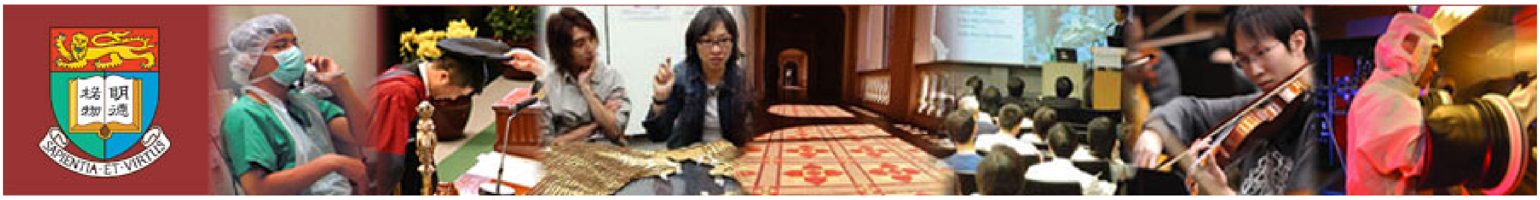

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Stabilizing Solution and Parameter Dependence of Modified Algebraic Riccati Equation with Application to Discrete-time Network Synchronization

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***Abstract*—This technical note deals with a modified algebraic Riccati equation (MARE) and its corresponding inequality and difference equation, which arise in modified optimal control and filtering problems and are introduced into the cooperative control problems recently. The *stabilizing property* of the solution to MARE is presented. Then, the uniqueness is proved for the almost stabilizing and positive semi-definite solution. Next, the *parameter dependence* of MARE is analyzed. An obtained parameter dependence result is finally applied to the study of semi-global synchronization of leader-following networks with discrete-time linear dynamics subject to actuator saturation.**

***Index Terms*—Riccati equation / inequality, stabilizing solution, parameter dependence, synchronization, input saturation.**

I. Introduction

This technical note considers the following modified algebraic Riccati equation (MARE):

$$P = A^T PA - \gamma A^T PB(B^T PB + R)^{-1}B^T PA + Q.$$

This kind of quadratic matrix equation and its corresponding inequality and difference equation have been studied in modified optimal control [1]–[3], modified filtering [4]–[7], and the control and estimation for networked systems [8]–[12]. The existence and uniqueness of a positive semi-definite solution are established for MARE in [9]. Recently, MARE and the modified algebraic Riccati inequality (MARI) are applied to the discrete-time cooperative control problems in [13]–[17].

The scalar parameter γ in MARE is called the *characteristic parameter* hereafter. If $\gamma \neq 1$, MARE cannot be transformed to ARE via scaling. However, some properties of the discrete-time ARE are still preserved for MARE even if $\gamma \neq 1$. If γ is greater than a critical value γ_c and some other conditions are satisfied, then there exists a unique positive semi-definite solution [12]. In this note, it is shown that this solution possesses the *stabilizing* property. Besides, a similar relationship between the solutions of ARE and of the algebraic Riccati inequality (ARI) is demonstrated for MARE and MARI. In addition, it is noted that the critical value γ_c non-decreasingly depends on the parameter matrices Q and R. Furthermore, the monotonic dependence of MARE solution on parameter matrices is shown to be the same as that for ARE; and the MARE solution is found to be non-increasingly dependent on the characteristic parameter. To the best of our knowledge, both stabilizing property and parameter dependence have not been studied in detail for MARE before. The parameter dependence results are important for studying discrete-time network synchronization, including the event-triggered cooperative control of linear dynamical networks in [18] and the input-saturated synchronization in this note.

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Noticeably, it is difficult to analyze the uniqueness of the *almost stabilizing* solution for MARE. When $\gamma \neq 1$, none of the existing methods in the literature for proving the uniqueness of an almost stabilizing solution to ARE is applicable to MARE. Nevertheless, for null controllable systems [19]–[23] containing single-integrator [24]–[26], double-integrator [26], [27] and multiple-integrator systems [28] as special cases, that is, (A, B) is stabilizable and the spectral radius of A is not larger than 1, the uniqueness of the almost stabilizing and positive semi-definite solution of MARE with $Q = 0$ is demonstrated in this note. Consequently, a key parameter dependence result for MARE can be established. Specifically, under some assumptions, the solution converges monotonically to a zero matrix as Q approaches zero.

This key MARE parameter dependence result is then applied to synchronization of dynamical networks with linear dynamics subject to input saturation. We are concerned with the discrete-time problem for leader-following networks on undirected switching graphs. In this setting, the network coupling makes it more difficult than the continuous-time one discussed in [20], since the ARE results in [23] for one single system will not work for multi-agent systems (MAS). In light of this, the MARE is explored in this note. The obtained MARE counterparts of the ARE results in [23] enable one to design low-gain feedback laws for leader-following networks with discrete-time higher-order dynamics subject to input saturation, so as to achieve semi-global exponential synchronization.

Nomenclature: For $X \in \mathbb{R}^{p\times p}$, its eigenvalues are denoted by $\lambda_1(X)$, $\lambda_2(X)$,..., $\lambda_p(X)$ satisfying that $|\lambda_1(X)| \leq ... \leq |\lambda_p(X)|$; and $\rho(X) = |\lambda_p(X)|$ denotes its spectral radius. A square matrix A is said to be Schur if $\rho(A) < 1$. I denotes an identity matrix with compatible dimension; diag$\{\cdot\}$ denotes a diagonal matrix; and $\otimes$ denotes the Kronecker product.

II. Modified Algebraic Riccati Equation

A. Preliminaries

Lemma 1. [9] Let $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, $Q \succ 0$, and $R \succ 0$. Assume that the pair (A, B) is stabilizable with $\rho(A) \geq 1$. Consider the modified algebraic Riccati equation (MARE)

$$P = g_\gamma(P) \triangleq A^T PA - \gamma A^T PB(B^T PB + R)^{-1}B^T PA + Q, \quad (1)$$

and the modified algebraic Riccati inequality (MARI)

$$P \succeq g_\gamma(P) \quad \text{for a symmetric matrix } P, \quad (2)$$

as well as the strict MARI

$$P \succ g_\gamma(P) \quad \text{for a symmetric matrix } P. \quad (3)$$

Then, there exists a critical value $\gamma_c \in [0, 1)$ satisfying that

$$\gamma_c \triangleq \inf\{\ \gamma \mid \exists\ P \succeq 0 \text{ solving MARI (3) } \} \quad (4)$$

and $\gamma_c \geq \gamma_1(A) \triangleq 1 - 1/(\rho(A))^2 \geq 0$. For any $\gamma \in (\gamma_1(A), 1]$, every positive semi-definite solution (if it exists) to MARI (2) is positive definite. For any $\gamma \in (\gamma_c, 1]$, MARE (1) has a unique positive semi-definite solution $P \succeq 0$, which is positive definite; furthermore, P is the limit of any sequence of matrices $\{P_k\}$ defined by the following modified Riccati difference equations:

$$P_{k+1} = g_\gamma(P_k), \quad k = 0, 1, 2, ..., \quad (5)$$

for any initial $P_0 \succeq 0$. The scalar parameter γ is referred to as the *characteristic parameter* of MARE (1) and of other corresponding Riccati equation/inequalities.

Lemma 2. [12, Lemma 5.4 (b)] The critical value γ_c defined in (4) satisfies that $0 \leq \gamma_1(A) \leq \gamma_c \leq \gamma_2(A)$, where $\gamma_1(A) = 1 - 1/(\rho(A))^2$ and $\gamma_2(A) \triangleq 1 - 1/(M(A))^2$, with $M(A) \triangleq \prod_{i=1}^{n} \max\{1, |\lambda_i(A)|\}$.

Corollary 1. *Assume that $Q \succ 0$, $R \succ 0$, and A is Schur. For any $\gamma \in (0, 1]$, every positive semi-definite solution to MARI (2) is positive definite; and MARE (1) has a unique positive semi-definite solution P, where $P \succ 0$; furthermore, this solution P is the limit of any sequence $\{P_k\}$ defined by (5) for any initial $P_0 \succeq 0$.*

Proof. If $P \succeq 0$ solves (2), then by [9, Lemma 1 (f)], $P \succeq g_\gamma(P) \succeq (1-\gamma)A^T P A + Q \succ 0$. Since A is Schur, there exists a $W \succ 0$ such that $W = A^T W A + Q + I$. Thus, W solves (3) and the condition of [9, Lemma 4] is satisfied. Denote $Z_1 = g_\gamma(0)$, $Z_{k+1} = g_\gamma(Z_k)$. By [9, Lemmas 1 (b,c), and 4], $\exists\ \breve{Z}$ *s.t.* $\breve{Z} \succeq ... \succeq Z_2 \succeq Z_1 = Q$. So $\bar{P} \triangleq \lim_{k\to\infty} Z_k$ exists *s.t.* $\bar{P} = g_\gamma(\bar{P})$. Denote $\bar{K} \triangleq -(B^T \bar{P} B + R)^{-1} B^T \bar{P} A$, $\bar{A} \triangleq A + B\bar{K}$. Consider the operator $\mathcal{L}(Y) = (1-\gamma)A^T Y A + \gamma \bar{A}^T Y \bar{A}$. One has $\mathcal{L}(\bar{P}) + Q + \gamma \bar{K}^T R \bar{K} = g_\gamma(\bar{P}) = \bar{P} \succ \mathcal{L}(\bar{P})$. By [9, Lemma 3], $\lim_{k\to\infty} \mathcal{L}^k(Y) = 0,\ \forall\ Y \succeq 0$. For any $\hat{Z}_0 \succeq \bar{P}$, denote $\hat{Z}_{k+1} = g_\gamma(\hat{Z}_k) \succeq g_\gamma(\bar{P}) = \bar{P}$. By [9, Lemma 1 (a,b)], $0 \preceq \hat{Z}_{k+1} - \bar{P} = g_\gamma(\hat{Z}_k) - g_\gamma(\bar{P}) \preceq (1-\gamma)A^T(\hat{Z}_k - \bar{P})A + \gamma \bar{A}^T(\hat{Z}_k - \bar{P})\bar{A} = \mathcal{L}(\hat{Z}_k - \bar{P}) \preceq ... \preceq \mathcal{L}^{k+1}(\hat{Z}_0 - \bar{P}) \to 0$. Consequently, $\lim_{k\to\infty} \hat{Z}_k = \bar{P}$. For any $P_0 \succeq 0$, $P_{k+1} = g_\gamma(P_k)$, let $\hat{Z}_0 = P_0 + \bar{P} \succeq P_0 \succeq Z_0 \triangleq 0$. By [9, Lemma 2], $\hat{Z}_k \succeq P_k \succeq Z_k$. Thus, $\lim_{k\to\infty} P_k = \bar{P}$. If $P \succeq 0$ solves (1), let $P_0 = P$, so finally one has $P = \lim_{k\to\infty} g_\gamma(P_k) = \lim_{k\to\infty} P_{k+1} = \bar{P} \succ 0$. □

B. Stabilizing Solution

Theorem 1. *Assume that $Q \succ 0$, $R \succ 0$, and the pair (A, B) is stabilizable. Let $\max\{0, 1 - 1/\rho(A)^2\} < \gamma \leq 1$. Denote $\alpha \triangleq 1 - \sqrt{1-\gamma} > 0$ and $\beta \triangleq 1 + \sqrt{1-\gamma}$ satisfying $(1-\alpha)^2 = (\beta - 1)^2 = 1 - \gamma$ and $1 - 1/\rho(A) < \alpha \leq \gamma \leq 1 \leq \beta$. For $\tau \in [\alpha, \beta]$, denote*

$$A_\tau \triangleq A - \tau B(B^T P B + R)^{-1} B^T P A. \quad (6)$$

Then, every positive semi-definite solution P (if it exists) to MARI (2) renders A_τ to be Schur for all $\tau \in [\alpha, \beta]$. This solution P is referred to as a stabilizing *solution of MARI (2). In particular, if this P also solves MARE (1), then it is referred to as a* stabilizing *solution of MARE (1).*

Proof. The stabilizing property with respect to $\tau = \alpha, \beta$ has been mentioned in [7, Theorem 2]. Similar to [7, Equation (10)], by straightforward manipulation, one obtains that

$$\begin{aligned} g_\gamma(X) = &(-\tau^2 + 2\tau - \gamma)A^T X B(B^T X B + R)^{-1} B^T X A \\ &+ A_\tau(X)^T X A_\tau(X) + K_\tau(X)^T R K_\tau(X) + Q, \end{aligned} \quad (7)$$

where $g_\gamma(X) \triangleq A^T X A - \gamma A^T X B(B^T X B + R)^{-1} B^T X A + Q$, $A_\tau(X) \triangleq A - B K_\tau(X)$, and $K_\tau(X) \triangleq \tau(B^T X B + R)^{-1} B^T X A$. For each $\tau \in [\alpha, \beta]$, one has $-\tau^2 + 2\tau - \gamma \geq 0$. Then, if $P \succeq 0$ solves MARI (2), since $Q \succ 0$, one has

$$P - A_\tau^T P A_\tau \succeq (K_\tau(P))^T R K_\tau(P) + Q \succ 0. \quad (8)$$

Due to the property of the Liapunov equation [33, Theorem 4], A_τ is Schur, and the positive definite solution P is a stabilizing solution to MARI (2). □

Remark 1. The stabilizing property with respect to $\tau = \alpha, \beta$ plays an important role in the modified optimal filtering problem [7]; the case of $\tau = 1$ can be verified following [29, Theorem 13.5.2]. If $\gamma = 1$, $\alpha = \gamma = 1 = \beta$, the MARI/MARE stabilizing property is the well-known one for ARI/ARE.

For MARI (2) or MARE (1) with general real symmetric Q and R, if R is nonsingular and a solution P satisfies that $(B^T P B + R)$ is nonsingular and $\rho(A_\gamma) \leq 1$ with $A_\gamma = A_\tau$ defined in (6) having $\tau = \gamma$, then this solution P is said to be *almost stabilizing*. If $\rho(A) \leq 1$, it is trivial to see that $P = 0$ is an almost stabilizing and positive semi-definite solution to MARI (2) and MARE (1) with $Q = 0$. Furthermore, we will demonstrate the uniqueness of an almost stabilizing and positive semi-definite solution to the following MARE:

$$X = A^T X A - \gamma A^T X B(B^T X B + R)^{-1} B^T X A. \quad (9)$$

Theorem 2. *Assume that $R \succ 0$, (A, B) is stabilizable, and $\rho(A) \leq 1$. Let $\gamma \in (0, 1]$. Then, $X = 0$ is the unique almost stabilizing and positive semi-definite solution to MARE (9).*

Proof. On the contrary, suppose that there exists a non-zero almost stabilizing solution $Y \succeq 0$. Denote $A_\gamma(X) \triangleq A - \gamma B(B^T X B + R)^{-1} B^T X A$. Then, $X = 0$ and $X = Y$ are both positive semi-definite solutions to the *linear matrix equation* (LME) [30]: $X = A^T X A_\gamma(Y) = (A_\gamma(Y))^T X A$. Applying [29, Theorem 5.2.3], there exists an eigenvalue λ_r of A and an eigenvalue λ_s of $A_\gamma(Y)$ such that $\lambda_r \lambda_s = 1$. Since $|\lambda_r| = |\lambda_s| \leq 1$, one has $|\lambda_r| = |\lambda_s| = 1$, and both A and $A_\gamma(Y)$ are not Schur. Thus, if $\rho(A) < 1$, a contradiction is obtained already. Now, assume that $\rho(A) = 1$.

Since the positive semi-definite matrix Y is orthogonally diagonalizable, there exists an orthogonal matrix U such that $Y = U^T \Lambda U$ with Λ being diagonal, and $\Lambda = \text{diag}\{0, \overline{Y}\}$, where $\overline{Y} \succ 0$ is a diagonal matrix with diagonal elements being the positive eigenvalues of Y. Denote $\bar{A} \triangleq U A U^T$, $\bar{B} \triangleq U B$, and $\bar{A}_\gamma \triangleq \bar{A} - \gamma \bar{B}(\bar{B}^T \Lambda \bar{B} + R)^{-1} \bar{B}^T \Lambda \bar{A}$. Then,

$$\Lambda = \bar{A}^T \Lambda \bar{A} - \gamma \bar{A}^T \Lambda \bar{B}(\bar{B}^T \Lambda \bar{B} + R)^{-1} \bar{B}^T \Lambda \bar{A}, \quad (10)$$

where $(\bar{A}, \bar{B})$ is stabilizable, and the eigenvalues of $\bar{A}$ and $\bar{A}_\gamma$ are all located within the closed unit disc. Now, partition

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$\bar{A}$ and $\bar{B}$ into block matrices with compatible dimensions as $\bar{A} = \begin{bmatrix} A_1 & A_4 \\ A_3 & A_2 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, such that (10) reduces to the following through straightforward manipulation:

$$\overline{Y} = A_2^T \overline{Y} A_2 - \gamma A_2^T \overline{Y} B_2 (B_2^T \overline{Y} B_2 + R)^{-1} B_2^T \overline{Y} A_2, \quad (11)$$

$$A_3^T \overline{Y} A_3 = \gamma A_3^T \overline{Y} B_2 (B_2^T \overline{Y} B_2 + R)^{-1} B_2^T \overline{Y} A_3,$$
$$A_3^T \overline{Y} A_2 = \gamma A_3^T \overline{Y} B_2 (B_2^T \overline{Y} B_2 + R)^{-1} B_2^T \overline{Y} A_2. \quad (12)$$

By (11) and noting that $\overline{Y} \succ 0$, one can easily verify that A_2 is nonsingular. By (12), $A_3 = \gamma B_2 (B_2^T \overline{Y} B_2 + R)^{-1} B_2^T \overline{Y} A_3$. Applying the Matrix Inversion Formulas [31], $B_2 (B_2^T \overline{Y} B_2 + R)^{-1} B_2^T \overline{Y} = I - (I + B_2 R^{-1} B_2^T \overline{Y})^{-1}$, thus one obtains that $(\gamma - 1) A_3 = \gamma (I + B_2 R^{-1} B_2^T \overline{Y})^{-1} A_3$. Therefore,

$$A_3 A_3^T = (\gamma - 1) B_2 R^{-1} B_2^T \overline{Y} A_3 A_3^T. \quad (13)$$

For any eigenvalue $\lambda(A_3 A_3^T)$ of $A_3 A_3^T \succeq 0$ with a real eigenvector $x \neq 0$, left-multiplying $x^T \overline{Y}$ and right-multiplying x on (13), one obtains $\lambda(A_3 A_3^T) = 0$; otherwise, one would have $0 < x^T \overline{Y} x = (\gamma - 1) x^T \overline{Y} B_2 R^{-1} B_2^T \overline{Y} x \leq 0$, which is impossible. Since the Jordan form of $A_3 A_3^T$ is diagonal, one concludes that $A_3 = 0$, and the eigenvalues of A_1 and A_2 are all located within the closed unit disc.

Next, applying the Popov-Belevitch-Hautus (PBH) rank test for stabilizability [32] to the pair $(\bar{A}, \bar{B})$, one obtains that $[A_2 - \lambda I, B_2]$ is of full row-rank for any λ satisfying that $|\lambda| \geq 1$, which in turn shows that (A_2, B_2) is stabilizable. Comparing (11) with (9), one obtains similarly that both A_2 and $A_{2\gamma}$ are not Schur, where $A_{2\gamma} \triangleq A_2 - \gamma B_2 (B_2^T \overline{Y} B_2 + R)^{-1} B_2^T \overline{Y} A_2$. Denote $\overline{F} = \gamma (B_2^T \overline{Y} B_2 + R)^{-1} B_2^T \overline{Y} A_2$. Then, $A_2 = A_{2\gamma} + B_2 \overline{F}$, and by (11),

$$\overline{Y} = A_{2\gamma}^T \overline{Y} A_{2\gamma} + A_{2\gamma}^T \overline{Y} B_2 \overline{F}. \quad (14)$$

Noting that $A_{2\gamma}^T \overline{Y} B_2 \overline{F} = A_2^T \overline{Y} B_2 \overline{F} - \overline{F}^T B_2^T \overline{Y} B_2 \overline{F} = (\frac{1}{\gamma} - 1) \overline{F}^T B_2^T \overline{Y} B_2 \overline{F} + \frac{1}{\gamma} \overline{F}^T R \overline{F} \succeq 0$, Equation (14) can be rewritten as $A_{2\gamma}^T \overline{Y} A_{2\gamma} + \overline{G}^T \overline{G} = \overline{Y} \succ 0$, where $\overline{G}^T \overline{G} \triangleq (\frac{1}{\gamma} - 1) \overline{F}^T B_2^T \overline{Y} B_2 \overline{F} + \frac{1}{\gamma} \overline{F}^T R \overline{F}$. Thus, by the property of Liapunov equation [33, Theorem 3], the eigenvalues of $A_{2\gamma}$ corresponding to the observable modes of the pair $(\overline{G}, A_{2\gamma})$ are all located inside the open unit disc. Since $A_{2\gamma}$ is not Schur, at least one eigenvalue μ exists such that $|\mu| = 1$ and the corresponding mode of $(\overline{G}, A_{2\gamma})$ is unobservable. Then, applying the eigenvector test for detectability [32], there exists an eigenvector $y \neq 0$ of $A_{2\gamma}$ corresponding to μ such that $A_{2\gamma} y = \mu y$ and $\overline{G} y = 0$.

Therefore, $\overline{G}^T \overline{G} y = 0$; $\overline{F} y = 0$, $B_2^T \overline{Y} A_2 \cdot y = 0$. Thus, $A_2 \cdot y = A_{2\gamma} y = \mu y$, $B_2^T \overline{Y} y = 0$, and $y = \bar{\mu} A_2 \cdot y$. Besides, by (11), $A_2^T \overline{Y} A_2 \cdot y = \overline{Y} y$. Since (A_2, B_2) is stabilizable, there exists K_2 such that both $(A_2 + B_2 K_2)$ and $(A_2^T + K_2^T B_2^T)$ are Schur. Denoting $\tilde{y} = \overline{Y} y$, one obtains that

$$(A_2^T + K_2^T B_2^T) \tilde{y} = \bar{\mu} A_2^T \overline{Y} A_2 \cdot y + K_2^T B_2^T \overline{Y} y = \bar{\mu} \tilde{y},$$

with $|\bar{\mu}| = 1$, $\tilde{y} = 0$. Thus, $y = \overline{Y}^{-1} \tilde{y} = 0$, which is a contradiction.

Therefore, $X = 0$ is the only positive semi-definite and almost stabilizing solution to MARE (9). □

Remark 2. When $\gamma = 1$, for any $Q \succeq 0$, $R \succ 0$ and stabilizable (A, B), the reduced ARE of MARE (1) has a unique almost stabilizing solution which is maximal and positive semi-definite [29, Theorem 13.5.2]. However, for MARE (1) with the parameter matrix $Q = 0$ and the characteristic parameter $\gamma \in (\gamma_c, 1)$, the available methods in the literature, *e.g.* [29], for proving the uniqueness of an almost stabilizing solution are all inapplicable. Nevertheless, the uniqueness as shown in Theorem 2 is obtained when $\rho(A) \leq 1$ and $Q = 0$.

C. Parameter Dependence

Proposition 1. *Assume that the pair (A, B) is stabilizable with $\rho(A) \geq 1$; $Q \succ 0$, $R \succ 0$; and $\gamma \in (\gamma_c, 1]$ with γ_c defined in (4). Consider MARI (2), MARE (1), and the following MARI:*

$$g_\gamma(P) \succeq P. \quad (15)$$

Let three positive semi-definite matrices $\breve{P}$, $\tilde{P}$ and $\hat{P}$ solve (2), (1) and (15), respectively. Then, $\breve{P} \succeq \tilde{P} \succeq \hat{P}$ and $\tilde{P} \succeq Q$.

Proof. First, using (7), one obtains that $g_\gamma(X) \succeq Q$ for any $X \succeq 0$. Therefore, the unique positive semi-definite solution $\tilde{P}$ to MARE (1) satisfies that $\tilde{P} \succeq Q$. Denote $\breve{P}_0 \triangleq \breve{P}$ and $\hat{P}_0 \triangleq \hat{P}$, and define two sequences of matrices through modified Riccati difference equation (5); that is, $\breve{P}_{k+1} \triangleq g_\gamma(\breve{P}_k)$, $\hat{P}_{k+1} \triangleq g_\gamma(\hat{P}_k)$, $k = 0, 1, 2, \ldots$. Then, $g_\gamma(\breve{P}_k) \succeq Q$, and using [9, Lemma 1 (c) and Lemma 4], one obtains that $\breve{P} \succeq \breve{P}_1 \succeq \breve{P}_2 \succeq \ldots \succeq Q \succ 0$, and $M_{\hat{P}} \succeq \ldots \succeq \hat{P}_2 \succeq \hat{P}_1 \succeq \hat{P}$, where $M_{\hat{P}}$ is a $\hat{P}$-dependent upper-bounded matrix, which exists as $\hat{P} \succeq 0$ and $\gamma > \gamma_c$. Meanwhile, since $\breve{P}_0 \succeq 0$ and $\hat{P}_0 \succeq 0$, Lemma 1 can be applied to show that the two sequences $\{\breve{P}_k\}$ and $\{\hat{P}_k\}$ both converge to $\tilde{P}$. Consequently, $\breve{P} \succeq \tilde{P} \succeq \hat{P}$. □

Proposition 2. *In Lemma 1, the critical value γ_c is non-decreasing with respect to the matrix Q in the sense that if $Q_2 \succeq Q_1$, then $\gamma_c(Q_2) \geq \gamma_c(Q_1)$, where the critical value $\gamma_c = \gamma_c(Q_2)$ for Q_2 and $\gamma_c = \gamma_c(Q_1)$ for Q_1. Meanwhile, γ_c is non-decreasing with respect to the matrix R; specifically, if $Q = \varepsilon I$ and $R = \delta I$ with $\varepsilon, \delta > 0$, then $\gamma_c = \gamma_c(\varepsilon, \delta)$ is a non-decreasing function with respect to both ε and δ.*

Proof. It follows from the definition of γ_c in (4). □

Theorem 3. *Assume that the pair (A, B) is stabilizable with $\rho(A) \geq 1$; $Q_0 \succ 0$, $R_0 \succ 0$; and $\gamma \in (\gamma_c, 1]$, where γ_c is defined in (4) with $Q = Q_0$ and $R = R_0$. Then, the unique positive semi-definite solution to MARE (1), denoted as $P(Q, R)$, is non-decreasing and, hence, continuous with respect to both parameter matrices Q and R in the sense that if $Q_0 \succeq Q_2 \succeq Q_1 \succ 0$, $R_0 \succeq R_2 \succeq R_1 \succ 0$, then $P(Q_2, R_2) \succeq P(Q_1, R_1)$. Specifically, if $Q_0 = \tilde{\varepsilon} I$, $R_0 = \tilde{\delta} I$ with $\tilde{\varepsilon}, \tilde{\delta} > 0$, and $Q = \varepsilon I$ and $R = \delta I$ with $\varepsilon \in (0, \tilde{\varepsilon}]$ and $\delta \in (0, \tilde{\delta}]$, then $P(\varepsilon, \delta) \triangleq P(\varepsilon I, \delta I)$ is non-decreasing and, hence, continuous with respect to both ε and δ. Besides, for fixed $Q = Q_0$ and $R = R_0$, the unique positive semi-definite solution to MARE (1), denoted as $P(\gamma)$, is non-increasing with respect to the MARE characteristic parameter $\gamma \in (\gamma_c, 1]$.*

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Proof. Denote γ_c with respect to (Q_1, R_1) and (Q_2, R_2) as $\gamma_c(Q_1, R_1)$ and $\gamma_c(Q_2, R_2)$, respectively. By Proposition 2, $\gamma_c(Q_1, R_1) \leq \gamma_c(Q_2, R_2)$. Let $\gamma > \gamma_c(Q_2, R_2)$, and denote the corresponding $g_\gamma(X)$ as $g_\gamma(X, Q_1, R_1)$ and $g_\gamma(X, Q_2, R_2)$, respectively. Then, $P_2 = g_\gamma(P_2, Q_2, R_2) \succeq g_\gamma(P_2, Q_1, R_1)$ and $P_1 = g_\gamma(P_1, Q_1, R_1)$, where $P_2 \triangleq P(Q_2, R_2)$ and $P_1 \triangleq P(Q_1, R_1)$. By Proposition 1, $P_2 \succeq P_1$. The non-increasing monotonicity of the solution with respect to γ can be similarly verified. □

Proposition 3. *Assume that $\tilde{\varepsilon}, \tilde{\delta} > 0$ and the pair (A, B) is stabilizable with $\rho(A) \geq 1$. Let $\gamma \in (\gamma_c, 1]$ with $\gamma_c = \gamma_c(\tilde{\varepsilon}, \tilde{\delta})$ defined in Proposition 2. Then, for any $\varepsilon \in (0, \tilde{\varepsilon}]$ and $\delta \in (0, \tilde{\delta}]$, there exists a unique positive semi-definite matrix $P(\varepsilon, \delta)$ solving the MARE: $P = A^TPA - \gamma A^TPB(B^TPB + \delta I)^{-1}B^TPA + \varepsilon I$; and $P(\varepsilon, \delta) \succ 0$. Therefore, there exists at least one positive definite solution P to the MARI*

$$P \succeq A^TPA - \gamma A^TPB(B^TPB + \delta I)^{-1}B^TPA + \varepsilon I. \quad (16)$$

Proof. It follows from Proposition 2 and Lemma 1. □

Remark 3. A special case of MARI (16) is the following:

$$P \succ A^TPA - \gamma A^TPB(B^TPB)^{-1}B^TPA, \quad (17)$$

which is presented in [8, (9)], and plays an important role in [14]–[17]. Any possible solution $P \succ 0$ to MARI (17) also solves MARI (16) for some sufficiently small $\varepsilon, \delta > 0$. In Theorem 3, the continuity of ARE solution with respect to parameter matrices (see [34]) is extended to MARE. The results in Theorem 3 can be applied to the discrete-time event-triggered synchronization problem [18].

D. Key Result on Parameter Dependence of MARE Solution

Assumption 1 below is standard for semi-global stabilization of control systems subject to actuator saturation [19]–[23].

Assumption 1. *The pair (A, B) is stabilizable and $\rho(A) = 1$.*

Theorem 4. *Let Assumption 1 hold and $\gamma \in (0, 1]$, $\tilde{\varepsilon} > 0$. Then, for each $\varepsilon \in (0, \tilde{\varepsilon}]$, there exists a unique positive definite matrix $P(\varepsilon)$ that solves the MARE*

$$P = A^TPA - \gamma A^TPB(B^TPB + I)^{-1}B^TPA + \varepsilon I. \quad (18)$$

Moreover, $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$ *monotonically.*

Proof. Combining Lemma 2, Proposition 3, and Theorems 1–3, one obtains the results. □

Remark 4. To prove $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$, it is necessary to establish the uniqueness of an almost stabilizing and positive semi-definite solution. The result in [23, Lemma 3.1] is merely a special case of Theorem 4. If $\gamma \neq 1$, no scaling manipulation can let γ be absorbed into P, B, A, R and Q.

III. Semi-global Discrete-time Synchronization via Low-Gain Feedback

In this section, the MARE results are applied to the semi-global synchronization problem for discrete-time linear MAS subject to input saturation. Consider a group of N agents, labeled as $1, 2, ..., N$. The motion of each agent is as follows:

$$x_i(t+1) = Ax_i(t) + B \cdot \text{sat}(u_i(t)), \ i = 1, 2, ..., N, \quad (19)$$

where $x_i \in \mathbb{R}^n$ is the state of agent i; $u_i \in \mathbb{R}^m$ is the control input for agent i; and $\text{sat} : \mathbb{R}^m \to \mathbb{R}^m$ is a *saturation operator* defined as $\text{sat}(u_i) \triangleq [\text{sat}(u_{i1}), \text{sat}(u_{i2}), ..., \text{sat}(u_{im})]^T$, with the *saturation function* $\text{sat}(u_{ij}) \triangleq \text{sgn}(u_{ij}) \min\{|u_{ij}|, \varpi\}$ for an *a priori* given input-saturation threshold $\varpi > 0$. Denote $u = [u_1^T, u_2^T, ..., u_N^T]^T$ and $\tilde{u} \triangleq [\text{sat}(u_1)^T, \text{sat}(u_2)^T, ..., \text{sat}(u_N)^T]^T$. The dynamics of the leader, labeled as $N + 1$, are described by

$$x_{N+1}(t+1) = Ax_{N+1}(t). \quad (20)$$

The problem of semi-global leader-following synchronization for the agents and leader described above is as follows: Design some linear feedback law u_i for each agent i, which uses only local information from neighboring agents, such that for any *a priori* given bounded set $\mathcal{X} \subset \mathbb{R}^n$, the synchronization $\lim_{t\to\infty} \|x_i(t) - x_{N+1}(t)\| = 0$, $\forall i = 1, 2, ..., N$, is exponentially achieved as long as $x_i(0) \in \mathcal{X}$, $\forall i = 1, 2, ..., N, N + 1$.

A communication network consisting of N agents is described by an undirected graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$ [20], [21]. Let $\overline{\mathcal{G}}(t) = \overline{\mathcal{G}}_{\delta(t)}$ be an extended graph generated by the leader and $\mathcal{G}(t)$, and $H(t) \triangleq \text{diag}\{h_1(t), h_2(t), ..., h_N(t)\} = H_{\delta(t)}$ be defined as $h_i(t) = 1$ if agent i is a neighbor of the leader at time step t, and $h_i(t) = 0$ otherwise, and $L(t) = L_{\delta(t)}$ be the Laplacian matrix ([20], [21]) of $\mathcal{G}(t)$; denote $\mathscr{L}(t) \triangleq L(t) + H(t) = \mathscr{L}_{\delta(t)}$. Here, $\delta : \mathbb{N} \to \Gamma$ is a switching signal whose value at time t equals the index of $\overline{\mathcal{G}}(t)$, and the index set Γ contains indexes of all extended graphs. Γ_{tree} denotes the set of indexes of extended graphs that contain a spanning tree with the leader as the root vertex. If $\delta(t) = s \in \Gamma_{tree}$ and $\mathscr{L}_s = L_s + H_s$, then $\mathscr{L}_s \succ 0$ [20], [21].

Lemma 3. Let $s \in \Gamma_{tree}$ and $\mathscr{L}_s = L_s + H_s$ with eigenvalues $0 < \lambda_1 \leq ... \leq \lambda_N$. Then, $\min_{\omega \in \mathbb{R}} \max_{i=1,...,N} |1 - \omega\lambda_i| = (\lambda_N - \lambda_1)/(\lambda_N + \lambda_1) < 1$, $\arg\min_\omega \max_{i=1,...,N} |1 - \omega\lambda_i| = 2/(\lambda_1 + \lambda_N)$.

Proof. It can be proved similarly to [15, Equation (14)]. □

Remark 5. For $s \in \Gamma_{tree}$, because the number of possible values of $\lambda_{1/N}(\mathscr{L}_s) \triangleq \lambda_1/\lambda_N$ is finite, one can find the minimum $\lambda_{1/N}(\mathscr{L}_s)$, denoted as $\min\{\lambda_{1/N}\}$, using an exhaustive search method; and there are a finite number of possible values of $(2/(\lambda_1 + \lambda_N))$, the set of which is denoted by $\overline{\Omega}$.

Assumption 2. *Extended graph $\overline{\mathcal{G}}(t)$ consisting of the N agents and the leader contains a spanning tree rooted at the leader all the time, that is, $\delta(t) \in \Gamma_{tree}$, $\forall t \in \mathbb{N}$.*

A low-gain feedback design for leader-following multi-agent systems (19) and (20) is carried out in three steps.

Step (i). Find $P = P(\varepsilon) \succ 0$ to solve the MARE

$$P = A^TPA - \theta A^TPB(B^TPB + I)^{-1}B^TPA + \varepsilon I, \quad (21)$$

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where $\varepsilon \in (0,1]$ is the low-gain parameter to be designed, and the MARE characteristic parameter θ is given by

$$\theta \triangleq \frac{4}{(\sqrt{\min\{\lambda_{1/N}\}} + \frac{1}{\sqrt{\min\{\lambda_{1/N}\}}})^2} \in (0,1], \tag{22}$$

with $\min\{\lambda_{1/N}\}$ defined in Remark 5.

Step (ii). Set the *controller parameter* $\omega(t)$ as

$$\omega(t) \triangleq \frac{2}{\lambda_1(t) + \lambda_N(t)} \in \overline{\Omega}, \tag{23}$$

where $\lambda_1(t)$ and $\lambda_N(t)$ denote the smallest and the largest eigenvalues, respectively, of $\mathscr{L}(t) = L(t) + H(t)$; and $\overline{\Omega}$ is defined in Remark 5.

Step (iii). Construct a linear feedback law as

$$u_i = K(h_i(t)(x_i - x_{N+1}) + \sum\nolimits_{j=1}^{N} a_{ij}(t)(x_i - x_j)) \tag{24}$$

for agents $i = 1, 2, ..., N$, where

$$K \triangleq -\omega(B^T P(\varepsilon) B + I)^{-1} B^T P(\varepsilon) A, \tag{25}$$

with $\omega = \omega(t)$ at time step t.

The numerical solution for $P(\varepsilon)$ is referred to [9, Theorem 6]. The key MARE result $\lim_{\varepsilon\to 0} P(\varepsilon) = 0$ in Theorem 4 guarantees the effectiveness of the bounded input technique.

Lemma 4. The unique positive definite solution $P(\varepsilon)$ to MARE (21) and the controller matrix K in (25) satisfy that

$$I_N \otimes P(\varepsilon) \succeq \left(\mathscr{L}(t) \otimes K\right)^T \left(\mathscr{L}(t) \otimes K\right), \tag{26}$$

where $\otimes$ denotes the Kronecker product [30].

Proof. Denote $\tilde{s} = \arg\min_{s\in\Gamma_{tree}} \lambda_1(\mathscr{L}_s)/\lambda_N(\mathscr{L}_s)$. Then, $\theta = 4\lambda_1(\mathscr{L}_{\tilde{s}})\lambda_N(\mathscr{L}_{\tilde{s}})/(\lambda_1(\mathscr{L}_{\tilde{s}}) + \lambda_N(\mathscr{L}_{\tilde{s}}))^2$. Denoting $\beta \triangleq 1 + \sqrt{1-\theta} = 2\lambda_N(\mathscr{L}_{\tilde{s}})/(\lambda_1(\mathscr{L}_{\tilde{s}}) + \lambda_N(\mathscr{L}_{\tilde{s}})) \geq 2\lambda_N(t)/(\lambda_1(t) + \lambda_N(t)) = \lambda_N(t)\omega(t)$ and by (8) with $(Q, R, \gamma, \tau) = (\varepsilon I, I, \theta, \beta)$, one has $P(\varepsilon) \succeq \lambda_N(t)^2 K^T K + \varepsilon I \succ \lambda_N(t)^2 K^T K$. Obviously, $\lambda_N(t)^2 I_N \succeq \mathscr{L}(t)^2$. Then, one has $I_N \otimes P(\varepsilon) \succeq \lambda_N(t)^2 I_N \otimes K^T K \succeq \mathscr{L}(t)^2 \otimes K^T K = (\mathscr{L}(t) \otimes K)^T (\mathscr{L}(t) \otimes K)$. □

Theorem 5. *Consider a multi-agent system consisting of N agents with linear dynamics (19) subject to an a priori given input-saturation threshold $\varpi > 0$, and a leader with dynamics (20). Let Assumptions 1 and 2 hold. Then, the controller given by (24) and (25) achieves semi-global synchronization of the multi-agent system. That is, for any a priori given bounded set $\mathcal{X} = \{x \in \mathbb{R}^n \mid \|x\|_\infty < R_{\mathcal{X}}\}$ with $R_{\mathcal{X}} > 0$, there exists an $\varepsilon^* \in (0,1]$ such that for the low-gain parameter $\varepsilon = \varepsilon^*$, $\lim_{t\to\infty} \|x_i(t) - x_{N+1}(t)\| = 0, \forall i = 1, 2, ..., N$, as long as $x_i(0) \in \mathcal{X}$ for all $i = 1, 2, ..., N, N+1$. Moreover, the convergence speed for the synchronization is exponential. Furthermore, an ε^* can be chosen such that*

$$\rho(P(\varepsilon^*)) \leq \varpi^2/(4NnR_{\mathcal{X}}^2). \tag{27}$$

Proof. Denote $\tilde{x}_i \triangleq x_i - x_{N+1}$, and $\tilde{x} = [\tilde{x}_1^T, \tilde{x}_2^T, ..., \tilde{x}_N^T]^T$. Applying the Kronecker product [30], $u_i = Kh_i(t)\tilde{x}_i + \sum_{j=1}^{N} K\left(a_{ij}(t)(\tilde{x}_i - \tilde{x}_j)\right)$; $u = (\mathscr{L}(t) \otimes K)\tilde{x}$. Therefore,

$$\tilde{x}_i(t+1) = A\tilde{x}_i(t) + B \cdot \mathrm{sat}\left((\mathscr{L}(t) \otimes K)\tilde{x}(t)\right), \tag{28}$$

for which the common quadratic Liapunov function $V(\tilde{x}) \triangleq \sum_{i=1}^{N} \tilde{x}_i^T P(\varepsilon)\tilde{x}_i = \tilde{x}^T \left(I_N \otimes P(\varepsilon)\right)\tilde{x}$ is used.

For any $\varepsilon_0 \in (0,1]$, since $\mathcal{X}$ is bounded, a *level set parameter* $c(\varepsilon_0) > 0$ can be defined as

$$c(\varepsilon_0) \triangleq \sup_{\varepsilon=\varepsilon_0, x_i(0)\in\mathcal{X}, i=1,2,...,N+1} V(\tilde{x}(0)). \tag{29}$$

And for $\varepsilon = \varepsilon_0$, define the level set

$$L_V(c(\varepsilon_0)) \triangleq \{\xi \in \mathbb{R}^{Nn} | V(\xi) \leq c(\varepsilon_0)\}, \tag{30}$$

which is bounded. Choose a sufficiently small ε^* such that when $\varepsilon_0 = \varepsilon^*$, conditions $\tilde{x}(t) \in L_V(c(\varepsilon^*))$ and $\omega(t) \in \overline{\Omega}$ imply that $\|(\mathscr{L}(t) \otimes K)\tilde{x}\|_\infty \leq \varpi$ for $i = 1, 2, ..., N$, where $\varpi > 0$ is the *a priori* given saturation threshold. The existence of an ε^* is guaranteed by the convergence $\lim_{\varepsilon\to 0} P(\varepsilon) = 0$ established in Theorem 4. In fact, if ε^* is chosen such that (27) holds, since $P(\varepsilon^*) \preceq \rho(P(\varepsilon^*))I$, by (29), one has $c(\varepsilon^*) \leq \rho(P(\varepsilon^*))\sum_{i=1}^{N} \|\tilde{x}_i(0)\|^2 \leq (\varpi^2/(4nNR_{\mathcal{X}}^2))Nn\|\tilde{x}_i(0)\|_\infty^2 \leq (\varpi^2/(4R_{\mathcal{X}}^2))(2R_{\mathcal{X}})^2 = \varpi^2$. By (26) and (30), if $\tilde{x}(t) \in L_V(c(\varepsilon^*))$ and $\omega(t) \in \overline{\Omega}$, then $\|(\mathscr{L}(t) \otimes K)\tilde{x}\|_\infty \leq \|(\mathscr{L}(t) \otimes K)\tilde{x}\| \leq \sqrt{c(\varepsilon^*)} \leq \varpi$. Within $L_V(c(\varepsilon^*))$, the dynamics of (28) remain linear for any $\delta(t) = s \in \Gamma_{tree}$, and can be equivalently expressed as $\tilde{x}(t+1) = (I_N \otimes A + \mathscr{L}(t) \otimes BK)\tilde{x}(t)$, where $\mathscr{L}(t) = L(t) + H(t)$, and K is given in (25).

For the remainder of the proof, $\varepsilon = \varepsilon^*$, and $P(\varepsilon^*)$ is denoted as P for short. Now, through straightforward manipulation, one can evaluate $\Delta V(t) \triangleq V(\tilde{x}(t+1)) - V(\tilde{x}(t))$, which is the variation of V along the discrete-time trajectories of $\tilde{x}$ within the set $L_V(c)$, as follows:

$$\begin{aligned}\Delta V(t) =&\tilde{x}^T \left(I_N \otimes \left(A^T PA - P\right) + 2\mathscr{L}(t) \otimes A^T PBK\right.\\ &\left. + \mathscr{L}(t)^2 \otimes K^T B^T PBK\right)\tilde{x}.\end{aligned} \tag{31}$$

Since $\mathscr{L}(t) \succ 0$, there exists an orthogonal matrix $U(t) = U_{\delta(t)}$ such that $\mathscr{L}(t) = \mathscr{L}_{\delta(t)} = U^T(t)\Lambda(t)U(t)$, $\Lambda(t) \triangleq \mathrm{diag}\{\lambda_1(t), \lambda_2(t), ..., \lambda_N(t)\} = \Lambda_{\delta(t)}$, where $\lambda_i(t) > 0$ are eigenvalues of $\mathscr{L}(t)$. Denote $z \triangleq (U(t) \otimes I_n)\tilde{x}$ and $z = [z_1^T, z_2^T, ..., z_N^T]^T$ with $z_i(t) \in \mathbb{R}^n$. Thus, (31) is continued as

$$\begin{aligned}&\Delta V(t) = z^T (I_N \otimes (A^T PA - P) + \Lambda(t) \otimes 2A^T PBK\\ &+\Lambda^2(t) \otimes K^T B^T PBK)z = -\sum\nolimits_{i=1}^{N} z_i^T \Phi(\lambda_i(t))z_i,\end{aligned} \tag{32}$$

where $\Phi(\phi) \triangleq -A^T PA + P - 2\phi A^T PBK - \phi^2 K^T B^T PBK$. Pre- and post-multiplying both sides of the inequality $B^T PB + I \succ B^T PB$ by $(B^T PB + I)^{-1}$, one can easily obtain that $-\omega A^T PBK \succeq K^T B^T PBK$. Therefore,

$$\Phi(\phi) \succeq \Psi(\psi), \quad \psi \triangleq 1 - (1 - \phi\omega)^2, \tag{33}$$

where $\Psi(\psi) \triangleq P - A^T PA + \psi A^T PB(B^T PB + I)^{-1} B^T PA$. Denote $\psi_i(t, \omega) \triangleq 1 - (1 - \lambda_i(t)\omega)^2$. Using Lemma 3, then $\max_\omega \min_{i=1,...,N} \psi_i(t, \omega) = 4\lambda_N(t)\lambda_1(t)/(\lambda_N(t) + \lambda_1(t))^2$ is achieved by taking $\omega = 2/(\lambda_1(t) + \lambda_N(t)) = \omega(t)$. Thus, for all $i = 1, ..., N$, one has $\psi_i(t, \omega(t)) \geq 4/(\sqrt{\lambda_1(t)/\lambda_N(t)} + 1/\sqrt{\lambda_1(t)/\lambda_N(t)})^2 \geq \theta > 0$, where θ is defined in (22). By (21) and (33), $\Phi(\lambda_i(t)) \succeq \Psi(\psi_i(t, \omega(t))) \succeq \Psi(\theta) = \varepsilon^* I$. By (32) and the fact that $z^T z = \tilde{x}^T\tilde{x}$, one has

$$\Delta V(t) \leq -\varepsilon^* \tilde{x}^T \tilde{x} < 0, \forall \tilde{x} \in L_V(c) \setminus \{0\}. \tag{34}$$

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Since $\tilde{x}(0) \in L_V(c)$, $\tilde{x}(t)$ will always stay in $L_V(c)$ and the agent dynamics will always remain linear. Consequently, by the Liapunov stability theory, the discrete-time trajectory $\tilde{x}$ starting from the level set $L_V(c)$ will converge exponentially to the origin $\tilde{x} = 0$ as t goes to infinity, which in turn implies that $\lim_{t\to\infty} \|x_i(t) - x_{N+1}(t)\| = 0,\ i = 1, 2, ..., N.$ □

Remark 6. The control design for the semi-global synchronization is dependent on the given bounded set $\mathcal{X}$, which can be arbitrarily large. The low-gain feedback law (24) depends on $\varepsilon = \varepsilon^*$ satisfying (27) to prevent input saturation. In view of (34), the low-gain parameter acts as a synchronization performance indicator. The larger the ε^* is, the faster the synchronization will be. While by (27), the setting of ε^* is dependent on the input-saturation threshold ϖ and the bounded set $\mathcal{X}$ for the initial states. The larger the ϖ is, the larger the ε^* can be; the smaller the $\mathcal{X}$ is, the larger the ε^* can be. Given ϖ and $\mathcal{X}$, the best low-gain setting is to make (27) an equality. In addition, because MAS (19) is allowed to contain high-order integrator dynamics, the global synchronization cannot be achieved via a saturated linear controller [22], [23].

IV. Conclusion

In this note, the stabilizing property and the parameter dependence have been studied for MARE. The uniqueness of an almost stabilizing and positive semi-definite solution has been established for MARE with $Q = 0$. The discrete-time semi-global synchronization problem has been solved for linear MAS subject to input saturation. Future studies include: MARE with general real symmetric parameter matrices Q and R; the global synchronization for networks containing high-order integrator dynamics subject to actuator saturation.

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