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An Inversion Theorem for the Singular Integral Poisson Equation

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Abstract

The one dimensional Poisson equation governing the electric potential and the charge distributions in a plasma composed of electrons and one species of fully ionized ions is reduced to a singular integral equation. We prove an inversion theorem which allow us to solve this equation in favour of the distribution function of one of the particle species, chosen according to need, once the electric potential and the distribution of the particle of the other species be given. At variance with previous results, the unknown distribution function is determined over its whole energy range and it is written as the boundary value of a suitable sectionally analytic function. This fact allows us to extend the distribution function thus found over the whole complex energy domain.

Key words: plasma, oscillations, BGK-waves, electrostatic solitary waves, double-layers

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1 Introduction

In this report, we address the problem of solving the Poisson equation, which, in electrostatic conditions, governs the electric potential and charge distributions in a plasma composed of electrons and one species of fully ionized ions. Since the very beginning of Plasma Physics, this equation was conceived as a way to determine the electric potential in the plasma, once the charge distributions of the electrons and of the ions are given (e.g. [1]).

It was later realized [2] that the reverse approach could also be used: the Poisson equation was reduced to an Abel equation and solved in favour of the energy distribution function of one of the particle species (usually electrons, e.g. [3,4,5]) once the electric potential and the energy distribution of the other particle species were given.

This latter approach turned out to be very profitable, chiefly in Space plasmas, where data about electric potential waveforms are more accurate than data about particle distributions (e.g. [6,7,8,9]). One limitation of this procedure consists in the fact that the reduction of Poisson equation to an Abel equation for — say — the electron distribution function, requires that this distribution be arbitrarily specified over a semi-infinite range of the electron energy. In Ref. [10], we showed that this constraint induces singularities of the logarithmic and of the jump type in the solution of the Abel equation.

Our present work aims at removing this limitation. Rather than reducing the Poisson equation to an Abel equation over a finite range we cast it as an integral equation over an infinite range. Several such integral equations are considered, according to the species of the particles whose distribution we wish to determine. Once the integral forms of Poisson equations are cast in this way, we prove an inversion theorem for singular integral equations. This theorem allows us to completely determine the distribution function of one of the particle species over its *whole* energy range and in fact over the whole complex energy domain.

2 Assumptions, notation and basic equations

We assume that the plasma in the tripolar region be fully ionized and that the particles in the plasma are electrons and one species of finite mass, mobile ions. We adopt a standard statistical description of the plasma, which is given in terms of the electron and ion "one particle" velocity distribution functions (or simply velocity distribution functions) and of the mean "self consistent" macroscopic force (cf. e.g. Ref. [11]). We further assume that: (a) the velocities of the particles be largely non relativistic, so that the mean force is mainly electrostatic; (b) the plasma be homogeneous along two cartesian coordinates, so that the distribution functions and the mean force depend on one only space cartesian coordinate called X. In principle, the distributions and the mean force may depend on time. However, under assumption (a), time enters the physical laws introduced below, only as a parameter. Therefore, any mention of time will be omitted in the following.

Under these conditions, the velocity distribution functions, which, in general, depend on three velocity coordinates, will first be integrated over the whole domain of those velocity coordinates orthogonal to X. The resulting integrated distributions will depend on X and on one velocity coordinate only, called $V_{\mathrm{e}X}$ for the electron distribution, and $V_{\mathrm{i}X}$ for the ion distribution. Under conditions, the mean force will be electrostatic, directed along X, and it may be evaluated by taking the X-derivative of a function of position. In particular, for the electrons, which have electric charge $-|e|$, this function is $|e|\Phi(X)$. The quantity $\Phi(X)$ is to be identified with the observed electric potential in the plasma.

Next, we proceed to a non dimensional representation of the plasma quantities. To do so, we use Gaussian units and we set the Boltzmann constant to unity. If the electrons have mass $m_{\rm e}$, if, in the limit $X \to +\infty$, they have boundary number density $n_{\rm e\infty}$ and, in the same limit, they have boundary kinetic temperature $T_{\rm e\infty}$, then we denote by

$$\lambda_{\rm De} = \surd\{T_{\rm e\infty}/[4\pi e^2 n_{\rm e\infty}]\} \tag{2.1}$$

the electron Debye length, by

$$x = X/\lambda_{\rm De} \tag{2.2}$$

the normalized space coordinate, by

$$v_{\rm T_e} = \surd[T_{\rm e\infty}/m_{\rm e}] \tag{2.3}$$

the electrons' boundary mean thermal speed, by

$$v_{\rm e} = V_{{\rm e}X}/v_{\rm T_e} \tag{2.4}$$

the normalized electron velocity coordinate along x and by

$$n_{\rm e\infty} f_{\rm e}(x, v_{\rm e}) {\rm d}x {\rm d}v_{\rm e} \tag{2.5}$$

the probability of finding any one of the electrons having a position within a distance dx from x and a velocity component parallel to x within a distance dv_e from v_e, irrespective of its velocity components orthogonal to x, irrespective of the position and velocity of all the other electrons, and irrespective of the position and velocity of all the ions.

Likewise, if the ions have mass m_i, atomic number Z_i and electric charge $+Z_\mathrm{i}|e|$, if charge neutrality is approached as $x \to \infty$, so that the ions' boundary number density there is

$$n_{\mathrm{i}\,\infty} = n_{\mathrm{e}\,\infty}/Z_\mathrm{i} \ , \tag{2.6}$$

and if, in the same limit, the ions have boundary kinetic temperature

$$T_{\mathrm{i}\,\infty} = \theta Z_\mathrm{i} T_{\mathrm{e}\,\infty} \ , \tag{2.7}$$

then we denote by

$$v_{\mathrm{T_i}} = \surd[T_{\mathrm{i}\infty}/m_\mathrm{i}] \tag{2.8}$$

the ions' boundary mean thermal velocity, by

$$v_\mathrm{i} = V_{\mathrm{i}\,X}/v_{\mathrm{T_i}} \tag{2.9}$$

the normalized ion velocity coordinate along x and by

$$[n_{\mathrm{e}\,\infty}/Z_\mathrm{i}] f_\mathrm{i}(x, v_\mathrm{i}) \mathrm{d}x \mathrm{d}v_\mathrm{i} \tag{2.10}$$

the probability finding any one of the ions having a position within a distance dx from x and a velocity component parallel to x within a distance dv_i from

v_i, irrespective of its velocity components orthogonal to x, irrespective of the position and velocity of all the other ions, and irrespective of the position and velocity of all the electrons.

Last, the electric potential will be normalized to the electron boundary kinetic temperature

$$\phi(x) = |e|\Phi(\lambda_\mathrm{De} x)/T_{\mathrm{e}\infty}. \tag{2.11}$$

A typical waveform of the electric potential in the tripolar region is shown in Fig. 1.

Denoting by $x = x_\mathrm{min}$ the position at which the electric potential has its absolute minimum (cf. Fig. 1), the potential energy of a test electron positioned at x will be suitably rescaled, normalized to the electrons' boundary kinetic temperature, and denoted by

$$-U_\mathrm{e}(\phi) = -[\phi - \phi(x_\mathrm{min})]. \tag{2.12}$$

Likewise, being the absolute maximum of the electric potential located at $x = \infty$ (cf. Fig. 1), and being θ the ion to electron temperature ratio there (cf. Eq. (2.7)), the potential energy of a test ion positioned at x will be suitably rescaled, normalized to the ions' boundary kinetic temperature, and denoted by

$$-U_\mathrm{i}(\phi) = -[\lim_{x\to+\infty} \phi(x) - \phi]/\theta. \tag{2.13}$$

We notice that, in Eqs. (2.12) and (2.13), the electron and ion electric potential energies U_e and U_i were judiciously defined in such a way that (cf. Fig. 1)

$$U_\mathrm{e}(\phi(x)) \geq 0 \text{ for all values of } x, \tag{2.14a}$$
$$\max_{-\infty<x<+\infty}(-U_\mathrm{e}(\phi(x))) = -U_\mathrm{e}(\phi(x_\mathrm{min})) = 0, \tag{2.14b}$$

and

$$U_\mathrm{i}(\phi(x)) \geq 0 \text{ for all values of } x, \tag{2.15a}$$
$$\max_{-\infty<x<+\infty}(-U_\mathrm{i}(\phi(x))) = -U_\mathrm{i}(\lim_{x\to+\infty}\phi(x)) = 0. \tag{2.15b}$$

Next, we introduce the electron and ion space charge densities, which we conveniently normalize and respectively denote by $|e|n_{\mathrm{e}\infty}\varrho_\mathrm{e}(x)$ and $|e|n_{\mathrm{e}\infty}\varrho_\mathrm{i}(x)$, and which, using the normalizations of the velocity distributions introduced in Eqs. (2.5) and (2.10), we write as

$$\varrho_\mathrm{e}(x) = -\int_{-\infty}^{\infty}\mathrm{d}v'_\mathrm{e} f_\mathrm{e}(x, v'_\mathrm{e}), \tag{2.16}$$

and

$$\varrho_\mathrm{i}(x) = +\int_{-\infty}^{\infty}\mathrm{d}v'_\mathrm{i} f_\mathrm{i}(x, v'_\mathrm{i}). \tag{2.17}$$

Using the assumption that the plasma be in electrostatic conditions, we relate the space charge distributions to the electric potential by Poisson's equation, which we write in the non dimensional form

$$\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} = -[\varrho_\mathrm{e}(x) + \varrho_\mathrm{i}(x)]. \tag{2.18}$$

Eq. (2.18) relates the electric potential and the particle velocity distribution functions f_e and f_i, through the charge densities given in Eqs. (2.16) and

(2.17). Now, in these latter two equations, we change the integration variable according to

$$v'_\alpha = \pm\sqrt{\{2[W'_\alpha + U_\alpha(\phi)]\}},\ \alpha = \mathrm{e}, \mathrm{i}, \tag{2.19}$$

the upper (respectively lower) sign holding in that part of the integrals, appearing in Eqs. (2.16) and (2.17), extending over the positive (respectively negative) range of v'_α. Given the bivariate distribution function $f_\alpha(x, v_\alpha)$, and given the variable W_α, the transformation given in Eq. (2.19) introduces, in the integrals appearing on the right hand side of Eqs. (2.16) and (2.17), the functions

$$\begin{aligned} &F_\alpha(W_\alpha) = f_\alpha(x, +\sqrt{\{2[W_\alpha + U_\alpha(\phi(x))]\}}) + \\ &f_\alpha(x, -\sqrt{\{2[W_\alpha + U_\alpha(\phi(x))]\}}),\ \alpha = \mathrm{e}, \mathrm{i}\ . \end{aligned} \tag{2.20}$$

and reduces those integrals to

$$\varrho_\mathrm{e}(x) = -n_\mathrm{e}(\phi(x)), \tag{2.21a}$$

$$n_\mathrm{e}(\phi) = \int_{-U_\mathrm{e}(\phi)}^{\infty} \mathrm{d}W'_\mathrm{e} \frac{F_\mathrm{e}(W'_\mathrm{e})}{\sqrt{\{2[W'_\mathrm{e} + U_\mathrm{e}(\phi)]\}}}, \tag{2.21b}$$

and

$$\varrho_\mathrm{i}(x) = +n_\mathrm{i}(\phi(x)), \tag{2.22a}$$

$$n_\mathrm{i}(\phi) = \int_{-U_\mathrm{i}(\phi)}^{\infty} \mathrm{d}W'_\mathrm{i} \frac{F_\mathrm{i}(W'_\mathrm{i})}{\sqrt{\{2[W'_\mathrm{i} + U_\mathrm{i}(\phi)]\}}}. \tag{2.22b}$$

Here, it is agreed that, in order for the integrals in Eqs. (2.21b) and (2.22b) to converge, the functions $F_\alpha(W_\alpha)$ vanish faster than $1/\sqrt{W_\alpha}$ as $W_\alpha \to +\infty$. These equations reveal that the quantity $F_\alpha(W_\alpha)\mathrm{d}x\mathrm{d}W_\alpha$ amounts to $\sqrt{\{2[W_\alpha +}$

$U_\alpha(\phi(x))]\}$ times the probability of finding a particle of species α having position within distance $\mathrm{d}x$ from x and total energy within distance $\mathrm{d}W_\alpha$ from W_α, irrespective of the sign of their velocity (cf. Eqs. (2.21b) and (2.22b)). In the following, the univariate functions F_e and F_i will be respectively known as the electron and ion *bi-directional energy distribution functions*.

An important property of the charge distributions written as in Eqs. (2.21) and (2.22) (rather than as in Eq. (2.16) and (2.17)) is that their values at position x are specified through the value of the potential ϕ at x. This is of course legitimate as long as $\phi(x)$ is a monotonic function of x. Wherever this condition fails, Eqs. (2.21) and (2.22) are still meaningful, in a piecewise sense, in each of the x-domains where $\phi(x)$ is a monotonic function of x (cf. Fig. 1).

An analogous transformation of the space variable from x to $\phi(x)$ may be conceived for the quantity $\mathrm{d}^2\phi(x)/\mathrm{d}x^2$, appearing on the left hand side of Poisson's equation (cf. Eq. (2.18)). To do so, we assume that the electric potential $\phi(x)$ be a continuous function of the position x. Then, in each of the domains of the tripolar region where $\phi(x)$ is a monotonic function of position (cf. Fig. 1), the inverse function ϕ^{-1} of ϕ certainly exists and the quantity $\mathrm{d}^2\phi(x)/\mathrm{d}x^2$ may well be conceived as a function of ϕ itself, which we call $\phi_{xx}(\phi)$:

$$\phi_{xx}(\psi) = [\mathrm{d}^2\phi(\xi)/\mathrm{d}\xi^2]|_{\xi=\phi^{-1}(\psi)}. \tag{2.23}$$

Practical ways of extracting ϕ_{xx} from $\mathrm{d}^2\phi(x)/\mathrm{d}x^2$ may rely on Lagrange's inversion formula (cf. e.g. [12]), which applies if the electric potential $\phi(x)$ is an analytic function of x.

In conclusion, the above transformations of the space charge densities $\varrho_\mathrm{e}(x)$

(cf. Eqs. (2.21)) and $\varrho_{\rm i}(x)$ (cf. Eqs. (2.22)) and of $\mathrm{d}^2\phi(x)/\mathrm{d}x^2$ (cf. Eq. (2.23)) allow us to rewrite Poisson's equation (cf. Eq. (2.18)) in the form

$$\phi_{xx}(\phi) = n_{\rm e}(\phi) - n_{\rm i}(\phi). \tag{2.24}$$

3 The inversion theorem

The relation between the electric potential and the ion and electron distribution functions established in Section 2 (cf. Eqs. (2.21b), (2.22b) and (2.24)) amounts to a fully fledged integro-differential equation, a fact already appreciated in Ref. [13]. As anticipated in Section 1, we need solve this set of equations for either the electron or the ion energy distribution functions over their own whole complex energy domain.

We divide this task into two parts. In the first part (cf. Section 4), Poisson's equation will be transformed from its fractional integral formulation (cf. Eqs. (2.21b), (2.22b) and (2.24)) to a pair of new, entirely equivalent fractional integral equations. In the second part (cf. Section 5), these fractional equations will be solved in favour of the electron or ion energy distribution functions and these latter will be extended over their whole respective complex energy planes.

To carry out the above task, we need two important theorems, to which we devote this section.

Theorem 3.1 *Let a and b be real constants, such that $a < b$ and let $f(w)$ and $g(w)$ be two functions which are Hölder continuous over the interval $a < w < b$. Provided all the integrals converge, we have the following results.*

(*i*) *For $a < w < b$, the identity*

$$\int_a^w \mathrm{d}u \frac{f(u)}{\sqrt{[w-u]}} = \int_w^b \mathrm{d}u \frac{g(u)}{\sqrt{[u-w]}} \tag{3.1}$$

holds if and only if, denoting by P *the Cauchy principal value of an integral,*

$$f(w) = \frac{1}{\pi}\mathrm{P}\int_a^b \mathrm{d}t \frac{\sqrt{[t-a]}}{\sqrt{[w-a]}} \frac{g(t)}{t-w}. \tag{3.2}$$

(*ii*) *For $a < w < b$, Eq. (3.2) is equivalent to*

$$f(w) = \frac{1}{\pi}\frac{A}{\sqrt{[w-a]}} + \frac{1}{\pi}\mathrm{P}\int_a^b \mathrm{d}t \frac{\sqrt{[w-a]}}{\sqrt{[t-a]}} \frac{g(t)}{t-w}, \tag{3.3a}$$

$$\text{where} \quad A = \int_a^b \mathrm{d}u \frac{g(u)}{\sqrt{[u-a]}}. \tag{3.3b}$$

(*iii*) *The results in points (i)–(ii) hold also for $b \to +\infty$; they hold also for $a \to -\infty$.*

Proof. To carry out the proof of the theorem, we introduce the function

$$\lambda(\zeta, \xi, \eta) = \int_\zeta^\xi \mathrm{d}\xi' \frac{1}{\sqrt{[\xi-\xi']}\sqrt{[\eta-\xi']}},$$
$$\text{for } \eta > \xi > \zeta, \tag{3.4}$$

whose properties we establish beforehand. Specifically, since in the integral on the right hand side of Eq. (3.4), $\eta > \xi$, we set $\xi' = \xi - [\eta - \xi]s^2$. In this way, Eq. (3.4) gives

$$\lambda(\zeta, \xi, \eta) = \int_0^{\frac{\sqrt{[\xi-\zeta]}}{\sqrt{[\eta-\xi]}}} \mathrm{d}s \frac{1}{\sqrt{[1+s^2]}} =$$

$$2 \operatorname{arcsinh}\left(\frac{\sqrt{[\xi-\zeta]}}{\sqrt{[\eta-\xi]}}\right), \quad \text{for } \eta > \xi > \zeta. \tag{3.5}$$

We now revert to the proof of the theorem and we show first that Eq. (3.2) necessarily follows from Eq. (3.1). Specifically, we regard Eq. (3.1) as an Abel equation for $f(w)$, whose solution is (cf. e.g. [14]))

$$f(w) = \frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}w} \int_a^w \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \int_u^b \mathrm{d}t \frac{g(t)}{\sqrt{[t-u]}}. \tag{3.6}$$

Since, by assumption, $w < b$, our idea is to split the inner t-integral, on the right hand side of Eq. (3.6), into the sum of an integral running from u to w and an integral running from w to b. Actually, this step must be taken with some care. To do so, we introduce a real number ϵ and we let the first of those two latter integrals run from u up to $w-\epsilon$, and the second one run from $w+\epsilon$ to b. Then, we develop our proof for $\epsilon \neq 0$ and we recover our final results by taking the limit for $\epsilon \to 0$. According to the above instructions, the double integral appearing on the right hand side of Eq. (3.6) splits as follows:

$$f(w) =$$

$$\frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}w} \left\{ \int_a^{w-\epsilon} \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \int_u^{w-\epsilon} \mathrm{d}t \frac{g(t)}{\sqrt{[t-u]}} + \right.$$

$$\int_{w-\epsilon}^{w} \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \int_u^{w-\epsilon} \mathrm{d}t \frac{g(t)}{\sqrt{[t-u]}} +$$

$$\left. \int_a^{w} \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \int_{w+\epsilon}^{b} \mathrm{d}t \frac{g(t)}{\sqrt{[t-u]}} \right\}. \tag{3.7}$$

In the first two integrals appearing on the right hand side of Eq. (3.7), we interchange the order of the t- and u-integration according to Fubini's theorem; in the third integral, such interchange is achieved in a straightforward way. Then, using the function λ, defined in Eq. (3.4), Eq. (3.7) reduces to

$$f(w) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}w} \left\{ \int_a^{w-\epsilon} \mathrm{d}t\, g(t) \lambda(a,t,w) + \right.$$

$$\left. \int_{w-\epsilon}^{w} \mathrm{d}t\, g(t)\lambda(w,t,w) + \int_{w+\epsilon}^{b} \mathrm{d}t\, g(t)\lambda(a,w,t) \right\}. \tag{3.8}$$

In the second integral appearing on the right hand side of Eq. (3.8), the third argument of the function λ always exceeds its second argument, and the value of λ given by Eq. (3.5) may be used, according to which $\lambda(w,t,w) = \mathrm{j}\pi$. Thus, since, by assumption, the function g is continuous, the contribution given by that integral to $f(w)$ is

$$\frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}w} \int_{w-\epsilon}^{w} \mathrm{d}t\, g(t)\lambda(w,t,w) = \mathrm{j} \lim_{\epsilon \to 0} [g(w) - g(w-\epsilon)] = 0. \tag{3.9}$$

On the other hand, taking the w-derivative of the first and third term on the right hand side of Eq. (3.8), we find

$$f(w) = \frac{1}{\pi} \lim_{\epsilon \to 0} \left\{ \int_a^{w-\epsilon} \mathrm{d}t\, g(t) \frac{\mathrm{d}}{\mathrm{d}w} \lambda(a,t,w) + \right.$$

$$g(w-\epsilon)\lambda(a, w-\epsilon, w) - g(w+\epsilon)\lambda(a,w,w+\epsilon) +$$

$$\left. \int_{w+\epsilon}^{b} \mathrm{d}t\, g(t) \frac{\mathrm{d}}{\mathrm{d}w} \lambda(a,w,t) \right\}. \tag{3.10}$$

Now, in the second and third term appearing on the right hand side of Eq. (3.10), the third argument of the function λ also exceeds its second argument, and the value of λ given by Eq. (3.5) may again be used. Also, since $\epsilon > 0$, we have

$$\lambda(a, w-\epsilon, w) = -\ln(\epsilon) + \ln(2[w-a]) + O(\epsilon),$$
$$\lambda(a, w, w+\epsilon) = -\ln(\epsilon) + \ln(2[w-a]) + O(\epsilon),$$
$$\text{for } \epsilon \simeq 0,\ \epsilon > 0. \tag{3.11}$$

We thus see that, since, by assumption, the function g is continuous, the contribution given by the second and third term appearing on the right hand side of Eq. (3.10) to $f(w)$ is

$$\frac{1}{\pi}\lim_{\epsilon\to 0}[g(w-\epsilon)\lambda(a, w-\epsilon, w) - g(w+\epsilon)\lambda(a, w, w+\epsilon)] = 0. \tag{3.12}$$

Furthermore, in both integrals appearing on the right hand side of Eq. (3.10) the third argument of the function λ also exceeds its second argument. Then, again using Eq. (3.5), the two w-derivatives of λ, appearing in those integrals, turn out to be

$$\frac{\mathrm{d}}{\mathrm{d}w}\lambda(a, t, w) = \frac{\mathrm{d}}{\mathrm{d}w} 2\,\mathrm{arcsinh}\left(\frac{\sqrt{[t-a]}}{\sqrt{[w-t]}}\right) =$$
$$2\frac{\sqrt{[w-t]}}{\sqrt{[w-a]}}\sqrt{[t-a]}\frac{-1/2}{[w-t]^{3/2}} =$$
$$\frac{\sqrt{[t-a]}}{\sqrt{[w-a]}}\frac{1}{t-w} \tag{3.13}$$

and

$$
\frac{\mathrm{d}}{\mathrm{d}w}\lambda(a,w,t) = \frac{\mathrm{d}}{\mathrm{d}w} 2\,\mathrm{arcsinh}\left(\frac{\sqrt{[w-a]}}{\sqrt{[t-w]}}\right) =
$$
$$
2\frac{\sqrt{[t-w]}}{\sqrt{[t-a]}}\frac{1}{2}\frac{\sqrt{[t-w]}}{\sqrt{[w-a]}}\frac{t-a}{[t-w]^2} =
$$
$$
\frac{\sqrt{[t-a]}}{\sqrt{[w-a]}}\frac{1}{t-w}. \tag{3.14}
$$

Finally, substituting Eqs. (3.12), (3.13) and (3.14) into the right hand side of Eq. (3.10), we recover Eq. (3.2), as desired:

$$
f(w) = \frac{1}{\pi}\lim_{\epsilon\to 0}\left\{\int_a^{w-\epsilon} \mathrm{d}t\, g(t)\frac{\sqrt{[t-a]}}{\sqrt{[w-a]}}\frac{1}{t-w} + \right.
$$
$$
\left.\int_{w+\epsilon}^{b} \mathrm{d}t\, g(t)\frac{\sqrt{[t-a]}}{\sqrt{[w-a]}}\frac{1}{t-w}\right\} =
$$
$$
\frac{1}{\pi}\mathrm{P}\int_a^b \mathrm{d}t\frac{\sqrt{[t-a]}}{\sqrt{[w-a]}}\frac{g(t)}{t-w}. \tag{3.15}
$$

This concludes the proof of the first part of point (i) of the theorem. The proof of the second part of that point consists in showing that Eq. (3.1) necessarily follows from Eq. (3.2). To do so, we substitute the right hand side of Eq. (3.2) directly into the left hand side of Eq. (3.1) and interchange the order of integration, which we can do, provided the integrals be understood as Cauchy's principal value integrals. In this way, we get

$$\int_a^w \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \frac{1}{\pi} \mathrm{P} \int_a^b \mathrm{d}t \frac{\sqrt{[t-a]}}{\sqrt{[u-a]}} \frac{g(t)}{t-u} =$$

$$\int_a^b \mathrm{d}t \sqrt{[t-a]} g(t) \frac{1}{\pi} \mathrm{P} \int_a^w \mathrm{d}u \frac{1}{\sqrt{[w-u]}\sqrt{[u-a]}} \frac{1}{t-u}. \tag{3.16}$$

In the inner integral appearing on the right hand side of Eq. (3.16), we change the integration variable according to $u = [w + as^2]/[1 + s^2]$, to get

$$\frac{1}{\pi} \mathrm{P} \int_a^w \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \frac{1}{\sqrt{[u-a]}} \frac{1}{t-u} =$$
$$\frac{1}{\pi} \mathrm{P} \int_0^\infty \mathrm{d}s \frac{2}{[t-w] + [t-a]s^2}. \tag{3.17}$$

In the integral on the right hand side of Eq. (3.17), we must take into account that the quantity t is the integration variable of the outer integral on the right hand side of Eq. (3.16), and thus t exceeds a, the lower limit of integration there. Therefore, in the denominator of the integrand in the integral on the right hand side of Eq. (3.17), two cases arise: if $t < w$, the signs of the two terms in the denominator disagree and the principal part of the integral itself obviously vanishes; if $t > w$, the two signs agree and the integral reduces to $1/\{\sqrt{[t-w]}\sqrt{[t-a]}\}$. Substituting this result into Eq. (3.16), only the contribution

$$\int_w^b \mathrm{d}t \sqrt{[t-a]} g(t) \frac{1}{\sqrt{[t-w]}\sqrt{[t-a]}} \tag{3.18}$$

survives, which coincides with the right hand side of Eq. (3.1), as desired.

This concludes the proof of point (i) of the theorem. The proof of point (ii) consists in reducing Eq. (3.2) to Eq. (3.3), which we do by means of the obvious identity

$$\frac{\sqrt{[u-a]}}{\sqrt{[w-a]}}\frac{1}{u-w}=\frac{1}{\sqrt{[u-a]}}\left\{\frac{1}{\sqrt{[w-a]}}+\frac{\sqrt{w-a]}}{u-w}\right\}. \tag{3.19}$$

To prove point (iii), we work out the above arguments using finite values for the constants a and b and then let them take arbitrarily large values in Eqs. (3.1), (3.2) and (3.3a). In so doing, we must ensure that, as $|w| \to +\infty$, the functions $f(w)$ and $g(w)$ vanish faster than a suitable negative power of $|w|$, in such a way that the integrals in the above formulæ converge, as assumed in the theorem.

□

Theorem 3.1 provide a practical means to extract the function f out of Eq. (3.1) once the function g is known. The following corollary precisely allows the reverse operation.

Corollary 3.1 *Let a and b be real constants, such that $a < b$ and let $f(w)$ and $g(w)$ be two functions which are Hölder continuous over the interval $a < w < b$. Then, provided all the integrals converge, we have the following results.*

(i) For $a < w < b$, the identity

$$\int_a^w \mathrm{d}u \frac{f(u)}{\sqrt{[w-u]}} = \int_w^b \mathrm{d}u \frac{g(u)}{\sqrt{[u-w]}} \tag{3.20}$$

holds if and only if, denoting by P *the Cauchy principal value of an integral,*

$$g(w) = -\frac{1}{\pi}\mathrm{P}\int_a^b \mathrm{d}t \frac{\sqrt{[b-t]}}{\sqrt{[b-w]}}\frac{f(t)}{t-w}. \tag{3.21}$$

(ii) For $a < w < b$, Eq. (3.21) is equivalent to

$$g(w) = \frac{1}{\pi} \frac{B}{\sqrt{[b-w]}} - \frac{1}{\pi} \mathrm{P} \int_a^b \mathrm{d}t \frac{\sqrt{[b-w]}}{\sqrt{[b-t]}} \frac{f(t)}{t-w}, \tag{3.22a}$$

$$\text{where} \quad B = \int_a^b \mathrm{d}u \frac{f(u)}{\sqrt{[b-u]}}. \tag{3.22b}$$

(iii) The statements in points (i)–(ii) hold also if $b \to +\infty$.

Proof.

The proof of the first part of point (i) consist in showing that Eq. (3.21) necessarily follows from Eq. (3.20). To do so, in the integrals appearing on each side of Eq. (3.20), we set $w = b + a - y$ and we change the integration variable according to $u = b + a - s$. In this way, Eq. (3.20) reduces to

$$\int_a^y \mathrm{d}s \frac{g(b+a-s)}{\sqrt{[y-s]}} = \int_y^b \mathrm{d}s \frac{f(b+a-s)}{\sqrt{[s-y]}}. \tag{3.23}$$

Now, this relation between the functions $g(b+a-y)$ and $f(b+a-y)$ has the same structure of the relation between $f(w)$ and $g(w)$ given in Theorem 3.1 (cf. Eq. (3.20)), provided the role of f and g be interchanged. Therefore, Eq. (3.23) directly leads to (cf. Eq. (3.2))

$$g(b+a-y) = \frac{1}{\pi} \mathrm{P} \int_a^b \mathrm{d}u \frac{\sqrt{[u-a]}}{\sqrt{[y-a]}} \frac{f(b+a-u)}{u-y}. \tag{3.24}$$

If, in the integral on the right hand side of Eq. (3.24), we set $y = b + a - w$ and if we change the integration variable according to $u = b+a-t$, we recover Eq. (3.21), as desired.

This concludes the proof of the first part of point (i) of the theorem. The proof of the second part of that point consists in showing that Eq. (3.20) necessarily follows from Eq. (3.21). But this is entirely equivalent to showing that Eq. (3.23) necessarily follows from Eq. (3.24), a task which is accomplished by the same procedure used in Theorem 3.1 to show that Eq. (3.1) follows from Eq. (3.2).

This concludes the proof of point (i) of the theorem. The proof of point (ii) consists in reducing Eq. (3.21) to Eq. (3.22), which we do by means of the obvious identity

$$\frac{\sqrt{[b-u]}}{\sqrt{[b-w]}}\frac{1}{u-w} = \frac{1}{\sqrt{[b-u]}}\left\{\frac{1}{\sqrt{[b-w]}} - \frac{\sqrt{[b-w]}}{u-w}\right\}. \tag{3.25}$$

To prove point (iii), we work out the above arguments using finite values for the constants a and b and then let them take arbitrarily large values in Eqs. (3.20), (3.21) and (3.22a). In so doing, we must ensure that, as $|w| \to +\infty$, the functions $f(w)$ and $g(w)$ vanish faster than a suitable negative power of $|w|$, in such a way that the integrals in the above formulæ converge, as assumed in the theorem.

□

4 The fractional distributions of the electric charge density of electrons and ions

In this section we carry out the first part of the task, set at the beginning of Section 3, of solving Poisson's equation (cf. (2.24)) in favour of the electron and ion distribution functions. This part consists in casting that equation into a pair of new, entirely equivalent and more convenient fractional equations.

To do so, we need reformulate the electron and ion number densities, given in Eqs. (2.21b) and (2.22b), in a more convenient form. In each of the following two pairs of transformations of these quantities, the first transformation concerns the electron number density, and the second one concerns the ion number density. A first pair of transformations is worked out using the obvious identity (cf. Eqs. (2.12) and (2.13))

$$U_{\rm e}(\phi) = [\lim_{x\to+\infty} \phi(x) - \phi(x_{\rm min})] - \theta U_{\rm i}(\phi), \tag{4.1}$$

and the change of the integration variable $W'_{\rm e} = \theta W' - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\rm min})]$, in the integral on the right hand side of Eq. (2.21b). These relations reduce the the electron number density to

$$n_{\rm e}(\phi) = \int_{U_{\rm i}(\phi)}^{\infty} {\rm d}W' \frac{F_{\rm e}(\theta W' - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\rm min})])\sqrt{\theta}}{\sqrt{\{2[W' - U_{\rm i}(\phi)]\}}}. \tag{4.2}$$

Likewise, the obvious identity (cf. Eqs. (2.12) and (2.13))

$$U_{\rm i}(\phi) = \{[\lim_{x\to+\infty} \phi(x) - \phi(x_{\rm min})] - U_{\rm e}(\phi)\}/\theta, \tag{4.3}$$

and the change of the integration variable $W'_{\rm i} = \{W' - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\rm min})]\}/\theta$ in the integral on the right hand side of Eq. (2.22b), reduce the ion number density to

$$n_{\rm i}(\phi) = \int_{U_{\rm e}(\phi)}^{\infty} {\rm d}W' \frac{F_{\rm i}(\{W' - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\rm min})]\}/\theta)/\sqrt{\theta}}{\sqrt{\{2[W' - U_{\rm e}(\phi)]\}}}. \tag{4.4}$$

The second pair of transformation has the electron number density written as

$$n_{\rm e}(\phi) = \int_{-U_{\rm i}(\phi)}^{\infty} {\rm d}W' \frac{K_{\rm e}(W')}{\sqrt{\{2[W' + U_{\rm i}(\phi)]\}}} =$$

$$\int_{-\infty}^{U_{\rm i}(\phi)} {\rm d}W'' \frac{K_{\rm e}(-W'')}{\sqrt{\{2[U_{\rm i}(\phi) - W'']\}}}. \tag{4.5}$$

Then, Theorem 3.1, which we proved in Section 3, prescribe that, for both Eqs. (4.2) and (4.5) to hold, the functions $K_{\rm e}(-W)$ and $F_{\rm e}(\theta W - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\rm min})])\sqrt{\theta}$ must be related according to Eq. (3.2 (now with $a \to -\infty$ and $b \to +\infty$)

$$K_{\rm e}(-W) = \frac{1}{\pi} {\rm P} \int_{-\infty}^{+\infty} {\rm d}W' \times$$

$$\frac{F_{\rm e}(\theta W' - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\rm min})])\sqrt{\theta}}{W' - W}. \tag{4.6}$$

Here, the symbol P prescribes that the integral on the right hand side of Eq. (4.7) must be understood as a Cauchy principal value. In the integral appearing on the right hand side of Eq. (4.6), we change the integration variable according to $\theta W' = W''$, and, upon setting $W \to -W/\theta$, we finally write

$$K_{\rm e}(W/\theta)/\sqrt{\theta} = \frac{1}{\pi} {\rm P} \int_{-\infty}^{+\infty} {\rm d}W' \times$$

$$\frac{F_{\rm e}(W' - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\rm min})])}{W' + W}. \tag{4.7}$$

Likewise, if we write the ion density as

$$n_{\rm i}(\phi) = \int_{-U_{\rm e}(\phi)}^{\infty} {\rm d}W' \frac{K_{\rm i}(W')}{\sqrt{\{2[W' + U_{\rm e}(\phi)]\}}} =$$

$$\int_{-\infty}^{U_{\rm e}(\phi)} {\rm d}W' \frac{K_{\rm i}(-W')}{\sqrt{\{2[U_{\rm e}(\phi) - W']\}}} \tag{4.8}$$

then, the same Theorem 3.1 prescribe that, for both Eqs. (4.4) and (4.8) to hold, the functions $K_{\rm i}(-W)$ and $F_{\rm i}(\{W - [\lim_{x\to+\infty}\phi(x) - \phi(x_{\rm min})]\}/\theta)/\sqrt{\theta}$ must be related according to Eq. (3.2) (now with $a \to -\infty$ and $b \to +\infty$). Upon changing the sign of the argument of the function $K_{\rm i}$, this relation reads

$$K_{\rm i}(W) = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{+\infty} \mathrm{d}W' \times$$
$$\frac{F_{\rm i}(\{W' - [\lim_{x\to+\infty}\phi(x) - \phi(x_{\rm min})]\}/\theta)/\sqrt{\theta}}{W' + W}. \tag{4.9}$$

Having performed the anticipated two pairs of transformations of the electron and ion number densities, we now use these quantities to rearrange Poisson's equation (cf. Eq. 2.24), as announced at the beginning of this section. Specifically, if, instead of the Eq. (2.22b) for the ion number density, we use Eq. (4.8), then, taking the electron number density from Eq. (2.21b), and introducing the function

$$H(W) = F_{\rm e}(W) - K_{\rm i}(W), \tag{4.10}$$

Poisson's equation (cf. Eq. (2.24)) takes the form

$$\phi_{xx}(\phi) = \int_{-U_{\rm e}(\phi)}^{\infty} \mathrm{d}W' \frac{H(W')}{\sqrt{\{2[W' + U_{\rm e}(\phi)]\}}}. \tag{4.11}$$

Likewise, if, instead of the Eq. (2.21b) for the electron number density, we use Eq. (4.5), then, taking the ion number density from Eq. (2.22b), and introducing the function

$$K(W) = F_{\rm i}(W) - K_{\rm e}(W), \tag{4.12}$$

Poisson's equation (cf. Eq. (2.24)) takes the form

$$\phi_{xx}(\phi) = -\int_{-U_{\rm i}(\phi)}^{\infty} {\rm d}W' \frac{K(W')}{\sqrt{\{2[W' + U_{\rm i}(\phi)]\}}}. \tag{4.13}$$

Eqs. (4.11) and (4.13) are the pair of fractional forms of Poisson's equation, announced at the beginning of this section.

In conclusion, the above analysis showed that, based on Poisson equation, it is always possible to write the second derivative of the electric potential ϕ_{xx} as a fractional Weil transform (cf. e.g. [15]) of a suitable single function and that this function may be constructively related to the electron and ion energy distribution functions. The convenience of writing the quantity ϕ_{xx} in this way will be made clear in Section 5.

5 The Hilbert solutions of the integral Poisson equation

In this section we carry out the second part of the task, set at the beginning of Section 3, of solving Poisson's equation (cf. (2.24)) in favour of the electron and ion distribution functions. First, using the definition of the electron potential energy $U_{\rm e}$ (cf. Eq. (2.12)), we cast Eq. (4.11)) as an Abel equation (cf. e.g. [14]) for the quantity $H(W)$

$$\phi_{xx}(U_{\rm e} + \phi(x_{\rm min})) = \int_{-\infty}^{U_{\rm e}} {\rm d}W' \frac{H(-W')}{\sqrt{\{2[U_{\rm e} - W']\}}}, \tag{5.1}$$

whose solution is

$$H(-W) = \frac{\sqrt{2}}{\pi}\frac{\mathrm{d}}{\mathrm{d}W}\int_{-\infty}^{W} \mathrm{d}W' \frac{\phi_{xx}(W' + \phi(x_{\min}))}{\sqrt{[W - W']}}. \tag{5.2}$$

In the integral appearing on the right hand side of Eq. (5.2), we change the integration variable according to $W' = -W''$ and then we change the sign of the argument of the function H, to get

$$H(W) = -\frac{\sqrt{2}}{\pi}\frac{\mathrm{d}}{\mathrm{d}W}\int_{W}^{+\infty} \mathrm{d}W'' \frac{\phi_{xx}(-W'' + \phi(x_{\min}))}{\sqrt{[W'' - W]}}. \tag{5.3}$$

Following an analogous procedure, Eq. (4.13) may be cast as an Abel equation for the quantity $K(W)$

$$\phi_{xx}(\lim_{x\to+\infty} \phi(x) - \theta U_{\mathrm{i}}) = -\int_{-\infty}^{U_{\mathrm{i}}} \mathrm{d}W' \frac{K(-W')}{\sqrt{\{2[U_{\mathrm{i}} - W']\}}}, \tag{5.4}$$

whose solution is

$$K(W) = \frac{\sqrt{2}}{\pi}\frac{\mathrm{d}}{\mathrm{d}W}\int_{W}^{+\infty} \mathrm{d}W' \frac{\phi_{xx}(\lim_{x\to+\infty} \phi(x) + \theta W')}{\sqrt{[W' - W]}}. \tag{5.5}$$

In this way, Eqs. (4.11) and (5.3) on one hand, or Eqs. (4.13) and (5.5) on the other, respectively establish a complete equivalence between the knowledge of the electric potential ϕ and that of the functions $H(W)$ or $K(W)$.

Because of this equivalence, Eqs. (4.10) may be used to extract the electron energy distribution function once the function ϕ_{xx} (i.e. F) and the ion distribution are given: specifically, using Eq. (4.9), we rewrite Eq. (4.10) as

$$F_{\rm e}(W) = \frac{1}{\pi}{\rm P}\int_{-\infty}^{+\infty} {\rm d}W' \times$$

$$\frac{F_{\rm i}(\{W' - [\lim_{x\to+\infty}\phi(x) - \phi(x_{\rm min})]\}/\theta)/\sqrt{\theta}}{W'+W} +$$

$$H(W), \tag{5.6}$$

Likewise,Eqs. (4.12) may be used to extract the ion energy distribution function once the function ϕ_{xx} (i.e. K) and the electron distribution are given: specifically, using Eq. (4.7), we rewrite Eq. (4.12) as

$$F_{\rm i}(W/\theta)/\sqrt{\theta} = \frac{1}{\pi}{\rm P}\int_{-\infty}^{+\infty} {\rm d}W' \times$$

$$\frac{F_{\rm e}(W' - [\lim_{x\to+\infty}\phi(x) - \phi(x_{\rm min})])}{W'+W} +$$

$$K(W/\theta)/\sqrt{\theta}. \tag{5.7}$$

Eqs. (5.6) and (5.7) are the solutions of Poisson's equation announced at the beginning of this section.

We now proceed to rearranging the solutions to Poisson's equation given in Eqs. (5.6) and (5.7) in an equivalent form. To do so, in the integral appearing on the right hand side of Eq. (4.11), we change the integration variable according to $W' = \phi(x_{\rm min}) - W''$, and, writing the electron potential energy $U_{\rm e}(\phi)$ as in Eq. (2.12), we get

$$\phi_{xx}(\phi) = \int_{-\infty}^{\phi} {\rm d}W'' \frac{H(\phi(x_{\rm min}) - W'')}{\sqrt{[2(\phi - W'')]}}. \tag{5.8}$$

Likewise, in the integral appearing on the right hand side of Eq. (4.13), we change the integration variable according to $W' = [W'' - \lim_{x\to+\infty}\phi(x)]/\theta$, and, writing the ion potential energy $U_\mathrm{i}(\phi)$ as in Eq. (2.13), we get

$$\phi_{xx}(\phi) = -\int_\phi^\infty \mathrm{d}W'' \frac{K([W'' - \lim_{x\to+\infty}\phi(x)]/\theta)/\sqrt{\theta}}{\sqrt{\{2[W''-\phi]\}}}. \tag{5.9}$$

Subtracting the respective sides of Eqs. (5.8) and (5.9), an integral relation is obtained between the functions $H(\phi(x_\mathrm{min}) - W)$ and $K([W - \lim_{x\to+\infty}\phi(x)]/\theta)/\sqrt{\theta}$, which is of the kind shown in Eq. (3.1) and in Eq. (3.20): there now, $a \to -\infty$ and $b \to \infty$ and $a < b$, in such a way that the hypotheses of Theorem 3.1 and of Corollary 3.1 apply. Then, after rearranging the arguments of the functions F and K, Theorem 3.1 prescribes that (cf. Eq. (3.2))

$$H(W) = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{+\infty} \mathrm{d}W' \times \frac{K(\{W' - [\lim_{x\to+\infty}\phi(x) - \phi(x_\mathrm{min})]\}/\theta)/\sqrt{\theta}}{W'+W}, \tag{5.10}$$

whereas Corollary 3.1 gives (cf. Eq. (3.21))

$$K(W/\theta)/\sqrt{\theta} = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{+\infty} \mathrm{d}W' \times \frac{H(W' - [\lim_{x\to+\infty}\phi(x) - \phi(x_\mathrm{min})])}{W'+W}. \tag{5.11}$$

Finally, substituting Eq. (5.10) into Eq. (5.6) and Eq. (5.11) into Eq. (5.7), we respectively get

$$F_\mathrm{e}(W) = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{+\infty} \mathrm{d}W' \frac{1}{W+W'} \times [F_\mathrm{i}(\{W' - [\lim_{x\to+\infty}\phi(x) - \phi(x_\mathrm{min})]\}/\theta) - K(W' - [\lim_{x\to+\infty}\phi(x) - \phi(x_\mathrm{min})]/\theta)]/\sqrt{\theta} \tag{5.12}$$

and

$$F_{\mathrm{i}}(W/\theta)/\sqrt{\theta} = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{+\infty} \mathrm{d}W' \frac{1}{W+W'} \times$$
$$[F_{\mathrm{e}}(W' - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\min})]) -$$
$$H(W' - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\min})])]. \tag{5.13}$$

To demonstrate the advantages of writing the solutions of Poisson's equation as in Eqs. (5.12) and (5.13), We introduce the complex variable z and, in the z-plane cut along the real axis, we define the two functions

$$\mathfrak{F}_{\mathrm{e}}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}z' \frac{1}{z'+z} \times$$
$$\{F_{\mathrm{e}}(z' - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\min})]) -$$
$$H(z' - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\min})])\} \tag{5.14}$$

and

$$\mathfrak{F}_{\mathrm{i}}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}z' \frac{1}{z'+z} \times$$
$$\{F_{\mathrm{i}}(z' - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\min})]) -$$
$$K(z' - [\lim_{x\to+\infty} \phi(x) - \phi(x_{\min})])\}. \tag{5.15}$$

Then, because of the Sokhotskyi-Plemelj formulæ (cf. e.g. [16]), denoting by j the imaginary unit, Eqs. (5.6) and (5.7) can be rewritten as

$$F_{\mathrm{e}}(W) = \lim_{\epsilon\to0^+} [\mathfrak{F}_{\mathrm{e}}(W + \mathrm{j}\epsilon) + \mathfrak{F}_{\mathrm{e}}(W - \mathrm{j}\epsilon)] \tag{5.16}$$

and

$$F_{\mathrm{i}}(W/\theta)/\sqrt{\theta} = \lim_{\epsilon\to 0^+}[\mathfrak{F}_{\mathrm{i}}(W+\mathrm{j}\epsilon)+\mathfrak{F}_{\mathrm{i}}(W-\mathrm{j}\epsilon)]. \quad (5.17)$$

Eqs. (5.16) and (5.17) show that the electron and ion energy distribution functions may be recovered as the boundary value of two suitable sectionally analytical functions (cf. e.g. [16]). Conversely, given the positive quantity y, the functions $\mathfrak{F}_{\mathrm{e}}(w+\mathrm{j}y)$ and $\mathfrak{F}_{\mathrm{e}}(w-\mathrm{j}y)$ are the extension of the electron energy distribution function in the upper and lower complex energy plane. Likewise, the functions $\mathfrak{F}_{\mathrm{i}}(w+\mathrm{j}y)$ and $\mathfrak{F}_{\mathrm{i}}(w-\mathrm{j}y)$ are the extensions of the ion energy distribution function.

6 Conclusions

In the present work, we consider a plasma made of electrons and one species of fully ionized ions in electrostatic conditions. The distributions of the electron and ion charge density and of the electric potential they generate are governed by Poisson's equation. This equation may be written as an integral relation between the energy distribution functions of the particle species and the electric potential in the plasma. Our aim is to determine the energy distribution function of one of the particle species — say the α-distribution function of species α, with α=electron or ion — once the electric potential and the energy distribution function of the other species — say the β-distribution function of species β, with β=ion or electron — are known.

In this framework, Poisson's integral equation has been extensively investigated for over half a century. In Ref. [2], this equation was reduced to an Abel equation for the α-distribution function over a finite energy domain, subject to the condition that the same α-distribution function be known over the remaining energy domain. Recently, in Ref. [10] we showed that, if the electric potential waveform is a skew function of the space coordinate, the solution of the integral Poisson equation proposed in Ref. [2] necessarily leads to a *singular* α-distribution function.

In our present treatment, we refrain from assigning the distribution function over a portion of its energy domain. Rather, our aim is to determine the α-distribution function 'en bloc', i.e. over its *whole* energy domain, starting only from the knowledge of the electric potential and of the β-energy distribution function of the other species. To do so, we still cast Poisson's equation as an integral equation, but different from the mentioned Abel equation considered so far.

To solve the integral Poisson equation in favour of the α-distribution function, in Section 3 we prove an inversion theorem. In so doing, we use basic notions of Calculus, without reverting to the theory of integral transforms. This theorem, gives the α-distribution function over its *whole* energy domain, directly from the distribution function of species β. Specifically, the α-distribution function is found as the boundary value of a suitable sectionally analytical function. This latter function precisely extends the α-distribution function over its complex energy domain.

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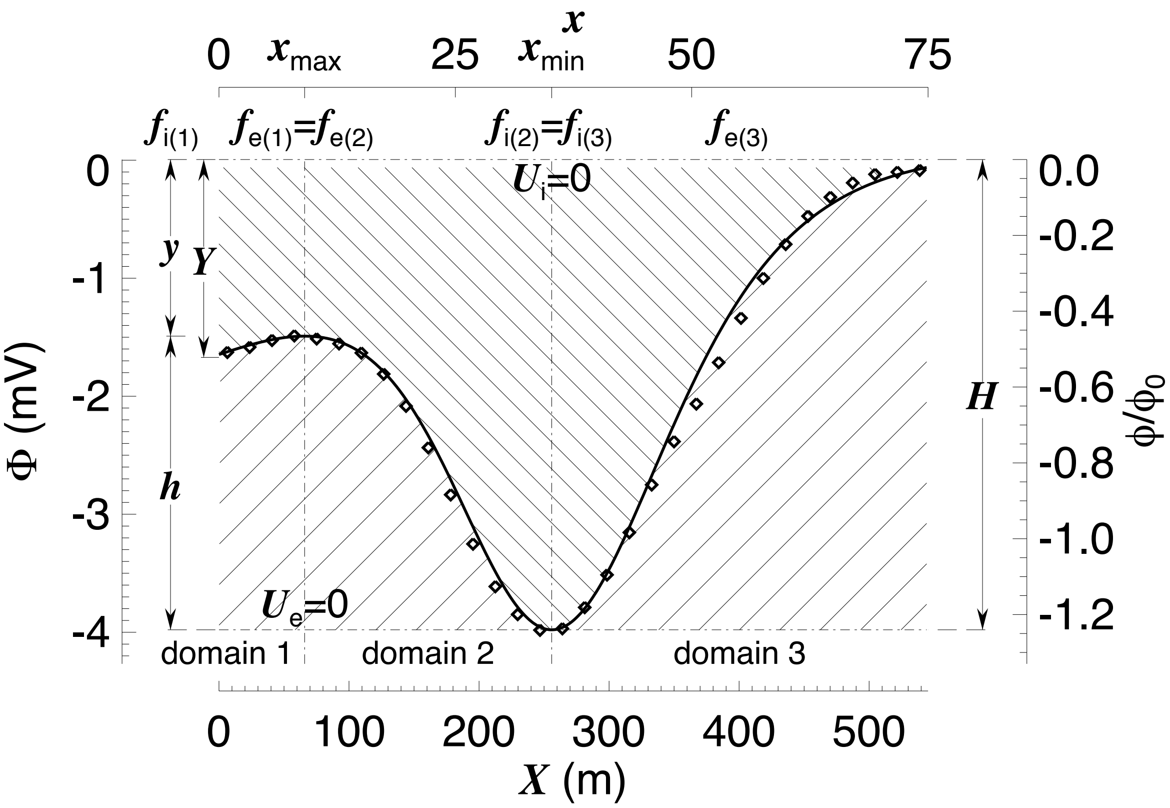
Figure caption

(1) Right and top axes: the observed (Φ) and normalized potential (ϕ) vs. the coordinate (X) and normalized coordinate (x). H, h, Y, y are potential jumps. The horizontal dash-dotted lines $U_\mathrm{e} = 0$ and $U_\mathrm{i} = 0$ denote the reference zero values of the electron and ion potential energies. The broad- and fine-hatched areas denote the position and energy values of the negative energy electrons and ions respectively. Also shown is the scheme for the electron and ion distributions f_e and f_i. Subscripts (1), (2) and (3) denote the domains where the potential is a monotonic function of position.



x
0
x_max
25
x_min
50
75
f_i(1)
f_e(1)=f_e(2)
f_i(2)=f_i(3)
f_e(3)
U_i=0
U_e=0
y
Y
h
H
Φ (mV)
φ/φ_0
0
-1
-2
-3
-4
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
domain 1
domain 2
domain 3
0
100
200
300
400
500
X (m)

Fig. 1.