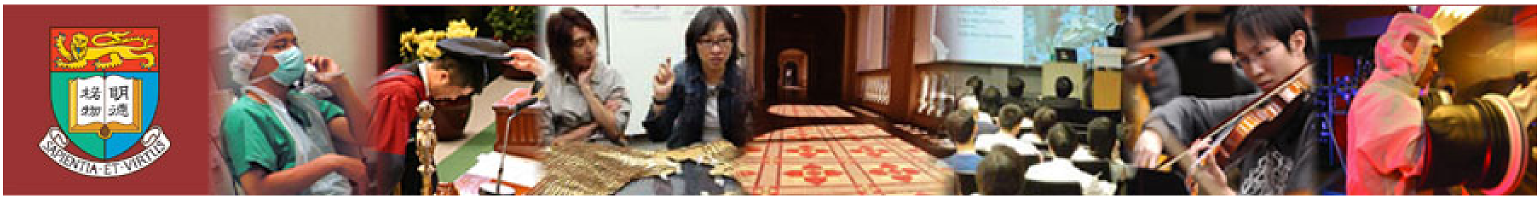

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A Nonconservative LMI Condition for Stability of Switched Systems With Guaranteed Dwell Time

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***Abstract*—Ensuring stability of switched linear systems with a guaranteed dwell time is an important problem in control systems. Several methods have been proposed in the literature to address this problem, but unfortunately they provide sufficient conditions only. This technical note proposes the use of homogeneous polynomial Lyapunov functions in the non-restrictive case where all the subsystems are Hurwitz, showing that a sufficient condition can be provided in terms of an LMI feasibility test by exploiting a key representation of polynomials. Several properties are proved for this condition, in particular that it is also necessary for a sufficiently large degree of these functions. As a result, the proposed condition provides a sequence of upper bounds of the minimum dwell time that approximate it arbitrarily well. Some examples illustrate the proposed approach.**

***Index Terms*—Dwell time, homogeneous polynomial, LMI, Lypaunov function, switched system.**

I. Introduction

An important problem in control systems consists of ensuring stability of switched linear systems under a dwell time constraint, see, e.g., [1]–[9]. Several methods have been proposed in the literature for addressing this problem, as in [10], [11] where a condition is provided on the basis of the norm of the transition matrices associated with the system matrices, and as in [12] where a condition is provided by exploiting quadratic Lyapunov functions and LMIs. Unfortunately, these methods provide conditions that are only sufficient.

This technical note addresses this problem by using homogeneous polynomial Lyapunov functions, which have been adopted in the study of uncertain systems [13]–[15], in the non-restrictive case where all the subsystems are Hurwitz. It is shown that a sufficient condition can be provided in terms of an LMI feasibility test by using a representation of polynomials in an extended space and the concept of sum of squares of polynomials (SOS). Several properties are proved for this condition, in particular that it is also necessary for a sufficiently large degree of the Lyapunov functions. As a result, the proposed condition provides a sequence of upper bounds of the minimum dwell time that approximate

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it arbitrary well. The proposed approach is illustrated by some numerical examples. A preliminary version of this technical note appeared in [16].

Before proceeding it is worth mentioning that SOS techniques have been proposed in the literature for investigating switched systems, as in [17] which addresses stability analysis of switched and hybrid systems under arbitrary switching using polynomial and piecewise polynomial Lyapunov functions. See also the recent survey [18].

The technical note is organized as follows. In Section II the problem is formulated and some preliminary results are given. Section III describes the proposed results. Section IV presents some examples that illustrate the use and benefits of the proposed approach. Lastly, Section V concludes the technical note with some final remarks. The proofs of the proposed results are reported in the Appendix.

II. Preliminaries

The notation used throughout the technical note is as follows: $\mathbb{N}, \mathbb{R}$: natural and real number sets; 0_n: origin of $\mathbb{R}^n$; $\mathbb{R}_0^n$: $\mathbb{R}^n \setminus \{0_n\}$; I_n: $n \times n$ identity matrix; A': transpose of A; $A > 0$ ($A \geq 0$): symmetric positive definite (semidefinite) matrix A; $\nabla v(x)$: first derivative row vector of the function $v(x)$.

We consider switched linear systems of the form

$$\dot{x}(t) = A_{\sigma(t)} x(t) \tag{1}$$

where $t \geq 0$, $x(t) \in \mathbb{R}^n$, and $\sigma(t)$ is a switching signal taking values in a finite set $\mathcal{S} = \{1, \ldots, M\}$ (note that the differential equation only holds almost everywhere), and $A_1, \ldots, A_M \in \mathbb{R}^{n \times n}$ are given matrices. The switching signal is assumed to belong to the set

$$D_T = \{\sigma : \mathbb{R}_+ \longrightarrow \mathcal{S} : t_{k+1} - t_k \geq T\} \tag{2}$$

where t_k are the switching instants and $T > 0$.

Problem . The problem we consider in this technical note is to establish whether (1) is exponentially stable for all $\sigma(\cdot) \in D_T$.

Solving this problem allows one to address the *minimum dwell time problem*, i.e., the computation of the minimum T ensuring exponential stability of (1) for all $\sigma(\cdot) \in D_T$. We define this time as

$$T_{\min} = \inf\{T > 0 : (1) \text{ is exponentially stable for all } \sigma(\cdot) \in D_T\}. \tag{3}$$

Let us observe that a necessary condition for (1) to be exponentially stable for all $\sigma(\cdot) \in D_T$ is that the matrices $A_1, \ldots, A_M$ are Hurwitz. Therefore, we assume without loss of generality that $A_1, \ldots, A_M$ are Hurwitz matrices.

Our starting point for addressing the considered problem is the following result that was given in [12] for guaranteeing an upper bound to the minimum dwell time.

Theorem 1 (see [12]): Assume that, for given $T > 0$

$$\exists P_i : \begin{cases} P_i > 0 \; \forall i \in \mathcal{S} \\ A_i' P_i + P_i A_i < 0 \quad \forall i \in \mathcal{S} \\ e^{A_i' T} P_j e^{A_i T} < P_i \quad \forall i, j \in \mathcal{S}, \; i \neq j. \end{cases} \tag{4}$$

Then, the system is exponentially stable for all $\sigma(\cdot) \in D_T$.

The above result deserves a few remarks.

1) For given Hurwitz matrices $A_1, \ldots, A_M$, there always exist $T > 0$ such that (4) holds.
2) The function $v(x,t) = x' P_{\sigma(t)} x$ is a piecewise quadratic Lyapunov function for (1) for all $\sigma(\cdot) \in D_T$.

The sufficient condition stated in Theorem 1 is not necessary for stability in D_T. This means that a system can be stable in D_T and no positive definite matrices P_i exist satisfying (4). The reason is that the inequalities define a Lyapunov function $v(x,t) = x' P_{\sigma(t)} x$, which is piecewise quadratic, whereas for stability in D_T, more complex Lyapunov functions are required. This latter observation is characterized by the following result which can be found in [19].

Theorem 2 (see [19]): The system is exponentially stable in D_T if and only if there exist continuous functions $v_i(x)$ such that

$$\begin{cases} v_i(x) > 0 \; \forall x \in \mathbb{R}_0^n \; \forall i \in \mathcal{S} \\ \nabla v_i(x) A_i x < 0 \; \forall x \in \mathbb{R}_0^n \; \forall i \in \mathcal{S} \\ v_j(e^{A_i T} x) < v_i(x) \; \forall x \in \mathbb{R}_0^n \; \forall i, j \in \mathcal{S}, \; i \neq j. \end{cases} \tag{5}$$

III. Proposed Results

The idea exploited in this technical note is to adopt homogeneous polynomials. Any homogeneous polynomial $h(x)$ of degree $2m$ in $x \in \mathbb{R}^n$ can be expressed as

$$h(x) = x^{\{m\}'} \left(H + L_m(\alpha)\right) x^{\{m\}} \tag{6}$$

where $x^{\{m\}} \in \mathbb{R}^{d(n,m)}$ contains all monomials of degree m in x, where

$$d(n,m) = \frac{(n+m-1)!}{(n-1)!m!} \tag{7}$$

$H \in \mathbb{R}^{d(n,m) \times d(n,m)}$ is a symmetric matrix, $L_m(\alpha)$ is a linear parametrization of the linear subspace

$$\mathcal{L}_m = \left\{ L = L' : x^{\{m\}'} L x^{\{m\}} = 0 \; \forall x \in \mathbb{R}^n \right\} \tag{8}$$

and $\alpha \in \mathbb{R}^{d_{par}(n,m)}$ is a free vector, where

$$d_{par}(n,m) = \frac{1}{2} d(n,m) \left(d(n,m) + 1\right) - d(n, 2m). \tag{9}$$

The representation (1) is known as square matrix representation (SMR) and Gram matrix method, and allows one to establish whether a polynomial is SOS via an LMI feasibility test, specifically $h(x)$ is SOS if and only if there exists α such that $H + L_m(\alpha) \geq 0.$, See e.g., [20] and references therein for details on SOS polynomials, and [21] for details on LMIs.

Now, there exists $\mathcal{A}_{i,m} \in \mathbb{R}^{d(n,m) \times d(n,m)}$ satisfying

$$\frac{dx^{\{m\}}}{dx} A_i x = \mathcal{A}_{i,m} x^{\{m\}} \; \forall x \in \mathbb{R}^n. \tag{10}$$

This matrix, called extended matrix of A_i with respect to $x^{\{m\}}$, can be computed as [15]

$$\mathcal{A}_{i,m} = (K_0' K_0)^{-1} K_0' \left(\sum_{i=0}^{m-1} I_{n^{m-1-i}} \otimes A \otimes I_{n^i} \right) K_0 \tag{11}$$

where K_0 is the matrix satisfying

$$x^{\otimes m} = K_0 x^{\{m\}} \tag{12}$$

and $x^{\otimes m}$ denotes the m-th Kronecker power of x. The following theorem provides a condition for guaranteeing exponential stability of (1) under a dwell time constraint based on homogeneous polynomial Lyapunov functions.

TABLE I
TOTAL NUMBER OF LMI VARIABLES IN (13) FOR $M = 2$ (LEFT) AND $M = 3$ (RIGHT)

$n \setminus m$	1	2	3	4
2	6	15	29	48
3	12	60	191	465
4	20	170	798	2655

$n \setminus m$	1	2	3	4
2	9	24	48	81
3	18	99	327	810
4	30	285	1386	4680

Theorem 3: Assume that, for some $T \in \mathbb{R}$ and $m \in \mathbb{N}$ with $T > 0$ and $m > 0$, the following condition holds:

$$\exists \Pi_i, \alpha_i, \alpha_{i,j}: \begin{cases} \Pi_i > 0 \; \forall i \in \mathcal{S} \\ \mathcal{A}_{i,m}' \Pi_i + \Pi_i \mathcal{A}_{i,m} + L_m(\alpha_i) < 0 \; \forall i \in \mathcal{S} \\ e^{\mathcal{A}_{i,m}' T} \Pi_j e^{\mathcal{A}_{i,m} T} < \Pi_i + L_m(\alpha_{i,j}) \; \forall i,j \in \mathcal{S}, \; i \neq j. \end{cases} \tag{13}$$

Then, (1) is exponentially stable for all $\sigma(\cdot) \in D_T$.

Theorem 3 states that one can establish whether there exists a homogeneous polynomial Lyapunov functions ensuring that (1) is exponentially stable for all $\sigma(\cdot) \in D_T$ through an LMI feasibility test.

Let us observe that (13) coincides with (4) for $m = 1$, i.e., in the case where the homogeneous polynomial Lyapunov functions are quadratic. Let us also observe that, as T tends to zero, the matrices Π_i tend to a common matrix Π, and (13) tends to the condition provided in [15] for robust stability of time-varying polytopic systems based on a common homogeneous polynomial Lyapunov function.

Table I shows the total number of LMI variables in (13) for some values of n, m and M.

The following result provides a monotonicity property of the condition (13) with respect to T.

Theorem 4: Assume that (13) holds for some $T = \bar{T}$ and m. Then, (13) holds also for $T = \bar{T} + \tau$ and m for all $\tau \in \mathbb{R}$, $\tau \geq 0$.

The following result provides a monotonicity property of the condition (13) with respect to m.

Theorem 5: Assume that (13) holds for some T and $m = \bar{m}$. Then, (13) also holds for T and $m = k\bar{m}$ for all $k \in \mathbb{N}$, $k \geq 1$. We now give an important result that states that the condition by Theorem 3 s not only sufficient but also necessary for some m.

Theorem 6: The system (1) is exponentially stable in D_T if and only if there exists m such that (13) holds.

Theorem 6 states that, whenever (1) is exponentially stable in D_T, there exists a homogeneous polynomial Lyapunov function of bounded degree that can be found by solving the LMI condition (13). Let us observe that this result does not contradict the result given in [22] which states that the degree of a polynomial Lyapunov function is not uniformly bounded over the class of asymptotically stable switched linear system.

Let us indicate with T_m the smallest upper bound of T_{min} guaranteed by Theorem 3 for a given m, i.e.

$$T_m = \inf \{T \in \mathbb{R}, \; T > 0 : (13) \text{ holds}\}. \tag{14}$$

Due to Theorem 6, one has that the minimum dwell time $T_{\min}$ can be approximated arbitrary well by the upper bound T_m, i.e.

$$\forall \varepsilon \in \mathbb{R}, \; \varepsilon > 0, \; \exists m \in \mathbb{N}, \; m \geq 1 : T_m - T_{min} < \varepsilon. \tag{15}$$

Moreover, due to Theorem 4, one can calculate T_m via a bisection search where at each iteration the condition (13) is tested. This search is conducted in an interval $[0, \hat{T}]$ where $\hat{T}$ is such that (13) holds for $T = \hat{T}$. Observe that such $\hat{T}$ is guaranteed to exist for all $m \geq 1$ since it exists for $m = 1$[12] and due to Theorem 5.

IV. EXAMPLES

In this section, examples are presented to illustrate the usefulness of the proposed approach. The upper bound T_m provided by condition (13) is computed by using Matlab and SeDuMi [23] on a personal computer with Windows XP, Pentium 4 3.20 GHz, 2 GB RAM. Each example shows the total number of LMI variables in the condition (3) and the average computational time (ACT) required for testing this condition in the bisection search used to find T_m. The matrix function $L_m(\alpha)$ is computed with the algorithm reported in [24] and available in the Matalb toolbox SMRSOFT [25].

For comparison we consider the upper bound provided in the pioneering paper [10], i.e.

$$T_{HM} = \max_i \inf_{\alpha>0,\beta>0} \left\{ \frac{\alpha}{\beta} : \left\| e^{A_i t} \right\| \leq e^{\alpha - \beta t}, \; \forall t > 0 \right\}.$$

In addition, we consider

$$T_{LB} = \inf \left\{ T > 0 \text{ s.t. } \max_q \left| \lambda_q \left(\prod_{p=1}^{M} e^{B_p \tau} \right) \right| < 1, \forall \tau > T \right\}$$

where λ_q denotes a generic eigenvalue and $\{B_1, B_2, \ldots, B_M\}$ are matrices corresponding to any permutation among those of the set $\{A_1, A_2, \ldots, A_M\}$. Of course $T_{LB} \leq T_{min}$, i.e., T_{LB} is a lower bound of the minimum dwell time.

A. Example 1

Consider for $n = 2$ and $M = 2$ the matrices

$$A_1 = \begin{pmatrix} 0 & 1 \\ -10 & -1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 1 \\ -0.1 & -4 \end{pmatrix}.$$

We have $T_{HM} = 2.4132$ and $T_{LB} = 0$. By using the proposed approach, we get the following upper bounds:

m	T_m	# LMI variables	ACT [s]
1	0.1700	6	0.1297
2	0.0451	15	0.1669
3	0.0000	29	0.1588
4	0.0000	48	0.2068

Since $T_3 = 0$, it clearly follows that $T_{\min} = T_3$ as T_3 is an upper bound of $T_{\min}$ and $T_{\min}$ is nonnegative.

In order to illustrate more clearly the use of the condition provided in Theorem 3, we report hereafter the matrices $\mathcal{A}_{i,m}$ and $L_m(\alpha)$ involved in (13) for the case $m = 2$ (homogeneous polynomial Lyapunov function of degree 4):

$$\mathcal{A}_{1,m} = \begin{pmatrix} 0 & 2 & 0 \\ -10 & -1 & 1 \\ 0 & -20 & -2 \end{pmatrix},$$

$$\mathcal{A}_{2,m} = \begin{pmatrix} 0 & 2 & 0 \\ -0.1 & -4 & 1 \\ 0 & -0.2 & -8 \end{pmatrix}$$

$$L_m(\alpha) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

B. Example 2

Here we consider an example with $n = 2$ and $M = 3$, specifically

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},\ A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix}.$$

We have $T_{HM} = 2.3024$ and $T_{LB} = 0.8006$. With the proposed approach we get the following upper bounds:

m	T_m	# LMI variables	ACT [s]
1	1.2989	6	0.1138
2	1.2459	20	0.2083
3	1.2427	72	0.2882
4	1.2427	272	0.4860

The minimum dwell time $T_{\min}$ coincides with T_3. This is confirmed by the fact that, taking the periodic signal of period $T = t_2 + t_3$ with $t_2 = t_3 = 1.2427$ as

$$\sigma(t) = \begin{cases} 1,\ t \in [kT, kT + t_2),\ k \in \mathbb{N} \\ 2,\ t \in [kT + t_2, (k+1)T),\ k \in \mathbb{N} \end{cases}$$

the associated periodic system $\dot{x}(t) = A_{\sigma(t)}x(t)$ is not asymptotically stable (the maximum modulus of the characteristic multipliers is equal to 1). Hence, $T_{min} = T_3 = 1.2427$.

C. Example 3

Next, we consider for $n = 3$ and $M = 3$ the matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & -4 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2 & -16 \end{pmatrix}$$
$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -2 & -9 \end{pmatrix}.$$

We have $T_{HM} = 18.5554$ and $T_{LB} = 5.0677$. With the proposed approach we get the following upper bounds:

m	T_m	# LMI variables	ACT [s]
1	11.7934	18	0.1441
2	11.2439	99	0.5649
3	11.2439	327	6.1792
4	11.2439	1386	69.9121

The minimum dwell time $T_{\min}$ coincides with T_2. This is confirmed by the fact that, taking the periodic signal of period $T = t_1 + t_2$ with $t_1 = 11.2439$ and $t_2 = 12.2849$ as

$$\sigma(t) = \begin{cases} 1,\ t \in [kT, kT + t_1),\ k \in \mathbb{N} \\ 2,\ t \in [kT + t_1, (k+1)T),\ k \in \mathbb{N} \end{cases}.$$

the associated periodic system $\dot{x}(t) = A_{\sigma(t)}x(t)$ is not asymptotically stable (the maximum modulus of the characteristic multipliers is equal to 1). Hence, $T_{\min} = T_2 = 11.2439$.

D. Example 4

Lastly, we consider for $n = 2$ and $M = 2$ the example in [26] given by

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},\ A_2 = \begin{pmatrix} -1 & -a \\ \frac{1}{a} & -1 \end{pmatrix}$$

where a is a parameter. The problem consists of determining the value a^* for which the system is exponentially stable for all switching signals in D_0 (arbitrary switching without dwell time) for all $a \in [1, a^*]$.

In [26] it has been shown analytically that $a^* \geq 10$, and that the set of values of a for which there exists a quadratic Lyapunov function is included in the interval $[1, 6]$.

The quantity a^* can be estimated with the proposed approach, which in the case $T = 0$ coincides with the condition provided in [15] for robust stability of time-varying polytopic systems based on a common homogeneous polynomial Lyapunov function. Specifically, a lower bound of a^* can be found through a bisection search where stability over $[1, a]$ is established by using Theorem 3. Let us denote this lower bound as a_m^*. The results obtained are as follows:

m	a_m^*
1	5.8283
2	9.2911
3	9.6825
4	10.4105

V. Conclusion

This technical note has proposed for the first time in the literature a nonconservative LMI condition for ensuring stability of switched linear systems with a guaranteed dwell time. This condition has been derived by exploiting homogeneous polynomial Lyapunov functions and a representation of polynomials in an extended space. As a result, the proposed condition provides a sequence of upper bounds of the minimum dwell time that approximate it arbitrarily well. Future work will investigate the possibility of determining upper bounds of the degree of the homogeneous polynomial Lyapunov functions required to prove stability.

Appendix

Proof of Theorem 3: Suppose that (13) holds, and define

$$v_i(x) = x^{\{m\}'}\Pi_i x^{\{m\}}.$$

The first LMI in (13) clearly implies that $v_i(x)$ is positive definite. Then, we have that

$$\nabla v_i(x)A_i x = x^{\{m\}'}\left(\mathcal{A}_{i,m}'\Pi_i + \Pi_i\mathcal{A}_{i,m} + L_m(\alpha_i)\right)x^{\{m\}}$$

and hence the second LMI in (13) implies that $\nabla v_i(x)A_i x$ is negative definite. Lastly, it turns out that

$$\left(e^{A_i t}x\right)^{\{m\}} = e^{\mathcal{A}_{i,m}t}x^{\{m\}}$$

which implies that

$$v_j(e^{A_i T}x) - v_i(x) = x^{\{m\}'} \times (e^{\mathcal{A}_{i,m}' T}\Pi_j e^{\mathcal{A}_{i,m}T} - \Pi_i - L_m(\alpha_{i,j}))x^{\{m\}}$$

and hence $v_j(e^{A_i T}x) - v_i(x)$ is negative definite. Therefore, (1) is exponentially stable for all $\sigma(\cdot) \in D_T$ since (5) holds.

Proof of Theorem 4: Suppose that (13) holds, and define $v_i(x) = x^{\{m\}'}\Pi_i x^{\{m\}}$. Consider any $\tau \geq 0$. From the second inequality one has that

$$v_i(x(\tau)) \leq v_i(x(0))$$

which implies that

$$e^{\mathcal{A}'_{i,m}\tau}\Pi_i e^{\mathcal{A}_{i,m}\tau} \leq \Pi_i.$$

Pre- and post-multiplying the third inequality of (13) by $e^{\mathcal{A}_{i,m}\tau}$ and $e^{\mathcal{A}'_{i,m}\tau}$, respectively, one gets that

$$e^{\mathcal{A}'_{i,m}(T+\tau)}\Pi_j e^{\mathcal{A}_{i,m}(T+\tau)} < \Pi_i + e^{\mathcal{A}'_{i,m}\tau}L_m(\alpha_{i,j})e^{\mathcal{A}_{i,m}\tau}.$$

Lastly, let us observe that

$$\begin{aligned} &x^{\{m\}'}e^{\mathcal{A}'_{i,m}\tau}L_m(\alpha_{i,j})e^{\mathcal{A}_{i,m}\tau}x^{\{m\}} \\ &= \left(e^{A_i\tau}x\right)^{\{m\}'}L_m(\alpha_{i,j})\left(e^{A_i\tau}x\right)^{\{m\}} \\ &= 0 \end{aligned}$$

which implies that

$$e^{\mathcal{A}'_{i,m}\tau}L_m(\alpha_{i,j})e^{\mathcal{A}_{i,m}\tau} \in \mathcal{L}_m$$

or, in other words

$$\exists\tilde{\alpha}_{i,j} : L_m(\tilde{\alpha}_{i,j}) = e^{\mathcal{A}'_{i,m}\tau}L_m(\alpha_{i,j})e^{\mathcal{A}_{i,m}\tau}.$$

Proof of Theorem 5: Let $\Pi_i, \alpha_i, \alpha_{i,j}$ be such that (13) holds for $T =$ and $m = \bar{m}$, and consider any $k \in \mathbb{N}$, $k \geq 1$. We now show that there exist $\tilde{\Pi}_i, \tilde{\alpha}_i, \tilde{\alpha}_{i,j}$ such that (13) holds for T and $m = k\bar{m}$.

Define $v_i(x) = x^{\{\bar{m}\}'}\Pi_i x^{\{\bar{m}\}}$. We have that (5) holds with these functions. Define also the homogeneous polynomial of degree $2k\bar{m}$

$$\tilde{v}_i(x) = v_i^k(x).$$

Clearly, (5) holds with $v_i(x), v_j(x)$ replaced by $\tilde{v}_i(x), \tilde{v}_j(x)$, respectively. Now, let us define

$$\tilde{\Pi}_i = K_1'\Pi_i{}^{\otimes k}K_1$$

where K_1 is the (full-column rank) matrix satisfying

$$x^{\{\bar{m}\}^{\otimes k}} = K_1 x^{\{k\bar{m}\}}.$$

We have that

$$\tilde{v}_i(x) = x^{\{k\bar{m}\}'}\tilde{\Pi}_i x^{\{k\bar{m}\}},\ \tilde{\Pi}_i > 0.$$

Then, let us define

$$\Phi_i = K_2'\left(\Pi_i{}^{\otimes k-1} \otimes \left(\mathcal{A}'_{i,\bar{m}}\Pi_i + \Pi_i\mathcal{A}_{i,\bar{m}} + L_{\bar{m}}(\alpha_i)\right)\right)K_2$$

where K_2 is the (full-column rank) matrix satisfying

$$\left(x^{\{\bar{m}\}^{\otimes k-1}} \otimes x^{\{\bar{m}\}}\right) = K_2 x^{\{k\bar{m}\}}.$$

We have that

$$\nabla\tilde{v}_i(x)A_i x = x^{\{k\bar{m}\}'}\Phi_i x^{\{k\bar{m}\}},\ \Phi_i < 0$$

and

$$\exists\tilde{\alpha}_i : \mathcal{A}'_{i,k\bar{m}}\tilde{\Pi}_i + \tilde{\Pi}_i\mathcal{A}_{i,k\bar{m}} + L_{k\bar{m}}(\tilde{\alpha}_i) = 7\,\Phi_i$$

because $\mathcal{A}'_{i,k\bar{m}}\tilde{\Pi}_i + \tilde{\Pi}_i\mathcal{A}_{i,k\bar{m}}$ and Φ_i are SMR matrices of the same homogeneous polynomials. Lastly, let us define

$$\Phi_{i,j} = K_1'\left(\left(e^{A_i'T}\Pi_j e^{A_iT}\right)^{\otimes k} - \Pi_i{}^{\otimes k}\right)K_1.$$

It can be verified that

$$\tilde{v}_j(e^{A_iT}x) - \tilde{v}_i(x) = x^{\{k\bar{m}\}'}\Phi_{i,j}x^{\{k\bar{m}\}},\ \Phi_{i,j} < 0$$

and

$$\exists\tilde{\alpha}_{i,j} : e^{\mathcal{A}'_{i,k\bar{m}}T}\tilde{\Pi}_j e^{\mathcal{A}_{i,k\bar{m}}T} - \tilde{\Pi}_i - L_{k\bar{m}}(\alpha_{i,j}) = \Phi_{i,j}$$

because $e^{\mathcal{A}'_{i,k\bar{m}}T}\tilde{\Pi}_j e^{\mathcal{A}_{i,k\bar{m}}T} - \tilde{\Pi}_i$ and $\Phi_{i,j}$ are SMR matrices of the same homogeneous polynomial.

Proof of Theorem 6: Assume that (1) is exponentially stable for all $\sigma(\cdot) \in D_T$. From Theorem 2 one has that there exist functions $v_i(x)$ satisfying (5). As shown in [27], these functions can be chosen of the form $\|M_i x\|_\infty$ for some matrix M_i, moreover as discussed in [28] $\|M_i x\|_{2p}$ converges uniformly on as $p \to \infty$ and the $v_i(x)$ can be chosen of the form $\|M_i x\|_{2p}$ for some finite p. Then, observe that if $v_i(x)$ satisfies (5) then also $v_i(x)^a$ satisfies (5) for all $a \in \mathbb{R}$, $a \geq 1$. Therefore, this implies that the $v_i(x)$ can be chosen of the form $\|M_i x\|_{2p}^{2p}$, which are homogeneous, positive definite, and sum of squares of polynomials (SOS).

Define

$$\begin{aligned} u_i(x) &= -\nabla v_i(x)A_i x \\ w_{i,j}(x) &= v_i(x) - v_j\left(e^{A_iT}x\right). \end{aligned}$$

The functions $u_i(x)$ and $w_{i,j}(x)$ are homogeneous polynomials since the $v_i(x)$ are homogeneous polynomials, and positive definite since (5) holds.

Now, suppose that each $v_i(x)$ is replaced by $\bar{v}_i(x) = v_i(x)^k$ with $k \in \mathbb{N}$, $k \geq 1$. We have that $u_i(x)$ and $w_{i,j}(x)$ become

$$\begin{aligned} \bar{u}_i(x) &= kv_i(x)^{k-1}u_i(x) \\ \bar{w}_{i,j}(x) &= v_i(x)^k - v_j\left(e^{A_iT}x\right)^k. \end{aligned}$$

Also

$$\bar{w}_{i,j}(x) = z_{i,j}(x)w_{i,j}(x)$$

where

$$z_{i,j}(x) = \sum_{l=0}^{k-1} v_i(x)^l v_j\left(e^{A_iT}x\right)^{k-1-l}.$$

For $\varepsilon \in \mathbb{R}$ define

$$\begin{aligned} \hat{u}_i(x) &= kv_i(x)^{k-1}(u_i(x) - \varepsilon\|x\|^{2p}) \\ \hat{w}_{i,j}(x) &= z_{i,j}(x)(w_{i,j}(x) - \varepsilon\|x\|^{2p}). \end{aligned}$$

Since $u_i(x)$ and $w_{i,j}(x)$ are positive definite, it follows that there exists $\varepsilon > 0$ such that $u_i(x) - \varepsilon\|x\|^{2p}$ and $w_{i,j}(x) - \varepsilon\|x\|^{2p}$ are positive definite, which implies that $\hat{u}_i(x)$ and $\hat{w}_{i,j}(x)$ are positive definite.

Let us observe that, since $v_i(x)$ is positive definite and SOS, then $\bar{v}_i(x)$ is positive definite and SOS for all $k \geq 1$. Also, since $u_i(x) - \varepsilon\|x\|^{2p}$ is positive definite, from [29] it follows that there exists a sufficiently large k (denoted as k_1) such that $\hat{u}_i(x)$ is positive definite and SOS. Similarly, one has that $(w_{i,j}(x) - \varepsilon\|x\|^{2p})v_i(x)^l v_j\left(e^{A_iT}x\right)^{k-1-l}$ is positive definite and SOS for all $l \in [0, k-1]$ for a sufficiently large k (denoted as k_2), which implies that $\hat{w}_{i,j}(x)$ is positive definite and SOS for all $k \geq k_2$.

Summarizing, one has that $\bar{v}_i(x)$, $\hat{u}_i(x)$ and $\hat{w}_{i,j}(x)$ are positive definite and SOS for all $k \geq k_3$, where $k_3 = \max\{k_1, k_2\}$, for some $\varepsilon > 0$.

Then, let us observe that any homogeneous polynomial $h(x)$ that is SOS can be expressed as in (6) with a positive semidefinite matrix H, see e.g., [20] and references therein. This means that one can write $\bar{v}_i(x) = x^{\{q\}'}\bar{V}_i x^{\{q\}}$ where $q = p + k_3$ and $\bar{V}_i \geq 0$. Since $\bar{v}_i(x) = \|M_i x\|_{2p}^{2q}$, it is not difficult to see that $\bar{V}_i$ can be chosen not only positive semidefinite but also positive definite (just observe that M_i must have full column rank). Similarly, one can write $\hat{u}_i(x) = x^{\{q\}'}\hat{U}_i x^{\{q\}}$ and $\hat{w}_{i,j}(x) = x^{\{q\}'}\hat{W}_{i,j} x^{\{q\}}$ where $\hat{U}_i \geq 0$ and $\hat{W}_{i,j} \geq 0$. Moreover, one can write $\bar{u}_i(x) = x^{\{q\}'}\bar{U}_i x^{\{q\}}$ and $\bar{w}_{i,j}(x) = x^{\{q\}'}\bar{W}_{i,j} x^{\{q\}}$. Since $\hat{u}_i(x) - \bar{u}_i(x) = -kv_i(x)^{k-1}\varepsilon\|x\|^{2p}$ and $\hat{w}_{i,j}(x) - \bar{w}_{i,j}(x) = -z_{i,j}(x)\varepsilon\|x\|^{2p}$, it is not difficult to see that $\bar{U}_i$ and $\bar{W}_{i,j}$ can be chosen positive definite (just observe that $\bar{U}_i = \hat{U}_i + \tilde{U}_i$ and $\bar{W}_{i,j} = \hat{W}_{i,j} + \tilde{W}_{i,j}$ with $\tilde{U}_i > 0$ and $\tilde{W}_{i,j} > 0$).

Lastly, since Π_i, $-\mathcal{A}'_{i,q}\Pi_i - \Pi_i\mathcal{A}_{i,q} - L_q(\alpha_i)$ and $\Pi_i + L_m(\alpha_{i,j}) - e^{\mathcal{A}'_{i,q}T}\Pi_j e^{\mathcal{A}_{i,q}T}$ are SMR matrices of $\bar{v}_i(x)$, $\bar{u}_i(x)$ and $\bar{w}_{i,j}(x)$, and since the parametrization of the SMR matrices in (13) is complete, it follows that there exist Π_i, α_i and $\alpha_{i,j}$ such that (13) holds.

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