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Analysis and optimisation for inerter-based isolators via fixed-point theory and algebraic solution

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Abstract

This paper is concerned with the problem of analysis and optimisation of the inerter-based isolators based on a “uni-axial” single-degree-of-freedom isolation system. In the first part, in order to gain an in-depth understanding of inerter from the prospective of vibration, the frequency responses of both parallel-connected and series-connected inerters are analysed. In the second part, three other inerter-based isolators are introduced and the tuning procedures in both the H_∞ optimisation and the H_2 optimisation are proposed in an analytical manner. The achieved H_2 and H_∞ performance of the inerter-based isolators is superior to that achieved by the traditional dynamic vibration absorber (DVA) when the same inertance-to-mass (or mass) ratio is considered. Moreover, the inerter-based isolators have two unique properties, which are more attractive than the traditional DVA: first, the inertance-to-mass ratio of the inerter-based isolators can easily be larger than the mass ratio of the traditional DVA without increasing the physical mass of the whole system; second, there is no need to mount an additional mass on the object to be isolated.

Keywords: Inerter, vibration isolation, H_∞ optimisation, H_2 optimisation.

1. Introduction

Inerter is a two-terminal mechanical device with the property that the applied force at its two terminals is proportional to the relative acceleration between them [1, 2], where the constant of proportionality is called inertance with a unit of kilogram. Since the initial application in Formula One racing car suspension systems [2], inerters have been applied to various mechanical systems mainly including vehicle suspensions [3, 4, 5, 6, 7, 8, 9] and vibration suppression [10, 11, 12, 13, 14]. The interest in passive network synthesis has also been rekindled [15, 16, 17, 18, 19, 20, 21, 22]. The influence of inerter on vibration systems' natural frequencies has been investigated in [23], where the fundamental property that inerter can reduce natural frequencies of vibration systems has been theoretically demonstrated.

In this paper, to further investigate the influence of inerter on vibration systems, the performance of the inerter-based isolators based on a “uni-axial” single-degree-of-freedom isolation system is studied. First, to gain an in-depth understanding of inerter from the perspective of vibration, the frequency responses of both parallel-connected and series-connected

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26 inerters are analysed. It is shown that an extra invariant point, which is independent of the
 27 damping ratio, can be introduced by using the series-connected inerter. Then, to further tune
 28 the invariant points, three other inerter-based isolators, each of which incorporates a spring,
 29 a damper and an inerter, are proposed. To facilitate the practical application, the optimal
 30 parameters of the inerter-based isolators in both H_∞ optimisation and H_2 optimisation are
 31 analytically derived. The H_∞ optimisation aims to minimise the maximum magnitude of
 32 the frequency response based on the fixed-point theory [24] which has been extensively used
 33 in tuning the parameters of dynamic vibration absorbers (DVA) (or tuned mass dampers
 34 (TMD)) [25, 26, 27]. While the H_2 optimisation aims to minimise the mean squared dis-
 35 placement of the object under random excitation [29]. An analytical method is employed
 36 to calculate the H_2 norm performance measures of the inerter-based isolators in this paper.
 37 In addition, the comparisons of the H_2 and H_∞ performances between the inerter-based
 38 isolators and the traditional DVA show the superiority of the inerter-based isolators. Two
 39 properties make the inerter-based isolators potentially more attractive than the traditional
 40 DVA: first, a relatively large inertance can easily be obtained without increasing the physical
 41 mass of the whole system [1]; second, there is no need to mount an additional mass on the
 42 object to be isolated, as an inerter is a built-in component in the inerter-based isolators.

43 In [12], one of the inerter-based isolators proposed in this paper (*C3* in Fig. 7 of this pa-
 44 per) has been employed to reduce vibrations in civil engineering structures, and a H_∞ tuning
 45 procedure for this configuration has been proposed by using the fixed-point theory [24]. The
 46 main difference between the procedures in [12] and the H_∞ optimisation proposed in this
 47 paper is that the optimal parameters in [12] are obtained through using iterative algorithms
 48 while the optimal parameters in this paper are obtained analytically. The analytical method
 49 alleviates possible numerical problems induced by iterations and reveals fundamental rela-
 50 tionship between tuning parameters and H_∞ performance. Detailed difference can be found
 51 in Section 4.

52 The organization of this paper is as follows. In Section 2, a “uni-axial” isolation system
 53 is introduced where the force and displacement transmissibilities are also derived. Section 3
 54 provides an in-depth analysis of the frequency response of two simple configurations with
 55 inerter to highlight the fundamental properties of inerter in vibration. Section 4 and Section 5
 56 derive the analytical solutions of the inerter-based isolators in H_∞ optimisation and H_2
 57 optimisation, respectively, where the comparisons with the traditional DVA are also given.
 58 Conclusions are drawn in Section 6.

59 2. Isolation system description

60 In this paper, a “uni-axial” isolation system is considered, as shown in Fig. 1, where
 61 the mass m is the object to be isolated, the mass m_f is the foundation, and $Q(s)$ is the
 62 isolator to be designed. In practice, two situations are commonly encountered depending
 63 on the circumstances. One is that the object must be isolated from the objectionable vi-
 64 bratory motions of the supporting surface, while the other is that the supporting surface
 65 must be protected from the dynamic load generated within the object. The former situation
 66 is called the displacement transmissibility problem and the later one is the force transmis-
 67 sibility problem [31]. In some cases, both tasks have to be addressed simultaneously [30].
 68 For linear isolators, the displacement transmissibility problem and the force transmissibility

Table 1: $W(s)$ for configurations in Fig. 2 and Fig. 7, where s denotes the Laplace variable.

$W_1(s) = bs + c$	$W_2(s) = \frac{1}{\frac{1}{c} + \frac{1}{bs}}$	$W_3(s) = \frac{1}{\frac{1}{\frac{k_1}{s+c}} + \frac{1}{bs}}$	$W_4(s) = \frac{1}{\frac{s}{k_1} + \frac{1}{bs} + \frac{1}{c}}$	$W_5(s) = \frac{1}{\frac{1}{bs+c} + \frac{s}{k_1}}$
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69 problem are equivalent if the mass of the foundation is sufficiently larger than that of the
70 object [31]. For brevity, in this paper, the assumption that $m_f = \infty$ is made and the *absolute*
71 *displacement transmissibility* and the *absolute force transmissibility* are identically treated as

$$\mu = \frac{|F_i|}{|F|} = \frac{|x_1|}{|x_2|} = \frac{|Q(j\omega)j\omega|}{|Q(j\omega)j\omega - m\omega^2|}, \quad (1)$$

72 where F is the force imposed on the object m , F_i is the force generated by the isolator, x_1 and
73 x_2 are the displacements of the object and the foundation, respectively. $Q(j\omega)$ is obtained by
74 replacing the Laplace variable s in $Q(s)$ with $j\omega$, where j is a complex variable with $j^2 = -1$
75 and $Q(s)$ is the admittance of the isolator, i.e. the ratio of the applied force F_i over the
76 relative velocity $\dot{x}_1 - \dot{x}_2$ in Laplace domain.

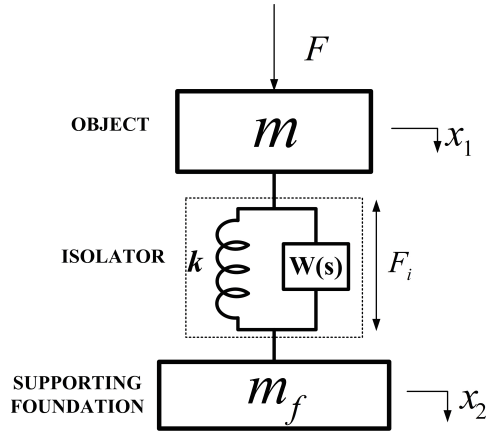


Figure 1: Uni-axial vibration isolation system.

77 As shown in Fig. 1, $Q(s) = \frac{k}{s} + W(s)$, where $W(s)$ denotes the admittances of passive
78 networks consisting of finite inter-connections of springs, dampers and inerters. In this pa-
79 per, five inerter-based isolators will be investigated, as shown in Fig. 2 and Fig. 7. Their
80 admittances are summarized in Table 1.

81 To obtain a dimensionless representation, $\omega_n = \sqrt{\frac{k}{m}}$ and $c_r = 2\omega_n m = 2\sqrt{mk}$ are used to
82 denote the natural frequency and the critical damping of the isolation system shown in Fig. 1
83 without $W(s)$, respectively. Also, $q = \frac{\omega}{\omega_n}$, $\zeta = \frac{c}{c_r}$, $\delta = \frac{b}{m}$ and $\lambda = \frac{k}{k_1}$ denote the frequency
84 ratio, the damping ratio, the inertance-to-mass ratio, and the stiffness ratio, respectively.

85 For the considered configurations as shown in Fig. 2 and Fig. 7, the transmissibility μ can
86 be obtained by substituting $Q_i(j\omega) = \frac{k}{j\omega} + W_i(j\omega)$, $i = 1, \dots, 5$, into (1), respectively, where
87 $W_i(j\omega)$ are given in Table 1 by replacing s with $j\omega$.

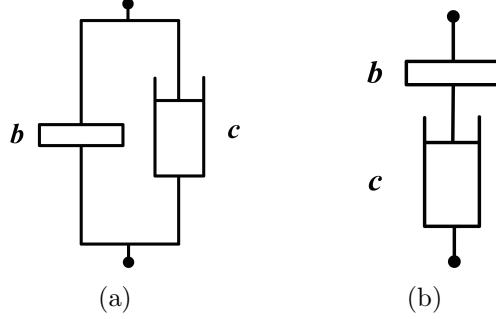


Figure 2: Two simple configurations as $W(s)$ of the isolators in Fig. 1. (a) C1; (b) C2.

3. Vibration analysis for two simple inerter-based isolators

This section is to analyse the fundamental properties of inerter from the perspective of vibration. Note that among all the applications of inerter, the main focus is to optimise some inerter-based mechanical networks possessing more complex structures than the conventional networks consisting of only springs and dampers. The proposed mechanical networks can be obtained either by using networks synthesis [8, 9, 22] or by giving some fixed-structure networks [3, 4, 5, 6, 7, 10, 12, 14]. Although the benefits of using inerter can be effectively demonstrated by these complex inerter-based mechanical networks, some fundamental properties of inerter in vibration are overlooked due to the complexity of the structure. Consequently, it lacks in-depth understanding of inerter from the perspective of vibration. In [23], the property that inerter can reduce vibration systems' natural frequencies is demonstrated. However, the influences of inerter on other aspects such as the invariant property in frequency domain are still unclear. This motivated the investigation of this section based on two simple inerter-based configurations, as shown in Fig. 2. The detailed analysis of the frequency responses of these configurations constitutes the main contribution of this section.

3.1. Analysis of C1

For this configuration, the transmissibility can be obtained as

$$\mu = \frac{|k - b\omega^2 + jc\omega|}{|k - (m + b)\omega^2 + jc\omega|} = \sqrt{\frac{(1 - \delta q^2)^2 + (2\zeta q)^2}{(1 - (1 + \delta)q^2)^2 + (2\zeta q)^2}}. \quad (2)$$

Fig. 3 shows the transmissibility μ with respect to different δ and ζ , where it is shown that an anti-resonant frequency (a particular frequency where minimum magnitude is obtained) and an invariant point (a particular frequency where the magnitude is independent of the damping ratio ζ) are introduced by using the parallel-connected inerter. For the undamped case, the anti-resonant frequency q_b can be obtained as $q_b = \sqrt{\frac{1}{\delta}}$, and the resonant frequency or natural frequency is $q_p = \sqrt{\frac{1}{1+\delta}}$. Note that the natural frequency q_p is a decreasing function with respect to δ , which is consistent with the result in [23].

The transmissibility μ in (2) can be rewritten as

$$\mu = \sqrt{\frac{A\zeta^2 + B}{C\zeta^2 + D}},$$

113 where $A = 4q^2$, $B = (1 - \delta q^2)^2$, $C = 4q^2$, and $D = (1 - (1 + \delta)q^2)^2$. To find the invariant
 114 points which are independent of damping, it requires

$$\frac{A}{C} = \frac{B}{D},$$

115 that is,

$$\frac{(1 - \delta q^2)^2}{(1 - (1 + \delta)q^2)^2} = 1.$$

116 Then, one obtains the nonzero invariant point q_i as

$$q_i = \sqrt{\frac{2}{1 + 2\delta}}.$$

117 Obviously, q_i is a decreasing function with respect to δ , which means that the parallel-
 118 connected inerter can effectively shift the invariant point left.

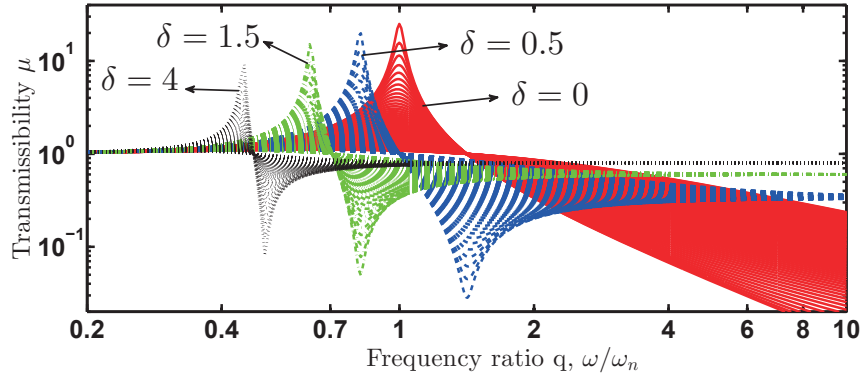


Figure 3: Transmissibility μ for the configuration $C1$ when ζ ranges from 0.02 to 1.2.

119 Fig. 4 depicts the transmissibility μ of configuration $C1$ when $\delta = 1$ with some typical ζ .
 120 The magnitudes at the natural frequency q_p , the anti-resonant frequency q_b , and infinity can

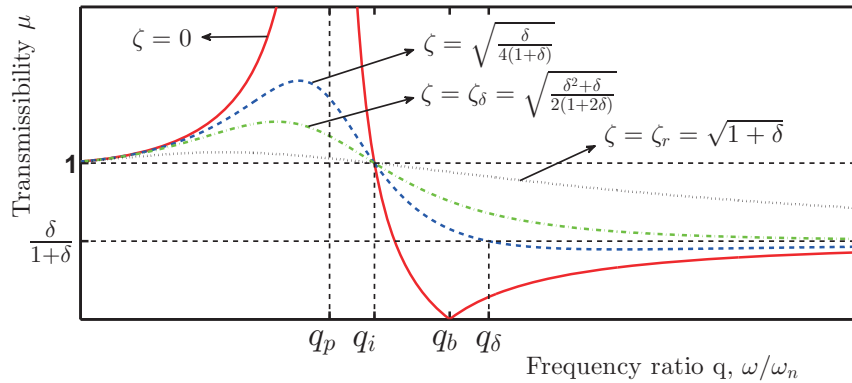


Figure 4: Transmissibility μ for the configuration $C1$ when $\delta = 1$.

121 be obtained as:

$$\mu|_{q=q_p} = \frac{1}{2} \sqrt{\frac{1}{\zeta^2(1+\delta)} + 4}, \quad (3)$$

$$\mu|_{q=q_b} = 2 \sqrt{\frac{1}{\frac{1}{\zeta^2\delta} + 4}}, \quad (4)$$

$$\mu|_{q \rightarrow \infty} = \frac{\delta}{1+\delta}, \quad (5)$$

122 where $\mu|_{q=q_j}$ means the value of μ when $q = q_j$, j denotes p , b or ∞ .

123 From (3) and (4), it is clear that $\mu|_{q=q_p}$ is a decreasing function with respect to both δ and
 124 ζ , and $\mu|_{q=q_b}$ is an increasing function with respect to both δ and ζ , as shown in Fig. 3. From
 125 (4), one obtains that for the undamped case, i.e., $c = 0$ or $\zeta = 0$, $\mu|_{q=q_b} = 0$, the effect of
 126 “dynamic absorption” of vibration occurs, which is uncommon for single-degree-of-freedom
 127 systems [30].

128 Equation (5) shows that the transmissibility approaches to an asymptote at the level of
 129 $\frac{\delta}{1+\delta}$ when q tends to ∞ . For a given δ , by solving the equation

$$\mu = \sqrt{\frac{(1 - \delta q^2)^2 + (2\zeta q)^2}{(1 - (1 + \delta)q^2)^2 + (2\zeta q)^2}} = \frac{\delta}{1 + \delta}, \quad (6)$$

130 one obtains that

$$q_\delta = \frac{\sqrt{2}}{2} \sqrt{\frac{1 + 2\delta}{\delta^2 + \delta - 2\zeta^2(1 + 2\delta)}}. \quad (7)$$

131 Note that q_δ is real if and only if $\zeta < \zeta_\delta = \sqrt{\frac{\delta^2 + \delta}{2(1 + 2\delta)}}$. Since the transmissibility tends to an
 132 asymptote at the level of $\frac{\delta}{1+\delta}$ when q tends to ∞ , ζ_δ is a critical value of ζ in the sense that:
 133 if $\zeta < \zeta_\delta$, there exists a finite q where the minimum of μ occurs; otherwise, μ is uniformly
 134 larger than $\frac{\delta}{1+\delta}$ and approaches $\frac{\delta}{1+\delta}$ when q tends to ∞ . The curve with $\zeta = \zeta_\delta$ is shown in
 135 Fig. 4.

136 Note that q_p and q_b are the natural frequency and the anti-resonant frequency of the
 137 undamped case, respectively. For the damped case, the real natural frequency q_{pr} and anti-
 138 resonant frequency q_{br} for a specific damping ratio ζ , can be obtained by setting the derivative
 139 of (2) to zero. Then, one obtains

$$q_{pr} = \sqrt{\frac{1 + 2\delta - \sqrt{1 + 8\zeta^2(1 + 2\delta)}}{2(\delta^2 + \delta - 2\zeta^2(1 + 2\delta))}}, \quad (8)$$

$$q_{br} = \sqrt{\frac{1 + 2\delta + \sqrt{1 + 8\zeta^2(1 + 2\delta)}}{2(\delta^2 + \delta - 2\zeta^2(1 + 2\delta))}}. \quad (9)$$

140 It is clear that if $\zeta \approx 0$, $q_{pr} \approx q_p$ and $q_{br} \approx q_b$ hold, but for a large ζ , it is not sufficient to
 141 use this estimation.

142 In summary, one obtains the following remarks.

- 143 **Remark 1.** 1. *The parallel-connected inerter can effectively lower the invariant point that*
144 *independent of the damping ratio ζ ;*
145 2. *The magnitude at the natural frequency is a decreasing function with respect to both*
146 *the damping ratio and the inertance-to-mass ratio; the magnitude at the anti-resonant*
147 *frequency is an increasing function with respect to both the damping ratio and the*
148 *inertance-to-mass ratio;*
149 3. *The isolation at high frequencies is weakened by using the parallel-connected inerter,*
150 *where the magnitude tends to $\frac{\delta}{1+\delta}$ when q tends to ∞ .*

151 *3.2. Analysis of $C2$*

152 For this configuration, the transmissibility can be obtained as

$$\begin{aligned} \mu &= \frac{\left| \frac{kc}{b} - c\omega^2 + kj\omega \right|}{\left| \frac{kc}{b} - c\omega^2 - \frac{mc}{b}\omega^2 + (k - m\omega^2)j\omega \right|}, \\ &= \sqrt{\frac{\delta^2 q^2 + 4(1 - \delta q^2)^2 \zeta^2}{\delta^2 (1 - q^2)^2 q^2 + 4(1 - (1 + \delta)q^2)^2 \zeta^2}}. \end{aligned} \quad (10)$$

153 By rewriting (10) as

$$\mu = \sqrt{\frac{A\zeta^2 + B}{C\zeta^2 + D}},$$

154 where $A = 4(1 - \delta q^2)^2$, $B = \delta^2 q^2$, $C = 4(1 - (1 + \delta)q^2)^2$, and $D = \delta^2 (1 - q^2)^2 q^2$, the invariant
155 points which are independent of damping can be similarly obtained by setting

$$\frac{A}{C} = \frac{B}{D},$$

156 that is,

$$\frac{1 - \delta q^2}{1 - (1 + \delta)q^2} = \pm \frac{1}{1 - q^2}.$$

157 For the case of plus sign, after simple calculation, one obtains $\delta q^4 = 0$, which leads to $q = 0$,
158 a trivial result. For the case of minus sign, one obtains

$$\delta q^4 - 2(1 + \delta)q^2 + 2 = 0.$$

159 Then, one can obtain the two nonzero invariant points as

$$q_{P,Q}^2 = 1 + \frac{1}{\delta} \pm \sqrt{1 + \frac{1}{\delta^2}}. \quad (11)$$

160 Denote $q_P < q_Q$. It is easy to show that $q_P^2 < 1$ and $q_Q^2 > 2$, and both q_P and q_Q are
161 decreasing functions with respect to δ . This indicates that, similar to the parallel-connected
162 inerter, the series-connected inerter can also effectively lower the invariant points. Note that
163 the magnitudes at P and Q are

$$\mu|_{q=q_P} = \left| \frac{1}{1 - q_P^2} \right|, \quad \mu|_{q=q_Q} = \left| \frac{1}{1 - q_Q^2} \right|.$$

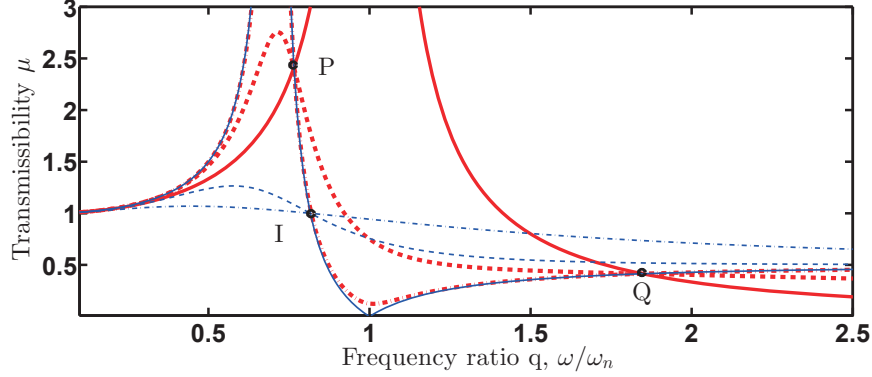


Figure 5: Comparison of the transmissibilities of configurations $C1$ and $C2$ when $\delta = 1$. Red bold lines denote $C2$ and blue thin lines denote $C1$. The solid lines denote $\zeta = 0$; the dash lines denote $\zeta = \zeta_\delta = 0.5774$; the dash-dot lines denote $\zeta = \zeta_r = \sqrt{1 + \delta} = \sqrt{2}$.

164 Since $q_P^2 < 1$ and $q_Q^2 > 2$, one obtains

$$\mu|_{q=q_P} > 1 > \mu|_{q=q_Q}, \quad (12)$$

165 which means that for a finite δ , it is impossible to equalise the ordinates at the two invariant
 166 points.

167 A comparison of the transmissibilities of configurations $C1$ and $C2$ is shown in Fig. 5,
 168 where two invariant points P and Q of configuration $C2$ are depicted. It is shown that for
 169 the same damping ratio ζ , the behaviors of configurations $C1$ and $C2$ are totally different.
 170 For example, for the case of $\zeta = \zeta_r = \sqrt{2}$ (dash-dot lines in Fig. 5), $C1$ is overdamped while
 171 $C2$ behaves similarly to the undamped case of $C1$. This is caused by the series structure of
 172 $C2$, as by varying the damping ratio ζ from 0 to ∞ , the configuration $C2$ is changed from the
 173 configuration with only a spring to the configuration with a parallel connection of a spring
 174 and an inerter.

175 In summary, one obtains the following remarks.

- 176 **Remark 2.** 1. *Two invariant points, which are independent of the damping ratio, can be*
 177 *introduced by using the series-connected inerter, and both the two invariant points are*
 178 *decreasing functions with respect to the inertance-to-mass ratio;*
 179 2. *For a finite inertance-to-mass ratio, the magnitude at the smaller invariant point is*
 180 *larger than 1 and the magnitude at the larger invariant point is smaller than 1;*
 181 3. *The series arrangement $C2$ behaves between the configuration with only a spring and*
 182 *the configuration with a parallel connection of a spring and an inerter.*

183 4. H_∞ optimisation for inerter-based isolators

184 In practice, in order to achieve good isolating performance, it is always desirable to
 185 minimise the maximum displacement of the object, which is known as H_∞ optimisation [26].
 186 In the previous section, it is shown that the invariant point, the resonant frequency and the
 187 anti-resonant frequency are directly determined by the inertance-to-mass ratio δ . Therefore,
 188 in this section, H_∞ tuning procedures for a given δ will be proposed.

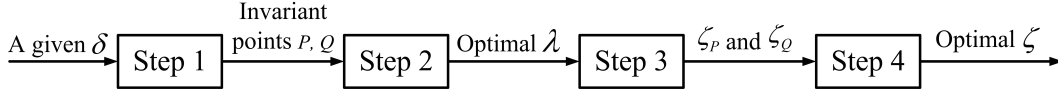


Figure 6: Graphical representation of Procedure 1.

189 For the configuration $C1$ in Fig. 2, the optimal damping in H_∞ optimisation for a given
 190 δ is ∞ , which is a trivial solution, as in this case the object and the foundation are stiffly
 191 connected. For the configuration $C2$ with a given inertance-to-mass ratio δ , the optimal
 192 damping ratio ζ for the H_∞ performance is the one making the curve horizontally pass through
 193 the invariant P , as shown in Fig. 5. The rationality is based on the fixed-point theory [24,
 194 Chapter 3.3]: the most favorable damping is the one making the curve horizontally pass
 195 through the highest invariant point. As demonstrated in Section 3, the magnitude of the
 196 invariant point P is always larger than that of the other invariant point Q . Therefore, based
 197 on this consideration, the optimal damping ratio ζ for configuration $C2$ can be obtained as
 198 follows:

199 **Proposition 1.** *For the configuration $C2$ with a given δ , the optimal damping ratio ζ in H_∞*
 200 *optimisation is*

$$\zeta_{opt} = \frac{1}{2} \sqrt{\delta(1 + \delta - \sqrt{1 + \delta^2})}. \quad (13)$$

201 *Proof.* See Appendix Appendix A. □

202 Note that two invariant points can be introduced by using the series-connected inerter, and
 203 in order to further tune the two invariant points, an extra spring k_1 is incorporated. Then,
 204 three inerter-based isolators are proposed as shown in Fig. 7. The fixed-point theory [24,
 205 Chapter 3.3] is employed to derive the optimal parameters for these three inerter-based
 206 isolators. The fixed-point theory can be summarised as follows [24, Chapter 3.3].

- 207 **Procedure 1.**
- 208 1. *For a given inertance-to-mass ratio δ , find the invariant points which*
 209 *are independent of the damping ratio ζ , and denote the two smaller invariant points as*
 210 *P and Q ;*
 - 211 2. *adjust the spring stiffness ratio λ so that the ordinates at the invariant points P and Q*
 212 *are equal;*
 - 213 3. *calculate the damping ratio ζ_P and ζ_Q so that the curves of transmissibility μ vs. q*
 214 *horizontally pass through P and Q , respectively;*
 - 215 4. *obtain the optimal damping ratio as $\zeta = \sqrt{\frac{\zeta_P^2 + \zeta_Q^2}{2}}$.*

216 A graphical representation of Procedure 1 is given in Fig. 6, indicating the required and
 217 output parameters in each step. According to this procedure, the optimal parameters λ and
 ζ for each configuration are derived subsequently.

218 **Remark 3.** *The fixed-point theory [24, Chapter 3.3] actually yields a suboptimal but highly*
 219 *precise solution as demonstrated in [33]. The merit of the fixed-point theory is that an ana-*
 220 *lytical solution can be easily derived, which makes it extensively employed in tuning dynamic*
 221 *vibration absorber (DVA) (or tuned mass damper (TMD)). See for example [25, 26, 27] and*

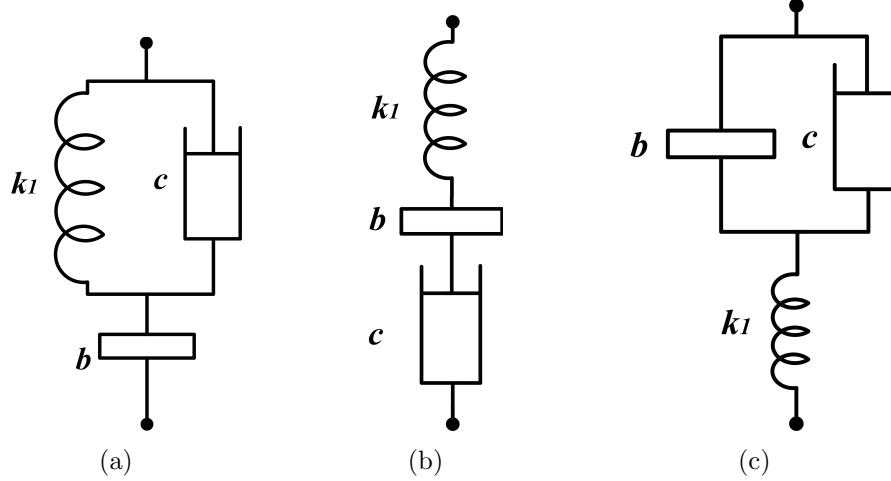


Figure 7: Three configurations as $W(s)$ of the isolators in Fig. 1. (a) C3; (b) C4; (c) C5.

222 references therein. This is also the reason why it is employed in this paper. Please note that
 223 the optimal parameters derived in this section are “optimal” in the sense of the fixed-point
 224 theory using Procedure 1, which would be suboptimal in practice.

225 **Proposition 2.** *The transmissibility for C3 can be obtained as*

$$\mu = \left| \frac{1 - \delta(1 + \lambda)q^2 + 2j\lambda(1 - \delta q^2)q\zeta}{1 - (\delta + 1 + \delta\lambda)q^2 + \delta\lambda q^4 + 2j\lambda(1 - (1 + \delta)q^2)q\zeta} \right|. \quad (14)$$

226 As shown in Appendix Appendix B, there are three invariant points for C3 which are
 227 denoted as P , Q and R ($q_P < q_Q < q_R$), respectively. Following Procedure 1, the largest
 228 invariant point R can be derived as

$$q_R^2 = \frac{1}{\delta} + \frac{3}{2} + \sqrt{\left(\frac{1}{\delta} - \frac{3}{2}\right)^2 + \frac{4}{\delta}}, \quad (15)$$

229 which possesses a relatively large value ($q_R^2 \geq 3$). The optimal stiffness ratio λ can be obtained
 230 as

$$\lambda = \frac{2(q_R^4\delta(1 + \delta) - (1 + 2\delta)q_R^2 + 1)}{\delta q_R^2(q_R^4\delta - 2(\delta + 1)q_R^2 + 2)} \text{ or } \frac{2((1 + 2\delta)(1 + \delta)q_R^2 - 2(1 + \delta))}{q_R^2(\delta(1 + 2\delta)q_R^2 - 2(1 + 2\delta + 2\delta^2))}. \quad (16)$$

231 The optimal damping ratio ζ can be obtained as

$$\zeta = \sqrt{\frac{\zeta_P^2 + \zeta_Q^2}{2}}, \quad (17)$$

232 where ζ_P^2 and ζ_Q^2 can be obtained as

$$\zeta_{P,Q}^2 = \left(\frac{1 - \delta(1 + \lambda)q_{P,Q}^2}{1 - \delta q_{P,Q}^2} \right) \left(\frac{\delta(1 + \lambda)(2 - (1 + 2\delta)q_{P,Q}^2) - (2\delta\lambda q_{P,Q}^2 - 1)(1 - \delta q_{P,Q}^2)}{4\lambda^2 q_{P,Q}^2} \right), \quad (18)$$

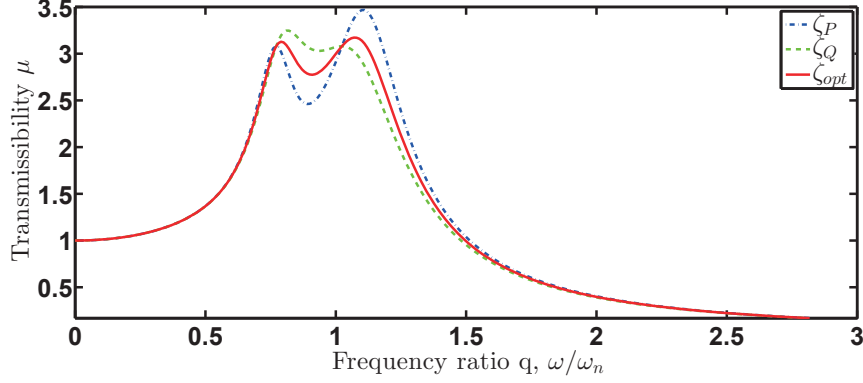


Figure 8: Transmissibility μ for $C3$ when $\delta = 0.2$.

233 q_P^2 and q_Q^2 are solutions of the following quadratic function with respect to q^2 :

$$q^4 - \left(\frac{2}{\delta\lambda}(1 + \lambda + \delta + \lambda\delta) - q_R^2 \right) q^2 + \frac{2}{\delta^2\lambda q_R^2} = 0. \quad (19)$$

234 *Proof.* See Appendix Appendix B. □

235 **Procedure 2.** In summary, the H_∞ tuning procedure for $C3$ is:

- 236 1. obtain q_R from (15);
- 237 2. obtain λ_{opt} by substituting q_R into (16);
- 238 3. obtain q_P and q_Q by solving (19);
- 239 4. obtain ζ_P^2 and ζ_Q^2 by substituting q_P and q_Q into (18), respectively;
- 240 5. obtain the optimal ζ_{opt} from (17).

241 Note that in [12, Section 3], a similar tuning procedure was given for the configuration $C3$
 242 by following the procedure given in [24, Chapter 3.3] as well. The main difference between
 243 the method in this paper and the one in [12] is the approach in calculating the optimal
 244 parameters λ and ζ : In this paper, the analytical solutions of the optimal λ and ζ are given,
 245 that is, (15), (16), and (18); while in [12], the optimal λ and ζ are obtained relying on
 246 numerical iterations. Hence, the procedure in this paper is more convenient and reliable.

247 The transmissibility μ of $C3$ for $\delta = 0.2$ is illustrated in Fig. 8.

248 **Proposition 3.** The transmissibility for $C4$ can be obtained as

$$\mu = \left| \frac{2(1 - \delta(1 + \lambda)q^2)\zeta + j\delta q}{2(\delta\lambda q^4 - (1 + \delta + \delta\lambda)q^2 + 1)\zeta + j\delta(1 - q^2)q} \right|. \quad (20)$$

249 Following Procedure 1, the optimal stiffness ratio λ can be obtained as

$$\lambda = \frac{1}{\delta}. \quad (21)$$

250 The optimal damping ratio ζ can be obtained as

$$\zeta_{opt} = \sqrt{\frac{\zeta_P^2 + \zeta_Q^2}{2}}, \quad (22)$$

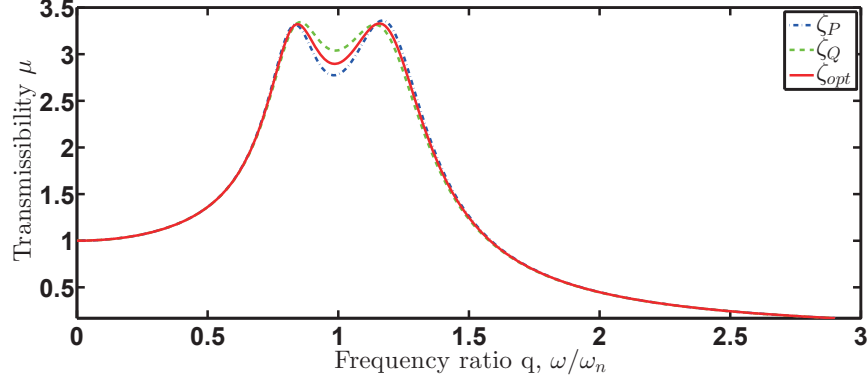


Figure 9: Transmissibility μ for $C4$ when $\delta = 0.2$.

251 where

$$\zeta_P^2 = \frac{\delta^2 \left(1 - \sqrt{\delta/(2+\delta)}\right)}{4 \left((1+\delta)\sqrt{\delta/(2+\delta)} - \delta \right) \left((\delta+3)\sqrt{\delta/(2+\delta)} + \delta \right)}, \quad (23)$$

$$\zeta_Q^2 = \frac{\delta^2 \left(1 + \sqrt{\delta/(2+\delta)}\right)}{4 \left((1+\delta)\sqrt{\delta/(2+\delta)} + \delta \right) \left((\delta+3)\sqrt{\delta/(2+\delta)} - \delta \right)}. \quad (24)$$

252 *Proof.* See Appendix Appendix C. □

253 The transmissibility μ of $C4$ for $\delta = 0.2$ is illustrated in Fig. 9.

254 **Proposition 4.** *The transmissibility for $C5$ can be obtained as*

$$\mu = \left| \frac{1 - \delta(1+\lambda)q^2 + j2(\lambda+1)\zeta q}{1 - (1+\delta+\delta\lambda)q^2 + \delta\lambda q^4 + j2\zeta(\lambda+1-\lambda q^2)q} \right|. \quad (25)$$

255 *Following Procedure 1, the optimal stiffness ratio λ can be obtained as*

$$\lambda = \frac{1}{2\delta} \left(1 - 2\delta + \sqrt{1 - 2\delta}\right), \quad (26)$$

256 *which requires $\delta < 1/2$. The optimal damping ratio ζ can be obtained as*

$$\zeta_{opt} = \sqrt{\frac{\zeta_P^2 + \zeta_Q^2}{2}}, \quad (27)$$

257 where

$$\zeta_{P,Q}^2 = \frac{(1 - \delta(1+\lambda)q_{P,Q}^2) (1 + 2\delta + 2\delta\lambda - 3\delta\lambda q_{P,Q}^2)}{4(\lambda+1)\lambda q_{P,Q}^2}, \quad (28)$$

258 and

$$q_{P,Q}^2 = \frac{1}{4\delta\lambda(\lambda+1)} \left(1 + 2\lambda + 2\delta(1+\lambda)^2 \pm \sqrt{(2\delta(1+\lambda)^2 + 1 - 2\lambda)^2 + 8\lambda}\right). \quad (29)$$

259 *Proof.* See Appendix Appendix D. □

260 The transmissibility μ of $C5$ for $\delta = 0.2$ is illustrated in Fig. 10.

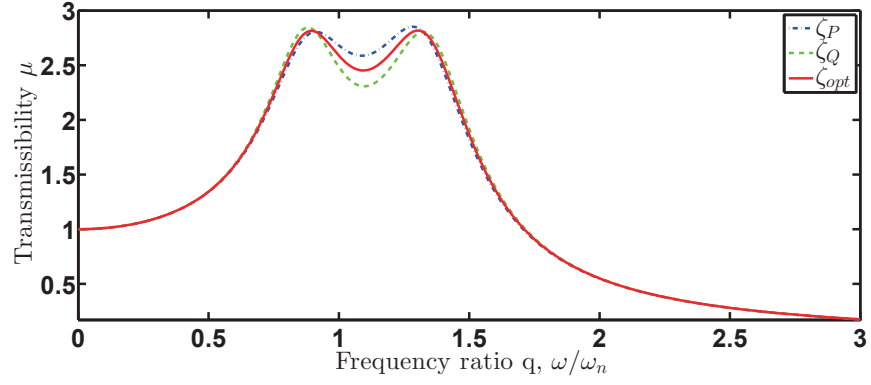


Figure 10: Transmissibility μ for $C5$ when $\delta = 0.2$.

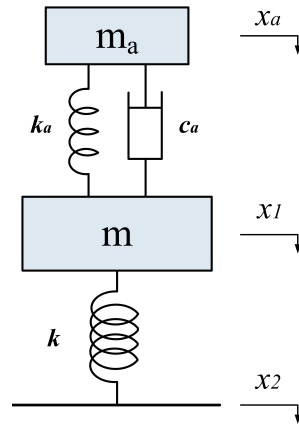


Figure 11: The dynamic vibration absorber attached to the object mass.

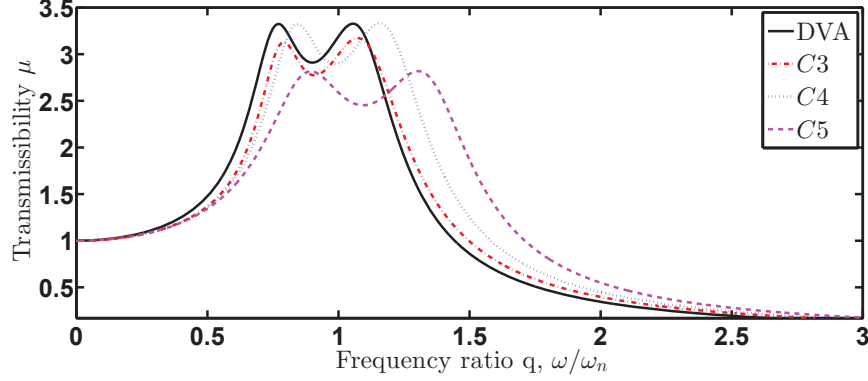


Figure 12: Comparison between traditional DVA and inerter-based isolators when $\delta = 0.2$.

4.1. Comparison between the traditional DVA and the inerter-based isolators

Now, all the optimal parameters for these inerter-based isolators in H_∞ optimisation have been derived. In this section, the performance of the inerter-based isolators will be compared with the traditional DVA as shown in Fig. 11. For the traditional DVA,

$$\mu = \sqrt{\frac{A\zeta^2 + B}{C\zeta^2 + D}},$$

where $A = 4\lambda^2 q^2$, $B = (1 - \delta\lambda q^2)^2$, $C = 4\lambda^2 (1 - (1 + \delta)q^2)^2 q^2 \zeta^2 + (1 - (1 + \delta + \delta\lambda)q^2 + \delta\lambda q^4)^2$, and the mass ratio δ and the stiffness ratio λ are defined as $\delta = \frac{m_a}{m}$ and $\lambda = \frac{k}{k_a}$, respectively.

It is well known that the optimal parameters for the traditional DVA [25, 26, 27] are

$$\lambda_{opt} = \frac{(\delta + 1)^2}{\delta}, \quad \zeta_{opt} = \frac{\delta}{1 + \delta} \sqrt{\frac{3\delta}{8(1 + \delta)}}.$$

Fig. 12 shows the comparison between the traditional DVA and the inerter-based isolators when the inertance-to-mass ratio (or mass ratio for traditional DVA) $\delta = 0.2$, where it is clearly shown that in terms of the same δ , the configuration C4 provides comparable performance compared with the traditional DVA; whereas both C3 and C5 perform better than the traditional DVA. Such an observation is confirmed by Fig. 13, where the comparison of the maximal μ with respect to different δ is shown. The comparison of the optimal stiffness ratio λ and damping ratio ζ with respect to different δ is shown in Fig. 14.

Note that the fundamental difference between the traditional DVA and the inerter-based isolators is that the inertance-to-mass ratio of the inerter-based isolators can easily be larger than the mass ratio of the traditional DVA, as large inertance can easily be obtained without increasing the physical mass of the whole system. For example, the inertance of a rack-pinion inerter or a ball-screw inerter can be significantly magnified by enlarging the gear ratios [1, 2]. However, the mass ratio δ for the traditional DVA is practically less than 0.25 [26, 28]. From this point of view, the performance of the inerter-based isolators can be further improved compared with the traditional DVA, and the inerter-based isolators are potentially more attractive than the traditional DVA.

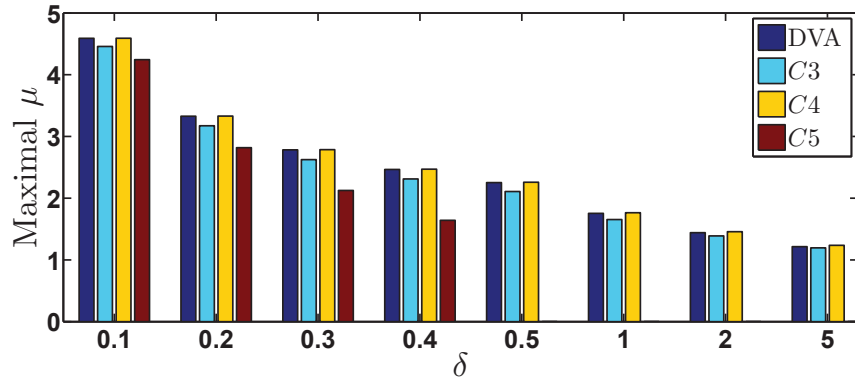
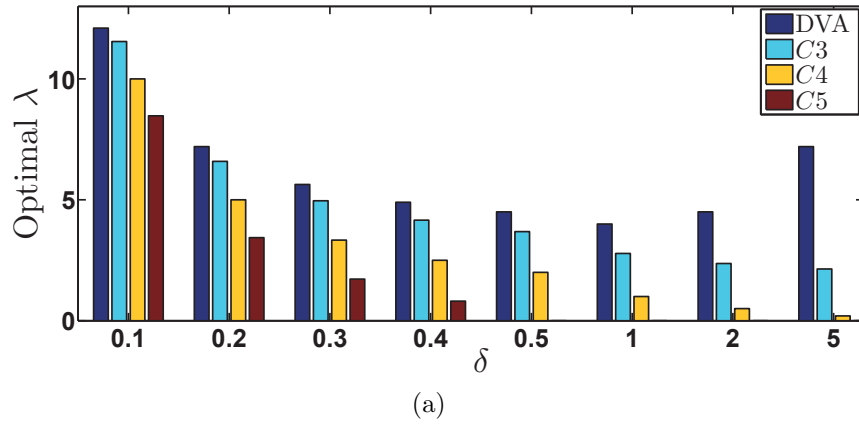
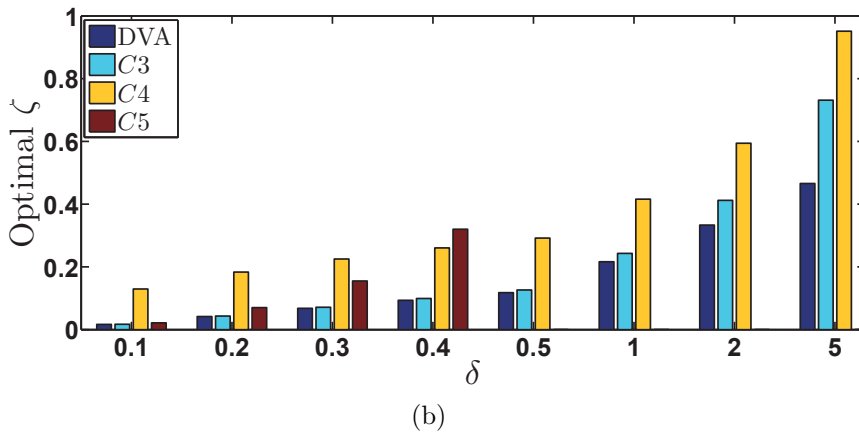


Figure 13: Comparison of the maximal μ in H_∞ optimisation.



(a)



(b)

Figure 14: Comparison of the optimal parameters in H_∞ optimization. (a) Optimal stiffness ratio λ ; (b) optimal damping ratio ζ .

284 **5. H_2 optimisation for inerter-based isolators**

285 H_2 optimisation aims to minimise the total vibration energy or the mean square motion
 286 of the object mass when white noise excitation is enforced [29]. In the case of random
 287 excitation such as wind loading instead of harmonic excitation, the H_2 optimisation would
 288 be more practical than the H_∞ optimisation. In this section, the analytical solutions for the
 289 inerter-based isolators in H_2 optimisation will be derived and compared with the traditional
 290 DVA.

291 The performance measure to be minimised in H_2 optimisation is defined as follows [29, 27]:

$$I = \frac{E[x_1^2]}{2\pi S_0 \omega_n}, \quad (30)$$

292 where S_0 is the uniform power spectrum density function. Denoting $\mu = |H(jq)|$, the mean
 293 square value of x_1 of the object mass m can be calculated as

$$E[x_1^2] = S_0 \int_{-\infty}^{\infty} |H(jq)|^2 d\omega = S_0 \omega_n \int_{-\infty}^{\infty} |H(jq)|^2 dq. \quad (31)$$

294 Substituting (31) into (30), one obtains

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(jq)|^2 dq, \quad (32)$$

295 which is exactly the definition of the H_2 norm of the transfer function $\hat{H}(s)$ by replacing jq
 296 in $H(jq)$ with the Laplace variable s .

297 Therefore, the H_2 performance measure is rewritten as

$$I = \left\| \hat{H}(s) \right\|_2^2. \quad (33)$$

298 In what follows, an analytical approach to calculating the H_2 norm of the transfer function
 299 $\hat{H}(s)$ will be presented according to [32, Chapter 2.6], which has been used to derive analytical
 300 solutions for vehicle suspensions in [6, 7].

301 For a stable transfer function $\hat{H}(s)$, its H_2 norm can be calculated as [32, Chapter 2.6]

$$\|\hat{H}(s)\|_2^2 = \|C(sI - A)^{-1}B\|_2^2 = CLC^T,$$

302 where A , B , C are the minimal state-space realization $\hat{H}(s) = C(sI - A)^{-1}B$ and L is the
 303 unique solution of the Lyapunov equation

$$AL + LA^T + BB^T = 0. \quad (34)$$

304 We can write $\hat{H}(s)$

$$\hat{H}(s) = \frac{b_{n-1}s^{n-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}$$

305 in its controllable canonical form below

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

306 where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, C = [b_0, b_1, b_2 \dots b_{n-1}].$$

307 Note that the analytical solution for the configuration $C1$ cannot be derived by using the
 308 above method, as the $\hat{H}(s)$ for $C1$ is not strictly proper. Actually, the H_2 norm of $\hat{H}(s)$
 309 for $C1$ is infinity which can be obtained by observing Fig. 4: the area under the frequency
 310 response curve of $C1$ which represents the H_2 norm of the transfer function is infinity.

311 The procedure to derive the optimal parameters for $C2$, $C3$, $C4$ and $C5$ can be sum-
 312 marised as:

313 **Procedure 3.**

- 314 1. *analytically calculate the H_2 performance measure I using the method discussed above.*
 315 *Denote the performance measure as $I = F(\lambda)\zeta + \frac{G(\lambda)}{\zeta}$, where $F(\lambda)$ and $G(\lambda)$ are func-*
 316 *tions of λ with $F(\lambda) > 0$, $G(\lambda) > 0$;*
- 317 2. *obtain the equations of optimal ζ and I as $\zeta_{opt} = \sqrt{\frac{F(\lambda)}{G(\lambda)}}$ and $I_{opt} = 2\sqrt{F(\lambda)G(\lambda)}$,*
 318 *respectively;*
- 319 3. *obtain the optimal λ as the one minimising $F(\lambda)G(\lambda)$, denoted as λ_{opt} ;*
- 320 4. *obtain the optimal ζ and I by substituting λ_{opt} into the equations obtained in Step 2,*
 321 *respectively.*

322 Note that in Step 1 of Procedure 3, it includes the case that $F(\lambda)$ and $G(\lambda)$ are constants
 323 with respect to λ . Following Procedure 3, the optimal parameters for $C2$, $C3$, $C4$, and $C5$
 324 in the H_2 optimisation will be derived subsequently.

325 **Proposition 5.** *For the configuration $C2$, the H_2 performance measure in (32) is*

$$I_{c2} = \frac{1 - \delta + \delta^2}{\delta^2} \zeta + \frac{1}{4\zeta}. \quad (35)$$

326 *For a given δ , the optimal ζ is*

$$\zeta_{opt} = \frac{\delta}{2\sqrt{1 - \delta + \delta^2}}.$$

327 *After substituting ζ_{opt} into (35), the optimal I_{c2} is*

$$I_{c2,opt} = \frac{\sqrt{1 - \delta + \delta^2}}{2\delta}.$$

328 *Proof.* Equation (35) can be obtained by direct calculation, and then the optimal ζ and $I_{c2,opt}$
 329 can be obtained subsequently. \square

330 **Proposition 6.** For the configuration C3, the H_2 performance measure in (32) is

$$I_{c3} = \frac{1 - \delta + \delta^2}{\delta^2} \zeta + \frac{1 - 2\delta\lambda + \delta^2\lambda^2 + \delta^2\lambda}{4\lambda^2\delta^2\zeta}. \quad (36)$$

331 For a given δ , the optimal λ can be obtained as

$$\lambda_{opt} = \begin{cases} \frac{2}{\delta(2-\delta)}, & \delta < 2, \\ \infty, & \delta \geq 2. \end{cases}$$

332 Note that in the case of $\delta \geq 2$, C3 reduces to C2. For a given δ and λ , the optimal ζ can be
333 obtained as

$$\zeta_{opt} = \frac{1}{2\lambda} \sqrt{\frac{1 - 2\delta\lambda + \delta^2\lambda}{1 - \delta + \delta^2}}.$$

334 Then, the optimal I_{c3} can be obtained by substituting ζ_{opt} and λ_{opt} into (36).

335 *Proof.* Equation (36) can be obtained by direct calculation. The optimal λ can be obtained
336 by checking the second part in (36). Since both parts in (36) are positive, the optimal ζ can
337 be obtained subsequently. \square

338 **Proposition 7.** For the configuration C4, the H_2 performance measure in (32) is

$$I_{c4} = \frac{1 - 2\delta\lambda + \delta^2\lambda^2 + 2\delta^2\lambda - \delta + \delta^2}{\delta^2} \zeta + \frac{1}{4\zeta}. \quad (37)$$

339 For a given δ , the optimal λ can be obtained as

$$\lambda_{opt} = \begin{cases} \frac{1-\delta}{\delta}, & \delta < 1, \\ 0, & \delta \geq 1. \end{cases}$$

340 Note that in the case of $\delta \geq 1$, C4 reduces to C2. For a given δ and λ , the optimal ζ can be
341 obtained as

$$\zeta_{opt} = \frac{1}{2} \sqrt{\frac{\delta^2}{1 - 2\delta\lambda + \delta^2\lambda^2 + 2\delta^2\lambda - \delta + \delta^2}}.$$

342 Then, the optimal I_{c4} can be obtained by substituting ζ_{opt} and λ_{opt} into (37).

343 *Proof.* The proof is omitted as it is similar to that of Proposition 6. \square

344 **Proposition 8.** For the configuration C5, the H_2 performance measure in (32) is

$$I_{c5} = (\lambda + 1)^2 \zeta + \frac{\delta^3\lambda^3 + \delta(3\delta - 2)\lambda^2 + (1 - 2\delta + 3\delta^3)\lambda + \delta^2}{4\lambda\zeta}. \quad (38)$$

345 For a given δ and λ , the optimal ζ and I_{c5} can be obtained as

$$\zeta_{opt} = \frac{1}{2(1 + \lambda)} \sqrt{\frac{\delta^3\lambda^3 + \delta(3\delta - 2)\lambda^2 + (1 - 2\delta + 3\delta^3)\lambda + \delta^2}{\lambda}}, \quad (39)$$

$$I_{c5,opt} = (\lambda + 1) \sqrt{\frac{\delta^3\lambda^3 + \delta(3\delta - 2)\lambda^2 + (1 - 2\delta + 3\delta^3)\lambda + \delta^2}{\lambda}}. \quad (40)$$

346 Let \mathcal{Q} be the set of real, positive solutions λ of the quartic equation

$$4\delta^2\lambda^4 + (11\delta - 6)\delta\lambda^3 + (2 - 6\delta + 9\delta^2)\lambda^2 + \delta^2\lambda - \delta^2 = 0. \quad (41)$$

347 The optimal λ is chosen from the elements of \mathcal{Q} as well as 0 that makes $I_{c5,opt}$ minimum. If
 348 the optimal λ is 0, configuration C5 reduces to C1.

349 *Proof.* Equation (38) can be obtained by direct calculation. Since both parts in (38) are
 350 positive, the optimal ζ and I_{c5} can be obtained as in (39) and (40) respectively in a straight-
 351 forward manner. In terms of (40), by making the derivative of $I_{c5,opt}$ with respect to λ zero,
 352 the quartic equation (41) can be obtained, and then the optimal λ can be selected from the
 353 real, positive solutions of the quartic equation as well as ∞ . \square

354 5.1. Comparison between the traditional DVA and the inerter-based isolators

355 Now, all the optimal parameters for the inerter-based isolators in H_2 optimisation have
 356 been derived. In this section, the performance of these inerter-based isolators will be com-
 357 pared with the traditional DVA as shown in Fig. 11.

358 For the traditional DVA shown in Fig. 11, the H_2 performance measure can be derived as

$$I_{DVA} = \frac{1 + \delta}{\delta}\zeta + \frac{(\delta + 1)^2 - \delta(\delta + 2)\lambda + \delta^2\lambda^2}{4\lambda^2\delta^2\zeta}, \quad (42)$$

359 where the mass ratio δ and the stiffness ratio λ are defined as $\delta = m_a/m$ and $\lambda = k/k_a$.

360 Similar to the inerter-based isolators, the optimal parameters can be obtained as:

$$\lambda_{opt} = \frac{2(\delta + 1)^2}{\delta(\delta + 2)},$$

361

$$\zeta_{opt} = 4\sqrt{\frac{\delta^3(3\delta + 4)}{(\delta + 1)^3}},$$

362

$$I_{DVA,opt} = \frac{1}{2}\sqrt{\frac{3\delta + 4}{\delta(\delta + 1)}}.$$

363 Fig. 15, Fig. 16 and Fig. 17 show the comparison between the traditional DVA and the
 364 inerter-based isolators in H_2 optimisation. As shown in Fig. 15, for the same δ , the inerter-
 365 based isolator C5 and C3 perform better than the traditional DVA when δ less than 0.44
 366 and 1.2, respectively, and the configuration C3 performs slightly worse than the traditional
 367 DVA. As shown in Fig. 15, when $\delta < 0.44$, the configuration C5 performs best among all
 368 the inerter-based isolators. From Fig. 16, it is shown that the damping ratios ζ of the
 369 inerter-based isolators are normally smaller than the traditional DVA. The detailed values
 370 of the parameters are given in Table 2, where it is shown that when $\delta = 0.2$, the inerter-
 371 based isolator C3 and C5 can provide 8.75% and 49.06% improvement compared with the
 372 traditional DVA.

373 Similar to the H_∞ optimisation, the fundamental difference between the traditional DVA
 374 and the inerter-based isolators is that relatively large value of inertance can easily be achieved
 375 without increasing the physical mass of the isolation system [1, 2]; whereas the attached mass

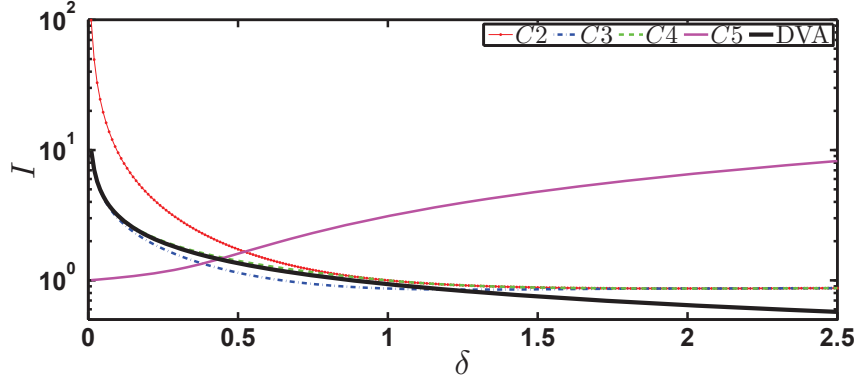


Figure 15: Comparison between traditional DVA and inerter-based isolators in H_2 optimisation.

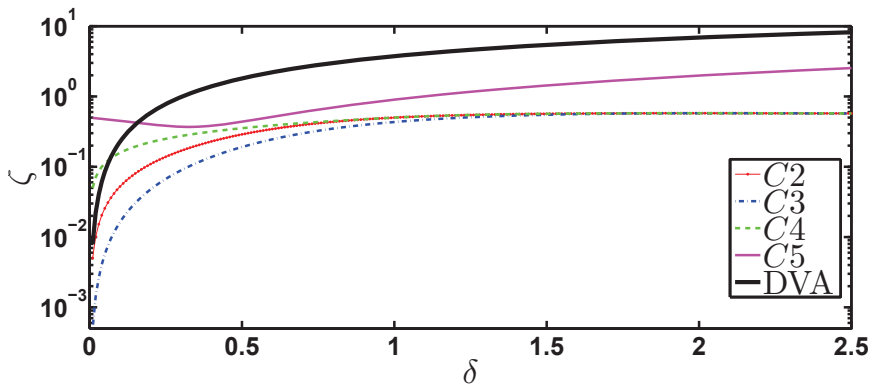


Figure 16: Optimal damping ratio ζ in H_2 optimisation.

376 m_a is normally quite small and the typical mass ratio δ for the traditional DVA is less than
 377 0.25 [26, 28]. In this sense, the performance of the inerter-based isolators can be further
 378 improved by increasing the inertance-to-mass ratio δ even $\delta > 0.25$, which is a potential
 379 advantage of the inerter-based isolators compared with the traditional DVA.

380 6. Conclusions

381 In this paper, the performance of inerter-based isolators has been investigated by applying
 382 five configurations with inerter in a “uni-axial” isolation system. In the first part of this paper,
 383 the frequency responses of the inerter in parallel connection and the one in series connection
 384 are analysed. It has been analytically demonstrated that both the parallel-connected inerter
 385 and the series-connected one can effectively lower the invariant points, and the isolation for
 386 high frequencies can be weakened by using inerter. In the second part of this paper, both
 387 H_∞ and H_2 performances have been considered for the proposed inerter-based isolators.
 388 The fixed-point theory and the analytical method in calculating H_2 norm are employed to
 389 analytically derive the optimal parameters in H_∞ and H_2 optimisation, respectively. The
 390 performances of the inerter-based isolators have also been compared with the traditional
 391 DVA to show the benefits of the inerter-based isolators. On one hand, it has been shown
 392 that for the same mass ratio or inertance-to-mass ratio, two inerter-based isolators perform
 393 better than the traditional DVA. On the other hand, two unique properties make the inerter-

Table 2: Comparison of optimal parameters in H_2 optimisation.(a) H_2 performance measure I

δ	DVA	$C2$	$C3$	$C4$	$C5$
0.1	3.1261	9.5394	2.9787	3.1623	1.0479
0.2	2.1890	4.5826	1.9975	2.2361	1.1152
0.3	1.7723	2.9627	1.5607	1.8257	1.2184
0.4	1.5236	2.1794	1.3077	1.5811	1.3798
0.5	1.3540	1.7321	1.1456	1.4142	1.6015
1	0.9354	1.0000	0.8660	1.0000	3.1087
2	0.6455	0.8660	0.8660	0.8660	6.5065
5	0.3979	0.9165	0.9165	0.9165	16.9393

(b) optimal stiffness ratio λ

δ	DVA	$C3$	$C4$	$C5$
0.1	11.5238	10.5263	9.0000	0.0796
0.2	6.5455	5.5556	4.0000	0.1787
0.3	4.8986	3.9216	2.3333	0.2824
0.4	4.0833	3.1250	1.5000	0.3426
0.5	3.6000	2.6667	1.0000	0.3542
1	2.6667	2.0000	0	0.3139
2	2.2500	∞	0	0.2815
5	2.0571	∞	0	0.2623

(c) optimal damping ratio ζ

δ	DVA	$C2$	$C3$	$C4$	$C5$
0.1	0.2274	0.0524	0.0164	0.1581	0.4495
0.2	0.5837	0.1091	0.0476	0.2236	0.4014
0.3	0.9816	0.1688	0.0889	0.2739	0.3704
0.4	1.3930	0.2294	0.1376	0.3162	0.3827
0.5	1.8053	0.2887	0.1909	0.3536	0.4367
1	3.7417	0.5000	0.4330	0.5000	0.9004
2	6.8853	0.5774	0.5774	0.5774	1.9810
5	13.2637	0.5455	0.5455	0.5455	5.3157

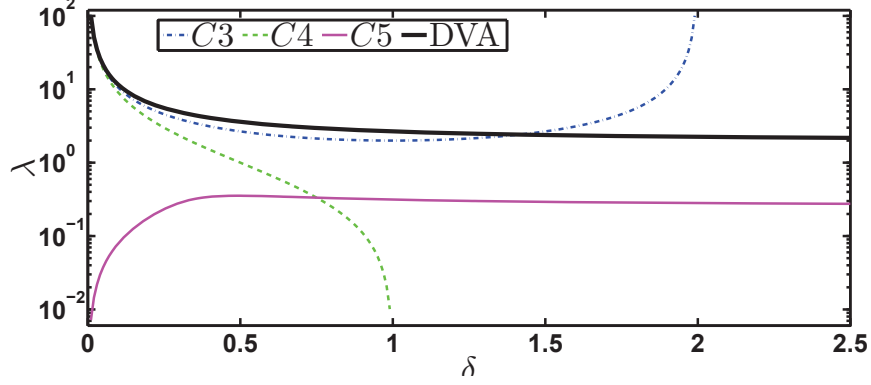


Figure 17: Optimal stiffness ratio λ in H_2 optimisation.

394 based isolators potentially more attractive than the traditional DVA: first, a large inertance
 395 can easily be obtained for inerter without increasing the physical mass of the whole system;
 396 second, the inerter is a built-in element and there is no need to mount an additional mass to
 397 the object to be isolated.

398 In practical applications of the inerter-based isolators, the large transmission ratios em-
 399 ployed in the physical embodiments of inerter will amplify the internal friction of the rotating
 400 device with a gain that is equal to the square of the transmission ratio. This could lead to
 401 an amount of damping at a system level larger than the optimal one, which may render the
 402 proposed inerter-based isolators far from an ideal design. More research work needs to be
 403 carried to find low-friction designs to be used with high amplification ratio.

404 Appendix A. Proof of Proposition 1

405 Observing Fig. 5, it is shown that the curve horizontally passing through P indicates the
 406 optimal damping. This optimal damping can be obtained by solving the following equation

$$\frac{\partial \mu^2}{\partial q^2} \Big|_{q=q_P} = 0. \quad (\text{A.1})$$

407 Denote $\mu = \sqrt{\frac{n}{m}}$, where $n = \delta^2 q^2 + 4(1 - \delta q^2)^2 \zeta^2$, $m = \delta^2(1 - q^2)^2 q^2 + 4(1 - (1 + \delta)q^2)^2 \zeta^2$.
 408 Equation (A.1) can be written in another form as

$$n'm - m'n = 0,$$

409 where $n' = \partial n / \partial q^2$, and $m' = \partial m / \partial q^2$. For the invariant point P ,

$$\frac{n}{m} = \frac{1}{(1 - q^2)^2} = \frac{(1 - \delta q^2)^2}{(1 - (1 + \delta)q^2)^2},$$

410 therefore,

$$(1 - q^2)^2 n' - m' = 0.$$

411 Since

$$n' = -8(1 - \delta q^2)\delta\zeta^2 + \delta^2,$$

412

$$m' = -8(1 - (1 + \delta)q^2)(\delta + 1)\zeta^2 + \delta^2(1 - q^2)(1 - 3q^2),$$

413 after substituting q_P into (11), one obtains

$$\zeta_{opt} = \frac{1}{2}\sqrt{\delta(1 + \delta - \sqrt{1 + \delta^2})}.$$

414 **Appendix B. Proof of Proposition 2**

415 Denote

$$A = 4\lambda^2(1 - \delta q^2)^2 q^2, \quad B = (1 - \delta(1 + \lambda)q^2)^2,$$

416

$$C = 4\lambda^2(1 - (1 + \delta)q^2)^2 q^2, \quad D = (1 - (\delta + 1 + \delta\lambda)q^2 + \delta\lambda q^4)^2.$$

417 Then, μ in (14) can be rewritten as

$$\mu = \sqrt{\frac{A\zeta^2 + B}{C\zeta^2 + D}}. \quad (\text{B.1})$$

418

To find the invariant points which are independent of damping, it requires

$$\frac{A}{C} = \frac{B}{D},$$

419 that is,

$$\frac{1 - \delta q^2}{1 - (1 + \delta)q^2} = \pm \frac{1 - \delta(1 + \lambda)q^2}{1 - (\delta + 1 + \delta\lambda)q^2 + \delta\lambda q^4}.$$

420

421 With the plus sign, after cross multiplication, one obtains $\delta^2\lambda q^6 = 0$, which leads to the trivial solution $q = 0$. With the minus sign, after simple calculation, one obtains

$$\delta^2\lambda q^6 - 2\delta(\lambda + \delta + 1 + \delta\lambda)q^4 + 2(2\delta + 1 + \delta\lambda)q^2 - 2 = 0, \quad (\text{B.2})$$

422

423 which is a cubic form in q^2 . Therefore, there are three invariant points for the configuration $C3$.

424

425 Denoting these three invariant points as P , Q and R ($q_P < q_Q < q_R$), separately, one obtains

$$q_P^2 + q_Q^2 + q_R^2 = \frac{2}{\delta\lambda}(\lambda + \delta + 1 + \lambda\delta), \quad (\text{B.3})$$

$$q_P^2 q_Q^2 q_R^2 = \frac{2}{\delta^2\lambda}, \quad (\text{B.4})$$

$$q_P^2 q_Q^2 + q_P^2 q_R^2 + q_Q^2 q_R^2 = \frac{2}{\delta^2\lambda}(2\delta + 1 + \delta\lambda). \quad (\text{B.5})$$

426

427 Since at points P and Q , the values of μ are independent of ζ , then in the case of $\zeta = \infty$, one obtains

$$\left| \frac{1 - \delta q_P^2}{1 - (1 + \delta)q_P^2} \right| = \left| \frac{1 - \delta q_Q^2}{1 - (1 + \delta)q_Q^2} \right|.$$

428 It can be checked that

$$\frac{1 - \delta q_P^2}{1 - (1 + \delta)q_P^2} > 0, \quad \frac{1 - \delta q_Q^2}{1 - (1 + \delta)q_Q^2} < 0.$$

429 Then, one obtains

$$\frac{1 - \delta q_P^2}{1 - (1 + \delta)q_P^2} = -\frac{1 - \delta q_Q^2}{1 - (1 + \delta)q_Q^2}.$$

430 After cross multiplication and simplification, one obtains

$$2\delta(1 + \delta)q_P^2q_Q^2 - (q_P^2 + q_Q^2)(1 + 2\delta) + 2 = 0. \quad (\text{B.6})$$

431 Substituting (B.4) and (B.5) into (B.6), one can obtain a quadratic equation with respect
432 to q_R^2 as

$$\delta\lambda(1 + 2\delta)q_R^4 - 2(\lambda + 2\delta\lambda + 3\delta + 2\delta^2 + 1 + 2\lambda\delta^2)q_R^2 + 4(1 + \delta) = 0. \quad (\text{B.7})$$

433 Note that q_R is the same solution as both (B.2) and (B.7) for the same δ and λ . Solving
434 λ from (B.2) and (B.7), separately, one obtains

$$\lambda = \frac{2(q_R^4\delta(1 + \delta) - (1 + 2\delta)q_R^2 + 1)}{\delta q_R^2(q_R^4\delta - 2(\delta + 1)q_R^2 + 2)}, \quad (\text{B.8})$$

$$\lambda = \frac{2((1 + 2\delta)(1 + \delta)q_R^2 - 2(1 + \delta))}{q_R^2(\delta(1 + 2\delta)q_R^2 - 2(1 + 2\delta + 2\delta^2))}. \quad (\text{B.9})$$

435 Equating the solutions and simplifying the results, one obtains

$$\delta q_R^4 - (2 + 3\delta)q_R^2 + 2 = 0. \quad (\text{B.10})$$

436 Then, one obtains q_R^2 as shown in (15).

437 From (15), it is easy to show that $q_R^2 \geq 3$, which is relatively large compared with the
438 natural frequency. This can explain why only invariant points P and Q are involved in the
439 H_∞ tuning of $C3$.

440 In this way, the optimal λ can be obtained by substituting q_R^2 in (15) into (B.8) or (B.9).
441 After obtaining λ , all the three invariant points can be obtained by solving

$$q^4 - \left(\frac{2}{\delta\lambda}(1 + \lambda + \delta + \lambda\delta) - q_R^2 \right) q^2 + \frac{2}{\delta^2\lambda q_R^2} = 0,$$

442 which is obtained from (B.4) and (B.5).

443 The procedure of calculating the optimal damping ratio ζ is similar to the procedure in
444 Appendix Appendix A, where the optimal ζ makes the gradients at invariant points P and
445 Q zero. After calculation and simplification, one obtains (18). Taking an average of ζ_P^2 and
446 ζ_Q^2 , one obtains the optimal ζ_{opt} as in (17).

447 **Appendix C. Proof of Proposition 3**

448 Denote

$$449 \quad A = 4(1 - \delta(1 + \lambda)q^2)^2, B = \delta^2q^2,$$

$$C = 4(1 - (1 + \delta + \delta\lambda)q^2 + \delta\lambda q^4)^2, D = \delta^2(1 - q^2)^2q^2,$$

450 and μ in (20) can be rewritten as

$$\mu = \sqrt{\frac{A\zeta^2 + B}{C\zeta^2 + D}}. \quad (\text{C.1})$$

451 To find the invariant points which are independent of damping, it requires

$$\frac{A}{C} = \frac{B}{D},$$

452 that is,

$$\frac{1 - \delta(1 + \lambda)q^2}{1 - (1 + \delta + \delta\lambda)q^2 + \delta\lambda q^4} = \pm \frac{1}{1 - q^2}.$$

453 Again, with the plus sign, one obtains the trivial solution zero, and with the minus sign, one
454 obtains

$$\delta(1 + 2\lambda)q^4 - 2(1 + \delta + \delta\lambda)q^2 + 2 = 0. \quad (\text{C.2})$$

455 Then, one obtains the two invariant points P and Q ($q_P < q_Q$) as

$$q_{P,Q}^2 = \frac{1 + \delta + \delta\lambda \pm \sqrt{(1 + \delta + \delta\lambda)^2 - 2\delta(1 + 2\lambda)}}{\delta(1 + 2\lambda)}. \quad (\text{C.3})$$

456 Letting the ordinates at invariant points P and Q equal, one has

$$\left| \frac{1}{1 - q_P^2} \right| = \left| \frac{1}{1 - q_Q^2} \right|.$$

457 It can be checked that $\frac{1}{1 - q_P^2} > 0$ and $\frac{1}{1 - q_Q^2} < 0$. Then, one obtains

$$\frac{1}{1 - q_P^2} = -\frac{1}{1 - q_Q^2}.$$

458 After cross multiplication and simplification, one has

$$q_P^2 + q_Q^2 = 2 \quad (\text{C.4})$$

459 Considering (C.2), one obtains

$$\frac{2(1 + \delta + \delta\lambda)}{\delta(1 + 2\lambda)} = 2,$$

460 which leads to (21).

461 Similar to the method in Appendix Appendix A, the optimal ζ can be obtained by making
462 μ have zero gradients at invariant points P and Q . After calculation and simplification, one
463 obtains

$$\zeta_{P,Q}^2 = \frac{q_{P,Q}^2 \delta^2}{4(1 - \delta(1 + \lambda)q_{P,Q}^2)(1 + 2\delta + 2\delta\lambda - \delta(1 + 3\lambda)q_{P,Q}^2)}.$$

464 After substituting (C.3) and (21), one obtains (23) and (24).

465 Taking an average of ζ_P^2 and ζ_Q^2 , one obtains the optimal ζ_{opt} as in (22).

466 **Appendix D. Proof of Proposition 4**

467 Denote

$$468 \quad A = 4(\lambda + 1)^2 q^2, B = (1 - \delta(1 + \lambda)q^2)^2,$$

$$C = 4(\lambda + 1 - \lambda q^2)^2 q^2, D = (1 - (1 + \delta + \delta\lambda)q^2 + \lambda\delta q^4)^2.$$

469 Then, μ in (25) can be rewritten as

$$\mu = \sqrt{\frac{A\zeta^2 + B}{C\zeta^2 + D}}. \quad (\text{D.1})$$

470 To find the invariant points which are independent of damping, it requires

$$\frac{A}{C} = \frac{B}{D},$$

471 that is,

$$\frac{\lambda + 1}{\lambda + 1 - \lambda q^2} = \pm \frac{1 - \delta(1 + \lambda)q^2}{1 - (1 + \delta + \delta\lambda)q^2 + \delta\lambda q^4}.$$

472 Similarly, with plus sign, one obtains the trivial solution zero, and with minus sign, one
473 obtains

$$2\delta\lambda(\lambda + 1)q^4 - (1 + 2\lambda + 2\delta(1 + \lambda)^2)q^2 + 2(\lambda + 1) = 0. \quad (\text{D.2})$$

474 Thus, one obtains the two invariant points P and Q ($q_P < q_Q$) as in (29).

475 Letting the ordinates at invariant points P and Q equal, one has

$$\left| \frac{\lambda + 1}{\lambda + 1 - \lambda q_P^2} \right| = \left| \frac{\lambda + 1}{\lambda + 1 - \lambda q_Q^2} \right|.$$

476 It can be checked that $\frac{\lambda+1}{\lambda+1-\lambda q_P^2} > 0$ and $\frac{\lambda+1}{\lambda+1-\lambda q_Q^2} < 0$. Then, one obtains

$$\frac{\lambda + 1}{\lambda + 1 - \lambda q_P^2} = -\frac{\lambda + 1}{\lambda + 1 - \lambda q_Q^2}.$$

477 After cross multiplication and simplification, one has

$$q_P^2 + q_Q^2 = \frac{2(\lambda + 1)}{\lambda}.$$

478 Comparing with (D.2), one obtains

$$\frac{1 + 2\lambda + 2\delta(1 + \lambda)^2}{2\delta\lambda(\lambda + 1)} = \frac{2(\lambda + 1)}{\lambda},$$

479 which leads to

$$2\delta\lambda^2 - 2(1 - 2\delta)\lambda + 2\delta - 1 = 0.$$

480 It can be checked that this equation has real solutions if and only if

$$\delta \leq 1/2.$$

481 Under this condition, the optimal λ can be obtained as in (26).

482 Note that if $\delta = \frac{1}{2}$, from (26), one has $\lambda = 0$, or $k = \infty$. In this case $C5$ reduces to $C1$.
483 Thus, the more reasonable assumption is $\delta < \frac{1}{2}$ rather than $\delta \leq \frac{1}{2}$.

484 Similar to the method in Appendix Appendix A, the optimal ζ can be obtained by making
485 μ have zero gradients at invariant points P and Q . After calculation and simplification, one
486 obtains ζ_P^2 and ζ_Q^2 as in (28).

487 Taking an average of ζ_P^2 and ζ_Q^2 , one obtains the optimal ζ_{opt} as in (27).

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