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Slip Flow due to a Stretching Cylinder

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Abstract The slip flow due to a stretching cylinder is studied. A similarity transform reduces the Navier–Stokes equations to a set of nonlinear ordinary differential equations. Asymptotic solutions for large Reynolds number and small slip show the problem can be related to the existing two-dimensional stretching cases. Due to algebraic decay, the equations are further transformed through a compressed variable, and then integrated numerically. It is found that slip greatly reduces the magnitudes of the velocities and the shear stress.

Keywords: Slip flow; Stretching Cylinder; Similarity

1. Introduction

The viscous flow due to a stretching boundary occurs in the extrusion of metal, plastic and food products. The stretching causes the entrainment of the adjacent fluid, which in turn affects the resistance and the solidification of the extruded material. If the extrusion velocity is linear with respect to distance [1], which usually occurs with material subjected to constant strain, similarity solutions of the fluid problem may exist [2]. The two-dimensional stretching of a flat sheet was solved by Crane [3] and the axisymmetric radial stretching of a surface by Wang [4]. Brady and Acrivos [5] considered the flow inside a stretching channel or tube, while Wang [6] studied the flow outside a stretching cylinder. The above sources, and their many extensions, applied the no-slip boundary condition between the fluid and the extrusate.

In certain cases, the no-slip condition does not hold and should be replaced by a partial slip condition. These occur when the fluid is a rarefied gas [7], or when it is particulate such as blood, foam, emulsion or suspension [8]. Slip also occurs on hydrophobic surfaces, especially in micro- and nano-fluidics [9].

Two-dimensional stretching surface with partial slip was studied by Andersson [10] and Wang [11], and the axisymmetric case by Ariel [12]. It was found that slip affects the velocities and the fluid resistances considerably. The purpose of the present paper is to investigate the axisymmetric similarity solution due to a stretching of a cylinder with a partial slip boundary. The results are relevant to the extrusion of filaments which are micron size and/or in a rarefied environment. It is also an exact similarity solution of the Navier–Stokes equations.

2. Formulation

Fig. 1 shows a cylinder of radius a being stretched longitudinally with a surface velocity of $Ws = 2bz$ where b is a constant and z is the axial direction. In addition, there is a longitudinal free stream velocity of W at infinity. We use a transformation similar to that of Wang [6]

$$u = -ab\frac{f(\eta)}{\sqrt{\eta}}, \quad w = 2bzf'(\eta) + Wg(\eta), \quad \eta = \left(\frac{r}{a}\right)^2 \tag{1}$$

Here (u, w) are velocities in the (r, z) directions respectively and continuity is satisfied. Since there is no longitudinal pressure gradient, using Eq. (1), the Navier–Stokes equations reduce to the nonlinear ordinary differential equations

$$\eta f'''(\eta) + f'' = R[(f')^2 - ff''] \tag{2}$$

$$\eta g''(\eta) + g' = R(gf' - fg') \tag{3}$$

where $R = ba^2/2v$ (v being the kinematic viscosity) is the Reynolds number.

On the cylinder surface, the partial slip condition [13] states that the slip velocity u_s is proportional to the local shear stress τ

$$u_s = N\tau \tag{4}$$

where N is the slip coefficient. Eqs. (1, 4) then yield

$$f'(1) - 1 = \lambda f''(1) \tag{5}$$

$$g(1) = \lambda g'(1) \tag{6}$$

Here $\lambda = 2N\rho\nu/a$ (ρ being the fluid density) is the normalized slip factor. Zero radial velocity on the surface requires

$$f(1) = 0 \tag{7}$$

And at infinity we have uniform flow W

$$f'(\infty) = 0 \tag{8}$$

$$g(\infty) = 1 \tag{9}$$

For given R and λ, Eqs. (2, 3, 5–9) are to be solved. Eqs. (2, 5) reduces to the no slip case of [6] only when $\lambda = 0$. Eqs. (3, 6) representing axial convection are entirely new.

3. Asymptotic solution for large R and small slip

At large R a boundary layer exists on the cylinder. The cylinder curvature is thus unimportant and the flow can be related to that on a two-dimensional stretching flat surface. Let $R \gg 1$ and $\lambda = \Lambda/\sqrt{R} \ll 1$. We expand as follows

$$f = \frac{1}{\sqrt{R}}h(\xi) + O(R^{-1}), \qquad g = k(\xi) + O(R^{-1/2}) \tag{10}$$

where

$$\xi = \sqrt{R}(\eta - 1) \tag{11}$$

is the boundary layer coordinate. Eqs. (2, 5, 7, 8) give

$$(h')^2 - hh'' = h''' \tag{12}$$

$$h'(0) - 1 = \Lambda h''(0), \quad h(0) = 0, \quad h'(\infty) = 0 \tag{13}$$

Eqs. (12, 13) represent the slip flow over a two dimensional stretching sheet. The solution was given by Andersson [10]

$$h = 1 - ce^{-\xi}, \quad c = 1/(1 + \Lambda) \tag{14}$$

Eqs. (3, 6, 9) yield

$$k'' + (1 - ce^{-\xi})k' + ce^{-\xi}k = 0, \quad k(0) = \Lambda k'(0), \quad k(\infty) = 1 \tag{15}$$

To solve Eq. (15), let

$$K(s) = k(\xi), \qquad s = e^{-\xi} \tag{16}$$

Eq. (15) then becomes

$$sK'' + csK' - cK = 0 \tag{17}$$

with the boundary conditions

$$K(0) = 1, \quad K(1) = -\Lambda K'(1) \tag{18}$$

The general solution to Eq. (17) is

$$K = c_1 s + c_2 s \int \frac{e^{-cs}}{s^2} ds = c_1 s - c_2 \left[e^{-cs} - csE(cs) \right] \tag{19}$$

where

$$E(z) = \int_z^\infty \frac{e^{-z}}{z} dz \tag{20}$$

is the exponential integral. The boundary conditions give

$$c_2 = 1, \quad c_1 = -e^{-c} + E(c) \tag{21}$$

Thus

$$k = \exp\left(-ce^{-\xi}\right) + e^{-\xi} \left[E(c) - cE(ce^{-\xi}) - e^{-c} \right] \tag{22}$$

Note that for no slip, $\Lambda = 0, c = 1$ and Eq. (22) reduces to the solution of the no slip flow over a stretching sheet found by Danberg and Fansler [14].

4. Algebraic decay at large distances

For large η, from Eqs. (8, 9) we can set

$$f \sim c + \phi(\eta), \quad g \sim 1 + \psi(\eta) \tag{23}$$

where $c = f(\infty)$ is a constant and ϕ, ψ decay to zero as $\eta \to \infty$. For large enough η, Eqs. (2, 3) linearize to

$$\eta\phi''' + (1 + Rc)\phi'' = 0 \tag{24}$$

$$\eta\psi'' + (1 + Rc)\psi' = R\phi' \tag{25}$$

The decay solutions to Eqs. (24, 25) are

$$\phi = c_1\eta^{1-Rc}, \quad \psi = c_2\eta^{-Rc} + c_1R\eta^{1-Rc} \tag{26}$$

where c_1, c_2 are constants. We see that the functions decay algebraically (instead of exponentially as in stretching flat surfaces). Also, a necessary condition for existence of a solution is $c > 1/R$.

5. Numerical method

Due to the algebraic decay, the domain is too large for direct numerical integration of Eqs. (2, 3). We greatly compress the domain by an exponential transform

$$\eta = e^x \tag{27}$$

After some work, Eqs. (2, 3) become

$$f_{xxx} - 2f_{xx} + f_x + R(ff_{xx} - ff_x - f_x^2) = 0 \tag{28}$$

$$g_{xx} + R(fg_x - f_xg) = 0 \tag{29}$$

The boundary conditions (in x) are

$$f(0) = 0, \quad (1 + \lambda)f_x(0) = 1 + \lambda f_{xx}(0) \tag{30}$$

$$f_x(\infty) = 0 \tag{31}$$

$$g(0) = \lambda g_x(0) \tag{32}$$

$$g(\infty) = 1 \tag{33}$$

Given R and λ, we guess $f_{xx}(0)$ and from Eq. (30) obtain $f(0)$ and $f_x(0)$. Then integrate Eq. (28) as an initial value problem by a standard Runge–Kutta algorithm. At large η (about 15) we check whether f_x decays to zero. Using shooting, the initial values can be accurately obtained. After f is determined, Eq. (29) is similarly integrated with a guessed $g_x(0)$ and $g(0)$ from Eq. (32). A solution is obtained when g-1 decays to zero at large enough x. Table 1 shows the results.

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Table 1(a) Initial and final values for $R = 0.1$

λ	$f_{xx}(0)$	$f_x(0) = f'(1)$	$f''(1)$	$c = f(\infty)$	$g_x(0) = g'(1)$
0	0.518195	1	-0.4818	15.061	0.1030
0.1	0.501976	0.9547	-0.4528	15.037	0.0965
0.5	0.449207	0.8164	-0.3672	14.984	0.0773
1	0.401035	0.7005	-0.2995	14.937	0.0621
2	0.336058	0.5574	-0.2213	14.879	0.0448
5	0.237711	0.3648	-0.1271	14.815	0.0250
10	0.168397	0.2440	-0.0756	14.769	0.0145

Table 1(b) Initial and final values for $R = 1$

λ	$f_{xx}(0)$	$f_x(0) = f'(1)$	$f''(1)$	$c = f(\infty)$	$g_x(0) = g'(1)$
0	-0.177752	1	-1.1778	1.8786	0.3379
0.1	-0.112773	0.8988	-1.0116	1.8401	0.2863
0.5	0.0042342	0.6681	-0.6638	1.7463	0.1815
1	0.0522636	0.5261	-0.4739	1.6862	0.1256
2	0.0790586	0.3860	-0.3070	1.6280	0.0777
5	0.0807401	0.2340	-0.1532	1.5585	0.0365
10	0.0671050	0.1519	-0.0848	1.5278	0.0190

Table 1(c) Initial and final values for $R = 10$

λ	$f_{xx}(0)$	$f_x(0) = f'(1)$	$f''(1)$	$c = f(\infty)$	$g_x(0) = g'(1)$
0	-2.344457	1	-3.3445	0.3858	1.1402
0.1	-1.512162	0.7716	-2.2838	0.3486	0.7700
0.5	-0.614427	0.4619	-1.0763	0.2886	0.3532
1	-0.334432	0.3328	-0.6672	0.2584	0.2143
2	-0.157647	0.2282	-0.3859	0.2302	0.1204
5	-0.042532	0.1312	-0.1738	0.1992	0.0514
10	-0.007778	0.0838	-0.0916	0.1818	0.0258

Table 1(d) Initial and final values for $R = 100$

λ	$f_{xx}(0)$	$f_x(0) = f'(1)$	$f''(1)$	$c = f(\infty)$	$g_x(0) = g'(1)$
0	-9.183421	1	-10.183	0.1064	3.661
0.1	-3.785225	0.5650	-4.3502	0.0816	1.551
0.5	-1.187700	0.2708	-1.4585	0.0587	0.5126
1	-0.633809	0.1831	-0.8169	0.0495	0.2843
2	-0.319381	0.1204	-0.4398	0.0416	0.1570
5	-0.119319	0.0672	-0.1866	0.0331	0.0625
10	-0.053108	0.0426	-0.0957	0.0281	0.0313

We see $c > 1/R$ as predicted in the previous section.

6. Discussions

Fig. 2 shows the similarity profile $f(\eta)/\sqrt{\eta} = f/\sqrt{e^x}$ for the radial velocity. Since the decay is too slow in η, we plotted the velocity against the compressed variable x. Because the radial velocity is zero on the surface and at infinity, there exists a maximum in the fluid. In contrast, for the two dimensional case the maximum is at infinity. The effect of slip is to decrease the velocity magnitude. Fig. 3 shows the similarity profile for the axial velocity $f'(\eta) = f_x(x)$ which is due to stretching of the cylinder. Fig. 4 shows the function g, affected by f, which is due to the axial convection at infinity. Both contribute to the axial velocity w.

The shear stress on the surface is

$$\tau_{rz} = \mu \frac{\partial w}{\partial r}\Big|_{r=a} = \frac{2\mu}{a}\left[2bzf''(1) + Wg'(1)\right] \tag{34}$$

where the initial values are given in Table 1. The location of zero shear is at

$$z = \frac{-Wg'(1)}{2bf''(1)} \tag{35}$$

This location is positive on the z axis if W is positive, and negative if W is negative. From Eq. (34) one can compute the total fluid dynamic drag for any segment of the stretching cylinder. Since both $|f''(1)|$ and $g'(1)$ decrease with increased slip, the magnitude of drag would also decrease with increased slip.

How large is the slip factor λ? For rarefied gasses in micron sized tubes, λ is of order 0.01. For fluids in superhydrophobic microtubes [15], λ could be order of 1. For lubricated surfaces, λ could reach 100 [16].

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Figure Captions

Fig. 1 The stretching cylinder with surface velocity *Ws* and a uniform flow *W* at infinity.

Fig. 2 The radial velocity $f(\eta)/\sqrt{\eta} = f/\sqrt{e^x}$. From top: $\lambda = 0, 1, 10$. (a) *R*=1 (b) *R*=10.

Fig. 3 The axial velocity $f'(\eta) = f_x(x)$. From top: $\lambda = 0, 1, 10$. (a) *R*=1 (b) *R*=10.

Fig. 4 The axial velocity g. From top for small *x*: $\lambda = 0, 1, 10$. (a) *R*=1 (b) *R*=10.
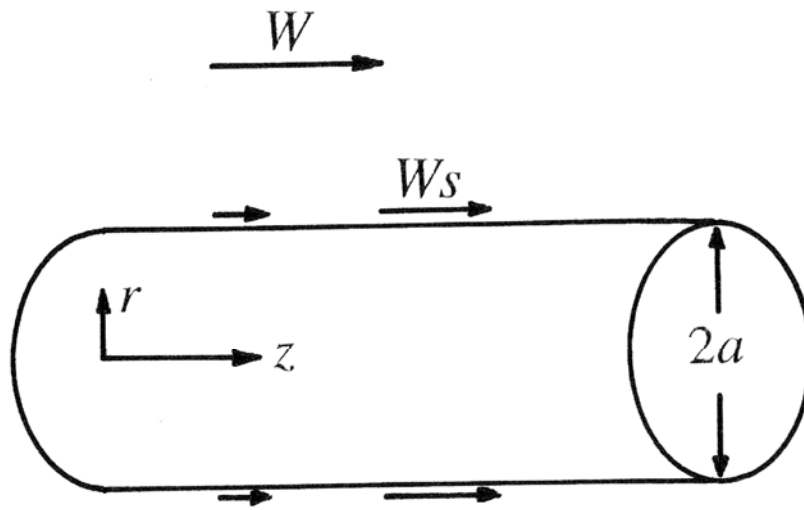
Figure 1
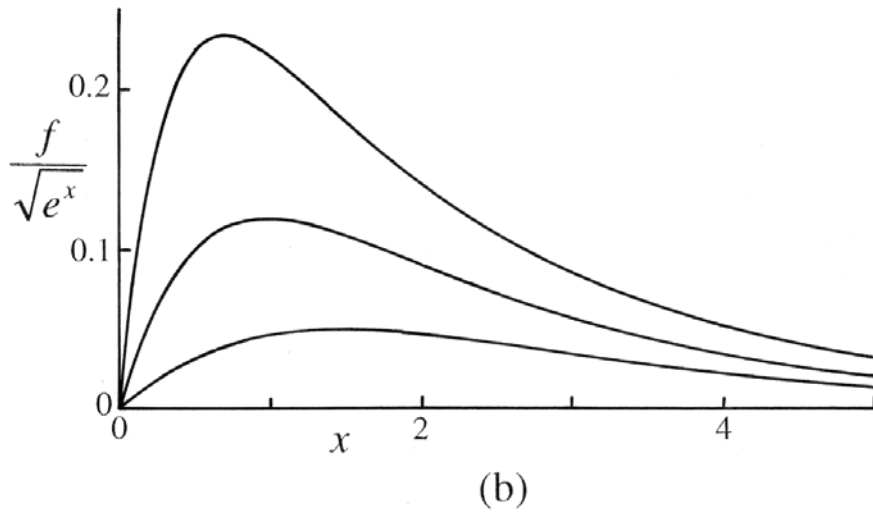
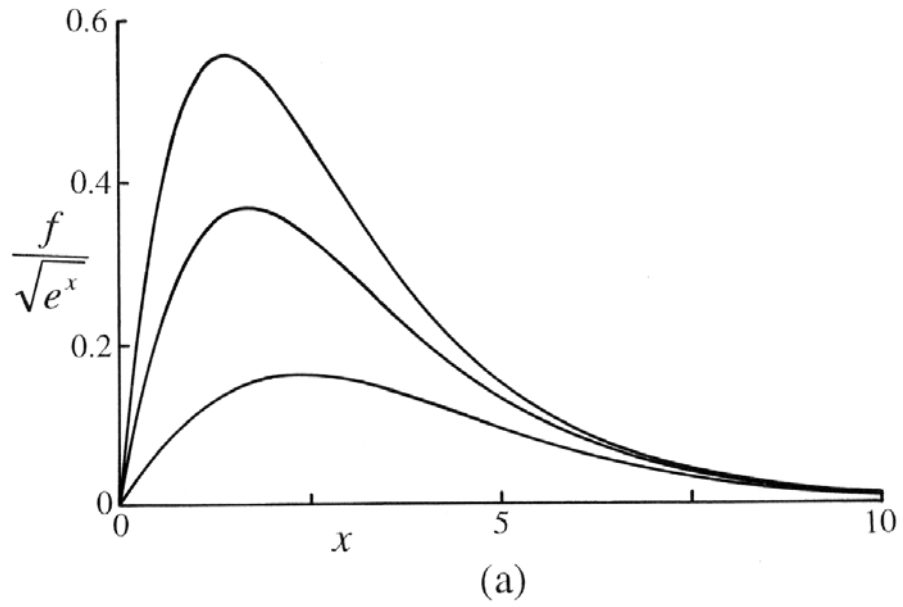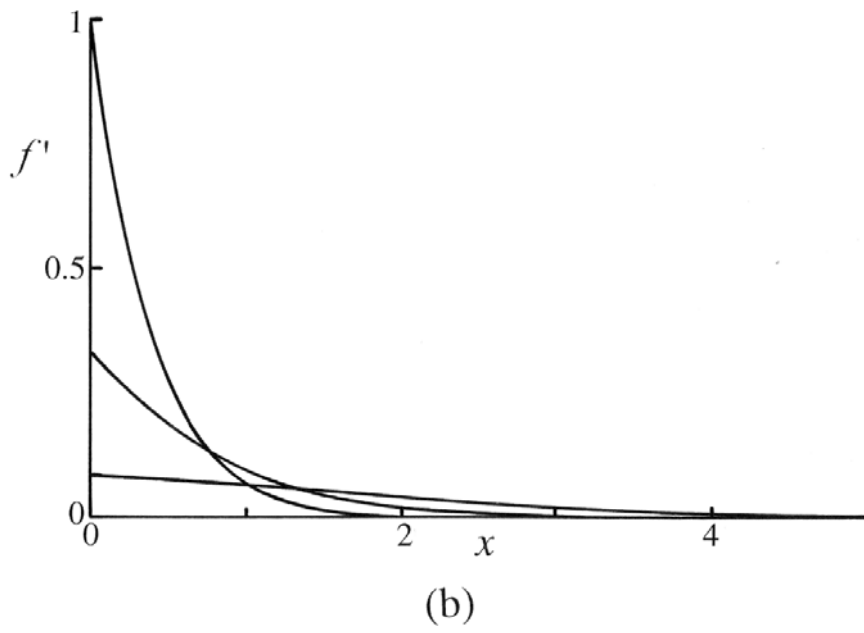Figure 2

Figure 3

Figure 4

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