Multi-Agent Systems with Dynamical Topologies: Consensus and Applications

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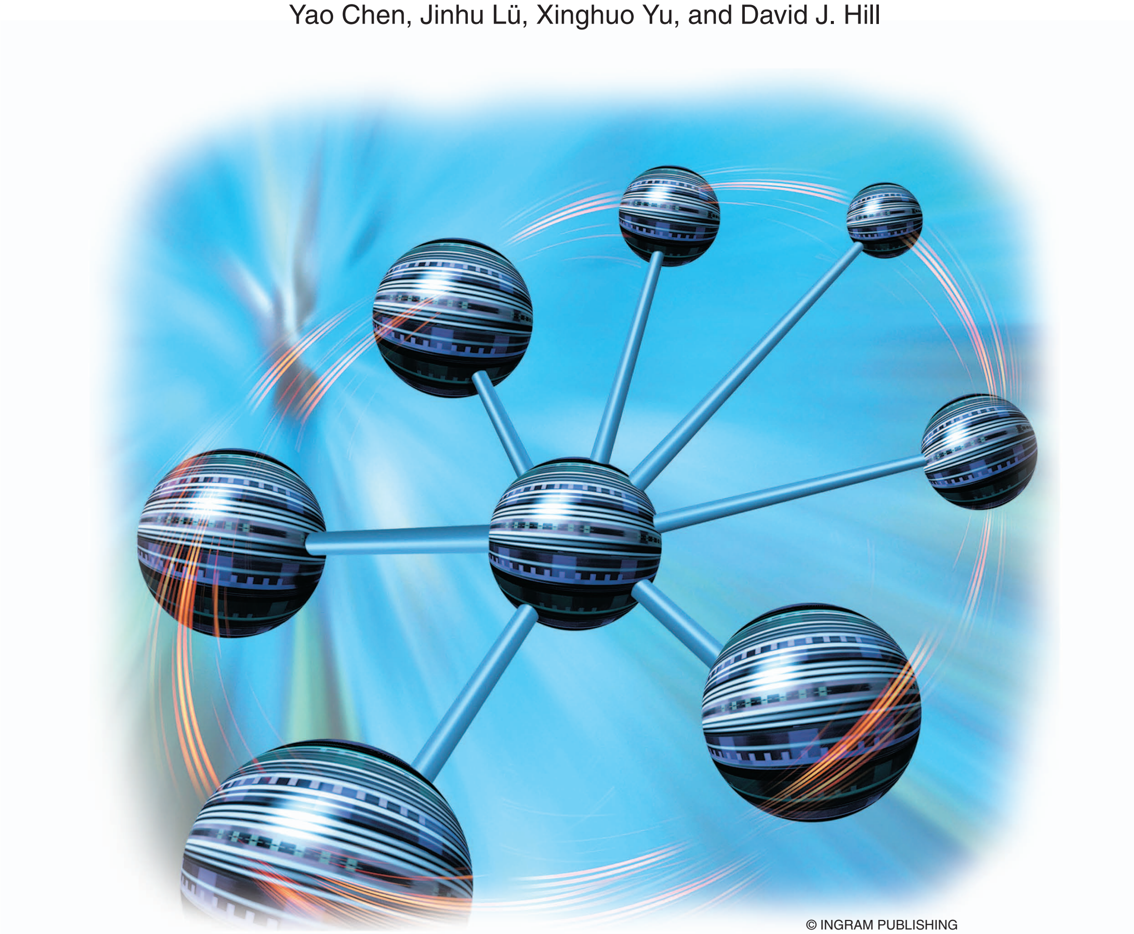

Abstract

It is well known that a multi-agent system (MAS) is a specific system consisting of multiple interacting autonomous agents. Consensus or synchronization, as one of the typical collective behaviors, is ubiquitous in nature. Over the last decades, consensus has been widely investigated in various disciplines, including mathematics, physics, biology, engineering, and social sciences. In particular, consensus of MAS with dynamical topology is an emerging new topic motivated by many real-world applications, such as wireless communication and sensor networks. However, the collective behavior of MAS with dynamical topology is very complex and cannot be easily analyzed by the traditional approaches. To resolve the issue of dynamical topology, various techniques and methods have been developed in the last decade. This paper aims to review the main advances in the consensus of MAS with dynamical topology, including several fundamental models and the corresponding methods. The main purpose is to promote this emerging topic on multi-agent systems, with emphasis on the interdisciplinary interest from the circuits and systems engineering communities.

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I. Introduction

Many practical engineering and social systems can be described by multi-agent systems (MAS). Here, an MAS is a system composed of multiple autonomous agents with mutual interactions among them. Over the last decades, MAS have been intensively investigated in various disciplines, such as mathematics, physics, biology, economics, computer science, and social sciences. There are some interesting topics in MAS, including cooperation and coordination, distributed problem solving, multi-agent learning, wireless communication, and so on [7], [28], [31], [35].

An MAS often demonstrates various typical collective behaviors guided by some simple interaction rules. Examples include pattern formation of bacteria colony, triangle formation of birds flocking, and synchronization of pendulum clocks. To investigate these collective behaviors quantitatively, some representative mathematical or physical models have been proposed for the last three decades. In 1987, Reynolds came up with the Boid model to describe the collective behavior of birds flocks by using three basic local rules: alignment, attraction, and repulsion [4]. In 1995, Vicsek and his colleagues introduced the famous Vicsek model [3] via a simple local alignment rule. In 2003, Jadbabie et al. [7], [52] analyzed the dynamics of the linearized Vicsek model based on the theory of infinite products of matrices [23]. In 2008, Lü and his colleagues improved the famous Couzin-Levin model [6] by merging the rules of attraction and repulsion.

As is known today, consensus or synchronization is a kind of basic collective behaviors. By consensus we mean a general agreement among all the members of an MAS. In particular, the essential implication of consensus indicates the coordination of the benefits of different group members. Several specific models have been used for the last ten years to further investigate the inner mechanism of consensus of MAS, such as the Vicsek model [3], the linear iteration model [14], [15], the first order continuous-time model [55], [57], and so on. In 2003, Jadbabie et al. investigated the consensus of a class of discrete-time MAS with undirected and jointly connected topology [7]. In 2004, Li and Wang further generalized the above work to the case of infinite connectivity [22]. In 2005, Lü and Chen explored the synchronization of time-varying complex networks [10]. In 2005, Ren and Beard considered the consensus problem under the condition of directed and jointly connected topology [8]. In 2008, Cao et al. introduced asynchronous communication into the consensus of MAS [13]. In 2007, Lin et al. analyzed the consensus of MAS with non-smooth dynamics [74].

Consensus of MAS has prospective applications in many different disciplines, such as biology [86], [87], [90], [91], sociology [85], parallel computation [64], power networks [84], sensor networks [88], [95], and robotics [89], [92]–[94]. Typical examples include: i) In [84], Xu et al. applied consensus of MAS to power systems and established an effective load scheduling algorithm; ii) In [85], Hegselmann and Krause introduced a consensus algorithm to sociology, describing the dynamics of opinion formation among different persons. The research of consensus has essential connections with biology: i) In [86], Evans and Patterson investigated the consensus behavior in flocks of estrildine finches; ii) In [87], Mirollo and Strogatz studied the synchronization of pulse-coupled biological oscillators; iii) In [91], Monus and Barta made some research on the degree of synchronization of sparrows under different predation risk. Moreover, sensor network is another important application area of consensus: i) In [88], Hong and Scaglione designed a synchronization protocol for large scale sensor networks; ii) In [35], Xiong and Liu designed a consensus-based time synchronization algorithm for wireless sensor networks; iii) In [95], Kar and Moura came up with a consensus algorithm for sensor networks with imperfect communication. Furthermore, consensus algorithm can also be applied to robotics: i) In [92], Lei and Zeng designed a consensus algorithm for swarm robotics cooperative control; ii) In [89], Igarashi et al. investigated the passivity-based attitude synchronization algorithm in SE(3); iii) In [93] and [94], Ren made a series studies on the attitude synchronization of robot manipulators.

Note that the consensus algorithm depends on the dynamics of agents and the topology structure among them. This particular problem has been a focal point for the past decade. In 2004, Olfati-Saber and Murray explored the consensus of first-order MAS with fixed and switching topology [57]. In 2006, Sarlette investigated the consensus of MAS over an unit cycle with a fixed topology [62]. In 2007, Ren and Atkins studied the consensus of second-order MAS with fixed topology [42]. In 2010, Yu et al. established some necessary and sufficient conditions for the consensus of second-order MAS [31]. In 2012, Zhu et al. investigated the flocking of non-holonomic MAS with proximity graphs [96]. Very recently, Lu et al. studied the finite-time control of MAS with a virtual leader [97].

Regarding the MAS with fixed topology, there are some conventional methods to analyze their dynamical behaviors [81]–[83], such as Lyapunov functions [10], [56] and eigenvalues techniques [28], [31]. It is well recognized that the above traditional approaches are hard to be generalized to the case of switching topology due to the switching dynamics of MAS. Moreover, many traditional methods of switching systems cannot be directly used to

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cope with the MAS with switching topology. To resolve the above problem, some effective approaches have been developed to deal with MAS with dynamical topology recently [51]. It includes and the method of infinite products of stochastic matrices [7], [8], [23], [29], [52], the graphical approach [13]–[15], the technique of set value analysis [16], and the method of a common quadratic Lyapunov function [55], [57]. This tutorial review paper will summarize the main advances in the consensus of MAS with dynamical topology, including modeling, analysis and control.

This paper is organized as follows. Section II briefly reviews several typical application examples of consensus. The representative models and the corresponding approaches are then given in detail in Section III, including discrete-time case and continuous-time case, respectively. Section IV makes some concluding remarks.

II. Prospective Application of Consensus

A. Application to Sociology

Opinion formation is one of the critical issues in sociology. By opinion formation we mean how a universal opinion forms in a group of people even if they have different opinions at the beginning, see Fig. 1 [98], [99].

In [85], Hegselmann and Krause proposed the following dynamical equation to describe the opinion formation process within a group people, which is called opinion-dynamics:

$$x_i(t+1) = \frac{1}{|\mathcal{I}_i(t)|} \sum_{j \in \mathcal{I}_i(t)} x_j(t),$$

where $\mathcal{I}_i(t)$ is given by

$$\mathcal{I}_i(t) = \{j : |x_j(t) - x_i(t)| \le \varepsilon\},$$

which is called the confidence region of agent i.

The above opinion dynamics can be viewed as a special kind of consensus algorithm and the evolution of opinion dynamics will lead to cluster consensus of the agents group.

B. Application to Power Systems

Consensus algorithm of MAS can be applied to load scheduling of power systems [84].

Consider a power network with N different nodes, each of which can generate power and consume power, see Fig. 2 [100]. Let P_{Gi} and P_{Li} be the active power generation and load of node i, respectively. Then the active power in balance net of the network is given by

$$P_{\text{net}} = \sum_{i=1}^{N} P_{Gi} - P_{Li}.$$

At the same time, let x_i^k be the net power of node i at time t_k, and the net power of each node i can be adaptively adjusted according to the net power of its neighbors:

$$x_i^{k+1} = x_i^k + \sum_{j=1}^{N} a_{ij}(x_j^k - x_i^k).$$

For the above updating process, if the parameters a_{ij} are appropriately chosen and the state of each node i initialized by $P_{Gi} - P_{Li}$, then the power load in the network will be balanced, that is $x_i^k \to (1/N)P_{\text{net}}$ as $k \to \infty$.

C. Application to Parallel Computing

Due to the distributed property of consensus algorithms, consensus can be applied to distributed or parallel computation, see Fig. 3 [101]. A typical application of consensus to distributed computation is given in [64], where Tsitsiklis considered a group of interconnected processors with each processor i receiving the information $s^i(n)$ at time n and evaluating the next value $x^i(n+1)$ by forming the convex combination of its old information and the received new messages, that is:

$$x^i(n+1) = \sum_{j=1}^{N} a_{ij}(n) x^j(t_{ij}(n)) + \gamma^i(n) s^i(n).$$

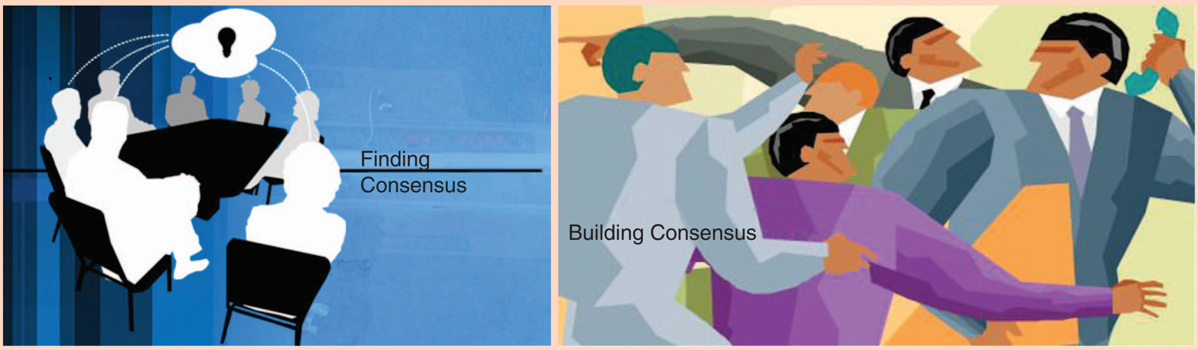

Figure 1. Reaching consensus among people [98], [99].

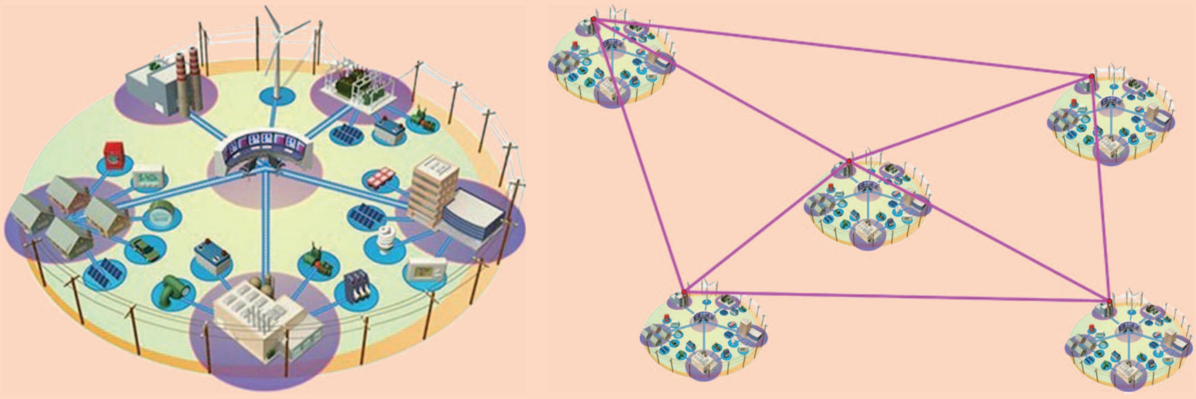

Figure 2. Consensus applications in power systems [100].

D. Application to Biology

Consensus or synchronization is also an important biological phenomenon. In [49], Cucker and Smale proposed the following flocking system:

$$\begin{cases} \dot{x}_i &= v_i, \\ \dot{v}_i &= \sum_{j=1}^{N} a_{ij}(v_j - v_i), \end{cases}$$

where the coupling coefficients are given by

$$a_{ij} = \frac{K}{(\delta^2 + \| x_i - x_j \|^2)^{\beta}}.$$

The above coupling coefficients mean that the influence between two birds is reciprocal to the distance between them, see Fig. 4 [102]. For this flocking system, Cucker and Smale have proved that the distance between any two birds satisfies

$$\limsup_{t \to \infty} \| x_i(t) - x_j(t) \| < D$$

for any two agents i, j and some constant $D > 0$ [49].

Figure 3. Application of consensus to parallel computation [101].

Figure 4. Consensus of birds flock [102].

E. Application to Sensor Networks

For real-world applications of sensor networks, the sensor nodes of the network are required to maintain accurate time synchronization, see Fig. 5 [103]. The so-called time synchronization algorithm plays an important role to align all the network nodes to a common notion of time. In [35], Xiong and Kishore proposed the following second-order discrete time-time synchronization (SO-DCTS) algorithm:

$$\begin{aligned} t_i(k) = t_i(k-1) + \varepsilon \sum_{j \in N_i} [t_j(k-1) - t_i(k-1)] \\ - \gamma\varepsilon \sum_{j \in N_i} [t_j(k-2) - t_i(k-2)], \end{aligned}$$

where $t_i(k)$ is the local time at node i during iteration k, N_i is the set of neighboring nodes of node i, ε and γ are positive constants. Time synchronization implies that all the local clocks will approach to a global clock, that is, $\lim_{k \to \infty} | t_i(k) - t_j(k) | = 0$ for any two different nodes i, j.

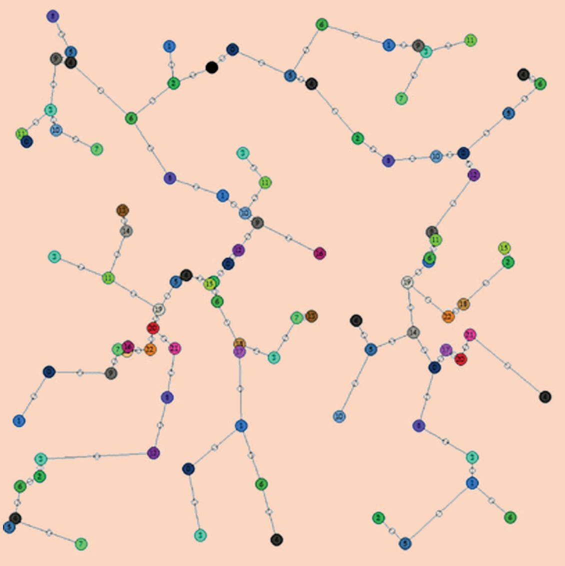

Figure 5. A sensor network [103].

F. Application to Robotics

Consensus of many robotics is an important application field of robotic systems. Since the Euler-Lagrange equation can be used to model a class of mechanical systems such as robotics and rigid bodies, see Fig. 6 [104], Ren investigated the following synchronization problem of Euler-Lagrange system [93]:

$$M_i(q_i)\ddot{q}_i + C_i(q_i,\dot{q}_i)\dot{q}_i = \tau_i, \quad i = 1, 2, \cdots, N,$$

where the controller τ_i is given by

$$\tau_i = -\sum_{j=1}^{N} a_{ij}(q_i - q_j) - \sum_{j=1}^{N} b_{ij}(\dot{q}_i - \dot{q}_j) - K_i\dot{q}_i.$$

Figure 6. Robot manipulator [104].

By using this controller, the states of the above system can reach synchronization, that is,

$$\lim_{t\to\infty} |q_i - q_j| = 0$$

for any $1 \le i < j \le N$.

III. Representative Models and Analysis Methods for MAS with Dynamical Topology

All notations in this Section are defined in [5], [21] [47], [72], [77]. To begin with, we briefly review several important algorithms.

There are two commonly used concepts of graph connectivity in consensus studies [5], [14], so-called Type-II joint connectivity, see Fig. 7, and sequential connectivity [20], see Fig 8. In the following, two kinds of graph factorization algorithms will be introduced.

Type-I graph factorization: For any graph $\mathcal{G}$ which contains a spanning tree, $\mathcal{G}$ can be factorized into several strongly connected components by using Algorithm 1, see Fig. 9. A fast algorithm with linear complexity is given in [70].

Type-II graph factorization: For any strongly connected graph $\mathcal{G}$ with period d, graph $\mathcal{G}$ can be factorized into d clusters by using Algorithm 2. An illustration graph is given in Fig. 10.

A. Discrete-Time MAS Model

1) First Order Linear MAS Model

In 1995, Vicsek et al. proposed the famous Vicsek model. Consider N particles on a plane, let $V = \{1, \cdots, N\}$ be the

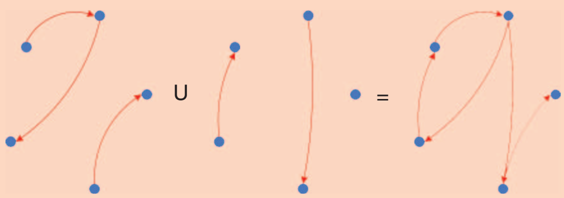

Figure 7. An example of Type-II jointly connectivity.

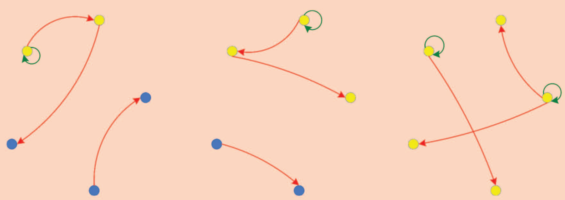

Figure 8. An example of sequential connectivity.

Algorithm 1. Factorizing a connected graph $\mathcal{G}$ into several strongly connected components.

Input:
A graph $\mathcal{G} = (V, E)$ which contains a spanning tree.
Output:
Subsets $\{V_i\}_{i=1}^{r}$ of V satisfying $V_i \bigcap V_j = \emptyset$ and $\bigcup_{i=1}^{r} V_i = V$.
Procedure:

```
V* := V;  r := 0;
s := 0;
V'_s := {The root of G};  V'_{s+1} := ∅;
while V'_s ≠ ∅ do
    Select i* ∈ V'_s;
    r := r + 1;
    Find the strongly connected component V_r ∈ V*
    which contains i*;
    Find the set of nodes V''_{s+1} ∈ V* − V_r which can be
    accessed from V_r by one step;
    V'_{s+1} := V'_{s+1} ⋃ V''_{s+1};
    V* := V* − V_r;
    V'_s := V'_s ⋂ V*;
    if V'_s = ∅ then
        s := s + 1;
        V'_s := V'_s ⋂ V*;
        V'_{s+1} := ∅;
    end if
end while
return {V_i}_{i=1}^r;
```

Algorithm 2. Factorization of a strongly connected graph $\mathcal{G}$ with period d.

Input:
A strongly connected graph $\mathcal{G} = (V, E)$.
Output:
Subsets $\{V_i\}_{i=0}^{d-1}$ of V satisfying $V_i \bigcap V_j = \emptyset$ and $\bigcup_{i=0}^{d-1} V_i = V$.
Procedure:

```
Select i* ∈ V;
for each i ∈ V do
    Find a path p with length l from node i* to i;
    if l ≡ r (mod d) then
        V_r := V_r ⋃ {i};
    end if
end for
return {V_i}_{i=0}^{d-1};
```

set of these particles. For particle $i \in V$, its position and heading are described by $p_i(t) = (x_i(t), y_i(t))^T$ and $\theta_i(t)$ $(\in [0, 2\pi))$, respectively. The updating rules are given by

$$p_i(t+1) = p_i(t) + v(\cos\theta_i(t), \sin\theta_i(t))^T,$$
$$\theta_i(t+1) = \arctan\left(\frac{\sum_{j\in\mathcal{N}_i(t)} \sin\theta_i(t)}{\sum_{j\in\mathcal{N}_i(t)} \cos\theta_i(t)}\right),$$

where $\mathcal{N}_i(t)$ is defined by

$$\mathcal{N}_i(t) = \{j : \| p_i(t) - p_j(t) \| < r\}.$$

Suppose that the region of motion of the particles is limited into a square area. Simulation results indicate that the headings of particles will reach consensus under the condition of high particle density and low noise amplitude. Fig. 11 [105] illustrates the above consensus phenomenon in the Vicsek model.

The nonlinearity of the Vicsek model makes the theoretical analysis very difficult. In 2003, to bypass the above difficultly, Jadbabie et al. introduced a linearized Vicsek model and gave the corresponding theoretical analysis [7].

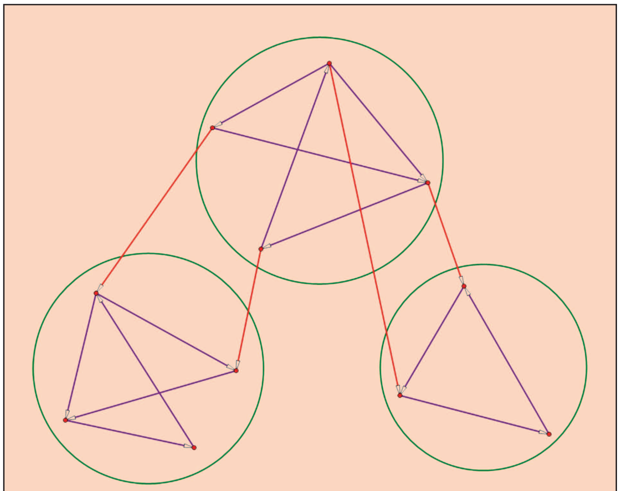

Figure 9. Factorize a weak connected graph into different components: each component is strongly connected.

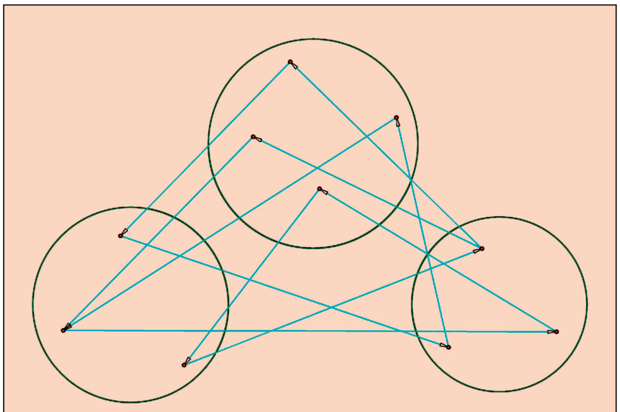

Figure 10. Factorize a strongly connected graph into different components: no links exists within each component.

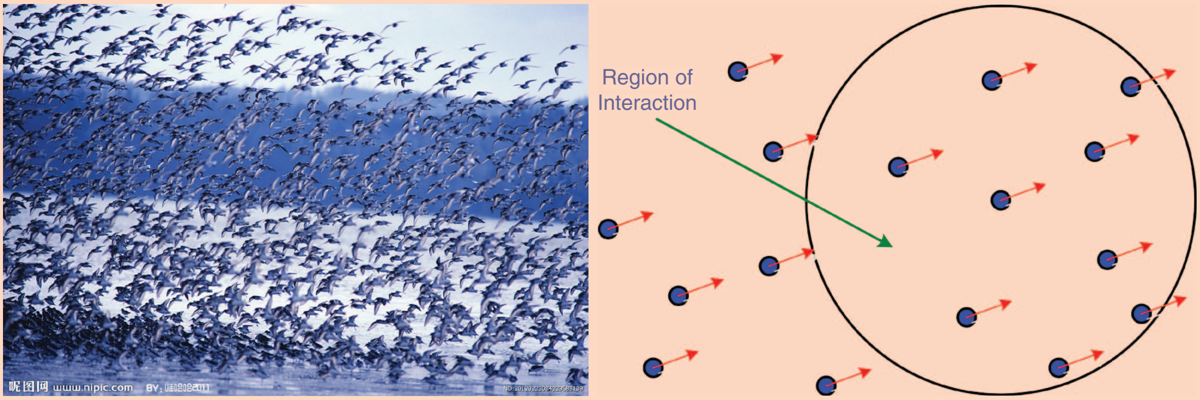

Figure 11. Consensus in Vicsek model [105].

The linearized Vicsek model can be rewritten into the following general form:

$$x_i(t+1) = \sum_{j=1}^{N} a_{ij}(t) x_j(t - \tau_j^i(t)), \tag{1}$$

where $x_i(t)$ is the state of agent i at time t, $a_{ij}(t) \geq 0$ is the coupling coefficient, $\sum_{j=1}^{N} a_{ij}(t) = 1$, and $\tau_j^i(t)$ is the transmission time delay from agent j to agent i.

For the case of fixed topology, Eq. (1) can be transformed into a discrete-time time-invariant linear system. However, for the case of switching topology, the following facts indicate to us that the traditional methods cannot be directly applied for the theoretical analysis of MAS (1).

In [58], Olshevsky and Tsitsiklis proved an important fact for the discrete-time consensus algorithm with switching topology. That is, there does not exist a common quadratic Lyapunov function for the consensus algorithm $x(t+1) = A(t)x(t)$, where $A(t) \in \mathcal{A}$ and $\mathcal{A}$ is the set satisfying the following conditions:

1) Each $A \in \mathcal{A}$ is stochastic, positive diagonal, and $\mathcal{G}(A)$ is strongly connected.
2) Each $A = (a_{ij})_{i,j=1}^{N} \in \mathcal{A}$ has the form of $A = D^{-1}\tilde{A}$, where $\tilde{A} = (\tilde{a}_{ij})$ is a 0-1 matrix, $D = \text{diag}\{d_1, d_2, \cdots, d_N\}$ with $d_i = \sum_{j=1}^{N} \tilde{a}_{ij}$.

In [25], Brayton and Tong proved the following fact: for two stochastic matrices A and B with different left Perron eigenvectors, there does not exist a convex, smooth, and closed common Lyapunonv function $V(t)$ satisfying $V(Mx) \leq V(x)$ for $M \in \{A, B\}$. It indicates that there does not exist a convex, smooth, and closed common Lyapunonv function for the consensus problem (10) with different left Perron eigenvectors.

Traditionally, the construction of Lyapunov candidate is viewed as an effective method for the stability analysis of dynamical systems. According to the above discussion, it is often difficult or even impossible to construct a traditional Lyapunov candidate for MAS with switching topology. Therefore, it is necessary to develop some new specific techniques to analyze the consensus of MAS with switching topology.

According to the existing literature, the analysis methods of the basic model (1) can be summarized into the following four classes:

i) The method of infinite products of stochastic matrices. This approach can be traced back to [23] and was applied to resolve the consensus of linearized Vicsek model in [7] under the condition of Type-I joint connectivity. Consequently, this method was applied to the case of Type-II joint connectivity in [8]. In detail, the basic idea of this method can be attributed to the construction of the following ergodic coefficient ([7], [8])

$$\lambda(A) = 1 - \min_{i_1, i_2} \sum_{j=1}^{N} \min\{a_{i_1, j}, a_{i_2, j}\}, \tag{2}$$

where $A = (a_{ij})_{i,j=1}^{N}$ is a stochastic matrix.

For the case of $\tau_j^i(t) \neq 0$, the method of infinite products of stochastic matrices cannot be directly applied to Eq. (1). In 2006, Xiao and Wang [18] generalized this method to the case of time-varying delays by using the following fact: For a stochastic matrix A, if $\mathcal{G}(A)$ contains a spanning tree and the root has a self-loop, then A is SIA.

ii) Reduction to absurdity. This method was introduced by Li and Wang [22]. For $\tau_j^i(t) = 0$, define

$$M(t) = \max_{i=1}^{N} x_i(t),$$

$$m(t) = \min_{i=1}^{N} x_i(t).$$

Li and Wang proved that if $\{\mathcal{G}_i\}_{i=1}^{\infty}$ is infinitely connected, then there is $\limsup_{t\to\infty} M(t) = \liminf_{t\to\infty} m(t)$, which leads to the consensus of MAS (1).

Moreover, this method is also effective for the case of Type-I joint connectivity.

iii) Graphical approach. This method was proposed by Cao et al. in 2008 [14], [15]. Firstly, they introduced two kinds of important graphs: "neighbor-shared graph" and "strongly rooted graph". Next, they established the inner relationships between these two kinds of graphs and the above ergodic coefficient defined in (2). Finally, they obtained the exponential convergence of MAS (1) by using the above relationships.

iv) The method of set construction. This method was proposed by Blondel et al. in [9]. They constructed the following set

$$V_{t,k}=\{i\in V: m_i(t)\geq \alpha^{kB}\}, \tag{3}$$

where B is the maximal time-delay, $m(0)=0$, $M(0)=1$. Then, they proved that there exists some t^* satisfying $V_{t^*,n}=V$. Repeating the above process, one can obtain the exponential convergence of (1) under the condition of Type-I jointly connectivity.

It should be pointed out that self-loops play an important role in the consensus of (1). Here, the existence of self-loops means $a_{ii}(t)>0$ for each $i\in V$ and $t\geq 0$. In [66], Xiao and Wang investigated the consensus algorithm (1) with $a_{ii}(t)$ not always greater than zero. The basic idea of [66] is to combine several graphs without self-loops into a composite graph with self-loops. Consequently, the traditional methods of MAS (1) with self-loops can be used to resolve the case without self-loops.

Moreover, if there does not exist a self-loop for any node in $\mathcal{G}(A(t))$, the MAS (1) can reach cluster consensus under some suitable conditions [24]. Hereafter, k-cluster consensus is defined as follows: there exist k sets $\{V_i\}_{i=1}^{k}$ satisfying $\bigcup_{i=1}^{k} V_i=V$, $V_i\bigcap V_j=\emptyset$ for $i\neq j$, and $\lim_{t\to\infty}|x_{i_1}(t)-x_{i_2}(t)|=0$ for $i_1,i_2\in V_i$. The corresponding factorization algorithm of cluster consensus is described by Algorithm 2.

2) Second Order Linear MAS Model

The corresponding second order linear MAS model of (1) is described by

$$\begin{cases} x_i(t+1) &= x_i(t)+hv_i(t),\\ v_i(t+1) &= v_i(t)+hu_i(t),\end{cases} \tag{4}$$

where $x_i(t)$ and $v_i(t)$ are the position and velocity of agent i, respectively.

In [29], Lin and Jia investigated the consensus of MAS (4) with the following controller

$$u_i(t)=-\gamma v_i(t)+\alpha\sum_{j=1}^{N}a_{ij}(t)(x_j(t-\tau_j^i)-x_i(t)) +\beta\sum_{j=1}^{N}a_{ij}(t)(v_j(t-\tau_j^i)-v_i(t)). \tag{5}$$

The key technique of [29] lies in the following linear transformation

$$\begin{pmatrix}\zeta_i(t)\\ \xi_i(t)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1 & \lambda\end{pmatrix}\begin{pmatrix}x_i(t)\\ v_i(t)\end{pmatrix}. \tag{6}$$

Using this linear transformation, the second order consensus problem (4) can be transformed into the corresponding first order consensus problem (1). Following this line, the methods for dealing with the first order MAS (1) can be applied to cope with the corresponding second order MAS (4). In particular, the damping term $-\gamma v_i(t)$ plays an key role for the application of the above linear transformation. For the case of $\gamma=0$, the above method proposed by Lin and Jia fails.

In [24], Chen et al. introduced a new method to analyze the consensus of MAS (4), where the controller is given by (5) with $\gamma=0$ and $\tau_j^i=0$. The basic idea of [24] lies in the construction of the polytope

$$\mathcal{P}_N=\{x: x\in\mathbf{R}^N, \mathbf{1}^T x=0, \|x\|_1\leq 2\}.$$

Let the set of vertices of $\mathcal{P}_N$ be $\mathcal{V}_N=\{\hat{e}_{N,i}\}_{i=1}^{\hat{N}}$. Consequently, construct the following convex polytope

$$\mathcal{P}_{N,1}=\mathrm{conv}\{\theta_{N,i},\theta'_{N,i}: 1\leq i\leq\hat{N}\}, \tag{7}$$

where $\theta_{N,i}=(\hat{e}_{N,i}^T,0^T)^T, \theta'_{N,i}=(0^T,\hat{e}_{N,i}^T)^T$. They proved that the polytope $\mathcal{P}_{N,1}$ is decreasing along (4) under some suitable conditions. Based on this idea, several consensus criteria were established.

Note that for the MAS (1), if the coefficients $a_{ij}(t)$ are not always nonnegative, then the traditional methods are not valid any more. In this case, one can use the following ergodic coefficient ([43])

$$\eta(A)=\frac{1}{2}\max_{i,j}\left(a_{ii}+a_{jj}-a_{ij}-a_{ji}+\sum_{k\neq i,j}|a_{ik}-a_{jk}|\right)$$

to resolve the above problem. In [67], Liu and Chen proved that $\Delta(Ax)\leq\eta(A)\Delta(x)$, where $\Delta(x)=\max_{i,j}|x_i-x_j|$ with $x=(x_1,x_2,\cdots,x_N)^T\in\mathbf{R}^N$. Consequently, the MAS (1) without delay reaches consensus under the condition of $\eta(A(t))<1-\delta$ with $t\geq 1$ and $\delta\in(0,1)$. In fact, the above $\eta(A)$ can be viewed as an estimate of the nontrivial eigenvalues of stochastic matrices [44]. That is, $|\nu|\leq\eta(A)\leq\lambda(A)$ for any stochastic matrix A, where ν is one of the nontrivial eigenvalues of A.

3) Nonlinear MAS Model

In 2005, Moreau investigated a general nonlinear MAS model [16], described by

$$x_i(t+1)=f_i(t,x(t)), \tag{8}$$

where $x(t)=(x_1(t),x_2(t),\cdots,x_N(t))^T$. To resolve the consensus of MAS (8), he introduced the following "convexity assumption":

$$x_i(t+1) \in \mathrm{relint}(\mathrm{conv}\{x_j(t)\}_{j\in N_i(t)}),$$

where $\mathrm{relint}(S)$ is the relative interior of S and $\mathrm{conv}(S)$ is the convex hull of S. By constructing a set-valued Lyapunov function, Moreau proved the consensus of (8) under the condition of the Type-II jointly connectivity or infinite connectivity.

In 2011, Chen et al. considered a class of nonlinear MAS with the following form [5]

$$x_i(t+1) = \sum_{j=1}^{n} a_{ij}(t) f_{ij}(x_j(t-\tau_j^i(t))), \tag{9}$$

where $f_{ij}(\cdot)$ is the transmission nonlinearity, $\tau_j^i(t)$ is the transmission delay, and $a_{ij}(t)$ are the coupling coefficients. To investigate the consensus of MAS (9), they introduced a class of nonlinear function $\mathcal{F}$ defined as follows.

A function f belongs to $\mathcal{F}$ if the following three conditions are satisfied:

1) f is continuous and $f \in \mathbf{R}^m \to \mathbf{R}^m$.
2) f is defined on some convex set $\mathcal{B} \subseteq \mathbf{R}^m$ and $f(x) \in \mathcal{B}$ for any $x \in \mathcal{B}$.
3) There exists a bounded convex set $\mathcal{U} \subseteq \mathcal{B}$ satisfying $f(x) = x$ for $x \in \mathcal{U}$ and $d(f(x), \mathcal{U}) < d(x, \mathcal{U})$ for any $f(x) \neq x$.

They proved that if $\{f_{ij}\}_{i,j}^n \subseteq \mathcal{F}$ and f_{ij} share two common sets $\mathcal{B}$ and $\mathcal{U}$ in the above definition, then (9) reaches consensus under the condition of Type-I jointly connectivity. However, infinite connectivity can not always guarantee the consensus of (9).

In [52], Liu and Guo proved that the Vicsek model reaches synchronization if the initial topology of Vicsek model is connected and the velocity of each agent is less than a given critical value. Moreover, the basic idea of the above work is based on the exponential convergence of headings. Consequently, there is $E(t) \subseteq E(t+1)$, where $E(t)$ is edge set of $G(t)$ and hence the topology connectivity can be naturally guaranteed.

For the MAS (1) without time-delays, one can rewrite (1) into the corresponding matrix form:

$$x(t+1) = A(t)\, x(t). \tag{10}$$

If $\xi^T A(t) = \xi^T$ with $\xi \geq 0$, $\xi^T \mathbf{1} = 1$ for any $t \geq 1$, one constructs the Lyapunov candidate $V(t) = \delta(t)^T \Xi \delta(t)$, where $\delta(t) = x(t) - \xi^T x(t)\mathbf{1}$ and $\Xi = \mathrm{diag}(\xi)$. According to $V(t+1) - V(t) \leq 0$, one obtains the consensus of (10). In [68], Nedić et al. considered the MAS (10) with $\mathbf{1}^T A(t) = \mathbf{1}^T$ for each $t \geq 1$. Under the condition of Type-II joint connectivity, they obtained a relatively accurate estimate of the convergence rate for (10). Furthermore, they investigated the corresponding quantized MAS model, described by

$$x_i(t+1) = \left\lfloor \sum_{j=1}^{N} a_{ij}(t)\, x_j(t) \right\rfloor,$$

where $Q\lfloor x \rfloor$ is the greatest integer which is not greater than Qx and $Q \in \mathbf{Z}^+$ is the level of quantization. They also obtained a quantization level independent convergence rate for this quantization MAS model.

4) Convergence Rate Analysis of MAS

Note that the convergence rate analysis of MAS (1) is another interesting issue. Define the convergence rate by

$$\rho = \sup_{x(0)\neq 0} \limsup_{t\to\infty} \frac{\Delta(x(t))}{\Delta(x(0))}. \tag{11}$$

When time-delays are not included and the topology is fixed, it is easy to derive $\rho = |\lambda_2|$, where λ_2 is the eigenvalue of A with modulus closest to 1. For Type-I joint connectivity, the method given in [9] can also be used to investigate MAS (1) with bounded time-delays. Furthermore, for MAS (1) with Type-II joint connected topology and without time-delays, an upper bound of ρ is obtained in [14], where

$$\rho \leq (1 - \alpha^{K(N-1)})^{\frac{1}{K(N-1)}}$$

with $\alpha = \inf_{a_{ij}(t)\neq 0} a_{ij}(t)$ and K is the interval of joint connectivity. In fact, the convergence rate analysis in [14] is mainly based on the following fact [60]: For positive diagonal stochastic matrices $\{A_i\}_{i=1}^N \subseteq \mathbb{R}^{N\times N}$ with each $\mathcal{G}(A_i)$ containing a spanning tree, there is $\lambda\left(\prod_{i=1}^N A_i\right) < 1$.

In [21], Chen et al. proposed a new method to estimate the convergence rate of (1) under the condition of Type-II joint connectivity. The critical techniques of [21] are based on the construction of some specific sets.

5) On the Lyapunov Exponent of MAS

Consider the following general discrete-time MAS model

$$y_{t+1} = A'_{\delta(t)}\, y_t,$$

where $\delta(t) \in \mathcal{P}$ is the switching signal. Let $y_t = (y_t^1, y_t^2, \cdots, y_t^N)$, $x_t^i = y_t^i - (1/N)\sum_{j=1}^N y_t^i$, and $x_t = (x_t^1, x_t^2, \cdots, x_t^N)$. Then the consensus of the above discrete-time linear MAS can be transformed into the stability of a class of basic linear switching systems, that is,

$$x_{t+1} = A_{\delta(t)}\, x_t, \tag{12}$$

where $\delta(t)$ is the switching signal and $\delta(t) \in \mathcal{P}$. Let $\mathcal{A} = \{A_\delta : \delta \in \mathcal{P}\}$ and consider the following linear inclusions

$$x_{t+1} \in \{Ax_t : A \in \mathcal{A}\}. \tag{13}$$

The Lyapunov exponent of the above inclusion is defined by

$$\rho(\mathcal{A}) = \limsup_{t\to\infty} \frac{\log\|x_t\|}{t}, \tag{14}$$

Algorithm 3. Stability analysis of a discrete inclusion.

Input:

A set $\mathcal{A}$ with finite number of $N \times N$ matrices; A predefined error ε.

Output:

The sign of Lyapunov exponent of the discrete inclusion $x_{t+1} \in \{Ax_t : A \in \mathcal{A}\}$.

Procedure:

1: Select a set $Z_0 = \{p_i\}_{i=1}^{m}$ satisfying $\mathbf{0} \in \mathrm{conv}(Z_0)$ and the interior of $\mathrm{conv}(Z_0)$ is non-empty.
2: Set $Z'_0 = Z_0$ and $k = 0$;
3: Set $Z'_{k+1} = \{A_j x : A \in M, x \in Z_k \bigcap Z'_k\}$ and $Z_{k+1} = \mathrm{ver}(\mathrm{conv}(Z_k \bigcup Z'_{k+1}))$. Here, $\mathrm{ver}(\cdot)$ is the vertices of a convex set.
4: **if** $Z_{k+1} \subseteq (1+\varepsilon)\mathrm{conv}(Z_k)$ **then**
5: $\quad \rho < \log(1+\varepsilon)$;
6: $\quad$ **return**
7: **end if**
8: **if** $Z_{k+1} \bigcap Z_0 = \emptyset$ **then**
9: $\quad$ **return** $\rho > 0$;
10: **end if**
11: $k := k+1$, **goto** 3;

where $0 \notin \{x_t\}_{t=1}^{\infty}$. If $\rho < 0$, the switching system (12) is asymptotically stable for any switching signal and any initial states. If $\rho > 0$, there exists a switching signal $\delta(t)$ and a initial state x_0 satisfying $x_{t+1} \to \infty$.

Based on the above deduction, it is very important to obtain the sign of the Lyapunov exponent for system (12). Consequently, the calculation techniques of Lyapunov exponent of linear discrete inclusion can be directly use to solve the consensus of MAS.

In [78]–[80], Barabanov developed a numerical approach to estimate the sign of the Lyapunov exponent of discrete time inclusion (13), given in Algorithm 3.

B. Continuous-Time MAS Model

1) First Order Linear MAS Model

In [57], Olfati-Saber and Murray investigated the following continuous-time MAS:

$$\dot{x}_i(t) = u_i(t), \tag{15}$$

where

$$u_i(t) = \sum_{j=1}^{N} a_{ij}(t)(x_j(t) - x_i(t)). \tag{16}$$

Let $x(t) = (x_1(t), x_2(t), \cdots, x_N(t))$, and $L(t) = D(t) - A(t)$ be the Laplacian matrix of the corresponding communication topology. Eqs. (15) and (16) can be transformed into the following matrix form:

$$\dot{x}(t) = L(t)x(t). \tag{17}$$

Consider the case of switching topology, that is, $L(t) = L_{s(k)}$ for $t \in [t_k, t_{k+1})$ and $s(k) \in \{1, 2, \cdots, K\}$. If each $L \in \{L_k\}_{k=1}^{K}$ satisfies $\xi^T L = 0$ with $\xi \geq 0$ and $\xi^T \mathbf{1} = 1$, then $\xi^T \dot{x} = 0$ and $\xi^T x(t) = \xi^T x(0)$. Denote $\delta(t) = x(t) - \xi^T x(t)\mathbf{1}$ and $\Xi = \mathrm{diag}(\xi)$. Consider the following Lyapunov candidate

$$V(t) = \delta(t)^T \Xi \delta(t).$$

Consequently, the consensus of MAS (17) can be obtained.

In [8], Ren and Beard investigated the following continuous time control protocol for MAS (15), described by

$$u_i(t) = \frac{1}{\sum_{j=1}^{N} a_{ij}^k} \sum_{j=1}^{N} a_{ij}^k (x_j(t) - x_i(t)), \tag{18}$$

where $t \in [t_k, t_{k+1})$ and a_{ij}^k are the piecewise constant coupling coefficients. By integrating (15) on each interval $[t_k, t_{k+1})$, the MAS (15) and (18) can be transformed into the corresponding discrete-time MAS. Consequently, based on the theory of infinite products of stochastic matrices [23], the corresponding consensus criteria for (15) and (18) can be obtained under the condition of jointly connectivity.

2) Second Order Linear MAS Model

The corresponding second order MAS of (15) is described by

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = u_i(t), \end{cases} \tag{19}$$

where $x_i(t)$, $v_i(t)$, and $u_i(t)$ are the position, velocity and control input of agent i, respectively.

In [31], Yu et al. obtained several sufficient and necessary conditions for the consensus of MAS (19) with the following controller

$$u_i(t) = \alpha \sum_{j=1}^{N} a_{ij}(x_j - x_i) + \beta \sum_{j=1}^{N} a_{ij}(v_j - v_i), \tag{20}$$

where the topology is connected, directed, and fixed. Furthermore, Wen et al. considered the MAS (19) with controller (20) and intermittent communication, where $u_i(t)$ is given by (20) for $t \in T$, otherwise, $u_i(t) = 0$ for $t \notin T$. Here, T is a subset of $[0, +\infty)$. Construct the Lyapunov candidate $V = \xi^T Q \xi$ with

$$Q = \begin{pmatrix} \alpha\beta(\Xi L + L^T \Xi) & \alpha\Xi \\ \alpha\Xi & \beta\Xi \end{pmatrix},$$

where $\Xi = \mathrm{diag}(\eta_1, \eta_2, \cdots, \eta_N)$, $\eta = (\eta_1, \eta_2, \cdots, \eta_N)^T$ is left perron eigenvector of L, and $L = (a_{ij})_{i,j=1}^{N}$ with $a_{ii} = -\sum_{j \neq i} a_{ij}$ is the Laplacian matrix. Several consensus criteria were obtained for the above MAS under the condition of intermittent communication.

In [26], Lin and Jia investigated the second order MAS (19) with controller $u_i(t)$ given by

$$u_i(t) = -\gamma v_i(t) + \sum_{j=1}^{N} a_{ij}(t)(x_j(t-\tau) - x_i(t-\tau)).$$

Under the condition of undirected jointly connected topology, they obtained the consensus criteria of (19) by using a linear transformation similar to (6) and constructing the following common Lyapunov candidate

$$V(t) = \tilde{\gamma}\delta(t)^T\delta(t) + \int_{-\tau}^{0}\int_{t+\alpha}^{t} \dot{\delta}(t)^T\dot{\delta}(t)\,\mathrm{d}s\mathrm{d}\alpha.$$

In [42], Ren and Atkins investigated the following consensus protocol for MAS (19):

$$u_i(t) = \sum_{j=1}^{N} g_{ij}k_{ij}[(x_j - x_i) + \gamma(v_j - v_i)], \qquad (21)$$

where $k_{ij} > 0$ are fixed coupling coefficients and $g_{ij} \in \{0,1\}$ characterize the topological structure, that is, $g_{ij} = 1$ if there exists a link from agent j to agent i. Supposed that the topology is undirected and fixed for each interval $[t_i, t_{i+1})$, and $k_{ij} = k_{ji}$ for any $1 \le i \le j \le N$. Construct the following Lyapunov candidate

$$V(t) = \frac{1}{2}\sum_{i=1}^{N}\left(\sum_{j=1}^{N} V_{ij} + v_i^T v_i\right),$$

where $V_{ij} = (1/2)k_{ij}(x_i - x_j)^2$. Using a similar method in [63], we obtain the generalized time derivative of Q satisfying $\dot{Q} \subseteq [e, 0]$ for some constant $e < 0$. Since the topology is always connected, then Q is always decreasing. It follows the consensus of MAS (19) and (21).

Furthermore, Ren and Atkins also considered the case of switching topology with each topology being a directed graph. Suppose that the topology is fixed and connected for each interval $[t_k, t_{k+1})$, and the topology is switching in the set $\mathcal{P}$ of finite graphs. By integrating (19) and (21) on each interval $[t_k, t_{k+1})$, there is

$$\dot{\xi} = D_{\delta(t)}\xi,$$

where $\xi = (x(t)^T, v(t)^T)^T$. Consequently, the consensus of MAS can be attributed to the following proposition ([75]).

Proposition 1: Consider a switching system $\dot{x} = A_{\delta(t)}x$, where $\delta(t)$ is the switching signal and $\delta(t) \in \Delta$. Then the above system is globally asymptotically stable under the following conditions:

1) For each $\delta \in \Delta$, the system $\dot{x} = A_\delta x$ is globally exponentially stable, i.e, $\|e^{A_\delta t}\| \le e^{a_\delta - \chi_\delta t}$.
2) The dwell time $\tau > \sup_{\delta\in\mathcal{P}}(a_\delta/\chi_\delta)$.

For the above results in continuous time ([42]), the corresponding results for discrete-time are given in [39].

In [76], Hong et al. investigated the following consensus protocol for MAS (19):

$$u_i(t) = -\kappa v_i(t) - b_i(t)(x_i - x_0) + \sum_{j=1}^{N} a_{ij}(t)(x_j - x_i), \qquad (22)$$

where $a_{ij}(t) = 1$ iff there exists a link from agent j to agent i, otherwise, $a_{ij}(t) = 0$. Furthermore, x_0 is the state of the leader and $b_i(t) = 1$ iff there exists a link from the leader to agent i. Suppose that the topology is jointly connected and fixed on each interval $[t_k, t_{k+1})$ with $\tau \le t_{k+1} - t_k \le T$. By constructing the Lyapunov candidate $V = \xi^T P\xi$ with

$$P = \begin{pmatrix} \kappa & 1 \\ 1 & 1 \end{pmatrix} \otimes I_N,$$

they proved that the MAS (19) and (22) will reach consensus for $\kappa \ge \kappa^*$, where $\kappa^* > 0$ is a given constant. Since the topology is fixed on each subinterval, the method given in [29] can also be used to investigate the MAS (19) and (22).

IV. Concluding Remarks

This paper has introduced several representative discrete-time and continuous-time MAS models. Based on these typical models, we reviewed the main methods and techniques for the consensus of MAS. In particular, the approaches for dealing with the consensus of discrete-time MAS are very different from those of the continuous-time MAS. Generally speaking, the basic methods for dealing with MAS with switching topology include the Lyapunov function method, the infinite products method of stochastic matrices, non-smooth analysis techniques, graphical approaches, and numerical approaches. Although the above methods provide us some effective tools to cope with the consensus of MAS with dynamical topology, there still remains some open problems in the consensus of MAS with switching topology. To resolve these open problems, it appears necessary to develop some new technical tools by merging the methods of switching systems, graph theory, and non-smooth analysis.

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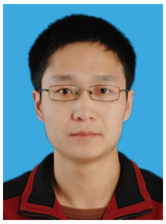

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