(\theta) = e$. Let $\chi$ be an irreducible constituent of $\theta^e$. Then $\theta(1)$ divides $\chi(1)$ and hence

$$e = e(\theta) \leq e(\chi) \leq e$$
for \( G \) has r.x.e. We have the equality throughout so that \( e(\chi) = e \) and \( \chi|_N = \theta \in \text{Irr}(N) \). Since \( G/N \) is nonabelian, there exists \( \varphi \in \text{Irr}(G/N) \) such that \( \varphi(1) > 1 \). Then \( \varphi \chi \in \text{Irr}(G) \) (see [2] Theorem 2), and hence

\[
e = e(\chi) < e(\varphi) + e(\chi) = e(\varphi \chi) \leq e.
\]

This is a contradiction.

We remark that in the proof of Lemma 3.3 above we obtained the following result.

**Corollary 3.4.** Let \( N \triangleleft G \) where \( G \) and \( N \) have r.x.e. Suppose \( \theta \in \text{Irr}(N) \) with \( e(\theta) = e \). If \( \chi \) is an irreducible constituent of \( \theta^G \), then \( e(\chi) = e \) and \( \chi|_N = \theta \in \text{Irr}(N) \).

The following proposition generalizes Lemma 2.7 in [5] however it will not be used in this paper.

**Proposition 3.5.** Let \( N \triangleleft G \) where \( G \) has c.d.\( \pi \). Suppose \( G/N \) is a \( \pi' \)-group. Then we have:

1. Any irreducible character of \( N \) is \( G \)-invariant and \( \chi|_N \in \text{Irr}(N) \) for any \( \chi \in \text{Irr}(G) \).
2. If \( N \) has r.x.e for \( \pi \), then so does \( G \).

Proof. Let \( \chi \in \text{Irr}(G) \). By Clifford's Theorem, \( \chi|_N = e \sum \theta_i \) where the \( \theta_i \) are distinct irreducible constituents and \( \chi(1) = e \theta_i(1) \). Then \( et \) is a \( \pi \)-number since \( G \) has c.d.\( \pi \). Now \( et \) divides \( |G : N| \) which is a \( \pi' \)-number. Thus we have \( e = t = 1 \). Since \( \chi \) is arbitrary, (1) and (2) follow from Frobenius Reciprocity Theorem.

Before going on to another result, we state here the result by Isaacs and Passman, which will be needed.

**Lemma 3.6 ([5] Proposition 2.5).** Let \( N \triangleleft G \) with \( G/N \) nilpotent. Suppose \( \chi \in \text{Irr}(G) \) with \( \chi|_N \) reducible. Then there exists a normal subgroup \( T \) of \( G \) of prime index such that \( N \trianglelefteq T \) and \( \chi = \psi^G \) for some \( \psi \in \text{Irr}(T) \).

The following lemma generalizes Lemma 2.8 in [5].

**Lemma 3.7.** Let \( N \triangleleft G \) with \( G/N \) nilpotent. Let \( G \) have r.x.e for \( \pi \) and \( N \) have r.x.\((e-1)\) for \( \pi \). If \( F \) is the inverse image of \( \Phi(G/N) \) in \( G \), then \( F \) has r.x.\((e-1)\) for \( \pi \).

Proof. \( F \triangleleft G \) and thus by Lemma 3.2 \( F \) has r.x.e for \( \pi \). Therefore it would be sufficient for our purpose to show that \( F \) has no irreducible character with \( T \)-exponent \( e \). Suppose \( \theta \in \text{Irr}(F) \) satisfies \( e(\theta) = e \). Let \( \chi \) be an irreducible constituent of \( \theta^G \). By Corollary 3.4, \( e(\chi) = e \) and \( \chi|_F \) is irreduc-
tible. Since $N$ has $r.x.(e-1)$ for $\pi$, $\chi|_N$ is reducible, and by Lemma 3.6 there exists a subgroup $T$ maximal in $G$ and containing $N$ with $\chi=\psi^G$ for some $\psi \in \text{Irr}(T)$. Therefore $\psi$ is a constituent of $\chi|_T$ which is thus reducible. Consequently $\chi|_F$ must be reducible for $F \subseteq T$. This is a contradiction and the result follows.

The following lemma is a part of the result appearing in [6], which is extremely useful in proving our main theorems. We will call it Isaacs-Passman’s Lemma in this paper.

**Lemma 3.8** (Isaacs-Passman’s Lemma). Let $E$ be a group such that $E''=1 < E'$ and $E' \leq K$ for all $K$ with $1 < K \triangleleft E$. Then we have one of the following.

**Case P.**
1. $E$ is a $p$-group for some prime $p$.
2. $Z(E)$ is cyclic.
3. Every nonlinear irreducible character has degree $|E: Z(E)|^{1/2}$.

**Case Q.**
4. $E$ is a Frobenius group with a cyclic complement and elementary abelian $q$-group $Q$ as kernel.
5. Every nonlinear irreducible character has degree $|E: Q|$.
6. For any $\lambda \in \hat{Q}$ and any $x \in E - Q$, there exists $\mu \in \hat{Q}$ with $\lambda = \mu^x \mu^{-1}$.

Let $N$ be normal and maximal with respect to $G/N$ being nonabelian. We note that if $G$ is solvable then $E=G/N$ satisfies of Isaacs-Passman’s Lemma. We are now ready for the proof of Theorem I.

**Proof of Theorem I.** We prove the result by induction on $e$. When $e=0$, the result is trivial. Suppose $e \geq 1$. It will be sufficient to show that $G$ has a normal series $G \triangleright B_{e-1} \triangleright A_{e-1}$ and there exists some prime $p_1 \in \pi$ such that

(1) $A_{e-1}$ has $r.x.(e-1)$ for $\pi$,

(2) $G/B_{e-1}$ is a cyclic $\pi_1$-group where $\pi_1 = \pi - \{p_1\}$,

(3) $B_{e-1}/A_{e-1}$ is an elementary abelian $p_1$-group, and

(4) $e(|G:A_{e-1}|) \leq 2e+1$.

We know that $G$ is solvable by Corollary 2.6. We may assume $G$ is nonabelian. Then there exists $N \triangleleft G$ which is maximal with $G/N$ nonabelian. Now $E=G/N$ satisfies the hypotheses of Isaacs-Passman’s Lemma. Thus $E$ has a unique nonlinear irreducible character degree $m$, which is also a character degree of $G$. So $m$ is a $\pi$-number with $e(m) \leq e$, because $G$ has $r.x.e$ for $\pi$. Since $E$ is nonabelian, $N$ has $r.x.(e-1)$ for $\pi$ by Lemma 3.3.

We consider two cases according to Isaacs-Passman’s Lemma, which we apply to $E$.

**Case P.** $E$ is a $p$-group for some prime $p$. Then $p$ divides $m$ and thus $p \in \pi$. Let $A_{e-1}$ be the inverse image of $\Phi(G/N)$ in $G$. By Lemma 3.7 $A_{e-1}$ has $r.x.(e-1)$ for $\pi$, and satisfies (1)' since $Z(E)$ is cyclic and $|E: Z(E)| = m^2$,
Thus we get (4)' . Let \( B_{e-1} = G \) and \( p_1 = p \). Then (2)' and (3)' hold, and the result follows for this case.

Case \( Q \). \( E \) is a Frobenius group with a cyclic complement and elementary abelian \( q \)-group \( Q \) as kernel. Let \( K \) be the inverse image of \( Q \) in \( G \). Since \( K \) has \( r.x.e \) by Lemma 3.2, we may consider the following two cases.

Case \( Q-1 \). \( K \) has \( r.x.(e-1) \) for \( \pi \). Let \( A_{e-1} \) be the inverse image of \( \Phi(G/K) \) in \( G \). Now \( G/K \cong E/Q \) is a cyclic group of order \( m \), therefore by Lemma 3.7 \( A_{e-1} \) has \( r.x.(e-1) \) for \( \pi \) and satisfies (1)' . Since \( |G: A_{e-1}| \) divides \( m \), (4)' follows for \( e(m) \leq e \leq 2e+1 \). Choose a prime divisor \( p_i \) of \( |G: A_{e-1}| \), which is a square-free \( \pi \)-number, and let \( B_{e-1} \) be the inverse image of a Sylow \( p_i \)-subgroup of \( G/A_{e-1} \) in \( G \). Then (2)' and (3)' follow.

Case \( Q-2 \). \( K \) has \( r.x.e \) for \( \pi \) but not \( r.x.(e-1) \) for \( \pi \). Then there exists \( \theta \in \text{Irr}(K) \) such that \( e(\theta) = e \). By Corollary 3.4 \( \theta \) is \( G \)-invariant. Let \( g \in G - K \).

For any \( \mu \in \hat{Q} \), \( \mu \theta \in \text{Irr}(K) \) and \( e(\mu \theta) = e(\theta) = e \). Thus similarly \( \mu \theta \) is \( G \)-invariant, so that

\[
\theta \mu = (\theta \mu)^g = \theta^g \mu^g = \theta \mu^g
\]

and \( \theta = \theta \mu^g \mu^{-1} \). Hence \( \theta \) vanishes off \( \text{Ker}(\mu^g \mu^{-1}) \). By (6) of Isaacs-Passman's Lemma, for any character \( \lambda \in \hat{Q} \) we can find a character \( \mu \in \hat{Q} \) and an element \( g \in G - K \) with \( \lambda = \mu^g \mu^{-1} \). Thus \( \theta \) vanishes \( \text{Ker} \lambda \). Now \( \hat{Q} = K/N \) has a subgroup of index \( q \). Let \( A_{e-1} \) be its inverse image in \( G \). \( A_{e-1} \) is the kernel of \( (1_{A_{e-1}})^x \) which is a sum of linear characters of \( Q \). So \( \theta \) vanishes off \( A_{e-1} \). Let

\[
a^2t = (\theta|_{A_{e-1}}, \theta|_{A_{e-1}})_{A_{e-1}} = \frac{|K|}{|A_{e-1}|}(\theta, \theta)_K = q .
\]

Hence \( a = 1 \) and \( t = q \). Thus \( q = \theta(1)/\phi(1) \in \pi \) and \( \theta|_{A_{e-1}} \) is reducible. For any irreducible character of \( K \) with \( T \)-exponent \( e \), similarly its restriction to \( K \) is reducible. Therefore we have (1)' . Let \( B_{e-1} = K \), \( p_1 = q \) and \( \pi_1 = \pi - \{p_1\} \). Since \( q \) is relatively prime to \( m = |E: Q| = |G: K| \), (2)' and (3)' are satisfied. Now

\[
e(|G: A_{e-1}|) = e(|G: K|) + e(|K: Q|) = e(m) + 1 \leq e + 1 \leq 2e + 1 ,
\]

and hence (4)' is also satisfied. This proves the theorem.

As consequences of Theorem I we have the following.

**Corollary 3.9.** Assume that \( G \) satisfies the hypotheses of Theorem I. Then we have:
(1) $G$ has the derived length $\leq 2e+1$, and Sylow $p$-subgroup of $G$ has the derived length $\leq e+1$.

(2) $G$ has a subnormal abelian subgroup $A_0$ with $|G:A_0| \leq r^{(e+2)}$, where $r$ is the biggest prime of $\pi(G) \cap \pi$.

(3) If $G$ has an abelian Hall $\pi$-subgroup, then $G$ has a normal series

$$G = A_s \triangleright A_{s-1} \triangleright \cdots \triangleright A_0$$

such that (i) $A_s$ has $r.x.i$ for $\pi$ and (ii) $A_s/A_{s-1}$ is a cyclic $\pi$-group of square-free order, whose $T$-exponent $\leq i$.

Proof. (1) and (2) immediately follow from Theorem I. We consider (3). Any section of $G$ which is a $\pi$-group must be abelian. Therefore only Case 2 in the proof of Theorem I can occur. Hence the result follows.

The above (1) may be of interest as the analogy to the following result appearing in [4]. A Sylow $p$-subgroup of a solvable group $G$ has the derived length $\leq 2m+1$, where $m$ is the biggest integer such that $p^m$ divides $\chi(1)$ for some $\chi \in \text{Irr}(G)$.

Let $G$ be a (not necessarily finite) group, and we suppose every irreducible $C[G]$-module is of finite dimension over $C$, where $C$ is the field of complex numbers. Then we may use the terminology "$r.x.e$ for $\pi$" as in the case of finite groups.

The following consequence of Theorem I generalizes Theorem I of [5].

**Corollary 3.10.** Let $G$ be (not necessarily finite) finitely generated group with $r.x.e$ for $\pi$. Moreover suppose $|\pi|$ is finite when $G$ is not finite. Then $G$ has a normal series

$$G = A_s \triangleright B_{s-1} \triangleright A_{s-1} \cdots \triangleright B_0 \triangleright A_0$$

and there exists some prime $p_i \in \pi$ for any $i$ such that

1. $A_0$ is abelian,
2. $A_i/B_{i-1}$ is a cyclic $\pi_i$-group where $\pi_i = \pi - \{p_i\}$,
3. $B_{i-1}/A_{i-1}$ is an elementary abelian $p_i$-group, and
4. $|A_i:A_{i-1}|$ is a $\pi$-number with $T$-exponent $\leq 2i+1$.

In particular $|G:A_0|$ is a $\pi$-number with $T$-exponent $\leq e(e+2)$ and hence $|G:A_0| \leq r^{r(e+2)}$, where $r = \max(\pi(G) \cap \pi)$.

Proof. Let $G$ be a finitely generated group which satisfies the above hypotheses. By the assumption there exists a prime $r$ such that $s \geq r$ for any $s \in \pi(G) \cap \pi$. There are only finitely many subgroups of $G$ with index $\leq r^{r(e+2)}$ by M. Hall’s Theorem (see [9] p. 56 or [6] p. 901). Suppose that $L_1, L_2, \cdots, L_t$ are all of those which are nonabelian. Choose $x_i, y_i \in L_i$ with the commutator $z_i = [x_i, y_i] + 1$. By Passman’s Theorem ([10] Theorem V), $G$ is a subdirect
product of finite groups. Thus we can find a normal subgroup $N$ of finite index in $G$ such that $z_i \in N$ for $i = 1, 2, \ldots, t$. Then $G/N$ is a finite group with $r.x.e$ for $\pi$ and thus there exists a normal series

$$G = A_t \triangleright B_{t-1} \triangleright A_{t-1} \triangleright \cdots \triangleright B_0 \triangleright A_0 \triangleright N$$

such that (2), (3) and (4) hold, $A_0/N$ is abelian and $|G: A_0| \leq r^{(e+2)}$ by Theorem I. By the choice of $N$, $A_0$ is abelian and hence the result is proved.

4. Large subnormal abelian subgroups. In this section we shall prove Theorem II.

We note that the function $f_s$ exists and satisfies $f_s(e) \leq e(e+2)$ by Theorem I. Thus there exists $f_s$ and clearly $f_s(e) \leq f_s(e)$.

In order to improve the upper bounds, we start with lemmas which correspond to the results in [6]. The following lemma is due ultimately to Isaacs and Passman.

**Lemma 4.1.** Let $G$ have $r.x.e$. Suppose $N \lhd G$ with $E = G/N$ being as in Case P of Isaacs-Passman’s Lemma. Let $Z$ be the complete inverse image of $Z(E)$ in $G$. Let $\beta \in \text{Irr}(E)$ with $\beta(1) > 1$. Then we have:

1. Given any character $\varphi \in \text{Irr}(Z)$, if $X_1$ is an irreducible constituent of $\varphi^G$ and if $X_1$ is an irreducible constituent of $X\beta$, then

$$e(\varphi) + e(X_1) \geq e(\beta) + e(t) + 2e(\varphi)$$

where $t$ is the number of distinct conjugates of $\varphi$.

2. $Z$ has $r.x.[e-e(\beta)/2]$.

3. Moreover if $e(\beta)$ is even, then $G$ has a normal subgroup $B$ with the following properties: $B > Z$, $e([B: Z]) = 1$ and $B$ has $r.x.(e-e(\beta)/2)$.

**Proof.** (1) Let $\chi$ be an irreducible constituent of $\varphi^G$. Then since $Z \lhd G$,

$$\chi|_Z = a \sum_{i=1}^{l-1} \varphi_i \varphi_i = \varphi.$$ Let $\beta|_Z = \beta(1)\lambda$, where $\lambda \in \widehat{Z/N}$. Let $(\varphi \lambda)^G = \sum a_i \chi_i$. By the proof of Lemma 3.5 of Isaacs-Passman [6], $a_i \lambda \beta(1) = (\chi \beta, \chi_i)$, $\chi(1) = at\varphi(1)$ and $\chi_i(1) = a_i t\varphi(1)$. Hence

$$\chi(1)\chi_i(1) = a_i at^2 \varphi(1)^2 = (\chi \beta, \chi_i) \beta(1) t\varphi(1)^2$$

and

$$e(\varphi) + e(X_1) \geq e(\beta) + e(t) + 2e(\varphi)$$

as desired.

(2) Since $G$ has $r.x.e$, $e(\chi)$ and $e(X_1)$ are $\leq e$. By (1), therefore, $e(\varphi) \leq e - e(\beta)/2$. Since $\varphi$ is an arbitrary character of $Z$, $Z$ has $r.x. [e-e(\beta)/2]$, and (2) follows.
(3) Let \( e(\beta) \) be even. Since \( G/Z \) is a \( p \)-group for some prime \( p \), there exists \( B \) such that \( Z < B < G \) and \( |B:Z| = p \). We may show \( B \) has \( r.x.(e-e(\beta)/2) \). Suppose that there exists an irreducible character \( \theta \) of \( B \) with \( e(\theta) > e-e(\beta)/2 \). By (2) \( Z \) has \( r.x.(e-e(\beta)/2) \) and hence \( \theta|_Z \) is reducible. By Lemma 3.6 there exists \( \varphi \in \text{Irr}(Z) \) with \( \theta = \varphi^p \). So \( e(\theta) = e(\varphi) + 1 \), and we have

\[
e-e(\beta)/2 \leq e(\theta) - 1 = e(\varphi) \leq e-e(\beta)/2.
\]

Thus we have \( e(\varphi) = e-e(\beta)/2 \). Now \( \varphi \) has \( p \) conjugates in \( B \). Hence if \( \varphi \) has \( t \) conjugates in \( G \), we have \( t \geq p > 1 \) and \( e(t) > 0 \). Thus by (1),

\[
2e-e(\beta) = 2e(\varphi) \leq 2e-e(\beta) - e(t) < 2e-e(\beta).
\]

This is a contradiction. Therefore \( B \) has \( r.x.(e-e(\beta)/2) \).

**Lemma 4.2.** \( f_\ell(0) = 0 \) and

\[
f_\ell(e) \leq \max \{ f_\ell(e-1) + e + 1, f_\ell(e-(m+1)/2) + 2m, f_\ell(e-n/2) + 2n - 1 \}
\]

\( m \) is an odd integer with \( 0 < m \leq e \) and \( n \) is an even integer with \( 0 < n \leq e \).

**Proof.** A group with \( r.x.0 \) is abelian and hence \( f_\ell(0) = 0 \). Let \( v \) be the right-hand side of the above inequality. The proof is by induction on \( |G| \).

We may assume that \( G \) is a nonabelian group with \( r.x.e \) and that \( e \geq 1 \). Since \( G \) is solvable, we can choose \( N \vartriangleleft G \) with \( E = G/N \) being a group as in Isaacs-Passman’s Lemma.

We consider three cases according to the cases of the proof of Theorem I. First we consider the case \( Q-1 \).

Case \( Q-1 \). \( K \) has \( r.x.(e-1) \), where \( K \) is as in the proof of Theorem I. Then \( K \) has a subnormal abelian subgroup \( A \) such that \( e(|K:A|) \leq f_\ell(e-1) \). Since \( K \vartriangleleft G \) and \( e(|G:K|) \leq e \), \( A \) is a subnormal abelian subgroup of \( G \) such that

\[
e(|G:A|) = e(|G:K|) + e(|K:A|) \leq e + f_\ell(e-1) < v.
\]

Case \( Q-2 \). \( K \) has \( r.x.e \) but not \( r.x.(e-1) \). Let \( A_{e-1} \) be as in the proof of that theorem. Then \( A_{e-1} \) is a subnormal subgroup with \( r.x.(e-1) \) and with \( e(|G:A_{e-1}|) \leq e+1 \). By induction \( A_{e-1} \) has a subnormal abelian subgroup \( A \) with \( e(|A_{e-1}:A|) \leq f_\ell(e-1) \). Therefore \( A \) is a subnormal abelian subgroup of \( G \) such that

\[
e(|G:A|) \leq e+1 + f_\ell(e-1) \leq v.
\]

Case \( P \). \( E \) is a \( p \)-group for some prime \( p \). Let \( Z \) be the inverse image of \( Z(E) \) in \( G \). Let \( \beta \in \text{Irr}(E) \) with \( \beta(1) > 1 \). We know that \( |G:Z| = \beta(1)^2 \) and that \( 0 < e(\beta) \leq e \).
Moreover there exist two cases to consider.

Case P-1. $e(\beta)$ is odd. Then

$$\left[ e-e(\beta)/2 \right] = e-(e(\beta)+1)/2 \leq e-1.$$ By Lemma 4.1 (2), $Z$ has $r.x.(e-(e(\beta)+1)/2)$. By induction $Z$ has a subnormal abelian subgroup $A$ with $e(|Z:A|) \leq f_e(e-(m+1)/2)$, where $m=e(\beta)$. Thus $A$ is a subnormal abelian subgroup of $G$ with

$$e(|G:A|) \leq 2m+f_e(e-(m+1)/2) \leq v.$$ By Lemma 4.1 (2), $Z$ has $r.x.(e-e(\beta)\beta)$.

Case P-2. $e(\beta)$ is even. Then let $B$ be as in Lemma 4.1 (3). Since $B$ has $r.x.(e-e(\beta)/2)$ and $e(\beta) \geq 2$, $B$ has a subnormal abelian subgroup $A$ with $e(|B:A|) \leq f_e(e-n/2)$, where $n=e(\beta)$. Thus $A$ is a subnormal abelian subgroup of $G$ with

$$e(|G:A|) \leq 2n-1+f_e(e-n/2) \leq v.$$ In any case $G$ has a subnormal abelian subgroup $A$ with $e(|G:A|) \leq v$, and hence $f_e(e) \leq v$. This completes the proof of our lemma.

From the proof of Theorem A in [6], we have immediately (2) of the following lemma.

**Lemma 4.3** (Isaacs-Passman). For any prime $p$ there exists $f_p(e)$, which satisfies

1. $f_p(0) = 0$, $f_p(1) = 2$, $f_p(2) = 4$ and
2. $2e \leq f_p(e) \leq \max\{f_p(e-(m+1)/2)+2m, f_p(e-n/2)+2n-1\}$

$m$ is an odd integer with $0 < m \leq e$ and $n$ is an even integer with $0 < n \leq e$.

The equality $f_p(2)=4$ of (1) is seen in [11], and the other equalities of (1) are seen in [6].

We remark that clearly $f_p(e) \leq f_p(e) \leq f_p(e)$ for any prime $p$.

**Corollary 4.4.** $f_p(e) = 2e$ for $e \leq 1$.

$$f_p(e) \leq 4e-[\log_2 8e]$$ for $e \geq 2$.

Proof. By Lemma 4.3 (1), $f_p(0) = 0$, $f_p(1) = 2$ and $f_p(2) = 4$, therefore by (2)

$$f_p(3) \leq 4 \cdot 3 - [\log_2 8 \cdot 3].$$

Thus the result holds for $e \leq 3$. We may suppose $e \geq 4$. Our inequality will be proved by induction on $e$. By Lemma 4.3 (2), it would be sufficient to show the following two inequalities.
(i) If \( m \) is any odd integer with \( 0 < m \leq e \), then
\[
\phi(p)(e-(m+1)/2) + 2m \leq 4e - \lfloor \log_2 8e \rfloor.
\]

(ii) If \( n \) is any even integer with \( 0 < n \leq e \), then
\[
\phi(p)(e-n/2) + 2n - 1 \leq 4e - \lfloor \log_2 8e \rfloor.
\]

Proof of (i). Let \( m \) be as in (i). Then since \( e \geq 4 \), \( 2 \leq e-(m+1)/2 \leq e-1 \). Hence induction is applicable. Write
\[
A = \{4e - \lfloor \log_2 8e \rfloor\} - \{\phi(p)(e-(m+1)/2) + 2m\}.
\]
By induction,
\[
A \geq 4e - \lfloor \log_2 8e \rfloor - \{4(e-(m+1)/2) - \lfloor \log_2 (e-(m+1)/2) \rfloor + 2m\}
\]
\[
= \lfloor \log_2 4(e-(m+1)/2) - \lfloor \log_2 e \rfloor \rfloor.
\]
Since \( m \leq e \), \( 4(e-(m+1)/2) \geq 2(e-1) \geq e \) for \( e \geq 4 \). Therefore we get \( A \geq 0 \), and hence (i) also follows.

Proof of (ii). Let \( n \) be as in (ii). Then \( 2 \leq e-n/2 \leq e-1 \) for \( e \geq 4 \). Thus by using induction we get
\[
\{4e - \lfloor \log_2 8e \rfloor\} - \{\phi(p)(e-n/2) + 2n - 1\}
\]
\[
\geq \lfloor \log_2 2(e-n/2) - \lfloor \log_2 e \rfloor \rfloor \geq 0,
\]
because \( n \leq e \), \( 2(e-n/2) \geq e \). Thus (ii) is proved, and hence the result follows.

We will need the following elementary inequality in (3).

Lemma 4.5. (1) \([x] + [y] + 1 \geq [x + y]\).
(2) \([x] - [y] \geq |x - y|\).
(3) We define a function \( z \) on all of nonnegative integers as follows.
\[
z(x) = \begin{cases} 2x & \text{if } x = 0 \text{ or } 1, \\ 4x - \lfloor \log_2 8x \rfloor & \text{if } x \geq 2. \end{cases}
\]
Then we have
\[
z(x+y) \geq z(x) + z(y) \text{ for any } x, y.
\]
and thus \( z(\sum_{i=1}^{r} x_i) \geq \sum_{i=1}^{r} z(x_i) \).

Proof. The inequalities (1) and (2) are well-known.
(3) By induction on \( r \) the last inequality follows from the first inequality.
We consider three cases.

Case 1. \( x \leq 1 \) and \( y \leq 1 \). Then since \( z(2) = 4 \), \( z(x+y) \geq z(x) + z(y) \).

Case 2. Either \( x \) or \( y \) is \( \leq 1 \). We may assume that \( x \geq 2 \) and \( y = 1 \). Then we have
The first inequality follows from (2) and the last inequality follows from the fact that \((4x)/(x+1) > 2\) for \(x \geq 2\).

Case 3. \(x \geq 2\) and \(y \geq 2\). Then we have

\[
\begin{align*}
\delta(x+y) - \delta(x) - \delta(y) &= \delta(x+1) - \delta(x) - 2 \\
&= 2 + [\log_2 x] - [\log_2 (x+1)] \geq [\log_2 (4x)/(x+1)] > 0.
\end{align*}
\]

The first (resp. the second) inequality follows from (1) (resp. (2)) and the last inequality follows from the fact that \((4xy)/(x+y) > 2\) for \(x \geq 2\) and \(y \geq 2\).

Now we are ready to prove our second main theorem.

Proof of Theorem II. By the first remark in this section, we may prove \((1)\) and \((2)\): \(f_s(e) \leq e(e+3)/2\).

We discuss \((2)\)' first. Use induction on \(e\). By Lemma 4.2, it would be sufficient to show that the following inequalities:

(i) \(f_s(e-1)+e+1 \leq e(e+3)/2\) for \(e \geq 1\).

(ii) If \(m\) is any odd integer with \(0 < m \leq e\), then

\[
f_s(e-(m+1)/2)+2m \leq e(e+3)/2.
\]

(iii) If \(n\) is any even integer with \(0 < n \leq e\), then

\[
f_s(e-n/2)+2n-1 \leq e(e+3)/2.
\]

Proof of (i). By induction,

\[
f_s(e-1)+e+1 \leq (e-1)((e-1)+3)/2 + e+1 = e(e+3)/2.
\]

Proof of (ii). Let \(m\) be as in (ii). Since \(0 \leq (m+1)/2 \leq e-1\), induction is applicable. Thus

\[
f_s(e-(m+1)/2)+2m
\leq \frac{1}{2} \left( e - \frac{m+1}{2} \right) \left( e - \frac{m+1}{2} + 3 \right) + 2m
\leq \frac{1}{2} e(e+3) + \frac{1}{8} (m+1)^2 - \frac{1}{4} (2e+3)(m+1) + 2m
\leq \frac{1}{2} e(e+3),
\]

because \(m\) and \(e\) are integers with \(0 < m \leq e\).
Proof of (iii). Let \( n \) be as in (iii). Since \( 0 < e - n \leq 1 \), by induction we can prove (iii) similarly.

Next we discuss (1). By Lemma 4.3 and the remark following that lemma, it would be sufficient to show that

\[
(1)' \quad f_n(0) = 0, \quad f_n(1) \leq 2 \quad \text{and} \quad f_n(e) \leq 4e - \lfloor \log_2 e \rfloor \quad \text{for} \quad e \geq 2.
\]

Now \( f_n(0) = 0 \) is trivial. Let \( G \) be a nilpotent group with \( r.x.e \), and write \( G = P_1 \times P_2 \times \cdots \times P_r \), where \( P_i \) is a Sylow \( p_i \)-subgroup of \( G \). Suppose that \( P_i \) has \( r.x.e_i \) but not \( r.x.(e_i - 1) \). Then \( G \) has \( r.x.\sum_{i=1}^r e_i \) and hence \( \sum_{i=1}^r e_i \leq e \). We define \( z(x) \) as in Lemma 4.5 (3). By Corollary 4.4 we know \( f_{\phi_2}(e_i) \leq z(e_i) \). Thus \( P_i \) has a subnormal abelian subgroup \( A_i \) with \( e(\{P_i : A_i\}) \leq z(e_i) \). If \( A = A_1 \times A_2 \times \cdots \times A_r \), then \( A \) is a subnormal abelian subgroup of \( G \), and

\[
e(\{G : A\}) = \sum_{i=1}^r e(\{P_i : A_i\}) \leq \sum_{i=1}^r z(e_i) \leq z(\sum_{i=1}^r e_i)
\]

\[
\leq z(e).
\]

The second and the last inequalities follow from Lemma 4.5 (3). We have, therefore, \( f_n(e) \leq z(e) \), and prove (1)'. This completes the proof of Theorem II.

5. A remark on a result of Issacs-Passman. A group \( G \) is said to have \( r.b.n \) (representation bound \( n \)) if \( \chi(1) \leq n \) for any \( \chi \in \text{Irr}(G) \).

The following result appears as Theorem D of [6]. Let \( h_2 \) be the function with the following property. If \( G \) is a solvable group with \( r.b.n \), then \( G \) has a subnormal abelian subgroup of index \( \leq h_2(n) \). Moreover we assume that \( h_2 \) is the smallest such function. Then

\[
h_2(n) \leq n^{3/2} \log_2 n.
\]

In this section we remark that the above upper bound may be slightly improved as follows.

**Theorem 5.1.** \( h_2(n) \leq n^{1/2} \log_2 n \).

Proof. If \( G \) is abelian, the result is trivial, so we may assume that \( G \) is nonabelian. As usual, choose \( N \triangleleft G \) with \( G/N \) being a group of Issacs-Passman’s Lemma. There are three cases in the proof of Theorem D of [6].

Case P. \( G \) has a normal subgroup of index \( \leq n^2 \) with \( r.b.(n/2) \). \( \cdots \cdots \cdots \cdots \cdots (1) \)

Case Q-1. \( G \) has a normal subgroup \( Q \) of index \( \leq n \) with \( r.b.(n/2) \). \( \cdots \cdots \cdots \cdots \cdots (2) \)

Case Q-2. \( G \) has a normal subgroup \( Q \) of index \( \leq n \) with \( r.b. \) \( (n/2) \), and \( Q/N \) is an abelian Sylow \( q \)-subgroup of \( G/N \) for some prime \( q \). In this case, moreover, it is known that if \( \theta \in \text{Irr}(Q) \) with \( \theta(1) > n/2 \) then \( \theta \) vanishes off \( N \). We consider this case more precisely.
Now \( Q/N \) has a subgroup of index \( q \). Let \( D \) be its inverse image in \( G \). Then \( \theta \) vanishes off \( D \) and \( D \triangleleft Q \). Let \( \theta|_D = a \sum_{i=1}^r \varphi_i \). Then

\[ a^t = (\theta|_D, \theta|_D) = \left| \frac{Q}{D} \right| (\theta, \theta) = q. \]

Hence \( a = 1 \) and \( t = q \). Thus

\[ q \leq at \varphi_i(1) = \theta(1) \leq n, \]

because \( Q \) has \( r.b.n \). So

\[ |G: D| \leq nq \leq n^2 \]

and \( \varphi_i(1) = \theta(1)/q \leq n/2 \). Since \( \theta \) is an arbitrary character of \( Q \) with \( \theta(1) > n/2 \), \( D \) has \( r.b.(n/2) \) by Frobenius Reciprocity Theorem. Thus we have:

\( G \) has a subnormal subgroup \( D \) of index \( \leq n^2 \) with \( r.b.(n/2) \). \( \cdots \cdots \cdots (3) \)

We now apply induction on \( n \). (1), (2) and (3) imply that \( G \) has a subnormal subgroup \( M \) of index \( \leq n^2 \) with \( r.b.(n/2) \). By induction \( M \) has a subnormal abelian subgroup \( A \) with

\[ |M: A| \leq (n/2)^{\log_2 n}. \]

Then \( A \) is subnormal in \( G \) with

\[ |G: A| \leq n^2(n/2)^{\log_2 n} = n^{\log_2 n}, \]

and the result follows.

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References

