



Title	Optimal proportional reinsurance with common shock dependence
Author(s)	Yuen, KC; Liang, Z; Zhou, M
Citation	Insurance: Mathematics and Economics, 2015, v. 64, p. 1-13
Issued Date	2015
URL	http://hdl.handle.net/10722/214577
Rights	This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License.

Optimal proportional reinsurance with common shock dependence

Kam Chuen Yuen¹, Zhibin Liang^{2*}, Ming Zhou³

¹ Department of Statistics and Actuarial Science,
The University of Hong Kong, Pokfulam Road, Hong Kong

² School of Mathematical Sciences and Institute of Finance and Statistics,
Nanjing Normal University, Jiangsu 210023, China

³ China Institute for Actuarial Science,
Central University of Finance and Economics, Beijing 100081, China

*Corresponding author: Zhibin Liang, Email: liangzhibin111@hotmail.com.

Abstract

In this paper, we consider the optimal proportional reinsurance strategy in a risk model with multiple dependent classes of insurance business, which extends the work of Liang and Yuen (2014) to the case with the reinsurance premium calculated under the expected value principle and to the model with two or more classes of dependent risks. Under the criterion of maximizing the expected exponential utility, closed-form expressions for the optimal strategies and value function are derived not only for the compound Poisson risk model but also for the diffusion approximation risk model. In particular, we find that the optimal reinsurance strategies under the expected value premium principle are very different from those under the variance premium principle in the diffusion risk model. The former depends not only on the safety loading, time and interest rate, but also on the claim size distributions and the counting processes, while the latter depends only on the safety loading, time and interest rate. Finally, numerical examples are presented to show the impact of model parameters on the optimal strategies.

Keywords: Dependent risks; Hamilton-Jacobi-Bellman equation; Compound Poisson process; Brownian motion; Exponential utility; Proportional reinsurance

1 Introduction

In the actuarial literature, the problem of optimal reinsurance has been studied by many authors in the past two decades. Among others, Browne (1995), Schmidli (2001, 2002), Promislow and Young (2005), and Bai et al. (2013) derive optimal strategies that minimize ruin probability; Kaluszka (2001, 2004) examine the optimal reinsurance problem under various mean-variance premium principles; Centeno (1986, 2002), Hald and Schmidli (2004), and Liang and Guo (2008) focus on constructing optimal contracts that maximize the adjustment coefficient by the martingale approach; and Cai and Tan (2007), Bernard and Tian (2009), Chi and Meng (2014), and Cheung et al. (2014) investigate the optimal problem by adopting the criteria of minimizing tail risk measures such as value at risk and conditional tail expectation. In this paper, we study the optimal reinsurance problem with the objective that maximizes the expected utility of terminal wealth, which is a popular criterion for various optimization problems in finance and modern risk theory. See, for example, Irgens and Paulsen (2004), Yang and Zhang (2005), Liang et al. (2011), Liang and Yuen (2014), and references therein.

Suppose that an insurance company that has a portfolio of $m \geq 2$ dependent classes of insurance business such as auto insurance, casualty insurance, health insurance, life insurance, and so on. For the l th ($l = 1, 2, \dots, m$) class of business, let $X_i^{(l)}$ ($i = 1, 2, \dots$) be the claim size random variables following a common distribution $F_l(x)$ with $F_l(x) = 0$ for $x \leq 0$ and $0 < F_l(x) < 1$ for $x > 0$. Their means are denoted by $\mu_l = E(X^{(l)})$. For each l , it is assumed that the moment generating function $M_l(r) = E(e^{rX^{(l)}})$ exists. Then, the aggregate claims process for the l th class of risk is given by

$$S_l(t) = \sum_{i=1}^{\tilde{N}_l(t)} X_i^{(l)},$$

where $\tilde{N}_l(t)$ is the claim number process for class l for $l = 1, 2, \dots, m$. As usual, it is assumed that $\{X_i^{(l)}; l = 1, 2, \dots, m, i = 1, 2, \dots\}$ are independent claim size random variables, and that they are independent of $\{\tilde{N}_l(t); l = 1, 2, \dots, m\}$.

In practice, the m classes of insurance business are usually dependent in some way. A typical example is that an earthquake, hurricane or tsunami, often leads to various kinds of insurance claims such as medical claims, death claims, household claims, etc. This example demonstrates a practical scenario that a single event generates claims from different lines of insurance. To depict such a dependence structure among several classes of business, the so-called common shock risk model may be of some practical relevance. In this paper, we consider the risk model with common

shock dependence in which the m claim number processes are correlated in the way that

$$\tilde{N}_l(t) = N_l(t) + N(t), \quad l = 1, 2, \dots, m,$$

with $N_l(t)$ and $N(t)$ being $m + 1$ independent Poisson processes with intensities $\lambda_1, \dots, \lambda_m$ and λ , respectively. It is obvious that the dependence among the m classes of business is due to a common shock governed by the counting process $N(t)$. Therefore, the aggregate claims process generated from m classes of business is given by

$$S_t = \sum_{l=1}^m S_l(t) = \sum_{l=1}^m \left(\sum_{i=1}^{N_l(t)+N(t)} X_i^{(l)} \right). \quad (1.1)$$

For each class, we further assume that $E(X^{(l)} e^{rX^{(l)}}) = M_l'(r)$ exists for $0 < r < \zeta^{(l)}$, and that $\lim_{r \rightarrow \zeta^{(l)}} E(X^{(l)} e^{rX^{(l)}}) = \infty$ for some $0 < \zeta^{(l)} \leq +\infty$.

Although research on risk models with dependent risk is increasing rapidly in recent years (see, for example, Wang (1998), Yuen et al. (2002, 2006), and the references therein), very few of these contributions deal with the problem in relation to optimal reinsurance. In the single period model, under the criterion of maximizing the expected utility of terminal wealth and that of maximizing the adjustment coefficient, Centeno (2005) studies the optimal excess of loss retention limits for two dependent classes of insurance risks. In the dynamic setting, Bai, Cai and Zhou (2013) also seek the optimal excess of loss reinsurance to minimize the ruin probability for the diffusion approximation risk model. Based on the aggregate claims process (1.1) with $m = 2$, Liang and Yuen (2014) adopt the variance premium principle to study the optimal proportional reinsurance problem for both the compound Poisson risk model and the diffusion approximation risk model. Interestingly, their optimal results for the diffusion risk model show that the optimal reinsurance strategies of the two classes are the same, and that the claim size distributions as well as the counting processes have no effect on the optimal reinsurance strategy. These results motivate us to investigate the same optimization problem for the risk model with two or more classes of dependent risks, and the reinsurance premium is calculated according to the expected value principle, which is another popular premium principle in practice.

Based on this model set-up, the main objective of this paper is to further study the optimal reinsurance problem of Liang and Yuen (2014) by means of the expected value premium principle instead of the variance premium principle. Both the model extension and the change in reinsurance premium principle certainly make the problem more meaningful but also make it more difficult to

tackle. Using the techniques of stochastic control theory and Hamilton-Jacobi-Bellman equation, we are still able to obtain explicit expressions for the optimal reinsurance strategies and the value function for both the compound Poisson case and the diffusion approximation case. In particular, we find that the optimal reinsurance strategies under the expected value principle are very different from those under variance principle in the diffusion approximation risk model. The former depends not only on the safety loading, time and interest rate, but also on the claim sizes and the claim number intensities, while the latter depends only on the safety loading, time and interest rate. It should be pointed out that the optimal reinsurance strategies with $m > 2$ in the latter case still have the same form (see Section 6 and Liang and Yuen (2014) for details). Furthermore, we show that, with normal constraints on the control variables, the value function is still a classical solution to the corresponding Hamilton-Jacobi-Bellman equation. From the numerical examples, we can see that the optimal reinsurance strategies are more sensitive to the claim size distribution than to the counting processes. Also, the numerical results show that the optimal reinsurance strategy of one class of insurance risk is independent of the claim sizes of the other class when the claim sizes are exponentially distributed.

The rest of the paper is organized as follows. In Section 2, we introduce the model of study and give a rigorous mathematical formulation of the problem. In Sections 3 and 4, under the expected value premium principle, we discuss the optimal reinsurance strategies for the compound Poisson case as well as the diffusion approximation case, respectively. In Section 5, we present some numerical examples which show the impact of some model parameters on the optimal strategies. In Section 6, we extend the work of Liang and Yuen (2014) for the diffusion risk model to the case with $m > 2$. Finally, we conclude the paper in Section 7.

2 Model and problem formulation

As usual, we define the surplus process

$$R_t = u + ct - S_t,$$

where u is the amount of initial surplus, c is the rate of premium, and S_t is the aggregate claims process given by (1.1).

It was shown in Yuen et al. (2002) or Wang and Yuen (2005) that S_t follows a compound Poisson process with parameter $\tilde{\lambda} = \sum_{l=1}^m \lambda_l + \lambda$, and that the common distribution of the transformed

claim size random variable X' is given by

$$F_{X'}(x) = \sum_{l=1}^m \frac{\lambda_l}{\bar{\lambda}} F_{X^{(l)}}(x) + \frac{\lambda}{\bar{\lambda}} F_{\sum_{l=1}^m X^{(l)}}(x).$$

Note that the compound Poisson process is a process with stationary and independent increment.

Let B_t be a standard Brownian motion. Then, the risk process

$$\hat{S}_t = \left(\sum_{l=1}^m a_l \right) t - \sqrt{\sum_{l=1}^m \sigma_l^2 + \sum_{i \neq j} \lambda \mu_i \mu_j} B_t$$

can be seen as a diffusion approximation to the compound Poisson process S_t (see Grandell (1991) for details), where

$$a_l = (\lambda_l + \lambda)E(X^{(l)}), \quad \sigma_l^2 = (\lambda_l + \lambda)E((X^{(l)})^2).$$

On the other hand, the Brownian motion with drift

$$\hat{S}_l(t) = a_l t - \sigma_l B_{lt}$$

can be seen as the diffusion approximation to compound Poisson process $S_l(t)$, where B_{lt} ($l = 1, 2, \dots, m$) are standard Brownian motions with correlation coefficient ρ_{ij} ($i \neq j$, $i, j = 1, 2, \dots, m$).

Furthermore,

$$\sum_{l=1}^m \hat{S}_l(t) = \left(\sum_{l=1}^m a_l \right) t - \sum_{l=1}^m \sigma_l B_{lt},$$

can be replaced by

$$\sum_{l=1}^m \hat{S}_l(t) = \left(\sum_{l=1}^m a_l \right) t - \sqrt{\sum_{l=1}^m \sigma_l^2 + \sum_{i \neq j} \sigma_i \sigma_j \rho_{ij}} B_t,$$

as the two forms have the same distributional properties. Hence, the sum $\sum_{l=1}^m \hat{S}_l(t)$ can also be regarded as a diffusion approximation to the compound Poisson process S_t when

$$\rho_{ij} = \frac{\lambda E(X^{(i)})E(X^{(j)})}{\sqrt{(\lambda_i + \lambda)E((X^{(i)})^2)(\lambda_j + \lambda)E((X^{(j)})^2)}} = \frac{\lambda \mu_i \mu_j}{\sigma_i \sigma_j},$$

i.e., $E[B_{it}B_{jt}] = \rho_{ij}t$ ($i \neq j$, $i, j = 1, 2, \dots, m$).

In this paper, we allow the insurance company to continuously cede a fraction of its claims with the retention levels $q_{lt} \in [0, 1]$ for each risk $X_i^{(l)}$ ($l = 1, 2, \dots, m$), and the reinsurance premium rate at time t is $\delta(q_t)$ with $q_t = (q_{1t}, q_{2t}, \dots, q_{mt})$. The company is also allowed to invest all its surplus in a risk-free asset (bond or bank account) with interest rate $r \geq 0$. Let $\{R_t^q, t \geq 0\}$ denote the

associated surplus process, i.e., R_t^q is the surplus of the insurer at time t , if the reinsurance strategy q_t is adopted. This process then evolves as

$$dR_t^q = [rR_t^q + (c - \delta(q_t))]dt - \sum_{l=1}^m q_{lt}dS_l(t). \quad (2.1)$$

Replacing $S_l(t)$ ($l = 1, 2, \dots, m$) of (2.1) respectively by $\hat{S}_l(t)$ ($l = 1, 2, \dots, m$) yield a new surplus process

$$d\hat{R}_t^q = [r\hat{R}_t^q + (c - \delta(q_t)) - \sum_{l=1}^m q_{lt}a_l]dt + \sum_{l=1}^m q_{lt}\sigma_l dB_{lt},$$

or equivalently,

$$\begin{aligned} d\hat{R}_t^q &= [r\hat{R}_t^q + (c - \delta(q_t)) - \sum_{l=1}^m q_{lt}a_l]dt \\ &\quad + \sqrt{\sum_{l=1}^m \sigma_l^2 q_{lt}^2 + \sum_{i \neq j} q_{it}q_{jt}\lambda\mu_i\mu_j}dB_t. \end{aligned} \quad (2.2)$$

Assume now that the insurer is interested in maximizing the expected utility of terminal wealth, say at time T . The utility function is $u(x)$, which satisfies $u' > 0$ and $u'' < 0$. Then, the objective function is

$$J^q(t, x) = E[u(R_T^q) | R_t^q = x], \quad (2.3)$$

or

$$J^q(t, x) = E[u(\hat{R}_T^q) | \hat{R}_t^q = x]. \quad (2.4)$$

We shall discuss (2.3) and (2.4) separately, and hence the use of the same notation $J^q(t, x)$ will not cause any confusion. The corresponding value function is then given by

$$V(t, x) = \sup_q J^q(t, x). \quad (2.5)$$

We further assume that the insurer has an exponential utility function

$$u(x) = -\frac{\varrho}{\nu}e^{-\nu x},$$

where $\varrho > 0$ and $\nu > 0$. This utility has constant absolute risk aversion (CARA) parameter ν . Such utility function plays an important role in insurance mathematics and actuarial practice, since they are the only functions under which the principle of “zero utility” gives a fair premium that is independent of the level of reserves of an insurance company (see Gerber (1979)).

Let $C^{1,2}$ denote the space of $\phi(t, x)$ such that ϕ and its partial derivatives ϕ_t , ϕ_x , ϕ_{xx} are continuous on $[0, T] \times R$. To solve the above problem, we use the dynamic programming approach

described in Fleming and Soner (2006). It follows from the standard arguments that if the value function $V \in C^{1,2}$, then V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\sup_q \mathcal{A}^q V(t, x) = 0, \quad (2.6)$$

for $t < T$, with the boundary condition

$$V(T, x) = u(x), \quad (2.7)$$

where

$$\begin{aligned} \mathcal{A}^q V(t, x) &= V_t + [rx + c - \delta(q)]V_x \\ &+ \sum_{l=1}^m \lambda_l E[V(t, x - q_l X^{(l)}) - V(t, x)] \\ &+ \lambda E[V(t, x - \sum_{l=1}^m q_l X^{(l)}) - V(t, x)], \end{aligned}$$

for the risk process (2.1), and

$$\begin{aligned} \mathcal{A}^q V(t, x) &= V_t + [rx + c - \delta(q) - \sum_{l=1}^m q_l a_l]V_x \\ &+ \frac{1}{2}(\sum_{l=1}^m \sigma_l^2 q_{lt}^2 + \sum_{i \neq j} q_{it} q_{jt} \lambda \mu_i \mu_j)V_{xx}, \end{aligned}$$

for the risk process (2.2).

Applying the methods of Fleming and Soner (2006) and Yang and Zhang (2005), we have the following verification theorem:

Theorem 2.1. *Let $W \in C^{1,2}$ be a classical solution to (2.6) that satisfies (2.7). Then, the value function V of (2.5) coincides with W . That is,*

$$W(t, x) = V(t, x).$$

Furthermore, let q^* be such that

$$\mathcal{A}^{q^*} V(t, x) = 0$$

holds for all $(t, x) \in [0, T) \times \mathcal{R}$, then

$$q^*(t, R_t^*) = (q_1^*(t, R_t^*), q_2^*(t, R_t^*), \dots, q_m^*(t, R_t^*))$$

is the optimal strategy, where R_t^* is the surplus process under the optimal strategy.

Remark 2.1. *In this paper, continuous trading is allowed; all assets are infinitely divisible; the process R_t^q is well defined on a complete probability space (Ω, \mathcal{F}, P) ; the information at time t is given by the complete filtration \mathcal{F}_t generated by R_t^q ; and the strategy q_t is \mathcal{F}_t -predictable.*

3 Optimal results for the compound Poisson risk model

In this section, we consider the optimization problem for the risk model (2.1). Throughout Sections 3 and 4, we assume that the reinsurance premium is calculated according to the expected value principle. That is,

$$\delta(q) = \sum_{l=1}^m (1 + \eta_l)(1 - q_l)a_l, \quad (3.1)$$

where $\eta_l (l = 1, 2, \dots, m)$ are the reinsurer's safety loading of the m classes of insurance business.

To solve the equation

$$\begin{aligned} \sup_q \{ & V_t + [rx + c - \delta(q)]V_x + \sum_{l=1}^m \lambda_l E[V(t, x - q_l X^{(l)}) - V(t, x)] \\ & + \lambda E[V(t, x - \sum_{l=1}^m q_l X^{(l)}) - V(t, x)] \} = 0, \end{aligned}$$

with the boundary condition $V(T, x) = u(x)$, we apply the method of Browne (1995) to fit a solution of the form

$$V(t, x) = -\frac{q}{\nu} \exp[-\nu x e^{r(T-t)} + h(T-t)], \quad (3.2)$$

where $h(\cdot)$ is a suitable function such that (3.2) is a solution to (2.6). The boundary condition $V(T, x) = u(x)$ implies $h(0) = 0$.

From (3.2), we get

$$\left\{ \begin{array}{l} V_t = V(t, x)[\nu x r e^{r(T-t)} - h'(T-t)], \\ V_x = V(t, x)[- \nu e^{r(T-t)}], \\ V_{xx} = V(t, x)[\nu^2 e^{2r(T-t)}], \\ E[V(t, x - q_l X^{(l)}) - V(t, x)] = V(t, x)[M_l(\nu q_l e^{r(T-t)}) - 1], \\ E[V(t, x - \sum_{l=1}^m q_l X^{(l)}) - V(t, x)] \\ = V(t, x)[\prod_{l=1}^m M_l(\nu q_l e^{r(T-t)}) - 1]. \end{array} \right. \quad (3.3)$$

Then, using (2.6) and (3.3), we obtain

$$\begin{aligned} \inf_q \{ & -h'(T-t) - c \nu e^{r(T-t)} - \sum_{l=1}^m \lambda_l - \lambda + \delta(q) \nu e^{r(T-t)} \\ & + \sum_{l=1}^m \lambda_l M_l(\nu q_l e^{r(T-t)}) + \lambda \prod_{l=1}^m M_l(\nu q_l e^{r(T-t)}) \} = 0, \end{aligned} \quad (3.4)$$

for $t < T$. Let

$$\tilde{f}(q) = \delta(q) \nu e^{r(T-t)} + \sum_{l=1}^m \lambda_l M_l(\nu q_l e^{r(T-t)}) + \lambda \prod_{l=1}^m M_l(\nu q_l e^{r(T-t)}). \quad (3.5)$$

For any $t \in [0, T]$, we have

$$\begin{cases} \frac{\partial \tilde{f}(q)}{\partial q_l} &= -(1 + \eta_l)a_l + M'_l(\nu q_l e^{r(T-t)})(\lambda_l + \lambda \prod_{j \neq l} M_j(\nu q_j e^{r(T-t)})) \cdot \nu e^{r(T-t)}, \\ \frac{\partial^2 \tilde{f}(q)}{\partial q_l^2} &= M''_l(\nu q_l e^{r(T-t)})(\lambda_l + \lambda \prod_{j \neq l} M_j(\nu q_j e^{r(T-t)})) \cdot \nu^2 e^{2r(T-t)} > 0, \\ \frac{\partial^2 \tilde{f}(q)}{\partial q_l \partial q_k} &= \lambda M'_l(\nu q_l e^{r(T-t)}) M'_k(\nu q_k e^{r(T-t)}) \prod_{j \neq l, k} M_j(\nu q_j e^{r(T-t)}) \cdot \nu^2 e^{2r(T-t)}, \end{cases}$$

for $l \neq k$, $l, k = 1, 2, \dots, m$, where $M''_l(r) = E((X^{(l)})^2 e^{rX^{(l)}})$.

Lemma 3.1. For $l = 1, 2, \dots, m$, $\tilde{f}(q)$ of (3.5) is a convex function with respect to q_l .

Proof: To prove that $\tilde{f}(q)$ is a convex function with respect to q_l for each l , it is sufficient to prove that the Hessian matrix of $\tilde{f}(q)$ is positive definite.

Let

$$\bar{\mathbf{A}} = \begin{pmatrix} \lambda_1 M''_1(\cdot) & 0 & \dots & 0 \\ 0 & \lambda_2 M''_2(\cdot) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m M''_m(\cdot) \end{pmatrix},$$

and

$$\bar{\mathbf{B}} = \begin{pmatrix} \frac{M''_1(\cdot)}{M_1(\cdot)} & \frac{M'_1(\cdot)M'_2(\cdot)}{M_1(\cdot)M_2(\cdot)} & \dots & \frac{M'_1(\cdot)M'_m(\cdot)}{M_1(\cdot)M_m(\cdot)} \\ \frac{M'_2(\cdot)M'_1(\cdot)}{M_2(\cdot)M_1(\cdot)} & \frac{M''_2(\cdot)}{M_2(\cdot)} & \dots & \frac{M'_2(\cdot)M'_m(\cdot)}{M_2(\cdot)M_m(\cdot)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{M'_m(\cdot)M'_1(\cdot)}{M_m(\cdot)M_1(\cdot)} & \frac{M'_m(\cdot)M'_2(\cdot)}{M_m(\cdot)M_2(\cdot)} & \dots & \frac{M''_m(\cdot)}{M_m(\cdot)} \end{pmatrix}.$$

Then, the Hessian matrix of $\tilde{f}(q)$ can be written as

$$\begin{pmatrix} \frac{\partial^2 \tilde{f}(q)}{\partial q_1^2} & \frac{\partial^2 \tilde{f}(q)}{\partial q_1 \partial q_2} & \dots & \frac{\partial^2 \tilde{f}(q)}{\partial q_1 \partial q_m} \\ \frac{\partial^2 \tilde{f}(q)}{\partial q_2 \partial q_1} & \frac{\partial^2 \tilde{f}(q)}{\partial q_2^2} & \dots & \frac{\partial^2 \tilde{f}(q)}{\partial q_2 \partial q_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \tilde{f}(q)}{\partial q_m \partial q_1} & \frac{\partial^2 \tilde{f}(q)}{\partial q_m \partial q_2} & \dots & \frac{\partial^2 \tilde{f}(q)}{\partial q_m^2} \end{pmatrix} = (\bar{\mathbf{A}} + \lambda \prod_{j=1}^m M_j(\cdot) \cdot \bar{\mathbf{B}}) \cdot \nu^2 e^{2r(T-t)}.$$

It is clear that $\bar{\mathbf{A}}$ is a positive definite matrix. Moreover, for any $C = (c_1, c_2, \dots, c_m) \in R^n$ and

$C \neq 0$, we have

$$\begin{aligned}
C \cdot \bar{\mathbf{B}} \cdot C' &= \sum_{i=1}^m c_i^2 \frac{M_i''(\cdot)}{M_i(\cdot)} + \sum_{i \neq k} 2c_i c_k \frac{M_i'(\cdot)M_k'(\cdot)}{M_i(\cdot)M_k(\cdot)} \\
&= \sum_{i=1}^m c_i^2 \frac{M_i''(\cdot)M_i(\cdot)}{(M_i(\cdot))^2} + \sum_{i \neq k} 2c_i c_k \frac{M_i'(\cdot)}{M_i(\cdot)} \cdot \frac{M_k'(\cdot)}{M_k(\cdot)} \\
&\geq \sum_{i=1}^m c_i^2 \left(\frac{M_i'(\cdot)}{M_i(\cdot)}\right)^2 + \sum_{i \neq k} 2c_i c_k \frac{M_i'(\cdot)}{M_i(\cdot)} \cdot \frac{M_k'(\cdot)}{M_k(\cdot)} \\
&= \left(\sum_{i=1}^m c_i \frac{M_i'(\cdot)}{M_i(\cdot)}\right)^2 > 0,
\end{aligned}$$

where C' is the transpose of C , and the first inequality follows from the Cauchy–Schwarz inequality.

Hence, $\bar{\mathbf{B}}$ is also a positive definite matrix. As a result, the Hessian matrix is positive definite. \square

It is easy to see that the minimizer $(q_1(T-t), q_2(T-t), \dots, q_m(T-t))$ of $\tilde{f}(q)$ satisfies the following equations

$$-(1 + \eta_l)a_l + M_l'(\nu q_l e^{r(T-t)})(\lambda_l + \lambda \prod_{j \neq l} M_j(\nu q_j e^{r(T-t)})) = 0, \quad l = 1, 2, \dots, m. \quad (3.6)$$

Moreover, we have

Lemma 3.2. *For any $t \in [0, T]$, if both $\hat{q} = (\hat{q}_1, \hat{q}_2, \dots, \hat{q}_m)$ and $\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_m)$ are the solutions to the equation (3.6), then $\hat{q} = \tilde{q}$.*

Proof: Assume that $\hat{q} \neq \tilde{q}$. Let

$$\tilde{f}'(q) = \left(\frac{\partial \tilde{f}(q)}{\partial q_1}, \frac{\partial \tilde{f}(q)}{\partial q_2}, \dots, \frac{\partial \tilde{f}(q)}{\partial q_m} \right),$$

and

$$\tilde{f}''(q) = \begin{pmatrix} \frac{\partial^2 \tilde{f}(q)}{\partial q_1^2} & \frac{\partial^2 \tilde{f}(q)}{\partial q_1 \partial q_2} & \cdots & \frac{\partial^2 \tilde{f}(q)}{\partial q_1 \partial q_m} \\ \frac{\partial^2 \tilde{f}(q)}{\partial q_2 \partial q_1} & \frac{\partial^2 \tilde{f}(q)}{\partial q_2^2} & \cdots & \frac{\partial^2 \tilde{f}(q)}{\partial q_2 \partial q_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \tilde{f}(q)}{\partial q_m \partial q_1} & \frac{\partial^2 \tilde{f}(q)}{\partial q_m \partial q_2} & \cdots & \frac{\partial^2 \tilde{f}(q)}{\partial q_m^2} \end{pmatrix}.$$

By Taylor's Theorem in higher dimensions, we have

$$\begin{aligned}
\tilde{f}(\hat{q}) &= \tilde{f}(\tilde{q}) + \tilde{f}'(\tilde{q}) \cdot (\hat{q} - \tilde{q})' + \frac{1}{2}(\hat{q} - \tilde{q}) \cdot \tilde{f}''(\tilde{q} + \theta(\hat{q} - \tilde{q})) \cdot (\hat{q} - \tilde{q})' \\
&= \tilde{f}(\tilde{q}) + \frac{1}{2}\Delta_1,
\end{aligned}$$

where $\theta \in [0, 1]$, and

$$\Delta_1 = (\hat{q} - \tilde{q}) \cdot \tilde{f}''(\tilde{q} + \theta(\hat{q} - \tilde{q})) \cdot (\hat{q} - \tilde{q})'.$$

From Lemma 3.1, we know that the Hessian matrix $f''(q)$ is a positive definite matrix. Thus, $\Delta_1 > 0$ for any $\hat{q} - \tilde{q} \neq 0$. This implies that

$$\tilde{f}(\hat{q}) = \tilde{f}(\tilde{q}) + \frac{1}{2}\Delta_1 > \tilde{f}(\tilde{q}). \quad (3.7)$$

Along the same lines, one can show that $\tilde{f}(\tilde{q}) > \tilde{f}(\hat{q})$, which is contrary to (3.7). \square

With the help of Lemma 3.2, and using the elementary fixed-point theorem (see for example, Ortega and Rheinboldt (1970)), one can show the uniqueness and the existence of the solution to the equation (3.6).

For $m = 2$, the equation (3.6) becomes

$$\begin{cases} -(1 + \eta_1)a_1 + M'_1(n_1)(\lambda_1 + \lambda M_2(n_2)) = 0, \\ -(1 + \eta_2)a_2 + M'_2(n_2)(\lambda_2 + \lambda M_1(n_1)) = 0, \end{cases} \quad (3.8)$$

where $n_1 = \nu q_1 e^{r(T-t)}$ and $n_2 = \nu q_2 e^{r(T-t)}$. In addition, we have

Lemma 3.3. *Let M_i^{-1} ($M_i'^{-1}$) be the inverse function of M_i (M_i') for $i = 1, 2$. If the following inequalities hold:*

$$\begin{cases} M_2^{-1}(1 + \eta_1 + \eta_1 \cdot \frac{\lambda_1}{\lambda}) > M_2'^{-1}((1 + \eta_2)\mu_2), \\ M_1^{-1}(1 + \eta_2 + \eta_2 \cdot \frac{\lambda_2}{\lambda}) > M_1'^{-1}((1 + \eta_1)\mu_1), \end{cases}$$

or

$$\begin{cases} M_2^{-1}(1 + \eta_1 + \eta_1 \cdot \frac{\lambda_1}{\lambda}) < M_2'^{-1}((1 + \eta_2)\mu_2), \\ M_1^{-1}(1 + \eta_2 + \eta_2 \cdot \frac{\lambda_2}{\lambda}) < M_1'^{-1}((1 + \eta_1)\mu_1), \end{cases}$$

then the equation (3.8) has a unique positive root (\bar{n}_1, \bar{n}_2) .

Proof: Let

$$H_1(n_1, n_2) = -(1 + \eta_1)a_1 + M'_1(n_1)(\lambda_1 + \lambda M_2(n_2)),$$

and

$$H_2(n_1, n_2) = -(1 + \eta_2)a_2 + M'_2(n_2)(\lambda_2 + \lambda M_1(n_1)).$$

Assume that the equation $H_1(n_1, n_2) = 0$ implies $n_2 = f_1(n_1)$, and that $H_2(n_1, n_2) = 0$ implies $n_2 = f_2(n_1)$. Denote by f_i^{-1} the inverse function of f_i for $i = 1, 2$. Differentiating both sides of the equation $H_1(n_1, n_2) = 0$ with respect to n_1 yields

$$\lambda f_1'(n_1)M_2'(n_2)M_1'(n_1) + (\lambda M_2(n_2) + \lambda_1)M_1''(n_1) = 0,$$

and thus

$$f_1'(n_1) = -\frac{(\lambda M_2(n_2) + \lambda_1)M_1''(n_1)}{\lambda M_2'(n_2)M_1'(n_1)} < 0.$$

Therefore, the function $f_1(n_1)$ is a decreasing function with

$$\begin{cases} f_1(0) &= M_2^{-1}(1 + \eta_1 + \eta_1 \cdot \frac{\lambda_1}{\lambda}) > M_2^{-1}(1) = 0, \\ f_1^{-1}(0) &= M_1'^{-1}((1 + \eta_1)\mu_1) > M_1'^{-1}(\mu_1) = 0. \end{cases}$$

By the same token, one can use $H_2(n_1, n_2) = 0$ to obtain

$$f_2'(n_1) = -\frac{\lambda M_2'(n_2)M_1'(n_1)}{(\lambda M_1(n_1) + \lambda_2)M_2''(n_2)} < 0,$$

and thus the function $f_2(n_1)$ is also a decreasing function with

$$\begin{cases} f_2(0) &= M_2'^{-1}((1 + \eta_2)\mu_2) > M_2'^{-1}(\mu_2) = 0, \\ f_2^{-1}(0) &= M_1^{-1}(1 + \eta_2 + \eta_2 \cdot \frac{\lambda_2}{\lambda}) > M_1^{-1}(1) = 0. \end{cases}$$

Therefore, if the following inequalities hold

$$\begin{cases} f_1(0) &> f_2(0), \\ f_1^{-1}(0) &< f_2^{-1}(0), \end{cases}$$

or

$$\begin{cases} f_1(0) &< f_2(0), \\ f_1^{-1}(0) &< f_2^{-1}(0), \end{cases}$$

the functions $f_1(n_1)$ and $f_2(n_1)$ have at least one point of intersection at some $\bar{n}_1 > 0$. Then, it follows from Lemma 3.2 that the equation (3.8) has a unique positive root. That is, the equation (3.8) has a unique positive root (\bar{n}_1, \bar{n}_2) with $\bar{n}_2 = f_1(\bar{n}_1) = f_2(\bar{n}_1)$. \square

From Lemma 3.3, we obtain

$$\bar{n}_1 = \nu q_1(T - t)e^{r(T-t)}, \quad \text{and} \quad \bar{n}_2 = \nu q_2(T - t)e^{r(T-t)},$$

which in turn give

$$\begin{cases} q_1(T - t) &= \frac{\bar{n}_1}{\nu}e^{-r(T-t)}, \\ q_2(T - t) &= \frac{\bar{n}_2}{\nu}e^{-r(T-t)}. \end{cases} \tag{3.9}$$

It is not difficult to see from (3.8) that \bar{n}_1 and \bar{n}_2 are constants depending on the safety loadings η_1 and η_2 , the claim size distributions as well as the counting processes.

Let

$$t_1 = T - \frac{1}{r} \ln \frac{\bar{n}_1}{\nu} \quad \text{for } \nu < \bar{n}_1 < \nu e^{rT}; \quad t_2 = T - \frac{1}{r} \ln \frac{\bar{n}_2}{\nu} \quad \text{for } \nu < \bar{n}_2 < \nu e^{rT}.$$

For $\bar{n}_1 \leq \nu$ ($\bar{n}_2 \leq \nu$), we set $t_1 = T$ ($t_2 = T$); and for $\bar{n}_1 \geq \nu e^{rT}$ ($\bar{n}_2 \geq \nu e^{rT}$), we set $t_1 = 0$ ($t_2 = 0$). To make sure that the retention level $q_{1t}, q_{2t} \in [0, 1]$, we need to discuss the optimal values in the following two cases: (1) $\bar{n}_1 \leq \bar{n}_2$; and (2) $\bar{n}_1 \geq \bar{n}_2$.

Case 1: In this case, we have $t_1 \geq t_2$. When $0 \leq t \leq t_2$, we have $(q_{1t}^*, q_{2t}^*) = (q_1(T-t), q_2(T-t))$. Denote the h function in (3.4) by h_1 . Substituting $(q_1(T-t), q_2(T-t))$ into (3.4), we get

$$h_1(T-t) = \tilde{h}_1(T-t) + C_1, \quad (3.10)$$

where

$$\begin{aligned} \tilde{h}_1(T-t) &= \frac{1}{r} [(1 + \eta_1)a_1 + (1 + \eta_2)a_2 - c] \nu e^{r(T-t)} \\ &\quad - (\lambda_1 + \lambda_2 + \lambda + (1 + \eta_1)a_1 \bar{n}_1 + (1 + \eta_2)a_2 \bar{n}_2)(T-t) \\ &\quad + (\lambda_1 M_1(\bar{n}_1) + \lambda_2 M_2(\bar{n}_2) + \lambda M_1(\bar{n}_1) M_2(\bar{n}_2))(T-t), \end{aligned}$$

and C_1 is a constant that will be determined later.

For $t \geq t_2$, $q_2(T-t) \geq 1$, and thus $q_{2t}^* = 1$. Substituting $q_{2t}^* = 1$ into (3.4), we get

$$\begin{aligned} \inf_{q_1} \{ &-h'(T-t) - c \nu e^{r(T-t)} - \lambda_1 - \lambda_2 - \lambda + \delta(q_1) \nu e^{r(T-t)} \\ &+ \lambda_2 M_2(\nu e^{r(T-t)}) + M_1(\nu q_1 e^{r(T-t)}) (\lambda_1 + \lambda M_2(\nu e^{r(T-t)})) \} = 0, \end{aligned} \quad (3.11)$$

for $t < T$, where $\delta(q_1) = (1 + \eta_1)(1 - q_1)a_1$. Therefore, the minimizer of the equation (3.11) is

$$\hat{q}_1(T-t) = M_1'^{-1} \left[\frac{(1 + \eta_1)a_1}{\lambda_1 + \lambda M_2(\nu e^{r(T-t)})} \right] \cdot \frac{1}{\nu} e^{-r(T-t)}. \quad (3.12)$$

As $M_1'(x)$ is an increasing function of x , it is not difficult to see that $\hat{q}_1(T-t)$ is an increasing function of t .

Denote by t_{01} the solution to the equation $\hat{q}_1(T-t) = 1$, and by h_2 the h function in (3.4). For $t_2 \leq t \leq t_{01}$, $(q_{1t}^*, q_{2t}^*) = (\hat{q}_1(T-t), 1)$. Substituting $(\hat{q}_1(T-t), 1)$ into (3.4), we get

$$h_2(T-t) = \tilde{h}_2(T-t) + C_2, \quad (3.13)$$

where

$$\begin{aligned}\tilde{h}_2(T-t) &= \frac{1}{r}[(1+\eta_1)a_1 - c]\nu e^{r(T-t)} - (\lambda_1 + \lambda_2 + \lambda)(T-t) - \int_0^{T-t} (1+\eta_1)a_1 \hat{q}_1(s) \nu e^{rs} ds \\ &\quad + \int_0^{T-t} \lambda_2 M_2(\nu e^{rs}) + M_1(\hat{q}_1(s) \nu e^{rs})(\lambda_1 + \lambda M_2(\nu e^{rs})) ds,\end{aligned}$$

and C_2 is also a constant that will be determined later.

For $t_{01} \leq t \leq T$, $(q_{1t}^*, q_{2t}^*) = (1, 1)$. Denote the h function in (3.4) by h_3 . Then, putting (1, 1) into (3.4), we get

$$\begin{aligned}h_3(T-t) &= -\frac{1}{r}c\nu(e^{r(T-t)} - 1) - (\lambda_1 + \lambda_2 + \lambda)(T-t) \\ &\quad + \int_0^{T-t} (\lambda_1 M_1(\nu e^{rs}) + \lambda_2 M_2(\nu e^{rs}) \\ &\quad + \lambda M_1(\nu e^{rs}) M_2(\nu e^{rs})) ds.\end{aligned}\tag{3.14}$$

Let

$$C_1 = h_2(T-t_2) - \tilde{h}_1(T-t_2), \quad \text{and} \quad C_2 = h_3(T-t_{01}) - \tilde{h}_2(T-t_{01}).$$

Then, we have

$$\begin{cases} h_2(T-t_2) &= \tilde{h}_1(T-t_2) + C_1 = h_1(T-t_2), \\ h_3(T-t_{01}) &= \tilde{h}_2(T-t_{01}) + C_2 = h_2(T-t_{01}). \end{cases}$$

□

Case 2: In this case, we have $t_1 \leq t_2$. For $0 \leq t \leq t_1$, $(q_{1t}^*, q_{12}^*) = (q_1(T-t), q_2(T-t))$ from which the h function in (3.4) can be written as

$$h_4(T-t) = \tilde{h}_1(T-t) + C_4.\tag{3.15}$$

For $t \geq t_1$, we have $q_1(T-t) \geq 1$, and thus $q_{1t}^* = 1$. Then, similar to the derivation of (3.12), we get the minimizer

$$\hat{q}_2(T-t) = M_2'^{-1} \left[\frac{(1+\eta_2)a_2}{\lambda_2 + \lambda M_1(\nu e^{r(T-t)})} \right] \cdot \frac{1}{\nu} e^{-r(T-t)},\tag{3.16}$$

which is increasing function of t .

Denote by t_{02} the solution to the equation $\hat{q}_2(T-t) = 1$, and by h_5 the h function in (3.4). For $t_1 \leq t \leq t_{02}$, $(q_{1t}^*, q_{1t}^*) = (1, \hat{q}_1(T-t))$. It follows from (3.4) and $(q_{1t}^*, q_{1t}^*) = (1, \hat{q}_1(T-t))$ that

$$h_5(T-t) = \tilde{h}_5(T-t) + C_5,\tag{3.17}$$

where

$$\begin{aligned}\tilde{h}_5(T-t) &= \frac{1}{r}[(1+\eta_2)a_2 - c]\nu e^{r(T-t)} - (\lambda_1 + \lambda_2 + \lambda)(T-t) - \int_0^{T-t} (1+\eta_2)a_2\hat{q}_2(s)\nu e^{rs} ds \\ &\quad + \int_0^{T-t} \lambda_1 M_1(\nu e^{rs}) + M_2(\hat{q}_2(s)\nu e^{rs})(\lambda_2 + \lambda M_1(\nu e^{rs})) ds.\end{aligned}$$

For $t_0 \leq t \leq T$, $(q_{1t}^*, q_{1t}^*) = (1, 1)$, and the corresponding h function in (3.4) is h_3 of (3.14).

Letting

$$C_4 = h_5(T-t_1) - \tilde{h}_1(T-t_1), \quad \text{and} \quad C_5 = h_3(T-t_0) - \tilde{h}_5(T-t_0),$$

yields

$$\begin{cases} h_5(T-t_1) &= \tilde{h}_1(T-t_1) + C_4 = h_4(T-t_1), \\ h_3(T-t_0) &= \tilde{h}_5(T-t_0) + C_5 = h_5(T-t_0). \end{cases}$$

□

The following theorem summarizes the above analysis.

Theorem 3.1. *Let (\bar{n}_1, \bar{n}_2) be the unique positive root of the equation (3.6), and $\hat{q}_1(T-t)$, $\hat{q}_2(T-t)$ and $(q_1(T-t), q_2(T-t))$ be given in (3.12), (3.16) and (3.9), respectively. Recall the functions $h_1(T-t)$, $h_2(T-t)$, $h_3(T-t)$, $h_4(T-t)$ and $h_5(T-t)$ defined in (3.10), (3.13), (3.14), (3.15) and (3.17), respectively. Then, we have*

(i) *If Case (1) holds, i.e., $\bar{n}_1 \leq \bar{n}_2$, for any $t \in [0, T]$, the optimal reinsurance strategies q_{1t}^* and q_{2t}^* under the model (2.1) are*

$$(q_{1t}^*, q_{2t}^*) = \begin{cases} (q_1(T-t), q_2(T-t)), & 0 \leq t \leq t_2, \\ (\hat{q}_1(T-t), 1), & t_2 \leq t \leq t_{01}, \\ (1, 1), & t_{01} \leq t \leq T, \end{cases}$$

and the value function $V(t, x)$ is given by

$$V(t, x) = \begin{cases} -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_1(T-t)\}, & 0 \leq t \leq t_2, \\ -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_2(T-t)\}, & t_2 \leq t \leq t_{01}, \\ -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_3(T-t)\}, & t_{01} \leq t \leq T; \end{cases}$$

(ii) If Case (2) holds, i.e., $\bar{n}_1 \geq \bar{n}_2$, for any $t \in [0, T]$, the optimal reinsurance strategies under the model (2.1) are

$$(q_{1t}^*, q_{2t}^*) = \begin{cases} (q_1(T-t), q_2(T-t)), & 0 \leq t \leq t_1, \\ (1, \hat{q}_2(T-t)), & t_1 \leq t \leq t_{02}, \\ (1, 1), & t_{02} \leq t \leq T, \end{cases}$$

and the value function is given by

$$V(t, x) = \begin{cases} -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_4(T-t)\}, & 0 \leq t \leq t_1, \\ -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_5(T-t)\}, & t_1 \leq t \leq t_{02}, \\ -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_3(T-t)\}, & t_{02} \leq t \leq T. \end{cases}$$

Remark 3.1. Since

$$\begin{cases} h_1(T-t_2) = h_2(T-t_2), \\ h_2(T-t_1) = h_3(T-t_1), \\ h_4(T-t_1) = h_5(T-t_1), \\ h_5(T-t_2) = h_3(T-t_2), \end{cases}$$

$V(t, x)$ is a continuous function on $[0, T] \times R$. Furthermore,

$$\left\{ \begin{array}{l} h'_1(T - t_2) = -c\bar{n}_2 - (\lambda_1 + \lambda_2 + \lambda + (1 + \eta_1)a_1(\bar{n}_1 - \bar{n}_2) \\ \quad + \lambda_1 M_1(\bar{n}_1) + M_2(\bar{n}_2)(\lambda_2 + \lambda M_1(\bar{n}_1)) \\ = h'_2(T - t_2), \\ h'_2(T - t_{01}) = -c\nu e^{r(T-t_{01})} - (\lambda_1 + \lambda_2 + \lambda) + \lambda_1 M_1(\nu e^{r(T-t_{01})}) \\ \quad + M_2(\nu e^{r(T-t_{01})})(\lambda_2 + \lambda M_1(\nu e^{r(T-t_{01})})) \\ = h'_3(T - t_{01}), \\ h'_4(T - t_1) = -c\bar{n}_1 - (\lambda_1 + \lambda_2 + \lambda + (1 + \eta_2)a_2(\bar{n}_2 - \bar{n}_1) \\ \quad + \lambda_1 M_1(\bar{n}_1) + M_2(\bar{n}_2)(\lambda_2 + \lambda M_1(\bar{n}_1)) \\ = h'_5(T - t_1), \\ h'_5(T - t_{02}) = -c\nu e^{r(T-t_{02})} - (\lambda_1 + \lambda_2 + \lambda) + \lambda_2 M_2(\nu e^{r(T-t_{02})}) \\ \quad + M_1(\nu e^{r(T-t_{02})})(\lambda_1 + \lambda M_2(\nu e^{r(T-t_{02})})) \\ = h'_3(T - t_{02}). \end{array} \right.$$

Therefore, we have $V(t, x) \in C^{1,2}$. That is, $V(t, x)$ is a classical solution to the HJB equation (2.6).

Remark 3.2. From (3.9) or the numerical examples in Section 5, we can see that the optimal reinsurance strategies in the compound Poisson risk model depend not only on the safety loading, time and interest rate, but also on the claim size distributions and the counting processes. Such a result is consistent with the one obtained under the variance principle (see Section 5 of Liang and Yuen (2014)). However, for the diffusion risk model, the result obtained in the following section is different from that obtained under the variance principle.

For $m > 2$, a note is given at the end of Section 4, i.e., Remark 4.3.

4 Optimal results for the diffusion model

In this section, we discuss the optimization problem for the diffusion approximation risk model. Under the expected value principle, the surplus process of the model (2.2) evolves as

$$\begin{aligned} d\hat{R}_t^q &= [r\hat{R}_t^q + (c - \delta(q_t)) - \sum_{l=1}^m q_{lt}a_l]dt \\ &\quad + \sqrt{\sum_{l=1}^m \sigma_l^2 q_{lt}^2 + \sum_{i \neq j} q_{it}q_{jt}\lambda\mu_i\mu_j}dB_t, \end{aligned}$$

and the HJB equation of this problem is

$$\begin{aligned} \sup_q \{ &V_t + [rx + c - \delta(q) - \sum_{l=1}^m q_l a_l]V_x \\ &+ \frac{1}{2}(\sum_{l=1}^m \sigma_l^2 q_l^2 + \sum_{i \neq j} q_i q_j \lambda \mu_i \mu_j)V_{xx} \} = 0, \end{aligned} \quad (4.1)$$

for $t < T$, with the boundary condition $V(T, x) = u(x)$. Again, we consider a solution with the form of (3.2). After substituting (3.3) into (4.1) and some algebraic manipulation, the equation (4.1) becomes

$$\begin{aligned} \inf_q \{ &-h'(T-t) - [c - \delta(q) - \sum_{l=1}^m q_l a_l]\nu e^{r(T-t)} \\ &+ \frac{1}{2}(\sum_{l=1}^m \sigma_l^2 q_l^2 + \sum_{i \neq j} q_i q_j \lambda \mu_i \mu_j)\nu^2 e^{2r(T-t)} \} = 0. \end{aligned} \quad (4.2)$$

Let

$$\begin{aligned} \tilde{g}(q) &= (\delta(q) + \sum_{l=1}^m q_l a_l)\nu e^{r(T-t)} \\ &\quad + \frac{1}{2}(\sum_{l=1}^m \sigma_l^2 q_l^2 + \sum_{i \neq j} q_i q_j \lambda \mu_i \mu_j)\nu^2 e^{2r(T-t)}. \end{aligned} \quad (4.3)$$

Then, for any $t \in [0, T]$, we have

$$\left\{ \begin{array}{l} \frac{\partial \tilde{g}(q)}{\partial q_l} = -\eta_l a_l \nu e^{r(T-t)} + (q_l \sigma_l^2 + \lambda \sum_{i \neq l} q_i \mu_l \mu_i) \nu^2 e^{2r(T-t)}, \\ \frac{\partial^2 \tilde{g}(q)}{\partial q_l^2} = \sigma_l^2 \nu^2 e^{2r(T-t)} > 0, \\ \frac{\partial^2 \tilde{g}(q)}{\partial q_l \partial q_k} = \lambda \mu_l \mu_k \nu^2 e^{2r(T-t)} > 0, \end{array} \right.$$

for $l \neq k$, $l, k = 1, 2, \dots, m$.

Lemma 4.1. For $l = 1, 2, \dots, m$, $\tilde{g}(q)$ of (4.3) is a convex function with respect to q_l .

Proof: To prove the lemma, it is sufficient to prove the Hessian matrix of $\tilde{g}(q)$ is positive definite.

Define

$$\mathbf{A} = \begin{pmatrix} \sigma_1^2 & \lambda\mu_1\mu_2 & \lambda\mu_1\mu_3 & \dots & \lambda\mu_1\mu_m \\ \lambda\mu_2\mu_1 & \sigma_2^2 & \lambda\mu_2\mu_3 & \dots & \lambda\mu_2\mu_m \\ \lambda\mu_3\mu_1 & \lambda\mu_3\mu_2 & \sigma_3^2 & \dots & \lambda\mu_3\mu_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda\mu_m\mu_1 & \lambda\mu_m\mu_2 & \lambda\mu_m\mu_3 & \dots & \sigma_m^2 \end{pmatrix}.$$

For any $C = (c_1, c_2, \dots, c_m) \in R^n$ and $C \neq 0$, we have

$$\begin{aligned} C \cdot \mathbf{A} \cdot C' &= \sum_{i=1}^m c_i^2 \sigma_i^2 + \sum_{i \neq k} 2c_i c_k \lambda \mu_i \mu_k \\ &= \sum_{i=1}^m c_i^2 (\lambda_i + \lambda) E(X^{(i)})^2 + \sum_{i \neq k} 2c_i c_k \lambda E(X^{(i)}) E(X^{(k)}) \\ &= \sum_{i=1}^m c_i^2 \lambda_i E(X^{(i)})^2 + \lambda \{ \sum_{i=1}^m c_i^2 E(X^{(i)})^2 + \sum_{i \neq k} 2c_i c_k E(X^{(i)}) E(X^{(k)}) \} \\ &\geq \sum_{i=1}^m c_i^2 \lambda_i E(X^{(i)})^2 + \lambda \{ \sum_{i=1}^m c_i^2 (E(X^{(i)}))^2 + \sum_{i \neq k} 2c_i c_k E(X^{(i)}) E(X^{(k)}) \} \\ &= \sum_{i=1}^m c_i^2 \lambda_i E(X^{(i)})^2 + \lambda (\sum_{i=1}^m c_i E(X^{(i)}))^2 > 0, \end{aligned}$$

which implies that \mathbf{A} is a positive definite matrix. Since the Hessian matrix in this case is given by

$$\begin{pmatrix} \frac{\partial^2 \tilde{g}(q)}{\partial q_1^2} & \frac{\partial^2 \tilde{g}(q)}{\partial q_1 \partial q_2} & \dots & \frac{\partial^2 \tilde{g}(q)}{\partial q_1 \partial q_m} \\ \frac{\partial^2 \tilde{g}(q)}{\partial q_2 \partial q_1} & \frac{\partial^2 \tilde{g}(q)}{\partial q_2^2} & \dots & \frac{\partial^2 \tilde{g}(q)}{\partial q_2 \partial q_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \tilde{g}(q)}{\partial q_m \partial q_1} & \frac{\partial^2 \tilde{g}(q)}{\partial q_m \partial q_2} & \dots & \frac{\partial^2 \tilde{g}(q)}{\partial q_m^2} \end{pmatrix} = \mathbf{A} \cdot \nu^2 e^{2r(T-t)},$$

the Hessian matrix is also a positive definite matrix. \square

It is clear that the minimizer $\bar{q}(T-t) = (\bar{q}_1(T-t), \bar{q}_2(T-t), \dots, \bar{q}_m(T-t))$ of (4.2) satisfies the following equations

$$-\eta_l a_l \nu e^{r(T-t)} + (q_l \sigma_l^2 + \lambda \sum_{i \neq l} q_i \mu_l \mu_i) \nu^2 e^{2r(T-t)} = 0, \quad l = 1, 2, \dots, m.$$

Or equivalently,

$$\mathbf{A}q' = \mathbf{B}, \tag{4.4}$$

where

$$\mathbf{B} = \begin{pmatrix} \eta_1 a_1 \\ \eta_2 a_2 \\ \eta_3 a_3 \\ \vdots \\ \eta_m a_m \end{pmatrix} \frac{1}{\nu} e^{-r(T-t)}.$$

Since the matrix \mathbf{A} is a positive definite matrix which is invertible, the equation (4.4) has a unique positive solution

$$\bar{q}(T-t) = (\mathbf{A}^{-1}\mathbf{B})'.$$

For $m = 2$, the equation (4.4) becomes

$$\begin{cases} q_1 \sigma_1^2 + \lambda q_2 \mu_1 \mu_2 = \frac{\eta_1 a_1}{\nu} e^{-r(T-t)}, \\ q_2 \sigma_2^2 + \lambda q_1 \mu_1 \mu_2 = \frac{\eta_2 a_2}{\nu} e^{-r(T-t)}, \end{cases}$$

it is easy to show that the solution to this equation is

$$\begin{cases} \bar{q}_1(T-t) = \frac{a_1 \eta_1 \sigma_2^2 - \lambda \mu_1 \mu_2 a_2 \eta_2}{\sigma_1^2 \sigma_2^2 - \lambda^2 \mu_1^2 \mu_2^2} \cdot \frac{1}{\nu} e^{-r(T-t)}, \\ \bar{q}_2(T-t) = \frac{a_2 \eta_2 \sigma_1^2 - \lambda \mu_1 \mu_2 a_1 \eta_1}{\sigma_1^2 \sigma_2^2 - \lambda^2 \mu_1^2 \mu_2^2} \cdot \frac{1}{\nu} e^{-r(T-t)}. \end{cases} \quad (4.5)$$

Let

$$\begin{cases} A_1 = \frac{a_1 \eta_1 \sigma_2^2 - \lambda \mu_1 \mu_2 a_2 \eta_2}{\sigma_1^2 \sigma_2^2 - \lambda^2 \mu_1^2 \mu_2^2}, \\ A_2 = \frac{a_2 \eta_2 \sigma_1^2 - \lambda \mu_1 \mu_2 a_1 \eta_1}{\sigma_1^2 \sigma_2^2 - \lambda^2 \mu_1^2 \mu_2^2}, \end{cases} \quad (4.6)$$

and

$$t_3 = T - \frac{1}{r} \ln \frac{A_1}{\nu} \quad \text{for } \nu < A_1 < \nu e^{rT}; \quad t_4 = T - \frac{1}{r} \ln \frac{A_2}{\nu} \quad \text{for } \nu < A_2 < \nu e^{rT}.$$

For $A_1 \leq \nu$ ($A_2 \leq \nu$), we set $t_3 = T$ ($t_4 = T$); and for $A_1 \geq \nu e^{rT}$ ($A_2 \geq \nu e^{rT}$), we set $t_3 = 0$ ($t_4 = 0$). To discuss the optimal values, we first prove the following lemma which plays a key role in this section.

Lemma 4.1. *Given λ and μ_i , a_i , σ_i for $i = 1, 2$, we have*

$$\frac{\lambda \mu_1 \mu_2 a_2}{\sigma_2^2 a_1} \leq \frac{\lambda \mu_1 \mu_2 a_2 + \sigma_1^2 a_2}{\sigma_2^2 a_1 + \lambda \mu_1 \mu_2 a_1} \leq \frac{\sigma_1^2 a_2}{\lambda \mu_1 \mu_2 a_1}. \quad (4.7)$$

Proof: Using Cauchy-Schwarz inequality, one can show that $\sigma_i^2 > \lambda\mu_1\mu_2$ for $i = 1, 2$, and thus

$$\frac{\lambda\mu_1\mu_2a_2}{\sigma_2^2a_1} \leq \frac{\sigma_1^2a_2}{\lambda\mu_1\mu_2a_1}.$$

In addition, for any positive parameters a, b, c and d , if

$$\frac{a}{b} \leq \frac{c}{d},$$

then we have

$$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d},$$

which completes the proof. \square

To make sure that the optimal reinsurance strategies belong to the interval $[0, 1]$, we need to investigate the optimal results in the following four cases:

(I)

$$\frac{\lambda\mu_1\mu_2a_2}{\sigma_2^2a_1}\eta_2 \leq \eta_1 \leq \frac{\lambda\mu_1\mu_2a_2 + \sigma_1^2a_2}{\sigma_2^2a_1 + \lambda\mu_1\mu_2a_1}\eta_2;$$

(II)

$$\frac{\lambda\mu_1\mu_2a_2 + \sigma_1^2a_2}{\sigma_2^2a_1 + \lambda\mu_1\mu_2a_1}\eta_2 \leq \eta_1 \leq \frac{\sigma_1^2a_2}{\lambda\mu_1\mu_2a_1}\eta_2;$$

(III)

$$\eta_1 \leq \frac{\lambda\mu_1\mu_2a_2}{\sigma_2^2a_1}\eta_2;$$

(IV)

$$\eta_1 \geq \frac{\sigma_1^2a_2}{\lambda\mu_1\mu_2a_1}\eta_2.$$

Case I: In this case, we have $\bar{q}_1(T-t) > 0$, $\bar{q}_2(T-t) > 0$, and $A_1 \leq A_2$. So, we have $t_3 \geq t_4$. Also, for $0 \leq t \leq t_4$, we have $(q_{1t}^*, q_{2t}^*) = (\bar{q}_1(T-t), \bar{q}_2(T-t))$. Then, substituting $(\bar{q}_1(T-t), \bar{q}_2(T-t))$ into (4.2), we get

$$h_6(T-t) = \tilde{h}_6(T-t) + C_6, \tag{4.8}$$

where

$$\begin{aligned} \tilde{h}_6(T-t) &= \frac{1}{r}((1+\eta_1)a_1 + (1+\eta_2)a_2 - c)\nu e^{r(T-t)} \\ &\quad + (\frac{1}{2}\sigma_1^2A_1^2 + \frac{1}{2}\sigma_2^2A_2^2 + A_1A_2\lambda\mu_1\mu_2 - a_1\eta_1A_1 - a_2\eta_2A_2)(T-t), \end{aligned}$$

and C_6 is a constant that will be determined later.

For $t \geq t_4$, $\bar{q}_2(T-t) \geq 1$, and thus $q_{2t}^* = 1$. Substituting $q_{2t}^* = 1$ into (4.2) yields

$$\begin{aligned} \inf_{q_1} \{ & -h'(T-t) - cv e^{r(T-t)} - \lambda_1 - \lambda_2 - \lambda + (\delta(q_1) + q_1 a_1 + a_2) \nu e^{r(T-t)} \\ & + \frac{1}{2}(\sigma_1^2 q_1^2 + \sigma_2^2 + 2q_1 \lambda \mu_1 \mu_2) \nu^2 e^{2r(T-t)} \} = 0, \end{aligned} \quad (4.9)$$

for $t < T$. It can be shown that the minimizer of the equation (4.9) has the form

$$\tilde{q}_1(T-t) = \frac{\eta_1 a_1 e^{-r(T-t)} - \lambda \mu_1 \mu_2 \nu}{\sigma_1^2 \nu}. \quad (4.10)$$

Let

$$\tilde{t}_{03} = T - \frac{1}{r} \ln\left(\frac{\eta_1 a_1}{\sigma_1^2 + \lambda \mu_1 \mu_2}\right).$$

For $t_4 \leq t \leq \tilde{t}_{03}$, it is easy to see that $(q_{1t}^*, q_{1t}^*) = (\tilde{q}_1(T-t), 1)$. Putting $(\tilde{q}_1(T-t), 1)$ into (4.2) gives

$$h_7(T-t) = \tilde{h}_7(T-t) + C_7, \quad (4.11)$$

where

$$\begin{aligned} \tilde{h}_7(T-t) &= \frac{1}{r}((1 + \eta_1)a_1 + a_2 - c)\nu e^{r(T-t)} + \frac{1}{4r}\sigma_2^2 \nu^2 e^{2r(T-t)} \\ &\quad - \int_0^{T-t} \eta_1 \tilde{q}_1(s) a_1 \nu e^{rs} - \frac{1}{2}(\sigma_1^2 \tilde{q}_1^2(s) + 2\tilde{q}_1(s)\lambda\mu_1\mu_2)\nu^2 e^{2rs} ds. \end{aligned}$$

For $\tilde{t}_{03} \leq t \leq T$, it follows that $(q_{1t}^*, q_{1t}^*) = (1, 1)$. Plugging this into (4.2), we get

$$\begin{aligned} h_8(T-t) &= \frac{1}{r}(a_1 + a_2 - c)\nu(e^{r(T-t)} - 1) \\ &\quad + \frac{1}{4r}(\sigma_1^2 + \sigma_2^2 + 2\lambda\mu_1\mu_2)\nu^2(e^{2r(T-t)} - 1). \end{aligned} \quad (4.12)$$

Let

$$C_6 = h_7(T-t_4) - \tilde{h}_6(T-t_4), \quad \text{and} \quad C_7 = h_8(T-\tilde{t}_{03}) - \tilde{h}_7(T-\tilde{t}_{03}).$$

Then, we have

$$\begin{cases} h_7(T-t_4) &= \tilde{h}_6(T-t_4) + C_6 = h_6(T-t_4), \\ h_8(T-\tilde{t}_{03}) &= \tilde{h}_7(T-\tilde{t}_{03}) + C_7 = h_7(T-\tilde{t}_{03}). \end{cases}$$

□

Case II: In this case, we still have $\bar{q}_1(T-t) > 0$ and $\bar{q}_2(T-t) > 0$ but $A_1 \geq A_2$. Thus, $t_3 \leq t_4$. For $0 \leq t \leq t_3$, $(q_{1t}^*, q_{2t}^*) = (\bar{q}_1(T-t), \bar{q}_2(T-t))$. This together with (4.2) gives

$$h_9(T-t) = \tilde{h}_6(T-t) + C_9. \quad (4.13)$$

For $t \geq t_3$, we have $\bar{q}_1(T-t) \geq 1$, and thus $q_{1t}^* = 1$. By substituting $q_{1t}^* = 1$ into (4.2) and mimicking the derivation of (4.10), we obtain the minimizer

$$\tilde{q}_2(T-t) = \frac{\eta_2 a_2 e^{-r(T-t)} - \lambda \mu_1 \mu_2 \nu}{\sigma_2^2 \nu}. \quad (4.14)$$

Write

$$\tilde{t}_{04} = T - \frac{1}{r} \ln\left(\frac{\eta_2 a_2}{(\sigma_2^2 + \lambda \mu_1 \mu_2) \nu}\right).$$

It is easy to check that $\tilde{t}_{04} \geq t_3$. For $t_3 \leq t \leq \tilde{t}_{04}$, $(q_{1t}^*, q_{1t}^*) = (1, \tilde{q}_2(T-t))$. Putting $(1, \tilde{q}_2(T-t))$ into (4.2), we obtain

$$h_{10}(T-t) = \tilde{h}_{10}(T-t) + C_{10}, \quad (4.15)$$

where

$$\begin{aligned} \tilde{h}_{10}(T-t) &= \frac{1}{r}((1 + \eta_2)a_2 + a_1 - c)\nu e^{r(T-t)} + \frac{1}{4r}\sigma_1^2\nu^2 e^{2r(T-t)} \\ &\quad - \int_0^{T-t} \eta_2 \tilde{q}_2(s) a_2 \nu e^{rs} - \frac{1}{2}(\sigma_2^2 \tilde{q}_2^2(s) + 2\tilde{q}_2(s)\lambda\mu_1\mu_2)\nu^2 e^{2rs} ds. \end{aligned}$$

For $\tilde{t}_{04} \leq t \leq T$, $(q_{1t}^*, q_{1t}^*) = (1, 1)$. Then, it can be shown that $h(T-t)$ function in (4.2) is given by h_8 of (4.12).

Let

$$C_9 = h_{10}(T-t_3) - \tilde{h}_6(T-t_3), \quad \text{and} \quad C_{10} = h_8(T-\tilde{t}_{04}) - \tilde{h}_{10}(T-\tilde{t}_{04}).$$

Then, we have

$$\begin{cases} h_{10}(T-t_3) &= \tilde{h}_6(T-t_3) + C_9 = h_9(T-t_3), \\ h_8(T-\tilde{t}_{04}) &= \tilde{h}_{10}(T-\tilde{t}_{04}) + C_{10} = h_{10}(T-\tilde{t}_{04}). \end{cases}$$

□

Case III: In this case, we have $\bar{q}_1(T-t) < 0$ and $\bar{q}_2(T-t) > 0$, and thus $q_{1t}^* = 0$. Substituting $q_{1t}^* = 0$ into (4.2) yields the minimizer

$$\check{q}_2(T-t) = \frac{\eta_2 a_2}{\sigma_2^2 \nu} \cdot e^{-r(T-t)}. \quad (4.16)$$

Let

$$t_{04} = T - \frac{1}{r} \ln\left(\frac{\eta_2 a_2}{\sigma_2^2 \nu}\right).$$

For $0 \leq t \leq t_{04}$, it follows that $(q_{1t}^*, q_{1t}^*) = (0, \check{q}_2(T-t))$. Plugging $(0, \check{q}_2(T-t))$ into (4.2), we get

$$h_{11}(T-t) = \tilde{h}_{11}(T-t) + C_{11}, \quad (4.17)$$

where

$$\tilde{h}_{11}(T-t) = \frac{1}{r}((1+\eta_2)a_2 + (1+\eta_1)a_1 - c)\nu e^{r(T-t)} - \frac{1}{4r} \frac{\eta_2^2 a_2^2}{\sigma_2^2} (T-t).$$

For $t_{04} \leq t \leq T$, $(q_{1t}^*, q_{2t}^*) = (0, 1)$. Using these optimal values and (4.2), we obtain

$$h_{12}(T-t) = \frac{1}{r}(a_2 + (1+\eta_1)a_1 - c)\nu(e^{r(T-t)} - 1) + \frac{1}{4r}\sigma_2^2\nu^2(e^{2r(T-t)} - 1). \quad (4.18)$$

Let $C_{11} = h_{12}(T-t_{04}) - \tilde{h}_{11}(T-t_{04})$. Then, we have

$$h_{11}(T-t_{04}) = \tilde{h}_{11}(T-t_{04}) + C_{11} = h_{12}(T-t_{04}).$$

□

Case IV: In this case, we have $\bar{q}_1(T-t) > 0$ and $\bar{q}_2(T-t) < 0$, and thus $q_{2t}^* = 0$. Substituting $q_{2t}^* = 0$ into (4.2) yields the minimizer

$$\check{q}_1(T-t) = \frac{\eta_1 a_1}{\sigma_1^2 \nu} \cdot e^{-r(T-t)}. \quad (4.19)$$

Write

$$t_{03} = T - \frac{1}{r} \ln\left(\frac{\eta_1 a_1}{\sigma_1^2 \nu}\right).$$

For $0 \leq t \leq t_{03}$, $(q_{1t}^*, q_{2t}^*) = (\check{q}_1(T-t), 0)$. This together with (4.2) yields

$$h_{13}(T-t) = \tilde{h}_{13}(T-t) + C_{13}, \quad (4.20)$$

where

$$\tilde{h}_{13}(T-t) = \frac{1}{r}((1+\eta_2)a_2 + (1+\eta_1)a_1 - c)\nu e^{r(T-t)} - \frac{1}{4r} \frac{\eta_1^2 a_1^2}{\sigma_1^2} (T-t).$$

For $t_{03} \leq t \leq T$, it follows that $(q_{1t}^*, q_{2t}^*) = (1, 0)$. Again, putting this into (4.2), we get

$$h_{14}(T-t) = \frac{1}{r}(a_1 + (1+\eta_2)a_2 - c)\nu(e^{r(T-t)} - 1) + \frac{1}{4r}\sigma_1^2\nu^2(e^{2r(T-t)} - 1). \quad (4.21)$$

Let $C_{13} = h_{14}(T-t_{03}) - \tilde{h}_{13}(T-t_{03})$. Then, we have

$$h_{13}(T-t_{03}) = \tilde{h}_{13}(T-t_{03}) + C_{13} = h_{14}(T-t_{03}).$$

□

To summarize, we state the results in the following theorem.

Theorem 4.1. *Let $(\bar{q}_1(T-t), \bar{q}_2(T-t))$, $\tilde{q}_1(T-t)$, $\tilde{q}_2(T-t)$, $\check{q}_2(T-t)$, and $\check{q}_1(T-t)$ be given in (4.5), (4.10), (4.14), (4.16) and (4.19), respectively. Also, recall the functions $h_i(T-t)$ for $i = 6, 7, \dots, 14$ given in (4.8), (4.11), (4.12), (4.13), (4.15), (4.17), (4.18), (4.20) and (4.21), respectively. Then, we have*

(i) If Case I holds, then the optimal reinsurance strategies under the model (2.2) are

$$(\bar{q}_{1t}^*, \bar{q}_{2t}^*) = \begin{cases} (\bar{q}_1(T-t), \bar{q}_2(T-t)), & 0 \leq t \leq t_4, \\ (\tilde{q}_1(T-t), 1), & t_4 \leq t \leq \tilde{t}_{03}, \\ (1, 1), & \tilde{t}_{03} \leq t \leq T, \end{cases}$$

for any $t \in [0, T]$, and the value function is given by

$$V(t, x) = \begin{cases} -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_6(T-t)\}, & 0 \leq t \leq t_4, \\ -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_7(T-t)\}, & t_4 \leq t \leq \tilde{t}_{03}, \\ -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_8(T-t)\}, & \tilde{t}_{03} \leq t \leq T; \end{cases}$$

(ii) If Case II holds, then the optimal reinsurance strategies under the model (2.2) are

$$(\bar{q}_{1t}^*, \bar{q}_{2t}^*) = \begin{cases} (\bar{q}_1(T-t), \bar{q}_2(T-t)), & 0 \leq t \leq t_3, \\ (1, \tilde{q}_2(T-t)), & t_3 \leq t \leq \tilde{t}_{04}, \\ (1, 1), & \tilde{t}_{04} \leq t \leq T, \end{cases}$$

for any $t \in [0, T]$, and the value function is given by

$$V(t, x) = \begin{cases} -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_9(T-t)\}, & 0 \leq t \leq t_3, \\ -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_{10}(T-t)\}, & t_3 \leq t \leq \tilde{t}_{04}, \\ -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_8(T-t)\}, & \tilde{t}_{04} \leq t \leq T; \end{cases}$$

(iii) If Case III holds, then the optimal reinsurance strategies under the model (2.2) are

$$(\bar{q}_{1t}^*, \bar{q}_{2t}^*) = \begin{cases} (0, \check{q}_2(T-t)), & 0 \leq t \leq t_{04}, \\ (0, 1), & t_{04} \leq T, \end{cases}$$

for any $t \in [0, T]$, and the value function is given by

$$V(t, x) = \begin{cases} -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_{11}(T-t)\}, & 0 \leq t \leq t_{04}, \\ -\frac{\rho}{\nu} \exp\{-\nu x e^{r(T-t)} + h_{12}(T-t)\}, & t_{04} \leq T; \end{cases}$$

(iv) If Case VI holds, then the optimal reinsurance strategies under the model (2.2) are

$$(\bar{q}_{1t}^*, \bar{q}_{2t}^*) = \begin{cases} (\check{q}_1(T-t), 0), & 0 \leq t \leq t_{03}, \\ (1, 0), & t_{03} \leq T, \end{cases}$$

for any $t \in [0, T]$, and the value function is given by

$$V(t, x) = \begin{cases} -\frac{g}{\nu} \exp\{-\nu x e^{r(T-t)} + h_{13}(T-t)\}, & 0 \leq t \leq t_{03}, \\ -\frac{g}{\nu} \exp\{-\nu x e^{r(T-t)} + h_{14}(T-t)\}, & t_{03} \leq T. \end{cases}$$

Remark 4.1. Since

$$\begin{cases} h_6(T-t_4) = h_7(T-t_4), & h_7(T-\tilde{t}_{03}) = h_8(T-\tilde{t}_{03}), \\ h_9(T-t_3) = h_{10}(T-t_3), & h_{10}(T-\tilde{t}_{04}) = h_8(T-\tilde{t}_{04}), \\ h_{11}(T-t_{04}) = h_{12}(T-t_{04}), & h_{13}(T-t_{03}) = h_{14}(T-t_{03}), \end{cases}$$

$V(t, x)$ is a continuous function for any $(t, x) \in [0, T] \times R$. Furthermore,

$$\begin{cases} h'_6(T-t_4) = h'_7(T-t_4), & h'_7(T-\tilde{t}_{03}) = h'_8(T-\tilde{t}_{03}), \\ h'_9(T-t_3) = h'_{10}(T-t_3), & h'_{10}(T-\tilde{t}_{04}) = h'_8(T-\tilde{t}_{04}), \\ h'_{11}(T-t_{04}) = h'_{12}(T-t_{04}), & h'_{13}(T-t_{03}) = h'_{14}(T-t_{03}). \end{cases}$$

That is, $V(t, x)$ is a classical solution to the HJB equation (4.1).

Remark 4.2. Similar to the optimal results in the compound Poisson risk model, we can see from (4.5) that the optimal reinsurance strategies under the expected value principle in the diffusion risk model depend on the safety loading, time, interest rate as well as the claim size distributions and the counting processes. However, when we consider the model (2.2) under the variance principle, the results are very different. The optimal reinsurance strategies under the variance principle are always the same, and depend only on the safety loading, time and interest rate only (see (6.3) or Section 4 of Liang and Yuen (2014) for details).

Remark 4.3. In Sections 3 and 4, we show how the optimal strategies and value functions can be obtained for $m = 2$. It is believed that the ideas and techniques shown here are still useful for deriving optimal results for $m > 2$. However, when m gets bigger, the number of cases needed to be considered may increase geometrically, and hence the optimal problem becomes very challenging.

5 Numerical examples

In this section, we restrict our attention to the model with two dependent classes of insurance business under the expected value principle. For numerical examples using the variance principle, we refer the readers to Liang and Yuen (2014).

In Examples 5.1-5.3, we assume that the claim sizes $\{X_i\}$ and $\{Y_i\}$ are exponentially distributed with parameters α_1 and α_2 , respectively. That is, $X \sim \exp(\alpha_1)$ and $Y \sim \exp(\alpha_2)$. For computational convenience, we set $\eta_1 = \eta_2 = \eta$, and hence

$$\begin{cases} M_1(\nu q_1 e^{r(T-t)}) = \frac{\alpha_1}{\alpha_1 - \nu q_1 e^{r(T-t)}}, \\ M_2(\nu q_2 e^{r(T-t)}) = \frac{\alpha_2}{\alpha_2 - \nu q_2 e^{r(T-t)}}. \end{cases}$$

The minimizer $(q_1(T-t), q_2(T-t))$ of (3.3) in the compound Poisson model satisfies the following equations

$$\begin{cases} -(1+\eta)(\lambda_1 + \lambda)/\alpha_1 + \frac{\alpha_1}{(\alpha_1 - \nu q_1 e^{r(T-t)})^2} \left(\lambda_1 + \frac{\lambda \alpha_2}{\alpha_2 - \nu q_2 e^{r(T-t)}} \right) = 0, \\ -(1+\eta)(\lambda_2 + \lambda)/\alpha_2 + \frac{\alpha_2}{(\alpha_2 - \nu q_2 e^{r(T-t)})^2} \left(\lambda_2 + \frac{\lambda \alpha_1}{\alpha_1 - \nu q_1 e^{r(T-t)}} \right) = 0, \end{cases}$$

and the minimizer $(\bar{q}_1(T-t), \bar{q}_2(T-t))$ of (4.2) in the diffusion case has the form as

$$\begin{cases} \bar{q}_1(T-t) = \frac{\alpha_1(2(\lambda_1 + \lambda)(\lambda_2 + \lambda) - \lambda(\lambda_2 + \lambda))}{4(\lambda_1 + \lambda)(\lambda_2 + \lambda) - \lambda^2} \cdot \frac{\eta}{\nu} e^{-r(T-t)}, \\ \bar{q}_2(T-t) = \frac{\alpha_2(2(\lambda_1 + \lambda)(\lambda_2 + \lambda) - \lambda(\lambda_1 + \lambda))}{4(\lambda_1 + \lambda)(\lambda_2 + \lambda) - \lambda^2} \cdot \frac{\eta}{\nu} e^{-r(T-t)}. \end{cases}$$

Besides, for the diffusion risk model, we have

$$\begin{cases} \bar{q}_2^* = \frac{\eta(\lambda_1 + \lambda)e^{-r(T-t)} - \frac{\lambda}{\alpha_2}\nu}{(\lambda_1 + \lambda)\frac{2}{\alpha_1}\nu}, & \text{when } \bar{q}_1^* = 1; \\ \bar{q}_1^* = \frac{\eta(\lambda_2 + \lambda)e^{-r(T-t)} - \frac{\lambda}{\alpha_1}\nu}{(\lambda_2 + \lambda)\frac{2}{\alpha_2}\nu}, & \text{when } \bar{q}_2^* = 1. \end{cases} \quad (5.1)$$

Example 5.1. Let $\eta = 2$, $r = 0.3$, $T = 10$, $\lambda_1 = 3$, $\lambda_2 = 4$, $\lambda = 2$, $\nu = 0.5$, and $t = 2.5$. The results are shown in Table 5.1.

Table 5.1 The effect of α_1 on the optimal reinsurance strategies

α_1	t_1	t_2	q_1^*	q_2^*	t_3	t_4	\bar{q}_1^*	\bar{q}_2^*
1	10	7.3303	0.0755	0.2348	8.3203	4.5222	0.1745	0.5452
2	8.8024	7.3303	0.1510	0.2348	6.0098	4.5222	0.3489	0.5452
3	7.4508	7.3303	0.2264	0.2348	4.6582	4.5222	0.5234	0.5452
4	6.4919	7.3303	0.3019	0.2348	3.6993	4.5222	0.6978	0.5452
5	5.7481	7.3303	0.3774	0.2348	2.9555	4.5222	0.8723	0.5452
6	5.1403	7.3303	0.4529	0.2348	2.3477	4.5222	1	0.5491
7	4.6265	7.3303	0.5284	0.2348	1.8339	4.5222	1	0.5610

From Tables 5.1 ($\alpha_2 = 3$), we can see that a greater value of α_1 yields greater values of the optimal reinsurance strategies q_1^* (\bar{q}_1^*). As expected, since a smaller value of α_1 implies “more risky” claim size with a larger expected value, the insurer would rather retain a less share of each claim. Also, it is easy to observe from the tables that q_2^* (\bar{q}_2^*) is independent of α_1 . That is, when the claim sizes are exponentially distributed, X (Y) has no effect on the optimal reinsurance strategies q_2^* and \bar{q}_2^* (q_1^* and \bar{q}_1^*). This interesting phenomenon may only appear in the case with exponential claim sizes and the reinsurance premium calculated under the expected value principle. This guess is partly due to the following two facts: (i) it was shown in Tables 4 and 5 in Liang and Yuen (2014) that the optimal results under the variance principle are indeed affected by both claim size distributions in the compound Poisson risk model; (ii) we see later in Example 5.4 and Table 5.4 that the claim sizes Y (X) does affect the values of the optimal reinsurance strategies q_2^* and \bar{q}_2^* as well as q_1^* and \bar{q}_1^* , when Y (X) follows a gamma distribution with parameters $\beta \neq 1$ and α . However, when one of the optimal strategies hits the boundary one, we see from the bold-faced values in Table 5.1 that the other optimal strategy is forced to change regardless to the choice of the claim size distributions. Moreover, we can see from Tables 5.1 that the value of q_1^* (\bar{q}_1^*) is always smaller than the value of q_2^* (\bar{q}_2^*) when the inequality $\alpha_1 < \alpha_2$ holds, and vice versa. This suggests that the values of the optimal reinsurance strategies are more sensitive to the claim size distribution than to the counting processes (see Table 5.3).

Remark 5.1. The bold-faced values in Tables 5.1-5.3 are calculated using (5.1) when one of the optimal reinsurance strategies is 1.

Example 5.2. Let $r = 0.3$, $T = 10$, $\lambda_2 = 4$, $\eta = 2$, $\nu = 0.5$, $\alpha_1 = 2$, $\alpha_2 = 3$, and $t = 4.6$. The results are shown in Table 5.2.

From Table 5.2, we see that the optimal reinsurance strategies decrease while λ increases. Since a greater value of λ implies a greater value of expected claim number, the insurer would rather retain a less share of each claim.

Table 5.2 The effect of λ on the optimal reinsurance strategies

λ	t_1	t_2	q_1^*	q_2^*	t_3	t_4	\bar{q}_1^*	\bar{q}_2^*
1	8.6176	7.1790	0.2996	0.4613	5.7821	4.3367	0.7083	1
2	8.8024	7.3303	0.2835	0.4408	6.0098	4.5222	0.6583	1
3	8.9134	7.3678	0.2742	0.4359	6.1542	4.6478	0.6273	0.9858
4	8.9871	7.4993	0.2682	0.4190	6.2537	4.7395	0.6089	0.9590
5	9.0391	7.5526	0.2640	0.4124	6.3251	4.8095	0.5960	0.9391
6	9.0778	7.5946	0.2610	0.4072	6.3796	4.8651	0.5863	0.9236
7	9.1075	7.6286	0.2587	0.4031	6.4212	4.9105	0.5791	0.9110

Example 5.3. Let $r = 0.3$, $T = 10$, $\lambda_2 = 4$, $\eta = 2$, $\lambda = 2$, $\nu = 0.5$, $\alpha_1 = 2$, $\alpha_2 = 3$, and $t = 4.5$. The results are shown in Table 5.3.

Table 5.3 The effect of λ_1 on the optimal reinsurance strategies

λ_1	t_1	t_2	q_1^*	q_2^*	t_3	t_4	\bar{q}_1^*	\bar{q}_2^*
1	9.2321	7.2525	0.2418	0.4379	6.5400	4.4448	0.5460	1
2	8.9572	7.2998	0.2626	0.4317	6.1961	4.4935	0.6015	1
3	8.8024	7.3303	0.2751	0.4278	6.0098	4.5222	0.6358	0.9934
5	8.6340	7.3672	0.2893	0.4231	5.8127	4.5549	0.6745	0.9837
6	8.5832	7.3792	0.2938	0.4216	5.7540	4.5650	0.6865	0.9807
7	8.5442	7.3886	0.2972	0.4204	5.7092	4.5729	0.6958	0.9784
8	8.5134	7.3963	0.3000	0.4194	5.6740	4.5791	0.7031	0.9765

From Table 5.3, we see that a greater value of λ_1 yields greater values of the optimal reinsurance strategies q_1^* and \bar{q}_1^* but smaller values of the optimal reinsurance strategy q_2^* and \bar{q}_2^* . Along the same lines, one can numerically show that a greater value of λ_2 yields greater values of q_2^* and \bar{q}_2^* but smaller values of q_1^* and \bar{q}_1^* .

In the following example, we show that q_1^* (q_2^*) does depend on the claim sizes Y (X) for some distributions. We assume that the claim sizes Y has a gamma distribution with parameters $\beta \neq 1$ and α , that is, $Y \sim \Gamma(\beta, \alpha)$. Then, we have

$$\begin{cases} \mu_2 & = E(Y) = \frac{\beta}{\alpha}, \\ \sigma_2^2 & = (\lambda_2 + \lambda)(\beta^2 + \beta)/\alpha^2, \\ M_2(r) & = \left(\frac{\alpha}{\alpha-r}\right)^\beta, \quad \text{for } r < \alpha. \end{cases}$$

The minimizer $(q_1(T-t), q_2(T-t))$ of (3.3) for the compound Poisson model satisfies the following equations

$$\begin{cases} -(1+\eta)(\lambda_1 + \lambda)/\alpha_1 + \frac{\alpha_1}{(\alpha_1 - \nu q_1 e^{r(T-t)})^2} \left(\lambda_1 + \lambda \left(\frac{\alpha}{\alpha - \nu q_2 e^{r(T-t)}} \right)^\beta \right) & = 0, \\ -(1+\eta)(\lambda_2 + \lambda)\beta/\alpha + \beta \left(\frac{\alpha}{\alpha - \nu q_2 e^{r(T-t)}} \right)^{\beta-1} \frac{\alpha}{(\alpha - \nu q_2 e^{r(T-t)})^2} \left(\lambda_2 + \frac{\lambda \alpha_1}{\alpha_1 - \nu q_1 e^{r(T-t)}} \right) & = 0, \end{cases}$$

and the minimizer $(\bar{q}_1(T-t), \bar{q}_2(T-t))$ of (4.2) for the diffusion model is given by

$$\begin{cases} \bar{q}_1(T-t) = \frac{\alpha_1((\lambda_1+\lambda)(\lambda_2+\lambda)(\beta+1)-\lambda\beta(\lambda_2+\lambda))}{2(\lambda_1+\lambda)(\lambda_2+\lambda)(\beta+1)-\lambda^2\beta} \cdot \frac{\eta}{\nu} e^{-r(T-t)}, \\ \bar{q}_2(T-t) = \frac{\alpha(2(\lambda_1+\lambda)(\lambda_2+\lambda)-\lambda(\lambda_1+\lambda))}{2(\lambda_1+\lambda)(\lambda_2+\lambda)(\beta+1)-\lambda^2\beta} \cdot \frac{\eta}{\nu} e^{-r(T-t)}. \end{cases}$$

Example 5.4. Assume that $X \sim \exp(\alpha_1)$ and $Y \sim \Gamma(\beta, \alpha)$. Let $\lambda = 2$, $\lambda_1 = 3$, $r = 0.3$, $T = 10$, $\lambda_2 = 4$, $\eta = 2$, $\nu = 0.5$, $\alpha_1 = 2$, $\alpha = 3$, and $t = 1.8$. The results are shown in Table 5.4.

Table 5.4 The effect of β on the optimal reinsurance strategies

β	t_1	t_2	q_1^*	q_2^*	t_3	t_4	\bar{q}_1^*	\bar{q}_2^*
0.6	8.6290	6.7988	0.1289	0.2232	5.8364	3.8070	0.2979	0.5477
0.7	8.6769	6.9395	0.1271	0.2140	5.8861	4.0007	0.2935	0.5167
0.8	8.7216	7.0747	0.1254	0.2055	5.9312	4.1837	0.2896	0.4891
0.9	8.7633	7.2049	0.1238	0.1976	5.9723	4.3573	0.2860	0.4643
1	8.8024	7.3303	0.1224	0.1903	6.0098	4.5222	0.2828	0.4419
2	9.0886	8.3819	0.1123	0.1388	6.2613	5.8352	0.2623	0.2980
3	9.2627	9.1830	0.1066	0.1092	6.3970	6.7747	0.2518	0.2248

From Table 5.4, we see that a greater value of β yields smaller values of the optimal reinsurance strategies q_i^* and \bar{q}_i^* ($i = 1, 2$). Since a greater value of β implies “more risky” claim sizes with a larger expected value, the insurer would rather retain a less share of each claim in general. It is also reasonable to expect that the optimal reinsurance strategy q_2^* (\bar{q}_2^*) decreases faster than q_1^* (\bar{q}_1^*) as the claim size distribution Y has a more direct impact on q_2^* (\bar{q}_2^*).

In the numerical examples, we find that the optimal reinsurance strategies are more sensitive to the claim size distribution than to the counting processes. This finding is consistent with that under the variance principle (see Tables 4-7 in Liang and Yuen (2014)). Moreover, when the claim

sizes are exponentially distributed, under the expected value principle, the optimal reinsurance strategy of one class of insurance risk is independent of the claim sizes of the other class. However, this is not true when the reinsurance premium is calculated using the variance principle, under which both claim size distributions do have impact on the optimal reinsurance strategies of the two classes (see Tables 4 and 5 in Liang and Yuen (2014)). Furthermore, in the diffusion risk model, the optimal reinsurance strategies under the expected value principle depend on the safety loading, time, interest rate, the claim size distributions and the counting processes, while those under variance principle depend only on the safety loading, time and interest rate (see (6.3) below).

6 Further results

For the diffusion risk model, we now extend the work of Liang and Yuen (2014) to the case with $m > 2$ classes of insurance business under the variance principle. In this case, the reinsurance premium rate becomes

$$\delta(q) = \sum_{l=1}^m (1 - q_l) a_l + \Lambda \left(\sum_{l=1}^m (1 - q_l)^2 \sigma_l^2 + \sum_{l=1}^m \sum_{k=1(k \neq l)}^m (1 - q_l)(1 - q_k) \lambda \mu_l \mu_k \right), \quad (6.1)$$

here Λ is the reinsurer's safety loading. Following the steps in Section 4, one can show that the optimal reinsurance strategies for the diffusion risk model satisfy the following equations

$$-2\Lambda((1 - q_l)\sigma_l^2 + \sum_{k=1(k \neq l)}^m (1 - q_k) \lambda \mu_l \mu_k) + (q_l \sigma_l^2 + \lambda \sum_{i \neq l} q_i \mu_l \mu_i) \nu e^{r(T-t)} = 0, \quad (6.2)$$

for $l = 1, 2, \dots, m$; or equivalently,

$$\mathbf{A}q' = \tilde{\mathbf{B}} \frac{2\Lambda}{2\Lambda + \nu e^{r(T-t)}},$$

where

$$\tilde{\mathbf{B}} = \begin{pmatrix} \sigma_1^2 + \sum_{k \neq 1} \lambda \mu_1 \mu_k \\ \sigma_2^2 + \sum_{k \neq 2} \lambda \mu_2 \mu_k \\ \sigma_3^2 + \sum_{k \neq 3} \lambda \mu_3 \mu_k \\ \vdots \\ \sigma_m^2 + \sum_{k \neq m} \lambda \mu_m \mu_k \end{pmatrix}.$$

Write $\mathbf{I} = (1, 1, \dots, 1)'_{1 \times m}$. Since $\tilde{\mathbf{B}} = \mathbf{A}\mathbf{I}$ and the matrix \mathbf{A} is invertible, the equation (6.2) has a unique solution

$$\begin{aligned} \bar{q}(T-t) &= (\mathbf{A}^{-1} \tilde{\mathbf{B}} \frac{2\Lambda}{2\Lambda + \nu e^{r(T-t)}})' \\ &= \mathbf{I}' \frac{2\Lambda}{2\Lambda + \nu e^{r(T-t)}}, \end{aligned}$$

which implies that, for any $t \in [0, T]$, the optimal reinsurance strategies under the variance principle in the diffusion model are

$$q_{1t}^* = q_{2t}^* = \cdots = q_{mt}^* = \frac{2\Lambda}{2\Lambda + \nu e^{r(T-t)}}, \quad (6.3)$$

which depend only on the safety loading, time and interest rate. This result is in line with the one in Theorems 4.1 and 5.2 of Liang et al. (2011). In addition, for the case with different safety loadings, the reader is referred to (6.1) of Liang and Yuen (2014) for which the optimal strategies also depend on the claim size distributions and the counting processes. Similar results for the optimal strategies should be expected in the case with $m > 2$.

For optimization problems under the variance principle, results similar to (6.3) (see Liang et al. (2011), Liang and Yuen (2014), and the reference therein) are often derived. That is, a more complicated premium principle, namely the variance premium principle, results in a simpler form of optimal value, at least compared to the results derived under the expected value principle. This interesting observation is really difficult to explain. A rough guess is that the variance or the dependence is considered both in the premium rate and surplus process, the impact of the claim size distributions and the counting processes is somehow cancelled out in the process of derivation.

7 Conclusion

The optimization reinsurance problem has been around for many years, and hence many interesting and useful results can be found in the literature. In this paper, we give a notable extension of the study to the risk model with common shock dependence.

From an insurer's point of view, we consider the optimal proportional reinsurance strategy in a risk model with $m \geq 2$ dependent classes of insurance business, which generalizes the results of Liang and Yuen (2014). Under the criterion of maximizing the expected exponential utility, closed-form expressions for the optimal strategies and value function are given not only for the compound Poisson risk model but also for the diffusion approximation risk model. Our results show that the optimal reinsurance strategies under the expected value principle are very different from those under the variance principle in the diffusion case. In particular, the former depends not only on the safety loading, time and interest rate, but also on the claim size distributions and the counting processes, and the optimal strategies of the two classes do not have the same form.

Acknowledgements

The authors would like to thank the anonymous reviewer and the Editor for their constructive comments and suggestions. The research of Kam Chuen Yuen was supported by the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. HKU 7057/13P), and the CAE 2013 research grant from the Society of Actuaries - any opinions, finding, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the SOA. The research of Zhibin Liang and Ming Zhou was supported by the National Natural Science Foundation of China (Grant No.11471165) and Jiangsu Natural Science Foundation (Grant No. BK20141442).

References

- [1] Bai, L. Cai, J. and Zhou, M., 2013. Optimal reinsurance policies for an insurer with a bivariate reserve risk process in a dynamic setting. *Insurance: Mathematics and Economics* 53, 664-670.
- [2] Bernard, C. and Tian, W. (2009). Optimal reinsurance arrangements under tail risk measures. *Journal of Risk and Insurance* 76(3), 709-725.
- [3] Browne, S., 1995. Optimal investment policies for a firm with random risk process: exponential utility and minimizing the probability of ruin. *Mathematics of Operations Research* 20, 937-958.
- [4] Cai, J. and Tan, K. (2007). Optimal retention for a stop-loss reinsurance under the VaR and CTE risk measures. *ASTIN Bulletin* 37(1), 93-112.
- [5] Centeno, M. (1986). Measuring the effects of reinsurance by the adjustment coefficient. *Insurance: Mathematics and Economics* 5, 169-182.
- [6] Centeno, M. (2002). Excess of loss reinsurance and Gerber's inequality in the Sparre Anderson model. *Insurance: Mathematics and Economics* 31, 415-427.
- [7] Centeno, M., 2005. Dependent risks and excess of loss reinsurance. *Insurance: Mathematics and Economics* 37, 229-238.
- [8] Cheung, K. C., Sung, K. C. J., Yam, S. C. P. and Yung, S. P., 2014. Optimal reinsurance under general law-invariant risk measures. *Scandinavian Actuarial Journal* 1, 72-91.
- [9] Chi, Y. and Meng, H., 2014. Optimal reinsurance arrangements in the presence of two reinsurers. *Scandinavian Actuarial Journal* 5, 424-438.
- [10] Fleming, W., Soner, H., 2006. *Controlled Markov Processes and Viscosity Solutions*. 2nd Ed. Springer-Verlag, NY.
- [11] Gerber, H., 1979. *An introduction to mathematical risk theory*. Huebner Foundation Monograph. No. 8.

- [12] Grandell, J., 1991. Aspects of risk theory. Springer-verlag, New York.
- [13] Hald, M. and Schmidli, H. (2004). On the maximization of the adjustment coefficient under proportional reinsurance. *ASTIN Bulletin* 34, 75-83.
- [14] Irgens, C. and Paulsen, J. (2004). Optimal control of risk exposure, reinsurance and investments for insurance portfolios. *Insurance: Mathematics and Economics* 35, 21-51.
- [15] Kaluszka, M. (2001). Optimal reinsurance under mean-variance premium principles. *Insurance: Mathematics and Economics* 28, 61-67.
- [16] Kaluszka, M. (2004). Mean-variance optimal reinsurance arrangements. *Scandinavian Actuarial Journal*, 28-41.
- [17] Liang, Z. and Guo, J. (2008). Upper bound for ruin probabilities under optimal investment and proportional reinsurance. *Applied Stochastic Models in Business and Industry* 24, 109-128.
- [18] Liang, Z., Yuen, Kam C. and Guo, J., 2011. Optimal proportional reinsurance and investment in a stock market with Ornstein-Uhlenbeck process. *Insurance: Mathematics and Economics* 49, 207-215.
- [19] Liang, Z., Yuen, Kam C., 2014. Optimal dynamic reinsurance with dependent risks: variance premium principle. *Scandinavian Actuarial Journal*, DOI: 10.1080/03461238.2014.892899.
- [20] Ortega, J. M. and Rheinboldt, W. C., 1970. Iterative solution of nonlinear equations in several variables. Academic Press, New York.
- [21] Promislow, D. and Young, V. (2005). Minimizing the probability of ruin when claims follow Brownian motion with drift. *North American Actuarial Journal* 9(3), 109-128.
- [22] Schmidli, H. (2001). Optimal proportional reinsurance policies in a dynamic setting. *Scandinavian Actuarial Journal*, 55-68.
- [23] Schmidli, H. (2002). On minimizing the ruin probability by investment and reinsurance. *Annals of Applied Probability* 12, 890-907.
- [24] Wang, G. and Yuen, Kam C., 2005. On a correlated aggregate claims model with thinning-dependence structure. *Insurance: Mathematics and Economics* 36, 456-468.
- [25] Wang, S., 1998. Aggregation of correlated risk portfolios: Models and algorithms, *Proceedings of the Casualty Actuarial Society*, vol. LXXXV, pp. 848-939.
- [26] Yang, H., Zhang, L., 2005. Optimal investment for insurer with jump-diffusion risk process. *Insurance: Mathematics and Economics* 37, 615-634.
- [27] Yuen, Kam C., Guo, J., Wu, X., 2002. On a correlated aggregate claim model with Poisson and Erlang risk process. *Insurance: Mathematics and Economics* 31, 205-214.
- [28] Yuen, Kam C., Guo, J., Wu, X., 2006. On the first time of ruin in the bivariate compound Poisson model. *Insurance: Mathematics and Economics* 38, 298-308.