Sectionally Analytic Distribution Functions of Electron and Ions associated with a BGK wave in a Collisionless Plasma

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Abstract

In this work, we invetigate further the singular nature [1] of the nonlinear stationary solutions [2] of the one dimensional Vlasov-Poisson system of equations, which governs a plasma made of electron and one species of fully ionised ions. First, we propose a new integral formulation of the Poisson equation and we prove two inversion lemmas for such equation. These lemmas allow us to write the solutions of the Poisson equation in such a way that the energy distribution of either of the particle species is related, in a straightforward way, to the energy distribution of the other species. Then, we show that these distribution functions are retrieved as boundary values of suitable sectionally analytic functions. These latter functions are shown to be the extension of the particle distributions into their respective complex energy domain.

Key words: plasma, oscillations, double layers, isolated electrostatic structures, electrostatic solitary waves, integral equations

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1 Introduction

In this report, we address the problem of solving Poisson's equation, which, in electrostatic conditions, relates the electric potential and charge distributions in a plasma composed of electrons and one species of fully ionised ions. In its simplest understanding, this equation is used as differential equation for the electric potential in the plasma and it is used to determine this latter, once the charge distributions of the electrons and of the ions are given.

The reverse approach may also be used [2]: Poisson's equation may be conceived as an integral equation for the energy distribution function of one of the particle species (say species β, usually electrons, e.g. [3]), whereas the electric potential and the distribution of the other particle species (say species α) are assumed to be known. In fact, only that fraction of the distribution corresponding to those particles of species β which are trapped by the electric field is found in this way, whereas the fraction corresponding to untrapped particles is assumed to be given.

This approach to Poisson's equation is appropriate e.g. when the energy distribution of the untrapped particles of species β, which are free to reach the plasma boundaries, is constrained by some boundary conditions. The solutions to Poisson's equation found in this way are known as BGK waves [2]. Usually, they are given in terms of the Abel transform of a quantity sometimes known as the "pseudopotential".

The BGK technique is used e.g. in Space Plasma Physics [4,5], where accu-

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rate data about electric potential waveforms are available (e.g. [6,7,8,9,10]) and where a judicious Ansätz may be formulated both about the energy distribution function of particles of species α and about the boundary energy distribution function of particles of species β.

Recently, in the attempt to interpret the electrostatic tripolar spikes observed in Refs. [6] as BGK waves, we proved that the smoothness of the waveform of the electric potential of the tripolar spike and the smoothness of the energy distributions of the particles are two incompatible requirements [1]: specifically, assuming that the potential waveform be smooth, we showed that both the electron and ion distribution functions are singular and that their singularities are of the jump and of the logarithmic type.

Our present work aims at furthering the understanding of these singularities. To do so, we first introduce a suitable integral representation of the electric charge distribution in the plasma and we establish "inversion rules" which allow us to do so. This representation permits us to write the BGK solutions of the integral Poisson equation directly in a simple and compact way by means of Hilbert transforms.

This approach has two advantages over the representation based on the Abel transform of the pseudopotential, which was used in all the works on BGK waves so far. On one hand, it allows the computation of the solution of Poisson's equation directly from the distribution function of the other particle species, without any need to compute the pseudopotential. Most important, the Hilbert transform representation allows a straightforward extension of the solution of the integral Poisson equation for *complex* values of the particle energy. The remarkable fact about this extended solution is that it is non singular: in fact it is a sectionally analytical function [11] of the complex energy, whose boundary value precisely gives the real-valued BGK solution of Poisson's equation.

2 Assumptions, notation and basic equations

According to the standard statistical treatment (cf. e.g. Ref. [12]), a complete description of the fully ionised plasma considered in our work may be given in terms of the electron and ion "one particle" velocity distribution functions and of the mean "self consistent" macroscopic force. Here we also assume that: (a) the velocities of the particles be largely non relativistic, so that the mean force is mainly electrostatic; (b) the plasma be homogeneous along two cartesian coordinates, so that the distribution functions and the mean force depend on one only space cartesian coordinate called X.

Under these conditions, we integrate the one particle velocity distribution functions (or velocity distribution functions for short) over the whole domain of the velocity coordinates orthogonal to X. The resulting integrated distributions will depend on X and on one velocity coordinate only — say $V_{\mathrm{e}X}$ for the electron velocity distribution, and $V_{\mathrm{i}X}$ for the ion velocity distribution. The mean electrostatic force may be evaluated by taking the X-derivative of a function of position. In particular, for the electrons, which have electric charge $-|e|$, this function is $|e|\Phi(X)$. The quantity $\Phi(X)$ is the observed electric potential in the plasma.

Now, let the electrons have mass m_{e} and, in the limit $X \to +\infty$, let them have boundary number density $n_{\mathrm{e}\infty}$ and boundary kinetic temperature $T_{\mathrm{e}\infty}$.

Then, adopting Gaussian units and setting the Boltzmann constant to one, we denote by

$$\lambda_{\mathrm{De}} = \surd\{T_{\mathrm{e}\infty}/[4\pi e^2 n_{\mathrm{e}\infty}]\} \tag{2.1}$$

the electron Debye length, by

$$x = X/\lambda_{\mathrm{De}} \tag{2.2}$$

the normalised space coordinate, by

$$v_{\mathrm{T_e}} = \surd[T_{\mathrm{e}\infty}/m_{\mathrm{e}}] \tag{2.3}$$

the electrons' boundary mean thermal speed, by

$$v_{\mathrm{e}} = V_{\mathrm{e}X}/v_{\mathrm{T_e}} \tag{2.4}$$

the normalised electron velocity coordinate along x and by

$$n_{\mathrm{e}\infty} f_{\mathrm{e}}(x, v_{\mathrm{e}})\mathrm{d}x\mathrm{d}v_{\mathrm{e}} \tag{2.5}$$

the probability of finding any one of the electrons having a position within a distance $\mathrm{d}x$ from x and a velocity component parallel to x within a distance $\mathrm{d}v_{\mathrm{e}}$ from v_{e}, irrespective of its velocity components orthogonal to x, irrespective of the position and velocity of all the other electrons, and irrespective of the position and velocity of all the ions.

Likewise, we assume that the ions have mass m_{i}, atomic number Z_{i} and electric charge $+Z_{\mathrm{i}}|e|$, and that charge neutrality is approached as $x \to \infty$, so that the ions' boundary number density there is

$$n_{\mathrm{i}\infty} = n_{\mathrm{e}\infty}/Z_{\mathrm{i}} \tag{2.6}$$

and, in the same limit, their boundary kinetic temperature is

$$T_{\mathrm{i}\infty} = \theta Z_{\mathrm{i}} T_{\mathrm{e}\infty} \; . \tag{2.7}$$

Then, we denote by

$$v_{\mathrm{T_i}} = \surd[T_{\mathrm{i}\infty}/m_{\mathrm{i}}] \tag{2.8}$$

the ions' boundary mean thermal velocity, by

$$v_{\mathrm{i}} = V_{\mathrm{i}X}/v_{\mathrm{T_i}} \tag{2.9}$$

the normalised ion velocity coordinate along x and by

$$[n_{\mathrm{e}\infty}/Z_{\mathrm{i}}] f_{\mathrm{i}}(x, v_{\mathrm{i}}) \mathrm{d}x \mathrm{d}v_{\mathrm{i}} \tag{2.10}$$

the probability finding any one of the ions having a position within a distance dx from x and a velocity component parallel to x within a distance dv_{i} from v_{i}, irrespective of its velocity components orthogonal to x, irrespective of the position and velocity of all the other ions, and irrespective of the position and velocity of all the electrons.

Last, we denote by

$$\phi(x) = |e|\Phi(\lambda_{\mathrm{De}} x)/T_{\mathrm{e}\infty}. \tag{2.11}$$

the electric potential normalised to the electron boundary kinetic temperature. A typical waveform of the electric potential in the is shown in Fig. 1.

Next we denote by $x = x_{\min}$ the position at which the electric potential has its absolute minimum (cf. Fig. 1) and by

$$-U_{\rm e}(\phi) = -[\phi - \phi(x_{\min})] \tag{2.12}$$

the rescaled potential energy of a test electron positioned at x to the electrons' boundary kinetic temperature. Likewise, being the absolute maximum of the electric potential located at $x = \infty$ (cf. Fig. 1), and being θ the ion to electron temperature ratio there (cf. Eq. (2.7)), we denote by

$$-U_{\rm i}(\phi) = -[\lim_{x\to+\infty} \phi(x) - \phi]/\theta \tag{2.13}$$

the rescaled potential energy of a test ion positioned at x normalised to the ions' boundary kinetic temperature.

In this way, the electron and ion electric potential energies $U_{\rm e}$ and $U_{\rm i}$ enjoy the following properties (cf. Eqs. (2.12) and (2.13) and Fig. 1)

$$U_{\rm e}(\phi(x)) \geq 0 \text{ for all values of } x, \tag{2.14a}$$

$$\max_{-\infty<x<+\infty}(-U_{\rm e}(\phi(x))) = -U_{\rm e}(\phi(x_{\min})) = 0, \tag{2.14b}$$

and

$$U_{\rm i}(\phi(x)) \geq 0 \text{ for all values of } x, \tag{2.15a}$$

$$\max_{-\infty<x<+\infty}(-U_{\rm i}(\phi(x))) = -U_{\rm i}(\lim_{x\to+\infty} \phi(x)) = 0. \tag{2.15b}$$

Next, we introduce the electron and ion space charge densities, which we denote by $|e|n_{\mathrm{e}\infty}\varrho_{\mathrm{e}}(x)$ and $|e|n_{\mathrm{e}\infty}\varrho_{\mathrm{i}}(x)$, and which, using the normalisations of the energy distributions introduced in Eqs. (2.5) and (2.10), we write as

$$\varrho_{\mathrm{e}}(x) = -\int_{-\infty}^{\infty} \mathrm{d}v'_{\mathrm{e}} f_{\mathrm{e}}(x, v'_{\mathrm{e}}), \tag{2.16}$$

and

$$\varrho_{\mathrm{i}}(x) = +\int_{-\infty}^{\infty} \mathrm{d}v'_{\mathrm{i}} f_{\mathrm{i}}(x, v'_{\mathrm{i}}). \tag{2.17}$$

Since we assumed that the plasma be in electrostatic conditions, Poisson's equation holds, which we write in the non dimensional form

$$\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} = -[\varrho_{\mathrm{e}}(x) + \varrho_{\mathrm{i}}(x)]. \tag{2.18}$$

Jointly with Eqs. (2.16) and (2.17), Eq. (2.18) relates the particle energy distribution functions f_{e} and f_{i} and the electric potential. Now, in the former two equations, we change the integration variable according to

$$v'_{\alpha} = \pm\sqrt{\{2[W'_{\alpha} + U_{\alpha}(\phi)]\}},\ \alpha = \mathrm{e}, \mathrm{i}, \tag{2.19}$$

the upper (respectively lower) sign holding in that part of the integrals, appearing in Eqs. (2.16) and (2.17), extending over the positive (respectively negative) range of v'_{α}. Given the bivariate distribution function $f_{\alpha}(x, v_{\alpha})$, and given the variable W_{α}, the transformation given in Eq. (2.19) introduces, in the integrals appearing on the right hand side of Eqs. (2.16) and (2.17), the quantities

$$F_\alpha(W_\alpha) = f_\alpha(x, +\surd\{2[W_\alpha + U_\alpha(\phi(x))]\}) +$$
$$f_\alpha(x, -\surd\{2[W_\alpha + U_\alpha(\phi(x))]\}),\ \alpha = \mathrm{e}, \mathrm{i}\ , \tag{2.20}$$

and reduces those integrals to

$$\varrho_\mathrm{e}(x) = -n_\mathrm{e}(\phi(x)), \tag{2.21a}$$
$$n_\mathrm{e}(\phi) = \int_{-U_\mathrm{e}(\phi)}^{\infty} \mathrm{d}W'_\mathrm{e} \frac{F_\mathrm{e}(W'_\mathrm{e})}{\surd\{2[W'_\mathrm{e} + U_\mathrm{e}(\phi)]\}}, \tag{2.21b}$$

and

$$\varrho_\mathrm{i}(x) = +n_\mathrm{i}(\phi(x)), \tag{2.22a}$$
$$n_\mathrm{i}(\phi) = \int_{-U_\mathrm{i}(\phi)}^{\infty} \mathrm{d}W'_\mathrm{i} \frac{F_\mathrm{i}(W'_\mathrm{i})}{\surd\{2[W'_\mathrm{i} + U_\mathrm{i}(\phi)]\}}. \tag{2.22b}$$

Here, it is agreed that, in order for the integrals in Eqs. (2.21b) and (2.22b) to converge, the functions $F_\alpha(W_\alpha)$ vanish faster than $1/\surd W_\alpha$ as $W_\alpha \to +\infty$. These equations reveal that the quantity $F_\alpha(W_\alpha)\mathrm{d}x\mathrm{d}W_\alpha$ amounts to $\surd\{2[W_\alpha + U_\alpha(\phi(x))]\}$ times the probability of finding a particle of species α having position within distance $\mathrm{d}x$ from x and total energy within distance $\mathrm{d}W_\alpha$ from W_α, irrespective of the sign of their velocity (cf. Eqs. (2.21b) and (2.22b)). In the following, the univariate functions F_e and F_i will be respectively known as the electron and ion *bi-directional energy distribution functions*.

An important property of the charge distributions written as in Eqs. (2.21) and (2.22) (rather than as in Eq. (2.16) and (2.17)) is that their values at position x are specified through the value of the potential ϕ at x. This is of course legitimate as long as $\phi(x)$ is a monotonic function of x. Wherever this condition fails, Eqs. (2.21) and (2.22) are still meaningful, in a piecewise sense, in each of the x-domains where $\phi(x)$ is a monotonic function of x (cf. Fig. 1).

If we also assume that, in these domains $\phi(x)$ is a be a continuous function of the position x, then, the inverse function ϕ^{-1} of ϕ certainly exists and the quantity $\mathrm{d}^2\phi(x)/\mathrm{d}x^2$ may well be conceived as a function of ϕ itself, which we call

$$\phi_{xx}(\psi) = [\mathrm{d}^2\phi(\xi)/\mathrm{d}\xi^2]|_{\xi=\phi^{-1}(\psi)}, \tag{2.23}$$

and which may be calculated by Lagrange's inversion formula (cf. e.g. [13]), provided the electric potential $\phi(x)$ is an analytic function of x.

In conclusion, the above transformations of the space charge densities $\varrho_{\mathrm{e}}(x)$ (cf. Eqs. (2.21)) and $\varrho_{\mathrm{i}}(x)$ (cf. Eqs. (2.22)) and of $\mathrm{d}^2\phi(x)/\mathrm{d}x^2$ (cf. Eq. (2.23)) allow us to rewrite Poisson's equation (cf. Eq. (2.18)) in the form

$$\phi_{xx}(\phi) = n_{\mathrm{e}}(\phi) - n_{\mathrm{i}}(\phi). \tag{2.24}$$

3 The inversion lemmas

The relation between the electron and ion distribution functions and the electric potential established in Section 2 (cf. Eqs. (2.21b), (2.22b) and (2.24)) amounts to a fully fledged integral equation which we need solve for the energy distribution function of one species subject to the following conditions: (a) the electric potential and the distribution function of the other species are given; (b) the distribution function to be determined is specified over the positive energy domain. Given these conditions, we wish to determine the distribution function over its own whole complex energy domain.

We divide this task into three parts. In the first part (cf. Section 4), Poisson's equation will be transformed from its fractional integral formulation (cf. Eqs. (2.21b), (2.22b) and (2.24)) to a pair of new, entirely equivalent fractional integral equations. In the second part (cf. Section 5), these fractional equations will be solved in favour of the electron or ion energy distribution functions. In the third part (cf. Section 6) these functions will be analytically continued over their whole respective complex energy planes.

To carry out the above tasks, we need two important lemmas, to which we devote this section.

Lemma 3.1 *Let a and b be real constants, such that $a < b$ and let $f(w)$ and $g(w)$ be two functions which are Hölder continuous over the interval $a < w < b$. Provided all the integrals converge, we have the following results.*

(i) For $a < w < b$, the identity

$$\int_a^w \mathrm{d}u \frac{f(u)}{\sqrt{[w-u]}} = \int_w^b \mathrm{d}u \frac{g(u)}{\sqrt{[u-w]}} \tag{3.1}$$

holds if and only if, denoting by P *the Cauchy principal value of an integral,*

$$f(w) = \frac{1}{\pi} \mathrm{P} \int_a^b \mathrm{d}t \frac{\sqrt{[t-a]}}{\sqrt{[w-a]}} \frac{g(t)}{t-w}. \tag{3.2}$$

(ii) For $a < w < b$, Eq. (3.2) is equivalent to

$$f(w) = \frac{1}{\pi} \frac{A}{\sqrt{[w-a]}} + \frac{1}{\pi} \mathrm{P} \int_a^b \mathrm{d}t \frac{\sqrt{[w-a]}}{\sqrt{[t-a]}} \frac{g(t)}{t-w}, \tag{3.3a}$$

$$\text{where} \quad A = \int_a^b \mathrm{d}u \frac{g(u)}{\sqrt{[u-a]}}. \tag{3.3b}$$

(iii) The results in points (i)–(ii) hold also for $b \to +\infty$; they hold also for $a \to -\infty$.

Proof. To carry out the proof of the lemma, we introduce the function

$$\lambda(\zeta,\xi,\eta) = \int_{\zeta}^{\xi} \mathrm{d}\xi' \frac{1}{\sqrt{[\xi-\xi']}\sqrt{[\eta-\xi']}},$$
$$\text{for } \eta > \xi > \zeta, \tag{3.4}$$

whose properties we establish beforehand. Specifically, since in the integral on the right hand side of Eq. (3.4), $\eta > \xi$, we set $\xi' = \xi - [\eta - \xi]s^2$. In this way, Eq. (3.4) gives

$$\lambda(\zeta,\xi,\eta) = \int_{0}^{\frac{\sqrt{[\xi-\zeta]}}{\sqrt{[\eta-\xi]}}} \mathrm{d}s \frac{1}{\sqrt{[1+s^2]}} =$$
$$2\,\mathrm{arcsinh}\left(\frac{\sqrt{[\xi-\zeta]}}{\sqrt{[\eta-\xi]}}\right), \quad \text{for } \eta > \xi > \zeta. \tag{3.5}$$

We now revert to the proof of the lemma and we show first that Eq. (3.2) necessarily follows from Eq. (3.1). Specifically, we regard Eq. (3.1) as an Abel equation for $f(w)$, whose solution is (cf. e.g. [14]))

$$f(w) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}w}\int_{a}^{w} \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \int_{u}^{b} \mathrm{d}t \frac{g(t)}{\sqrt{[t-u]}}. \tag{3.6}$$

Since, by assumption, $w < b$, our idea is to split the inner t-integral, on the right hand side of Eq. (3.6), into the sum of an integral running from u to w and an integral running from w to b. Actually, this step must be taken with some care. To do so, we introduce a real number ϵ and we let the first of those two latter integrals run from u up to $w-\epsilon$, and the second one run from $w+\epsilon$

to b. Then, we develop our proof for $\epsilon \neq 0$ and we recover our final results by taking the limit for $\epsilon \to 0$. According to the above instructions, the double integral appearing on the right hand side of Eq. (3.6) splits as follows:

$$f(w) =$$
$$\frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}w} \left\{ \int_a^{w-\epsilon} \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \int_u^{w-\epsilon} \mathrm{d}t \frac{g(t)}{\sqrt{[t-u]}} + \right.$$
$$\int_{w-\epsilon}^{w} \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \int_u^{w-\epsilon} \mathrm{d}t \frac{g(t)}{\sqrt{[t-u]}} +$$
$$\left. \int_a^{w} \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \int_{w+\epsilon}^{b} \mathrm{d}t \frac{g(t)}{\sqrt{[t-u]}} \right\}. \tag{3.7}$$

In the first two integrals appearing on the right hand side of Eq. (3.7), we interchange the order of the t- and u-integration according to Fubini's theorem; in the third integral, such interchange is achieved in a straightforward way. Then, using the function λ, defined in Eq. (3.4), Eq. (3.7) reduces to

$$f(w) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}w} \left\{ \int_a^{w-\epsilon} \mathrm{d}t\, g(t) \lambda(a,t,w) + \right.$$
$$\left. \int_{w-\epsilon}^{w} \mathrm{d}t\, g(t) \lambda(w,t,w) + \int_{w+\epsilon}^{b} \mathrm{d}t\, g(t) \lambda(a,w,t) \right\}. \tag{3.8}$$

In the second integral appearing on the right hand side of Eq. (3.8), the third argument of the function λ always exceeds its second argument, and the value of λ given by Eq. (3.5) may be used, according to which $\lambda(w,t,w) = \mathrm{j}\pi$. Thus, since, by assumption, the function g is continuous, the contribution given by that integral to $f(w)$ is

$$\frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}w} \int_{w-\epsilon}^{w} \mathrm{d}t\, g(t) \lambda(w,t,w) = \mathrm{j} \lim_{\epsilon \to 0} [g(w) - g(w-\epsilon)] = 0. \tag{3.9}$$

On the other hand, taking the w-derivative of the first and third term on the right hand side of Eq. (3.8), we find

$$f(w) = \frac{1}{\pi} \lim_{\epsilon \to 0} \left\{ \int_a^{w-\epsilon} \mathrm{d}t\, g(t) \frac{\mathrm{d}}{\mathrm{d}w} \lambda(a,t,w) + \right.$$
$$g(w-\epsilon)\lambda(a,w-\epsilon,w) - g(w+\epsilon)\lambda(a,w,w+\epsilon) +$$
$$\left. \int_{w+\epsilon}^{b} \mathrm{d}t\, g(t) \frac{\mathrm{d}}{\mathrm{d}w} \lambda(a,w,t) \right\}. \tag{3.10}$$

Now, in the second and third term appearing on the right hand side of Eq. (3.10), the third argument of the function λ also exceeds its second argument, and the value of λ given by Eq. (3.5) may again be used. Also, since $\epsilon > 0$, we have

$$\lambda(a,w-\epsilon,w) = -\ln(\epsilon) + \ln(2[w-a]) + O(\epsilon),$$
$$\lambda(a,w,w+\epsilon) = -\ln(\epsilon) + \ln(2[w-a]) + O(\epsilon),$$
$$\text{for } \epsilon \simeq 0,\ \epsilon > 0. \tag{3.11}$$

We thus see that, since, by assumption, the function g is continuous, the contribution given by the second and third term appearing on the right hand side of Eq. (3.10) to $f(w)$ is

$$\frac{1}{\pi} \lim_{\epsilon \to 0} [g(w-\epsilon)\lambda(a,w-\epsilon,w) - g(w+\epsilon)\lambda(a,w,w+\epsilon)] = 0. \tag{3.12}$$

Furthermore, in both integrals appearing on the right hand side of Eq. (3.10) the third argument of the function λ also exceeds its second argument. Then, again using Eq. (3.5), the two w-derivatives of λ, appearing in those integrals, turn out to be

$$\frac{\mathrm{d}}{\mathrm{d}w}\lambda(a,t,w) = \frac{\mathrm{d}}{\mathrm{d}w} 2\,\mathrm{arcsinh}\left(\frac{\sqrt{[t-a]}}{\sqrt{[w-t]}}\right) =$$

$$2\frac{\sqrt{[w-t]}}{\sqrt{[w-a]}}\sqrt{[t-a]}\frac{-1/2}{[w-t]^{3/2}} =$$

$$\frac{\sqrt{[t-a]}}{\sqrt{[w-a]}}\frac{1}{t-w} \tag{3.13}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}w}\lambda(a,w,t) = \frac{\mathrm{d}}{\mathrm{d}w} 2\,\mathrm{arcsinh}\left(\frac{\sqrt{[w-a]}}{\sqrt{[t-w]}}\right) =$$

$$2\frac{\sqrt{[t-w]}}{\sqrt{[t-a]}}\frac{1}{2}\frac{\sqrt{[t-w]}}{\sqrt{[w-a]}}\frac{t-a}{[t-w]^2} =$$

$$\frac{\sqrt{[t-a]}}{\sqrt{[w-a]}}\frac{1}{t-w}. \tag{3.14}$$

Finally, substituting Eqs. (3.12), (3.13) and (3.14) into the right hand side of Eq. (3.10), we recover Eq. (3.2), as desired:

$$f(w) = \frac{1}{\pi}\lim_{\epsilon\to 0}\left\{\int_a^{w-\epsilon}\mathrm{d}t\, g(t)\frac{\sqrt{[t-a]}}{\sqrt{[w-a]}}\frac{1}{t-w}+\right.$$

$$\left.\int_{w+\epsilon}^{b}\mathrm{d}t\, g(t)\frac{\sqrt{[t-a]}}{\sqrt{[w-a]}}\frac{1}{t-w}\right\} =$$

$$\frac{1}{\pi}\mathrm{P}\int_a^b \mathrm{d}t\frac{\sqrt{[t-a]}}{\sqrt{[w-a]}}\frac{g(t)}{t-w}. \tag{3.15}$$

This concludes the proof of the first part of point (i) of the lemma. The proof of the second part of that point consists in showing that Eq. (3.1) necessarily follows from Eq. (3.2). To do so, we substitute the right hand side of Eq.

(3.2) directly into the left hand side of Eq. (3.1) and interchange the order of integration, which we can do, provided the integrals be understood as Cauchy's principal value integrals. In this way, we get

$$\int_a^w \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \frac{1}{\pi} \mathrm{P} \int_a^b \mathrm{d}t \frac{\sqrt{[t-a]}}{\sqrt{[u-a]}} \frac{g(t)}{t-u} =$$

$$\int_a^b \mathrm{d}t \sqrt{[t-a]} g(t) \frac{1}{\pi} \mathrm{P} \int_a^w \mathrm{d}u \frac{1}{\sqrt{[w-u]}\sqrt{[u-a]}} \frac{1}{t-u}. \tag{3.16}$$

In the inner integral appearing on the right hand side of Eq. (3.16), we change the integration variable according to $u = [w + as^2]/[1 + s^2]$, to get

$$\frac{1}{\pi} \mathrm{P} \int_a^w \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \frac{1}{\sqrt{[u-a]}} \frac{1}{t-u} =$$

$$\frac{1}{\pi} \mathrm{P} \int_0^\infty \mathrm{d}s \frac{2}{[t-w] + [t-a]s^2}. \tag{3.17}$$

In the integral on the right hand side of Eq. (3.17), we must take into account that the quantity t is the integration variable of the outer integral on the right hand side of Eq. (3.16), and thus t exceeds a, the lower limit of integration there. Therefore, in the denominator of the integrand in the integral on the right hand side of Eq. (3.17), two cases arise: if $t < w$, the signs of the two terms in the denominator disagree and the principal part of the integral itself obviously vanishes; if $t > w$, the two signs agree and the integral reduces to $1/\{\sqrt{[t-w]}\sqrt{[t-a]}\}$. Substituting this result into Eq. (3.16), only the contribution

$$\int_w^b \mathrm{d}t \sqrt{[t-a]} g(t) \frac{1}{\sqrt{[t-w]}\sqrt{[t-a]}} \tag{3.18}$$

survives, which coincides with the right hand side of Eq. (3.1), as desired.

This concludes the proof of point (i) of the lemma. The proof of point (ii) consists in reducing Eq. (3.2) to Eq. (3.3), which we do by means of the obvious identity

$$\frac{\sqrt{[u-a]}}{\sqrt{[w-a]}}\frac{1}{u-w} = \frac{1}{\sqrt{[u-a]}}\left\{\frac{1}{\sqrt{[w-a]}} + \frac{\sqrt{w-a]}}{u-w}\right\}. \tag{3.19}$$

To prove point (iii), we work out the above arguments using finite values for the constants a and b and then let them take arbitrarily large values in Eqs. (3.1), (3.2) and (3.3a). In so doing, we must ensure that, as $|w| \to +\infty$, the functions $f(w)$ and $g(w)$ vanish faster than a suitable negative power of $|w|$, in such a way that the integrals in the above formulæ converge, as assumed in the lemma.

□

Lemma 3.1 provide a practical means to extract the function f out of Eq. (3.1) once the function g is known. The following corollary precisely allows the reverse operation.

Corollary 3.1 *Let a and b be real constants, such that $a < b$ and let $f(w)$ and $g(w)$ be two functions which are Hölder continuous over the interval $a < w < b$. Then, provided all the integrals converge, we have the following results.*

(i) For $a < w < b$, the identity

$$\int_a^w \mathrm{d}u \frac{f(u)}{\sqrt{[w-u]}} = \int_w^b \mathrm{d}u \frac{g(u)}{\sqrt{[u-w]}} \tag{3.20}$$

holds if and only if, denoting by P *the Cauchy principal value of an inte-*

gral,

$$g(w) = -\frac{1}{\pi} \mathrm{P} \int_a^b \mathrm{d}t \frac{\sqrt{[b-t]}}{\sqrt{[b-w]}} \frac{f(t)}{t-w}. \tag{3.21}$$

(ii) For $a < w < b$, Eq. (3.21) is equivalent to

$$g(w) = \frac{1}{\pi} \frac{B}{\sqrt{[b-w]}} - \frac{1}{\pi} \mathrm{P} \int_a^b \mathrm{d}t \frac{\sqrt{[b-w]}}{\sqrt{[b-t]}} \frac{f(t)}{t-w}, \tag{3.22a}$$

$$\text{where} \quad B = \int_a^b \mathrm{d}u \frac{f(u)}{\sqrt{[b-u]}}. \tag{3.22b}$$

(iii) The statements in points (i)–(ii) also hold if $a \to -\infty$; they also hold if $b \to +\infty$.

Proof.

The proof of the first part of point (i) consist in showing that Eq. (3.21) necessarily follows from Eq. (3.20). To do so, in the integrals appearing on each side of Eq. (3.20), we set $w = b + a - y$ and we change the integration variable according to $u = b + a - s$. In this way, Eq. (3.20) reduces to

$$\int_a^y \mathrm{d}s \frac{g(b+a-s)}{\sqrt{[y-s]}} = \int_y^b \mathrm{d}s \frac{f(b+a-s)}{\sqrt{[s-y]}}. \tag{3.23}$$

Now, this relation between the functions $g(b+a-y)$ and $f(b+a-y)$ has the same structure of the relation between $f(w)$ and $g(w)$ given in Lemma 3.1 (cf. Eq. (3.20)), provided the role of f and g be interchanged. Therefore, Eq. (3.23) directly leads to (cf. Eq. (3.2))

$$g(b+a-y) = \frac{1}{\pi} \mathrm{P} \int_a^b \mathrm{d}u \frac{\sqrt{[u-a]}}{\sqrt{[y-a]}} \frac{f(b+a-u)}{u-y}. \tag{3.24}$$

If, in the integral on the right hand side of Eq. (3.24), we set $y = b + a - w$ and if we change the integration variable according to $u = b + a - t$, we recover Eq. (3.21), as desired.

This concludes the proof of the first part of point (i) of the lemma. The proof of the second part of that point consists in showing that Eq. (3.20) necessarily follows from Eq. (3.21). But this is entirely equivalent to showing that Eq. (3.23) necessarily follows from Eq. (3.24), a task which is accomplished by the same procedure used in Lemma 3.1 to show that Eq. (3.1) follows from Eq. (3.2).

This concludes the proof of point (i) of the lemma. The proof of point (ii) consists in reducing Eq. (3.21) to Eq. (3.22), which we do by means of the obvious identity

$$\frac{\sqrt{[b-u]}}{\sqrt{[b-w]}}\frac{1}{u-w} = \frac{1}{\sqrt{[b-u]}}\left\{\frac{1}{\sqrt{[b-w]}} - \frac{\sqrt{[b-w]}}{u-w}\right\}. \tag{3.25}$$

To prove point (iii), we work out the above arguments using finite values for the constants a and b and then let them take arbitrarily large values in Eqs. (3.20), (3.21) and (3.22a). In so doing, we must ensure that, as $|w| \to +\infty$, the functions $f(w)$ and $g(w)$ vanish faster than a suitable negative power of $|w|$, in such a way that the integrals in the above formulæ converge, as assumed in the lemma.

□

Our next lemma is the following

Lemma 3.2 *Let a, b and c be real constants, such that $-c < a < b$. Let* *the function $f(w)$ be integrable for $-a < w < c$ and let the function $g(w)$ be integrable for $-c < w < b$. Provided the integrals in Eq. (3.26) below converge, we have the following results.*

(i) For $-a < w < c$, the identity

$$\int_{-a}^{w} \mathrm{d}u \frac{f(u)}{\sqrt{[w-u]}} = \int_{-w}^{b} \mathrm{d}u \frac{g(u)}{\sqrt{[w+u]}} \tag{3.26}$$

holds if and only if

$$f(w) = g(-w) + \frac{1}{\pi} \int_{a}^{b} \mathrm{d}u \frac{\sqrt{[u-a]}}{\sqrt{[w+a]}} \frac{g(u)}{u+w}. \tag{3.27}$$

(ii) For $-a < w < c$, Eq. (3.27) is equivalent to

$$f(w) = \frac{1}{\pi} \frac{C}{\sqrt{[w+a]}} + g(-w) - \frac{1}{\pi} \int_{a}^{b} \mathrm{d}u \frac{\sqrt{[w+a]}}{\sqrt{[u-a]}} \frac{g(u)}{u+w}, \tag{3.28a}$$

$$\text{where} \quad C = \int_{a}^{b} \mathrm{d}u \frac{g(u)}{\sqrt{[u-a]}}. \tag{3.28b}$$

(iii) The results in points (i)–(ii) also hold if $b \to +\infty$; they hold also if $c \to +\infty$.

Proof. To carry out the proof of the lemma, we introduce the function

$$\mu(w,u) = \int_{-a}^{w} \mathrm{d}t \frac{1}{\sqrt{[w-t]}\sqrt{[u+t]}},$$

$$\text{for } w > -a, \; u > a, \tag{3.29}$$

whose properties we establish beforehand. Specifically, since in the integral appearing on the right hand side of Eq. (3.29) $w + u > 0$, we change the integration variable according to $t = w - [w+u]s^2$, to get

$$\mu(w,u) = 2\int_0^{\frac{\sqrt{[w+a]}}{\sqrt{[w+u]}}} \mathrm{d}s \frac{1}{\sqrt{[1-s^2]}} =$$

$$2\arcsin\left(\frac{\sqrt{[w+a]}}{\sqrt{[w+u]}}\right), \quad \text{for } w > -a,\ u > a. \tag{3.30}$$

In the following, the w-derivative of $\mu(w,u)$ will be needed, which turns out to be

$$\frac{\mathrm{d}}{\mathrm{d}w}\mu(w,u) = \frac{\mathrm{d}}{\mathrm{d}w} 2\arcsin\left(\frac{\sqrt{[w+a]}}{\sqrt{[w+u]}}\right) =$$

$$2\frac{\sqrt{[w+u]}}{\sqrt{[u-a]}} \frac{1}{2} \frac{\sqrt{w+u]}}{\sqrt{[w+a]}} \frac{u-a}{[w+u]^2} = \frac{\sqrt{[u-a]}}{\sqrt{[w+a]}} \frac{1}{w+u}. \tag{3.31}$$

We now revert to the proof of the lemma and we show first that Eq. (3.27) necessarily follows from Eq. (3.26). Specifically, since, by assumption, $b > a$, we write the integral on the right hand side of Eq. (3.26) as the sum of an integral running from $-w$ to a and an integral running from a to b. After changing the integration variable in the first of these two latter integrals, according to $u \mapsto -u$, we subtract it from both sides of Eq. (3.26) itself, to get

$$\int_{-a}^{w} \mathrm{d}u \frac{f(u) - g(-u)}{\sqrt{[w-u]}} = \int_a^b \mathrm{d}u \frac{g(u)}{\sqrt{[w+u]}}. \tag{3.32}$$

Eq. (3.32) is an Abel equation for the quantity $f(w) - g(-w)$ and its solution is (cf. e.g. [14]))

$$f(w) - g(-w) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}w}\int_{-a}^{w} \mathrm{d}t \frac{1}{\sqrt{[w-t]}} \int_a^b \mathrm{d}u \frac{g(u)}{\sqrt{[t+u]}}. \tag{3.33}$$

Since, by assumption, $w > -a$, in the outer integral appearing on the right hand side of Eq. (3.33), t exceeds $-a$, the lower limit of integration there; also, the assumption $b > a$, implies that, in the inner integral, u exceeds a, the lower limit of integration there: thus we have $t + u > 0$, and, in the double integral appearing on the right hand side of Eq. (3.33) we may interchange the order of the t- and u-integration. Further using the function $\mu(w, u)$ defined in Eq. (3.29), and interchanging the order of the w-differentiation and u-integration operations, Eq. (3.33) reduces to

$$f(w) - g(-w) = \frac{1}{\pi} \int_a^b \mathrm{d}u \, g(u) \frac{\mathrm{d}}{\mathrm{d}w} \mu(w, u). \tag{3.34}$$

In the integral appearing on the right hand side of Eq. (3.34), the assumption $b > a$ implies that $u > a$, and since, also by assumption, $w > -a$, the value of $\mathrm{d}\mu(w, u)/\mathrm{d}w$ given by Eq. (3.31) can be used. Substituting this value in the right hand side of Eq. (3.34), this latter equation precisely reduces to Eq. (3.27), as desired.

This concludes the proof of the first part of point (i) of the lemma. The proof of the second part of that point consists in showing that Eq. (3.26) necessarily follows from Eq. (3.27). To do so, we substitute the quantity $f(w)$, given by Eq. (3.27), in the left hand side of Eq. (3.26)), to get

$$\int_{-a}^{w} \mathrm{d}u \frac{f(u)}{\sqrt{[w-u]}} = \int_{-a}^{w} \mathrm{d}u \frac{g(-u)}{\sqrt{[w-u]}} + \\ \frac{1}{\pi} \int_{-a}^{w} \mathrm{d}u \frac{1}{\sqrt{[w-u]}} \int_a^b \mathrm{d}t \frac{\sqrt{[t-a]}}{\sqrt{[u+a]}} \frac{g(t)}{t+u}. \tag{3.35}$$

In the second double integral on the right hand side of Eq. (3.35), the assumptions $w > -a$ and $b > a$ respectively imply that $u > -a$ and $t > a$, so that

$t+u>0$: thus we may interchange the order of the t- and u-integration, which reduces Eq. (3.35) to

$$\frac{1}{\pi}\int_{-a}^{w}\mathrm{d}u\frac{1}{\sqrt{[w-u]}}\int_{a}^{b}\mathrm{d}t\frac{\sqrt{[t-a]}}{\sqrt{[u+a]}}\frac{g(t)}{t+u}=$$
$$\frac{1}{\pi}\int_{a}^{b}\mathrm{d}t\,g(t)\sqrt{[t-a]}\int_{-a}^{w}\mathrm{d}u\frac{1}{\sqrt{[w-u]}}\frac{1}{\sqrt{[u+a]}}\frac{1}{t+u}. \tag{3.36}$$

In the inner u-integral on the right hand side of Eq. (3.36), we change the integration variable according to $u=[w-as^2]/[1+s^2]$, to get

$$\int_{-a}^{w}\mathrm{d}u\frac{1}{\sqrt{[w-u]}}\frac{1}{\sqrt{[u+a]}}\frac{1}{t+u}=$$
$$\int_{0}^{\infty}\mathrm{d}s\frac{2}{[t+w]+[t-a]s^2}. \tag{3.37}$$

In the integral appearing on the right hand side of Eq. (3.37), we must take into account that the quantity t is the integration variable of the outer integral appearing on the right hand side of Eq. (3.36) and thus t exceeds a, the lower integration bound there; also, since, by assumption, $w>-a$, we have $t+w>0$, and the signs of the two terms in the denominator of the integrand in the integral appearing on the right hand side of Eq. (3.37) agree: that integral thus reduces to $\pi/\{\sqrt{[t+w]}\sqrt{[t-a]}\}$. Substituting this result into the right hand side of Eq. (3.36), and this latter into Eq. (3.35), we find

$$\int_{-a}^{w}\mathrm{d}u\frac{f(u)}{\sqrt{[w-u]}}=\int_{-a}^{w}\mathrm{d}u\frac{g(-u)}{\sqrt{[w-u]}}+$$
$$\int_{a}^{b}\mathrm{d}t\frac{g(t)}{\sqrt{[t+w]}}=\int_{-w}^{b}\mathrm{d}t\frac{g(t)}{\sqrt{[t+w]}}, \tag{3.38}$$

which precisely reproduces Eq. (3.26), as desired.

This concludes the proof of point (i) of the lemma. The proof of point (ii) consists in reducing Eq. (3.27) to Eq. (3.28), which we do by means of the obvious identity

$$\frac{\sqrt{[u-a]}}{\sqrt{[w+a]}}\frac{1}{u+w} = \frac{1}{\sqrt{[u-a]}}\left\{\frac{1}{\sqrt{[w+a]}} - \frac{\sqrt{[w+a]}}{u+w}\right\}. \tag{3.39}$$

To prove point (iii), we work out the above arguments using finite values for the constants b and c and then let them take arbitrarily large values in Eqs. (3.26), (3.27) and (3.28a). In so doing, we must ensure that, as $|w| \to +\infty$, the functions $f(w)$ and $g(w)$ vanish faster than a suitable negative power of $|w|$, in such a way that the integrals in the above formulæ converge, as assumed in the lemma.

□

We further notice that an immediate consequence of point (ii) of Lemma 3.2 is that

$$f(w) \sim \frac{C}{\sqrt{[w+a]}}, \quad \text{for } w \simeq -a,$$
$$\text{if} \quad C = \int_a^b \mathrm{d}u \frac{g(u)}{\sqrt{[u-a]}} \neq 0. \tag{3.40}$$

Indeed, the assumption that the integral on the right hand side of Eq. (3.28a) converge, implies that the function $g(-w)$ may diverge at most as $1/|w+a|^p$, for $w \simeq -a$, with $p < 1/2$. This ensures that, on the right hand side of that equation, the first term is the leading one for $w \simeq -a$.

Lemma 3.2 provides a practical means to extract the function f out of Eq.

(3.26) once the function g is known. The following corollary allows the reverse operation.

Corollary 3.2 *Let a, b and c be real constants, such that $-c < a < b$. Let the function $f(w)$ be integrable for $-b < w < c$ and let the function $g(w)$ be integrable for $-c < w < b$. Provided the integrals in Eq. (3.41) below converge, we have the following results.*

(i) For $-w < b$, The identity

$$\int_{-a}^{w} \mathrm{d}u \frac{f(u)}{\sqrt{[w-u]}} = \int_{-w}^{b} \mathrm{d}u \frac{g(u)}{\sqrt{[w+u]}} \tag{3.41}$$

holds if and only if, denoting by j the imaginary unit,

$$g(w) = -f(-w) + \frac{1}{\mathrm{j}\pi} \int_{-b}^{-a} \mathrm{d}u \frac{\sqrt{[u+b]}}{\sqrt{[b-w]}} \frac{f(u)}{u+w},$$

$$\text{for } -c < w < a, \tag{3.42a}$$

$$g(w) = \frac{1}{\mathrm{j}\pi} \mathrm{P} \int_{-b}^{-a} \mathrm{d}u \frac{\sqrt{[u+b]}}{\sqrt{[b-w]}} \frac{f(u)}{u+w},$$

$$\text{for } a < w < b. \tag{3.42b}$$

(ii) Eq. (3.42) is equivalent to

$$g(w) = \frac{1}{\pi} \frac{D}{\sqrt{[b-w]}} - f(-w) + \frac{1}{\mathrm{j}\pi} \int_{-b}^{-a} \mathrm{d}u \frac{\sqrt{[b-w]}}{\sqrt{[u+b]}} \frac{f(u)}{u+w},$$

$$\text{for } -c < w < a, \tag{3.43a}$$

$$g(w) = \frac{1}{\pi} \frac{D}{\sqrt{[b-w]}} + \frac{1}{\mathrm{j}\pi} \mathrm{P} \int_{-b}^{-a} \mathrm{d}u \frac{\sqrt{[b-w]}}{\sqrt{[u+b]}} \frac{f(u)}{u+w},$$

$$\text{for } a < w < b, \tag{3.43b}$$

$$\text{where} \quad D = \frac{1}{\mathrm{j}} \int_{-b}^{-a} \mathrm{d}u \frac{f(u)}{\sqrt{[u+b]}}. \tag{3.43c}$$

(iii) The results in points (i)–(ii) hold also if $b \to +\infty$; they hold also if $c \to +\infty$.

Proof. We preliminary rearrange Eq. (3.41) as follows. In the integral appearing on its left hand side, we change the integration variable according to the Möbius transformation $u = -[b + at]/[t + 1]$; likewise in the integral appearing on its right hand side, we change the integration variable according to $u = -[b - at]/[t - 1]$; we also introduce the quantity

$$y = \frac{w + b}{w + a}. \tag{3.44}$$

In this way, denoting by j the imaginary unit, Eq. (3.41) takes the form

$$\int_0^y \mathrm{d}t \frac{g(-[b - at]/[t - 1])\sqrt{[t - 1]}}{\mathrm{j}[t - 1]^2\sqrt{[y - t]}} =$$

$$\int_{-y}^{+\infty} \mathrm{d}t \frac{f(-[b + at]/[t + 1])\sqrt{[t + 1]}}{[t + 1]^2\sqrt{[t + y]}}. \tag{3.45}$$

In Eq. (3.45), two cases arise, according to the sign of y. If $y > 0$, Eq. (3.45) precisely appears in the form considered in Lemma 3.2 (cf. Eq. (3.26), now with $a = 0$ and $b = +\infty$), provided, of course, the role of f and g be interchanged. Its solution is (cf. Eq. (3.27))

$$\frac{g([-[b-ay]/[y-1])\sqrt{[y-1]}}{\mathrm{j}[y-1]^2} -$$

$$\frac{f(-[b-ay]/[-y+1])\sqrt{[-y+1]}}{[-y+1]^2} =$$

$$\frac{1}{\pi}\int_0^{\infty} \mathrm{d}t \frac{\sqrt{t}}{\sqrt{y}} \frac{f(-[b+at]/[t+1])\sqrt{[t+1]}}{[t+1]^2} \frac{1}{t+y},$$

for $y > 0$. (3.46)

In the integral appearing on the right hand side of Eq. (3.46), we change the integration variable according to $t = -[u+b]/[u+a]$ and we take into account that: (*i*) $y = [w+b]/[w+a]$ (cf. Eq. (3.44)); (*ii*) Eq. (3.46) holds only for $y > 0$; (*iii*) since, by assumption, $-c < a < b$ and $-w < b$, then $y > 0 \Leftrightarrow -c < -w < a < b$. In this way, Eq. (3.46) reduces to

$$g(-w) + f(w) = \frac{1}{\mathrm{j}\pi}\int_{-b}^{-a} \mathrm{d}u \frac{\sqrt{[b+u]}}{\sqrt{[b+w]}} \frac{f(u)}{u-w},$$

for $-c < -w < a$. (3.47)

This shows that Eq. (3.42a), which we recover from Eq. (3.47) by simply letting $w \mapsto -w$, necessarily follows from Eq. (3.46) and in turn from Eq. (3.41), as desired.

On the other hand, if $y < 0$, we again use Eq. (3.45) and, on its right hand side, we change the integration variable according to $t \mapsto -t$ to get

$$-\int_{y}^{0} \mathrm{d}t \frac{g(-[b-at]/[t-1])\sqrt{[t-1]}}{[t-1]^2\sqrt{[t-y]}} =$$

$$\int_{-\infty}^{y} \mathrm{d}t \frac{f(-[b-at]/[1-t])/\sqrt{[1-t]}}{[1-t]^2\sqrt{[y-t]}}. \tag{3.48}$$

This equation appears exactly in the form considered in Corollary 3.1 (cf. Eq. (3.20), now with $a \to -\infty$ and $b = 0$). Its solution is (cf. Eq. (3.21))

$$-\frac{g(-[b-ay]/[y-1])\sqrt{[y-1]}}{[y-1]^2} =$$

$$-\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{0} \mathrm{d}t \frac{\sqrt{t}}{\sqrt{y}} \frac{f(-[b-at]/[1-t])\sqrt{[1-t]}}{[1-t]^2} \frac{1}{t-y},$$

$$\text{for } y < 0. \tag{3.49}$$

In the integral appearing on the right hand side of Eq. (3.49), we change the integration variable according to $t = [u+b]/[u+a]$ and we take into account that: (*i*) $y = [w+b]/[w+a]$ (cf. Eq. (3.44)); (*ii*) Eq. (3.49) holds only for $y < 0$; (*iii*) since, by assumption, $-c < a < b$ and $-w < b$, then $y < 0 \Leftrightarrow -c < a < -w < b$. In this way, Eq. (3.49) reduces to

$$g(-w) = \frac{1}{\mathrm{j}\pi}\mathrm{P}\int_{-b}^{-a} \mathrm{d}u \frac{\sqrt{[u+b]}}{\sqrt{[b+w]}} \frac{f(u)}{u-w},$$

$$\text{for } a < -w < b. \tag{3.50}$$

This shows that Eq. (3.42b), which we recover from Eq. (3.50) by simply letting $w \mapsto -w$, necessarily follows from Eq. (3.48) and in turn from Eq. (3.41), as desired.

This concludes the proof of the first part of point (i) of the corollary. The proof of the second part of that point consists in showing that Eq. (3.41) necessarily follows from Eq. (3.42). But this is entirely equivalent to showing that: (*i*) for $y > 0$, Eq. (3.45) necessarily follows from Eq. (3.46); (*ii*) for $y < 0$, Eq. (3.48) necessarily follows from Eq. (3.49). Part (*i*) of this task is accomplished by the same procedure used in Lemma 3.2 to show that Eq. (3.26) follows from Eq. (3.27); part (*ii*) of the task is accomplished by the same procedure used in Corollary 3.1 to show that Eq. (3.20) follows from Eq. (3.21).

This concludes the proof of point (i) of the corollary. The proof of point (ii) consists in reducing Eq. (3.42) to Eq. (3.43), which we do by means of the obvious identity

$$\frac{\sqrt{[u+b]}}{\sqrt{[b-w]}}\frac{1}{u+w} = \frac{1}{\sqrt{[u+b]}}\left\{\frac{1}{\sqrt{[b-w]}} + \frac{\sqrt{[b-w]}}{u+w}\right\}. \tag{3.51}$$

To prove point (iii) of the corollary, we work out the above arguments using finite values for the constants b and c and then let them take arbitrarily large values in Eqs. (3.41), (3.42a), (3.42b), (3.43a) and (3.43b). In so doing, we must ensure that, as $|w| \to +\infty$, the functions $f(w)$ and $g(w)$ vanish faster than a suitable negative power of $|w|$, in such a way that the integrals in the above formulæ converge, as assumed in the corollary. □

We notice notice that an immediate consequence of point (ii) of Corollary 3.2 is that, provided b be finite, then

$$g(w) \sim \frac{D}{\sqrt{[b-w]}}, \quad \text{for } w \simeq b < \infty,$$
$$\text{if} \quad D = \frac{1}{\mathrm{j}}\int_{-b}^{-a} \mathrm{d}u \frac{f(u)}{\sqrt{[u+b]}} \neq 0. \tag{3.52}$$

Indeed, the assumptions that b that the integral on the right hand sides of Eq. (3.43a) converge, implies that the function $f(-w)$ may diverge at most as $1/|b-w|^q$, for $w \simeq b$, with $q < 1/2$. This ensures that, on the right hand side of that equation, the first term is the leading one for $w \simeq b$.

We now show that the two expressions given in Eqs. (3.42a) and (3.42b) for the function $g(w)$ may be recovered as limit values of a suitable complex-valued function. To do so, we introduce the complex variable z and, in the complex z-plane cut along the line $b < \Re z < +\infty$, we consider the double-valued function $\sqrt{[b-z]}$, which has the following boundary values

for $z = w \pm \mathrm{j}\epsilon$,

$$\lim_{\epsilon\to 0^+} [b-z]^{1/2} = \sqrt{[b-w]}, \quad \text{if } w < b, \tag{3.53a}$$

$$\lim_{\epsilon\to 0^+} [b-z]^{1/2} = \pm\mathrm{j}\sqrt{[w-b]}, \quad \text{if } w > b. \tag{3.53b}$$

Next we consider the function

$$\mathfrak{g}(z) = -f(-z) + \frac{1}{\mathrm{j}\pi}\int_{-b}^{-a} \mathrm{d}u \frac{\sqrt{[u+b]}}{[b-z]^{1/2}} \frac{f(u)}{u-(-z)}, \tag{3.54}$$

Now, in the above introduced complex z-plane, a complex number z approaching a real number w, with $w < a$, definitively lies outside the branch cut of the function $[b-z]^{1/2}$, because, by assumption, in Corollary 3.2, $a < b$; also, the number $(-z)$, appearing in the denominator of the integrand of the integral on the right hand sides of Eqs. (3.54), definitively lies outside the integration path of that integral, which, therefore, is non singular. Then, using Eq. (3.53b), and taking the boundary value of both sides of Eq. (3.54), we precisely recover Eq. (3.42a):

$$\lim_{\epsilon\to 0^+} \mathfrak{g}(w-\mathrm{j}\epsilon) = -f(-w) + \frac{1}{\mathrm{j}\pi}\int_{-b}^{-a} \mathrm{d}u \frac{\sqrt{[u+b]}}{\sqrt{[b-w]}}\frac{f(u)}{u+w},$$

$$\text{for } -c < w < a. \tag{3.55}$$

On the other hand, a complex number z approaching a real number w, with $a < w < b$, definitively lies outside the branch cut of the function $[b-z]^{1/2}$; also, the number $(-z)$, appearing in the denominator of the integrand of the integral on the right hand sides of Eqs. (3.54), lies arbitrarily close to the integration path of that integral, which, therefore, is singular. Because of Eq. (3.53b) and of the Sokhotskyi-Plemelj formulæ (e.g. [11]), denoting by the symbols $\frac{\mathrm{P}}{\xi}$ and $\delta(\xi)$ respectively Cauchy's principal value and Dirac's delta distributions, the following identity may be used in that integral:

$$\lim_{\epsilon=0^+} \frac{1}{[b-(w\pm \mathrm{j}\epsilon)]^{1/2}}\frac{1}{u-[-(w\pm \mathrm{j}\epsilon)]} =$$

$$\frac{1}{\sqrt{[b-w]}}\left[\frac{\mathrm{P}}{u+w} \mp \mathrm{j}\pi\delta(u+w)\right],$$

$$\text{for } -b < u < -a \quad \text{and} \quad a < w < b. \tag{3.56}$$

Then, taking the boundary values of both sides of Eq. (3.54), we precisely recover Eq. (3.42b):

$$\lim_{\epsilon\to 0^+} \mathfrak{g}(w-\mathrm{j}\epsilon) = \frac{1}{\mathrm{j}\pi}\mathrm{P}\int_{-b}^{-a} \mathrm{d}u \frac{\sqrt{[u+b]}}{\sqrt{[b-w]}}\frac{f(u)}{u+w},$$

$$\text{for } a < w < b. \tag{3.57}$$

In conclusion, Eqs. (3.55) and (3.57) show that the complex-valued function $\mathfrak{g}$, defined in Eq. (3.54), is the solution of Eq. (3.41) in the whole complex plane:

the solution to that equation, evaluated at real values of its argument, may be recovered as the boundary value of the complex-valued function $\mathfrak{g}$ as its complex argument approaches the real axis in the lower half of the complex plane.

4 The fractional distributions of the electric charge density of electrons and ions

In this section we carry out the first part of the task, set at the beginning of Section 3, of solving Poisson's equation (cf. (2.24)) in favour of the electron and ion distribution functions. This part consists in casting that equation into two pairs of new, entirely equivalent and more convenient fractional equations.

To do so, we need reformulate the electron and ion number densities, given in Eqs. (2.21b) and (2.22b), in a more convenient form. In each of the following two pairs of transformations of these quantities, the first transformation concerns the electron number density, and the second one concerns the ion number density. A first pair of transformations is worked out using the obvious identity (cf. Eqs. (2.12) and (2.13))

$$U_{\mathrm{e}}(\phi) = [\lim_{x\to+\infty}\phi(x) - \phi(x_{\min})] - \theta U_{\mathrm{i}}(\phi), \tag{4.1}$$

and the change of the integration variable $W'_{\mathrm{e}} = \theta W' - [\lim_{x\to+\infty}\phi(x) - \phi(x_{\min})]$, in the integral on the right hand side of Eq. (2.21b). These relations reduce the the electron number density to

$$n_{\rm e}(\phi) = \int_{U_{\rm i}(\phi)}^{\infty} {\rm d}W' \frac{F_{\rm e}(\theta W' - [\lim_{x\to+\infty}\phi(x) - \phi(x_{\rm min})])\sqrt{\theta}}{\sqrt{\{2[W' - U_{\rm i}(\phi)]\}}}. \tag{4.2}$$

Likewise, the obvious identity (cf. Eqs. (2.12) and (2.13))

$$U_{\rm i}(\phi) = \{[\lim_{x\to+\infty}\phi(x) - \phi(x_{\rm min})] - U_{\rm e}(\phi)\}/\theta, \tag{4.3}$$

and the change of the integration variable $W'_{\rm i} = \{W' - [\lim_{x\to+\infty}\phi(x) - \phi(x_{\rm min})]\}/\theta$ in the integral on the right hand side of Eq. (2.22b), reduce the ion number density to

$$n_{\rm i}(\phi) = \int_{U_{\rm e}(\phi)}^{\infty} {\rm d}W' \frac{F_{\rm i}(\{W' - [\lim_{x\to+\infty}\phi(x) - \phi(x_{\rm min})]\}/\theta)/\sqrt{\theta}}{\sqrt{\{2[W' - U_{\rm e}(\phi)]\}}}. \tag{4.4}$$

To work out the second pair of transformations, we need the quantities

$$U_{\rm e0} = \max_{x\in\text{domain } j}(-U_{\rm e}(\phi(x))) = \min_{x\in\text{domain } j}(U_{\rm e}(\phi(x))), \tag{4.5}$$

and

$$U_{\rm i0} = \max_{x\in\text{domain } j}(-U_{\rm i}(\phi(x))) = \min_{x\in\text{domain } j}(U_{\rm i}(\phi(x))), \tag{4.6}$$

i.e. the maximum respectively of the electron and ion potential energy in each of the domains where $\phi(x)$ is monotonic (cf. Fig. (1)). We also need the quantity

$$U_0 = [\lim_{x\to+\infty}\phi(x) - \theta U_{i0}] - [U_{e0} + \phi(x_{\min})] =$$
$$\max_{x\in\text{domain } j}(\phi(x)) - \min_{x\in\text{domain } j}(\phi(x)) > 0, \tag{4.7}$$

i.e. the strictly positive, maximum electric potential difference in each of those domains. Using these quantities, we represent the electron number density n_e (cf. Eq. (2.21b)) in the form

$$n_e(\phi) = \int_{U_{e0}}^{U_e(\phi)} dW' \frac{H_e(W')}{\sqrt{\{2[U_e(\phi) - W']\}}}. \tag{4.8}$$

We notice that, because of the definition of U_{e0} (cf. Eq (4.5)), the upper limit of integration in the integral appearing in Eq. (4.8) always exceeds the lower limit. Therefore, Lemma 3.2, which we proved in Section 3, prescribes that, for both Eqs. (4.8) and (2.21b) to hold, the function H_e must be related to the electron energy distribution function F_e according to (cf. Eq. (3.27))

$$H_e(W + U_{e0}) = F_e(-[W + U_{e0}]) +$$
$$\frac{1}{\pi}\int_0^\infty dW' \frac{\sqrt{W'}}{\sqrt{W}} \frac{F_e(W' - U_{e0})}{W' + W},$$
$$\text{for } W \geq 0. \tag{4.9}$$

Given that the quantity $n_e(\phi)$ is the normalised electron electric charge density distribution (cf. Eq. 2.21a), and given that the integral appearing on the right hand side of Eq. (4.8) is fractional, we call the function $H_e(W)$ the "U_{e0}-based fractional distribution of the electron electric charge density".

Likewise, given the same real number U_{e0}, we wish to represent the ion number density (cf. Eq. (4.4)) in the form

$$n_{\mathrm{i}}(\phi) = \int_{U_{\mathrm{e}0}}^{U_{\mathrm{e}}(\phi)} \mathrm{d}W' \frac{K_{\mathrm{i}}(W')}{\sqrt{\{2[U_{\mathrm{e}}(\phi) - W']\}}}. \qquad (4.10)$$

Now, Lemma 3.1 prescribes that, for both Eqs. (4.10) and (4.4) to hold, the function K_{i} must be related to the ion energy distribution F_{i} according to (cf. Eq. (3.2)):

$$K_{\mathrm{i}}(W + U_{\mathrm{e}0}) = \frac{1}{\pi} \mathrm{P} \int_0^\infty \mathrm{d}W' \frac{\sqrt{W'}}{\sqrt{W}} \times$$
$$\frac{F_{\mathrm{i}}([W'/\theta + U_{\mathrm{i}0}] - U_0/\theta)/\sqrt{\theta}}{W' - W},$$
$$\text{for } W \geq 0. \qquad (4.11)$$

Given that the quantity $-n_{\mathrm{i}}(\phi)$ is the normalised ion electric charge density distribution (cf. Eq. 2.22a), we call the function $-K_{\mathrm{i}}(W)$, appearing on the right hand side of Eq. (4.10), the "$U_{\mathrm{e}0}$-based fractional distribution of the ion electric charge density".

To work out the second pair of transformations, we represent the electron number density n_{e} (cf. Eq. (4.2)) in the form

$$n_{\mathrm{e}}(\phi) = \int_{U_{\mathrm{i}0}}^{U_{\mathrm{i}}(\phi)} \mathrm{d}W' \frac{K_{\mathrm{e}}(W')}{\sqrt{\{2[U_{\mathrm{i}}(\phi) - W']\}}}. \qquad (4.12)$$

Using the same terminology as for Eq. (4.8), we call the function $-K_{\mathrm{e}}(W)$ the "$U_{\mathrm{i}0}$-based fractional distribution of the electron electric charge density". Lemma 3.1 (cf. Section 3) ensures that both representations of the quantity $n_{\mathrm{e}}(\phi)$ given in Eqs. (4.12) and (4.2) are legitimate, provided the function K_{e} be related to the electron energy distribution function F_{e} according to (cf. Eq.

(3.2))

$$K_{\rm e}(W/\theta+U_{\rm i0})/\surd\theta = \frac{1}{\pi}{\rm P}\int_0^\infty {\rm d}W'\frac{\surd W'}{\surd W}\frac{F_{\rm e}([W'+U_{\rm e0}]-U_0)}{W'-W},$$

for $W \geq 0$. (4.13)

Likewise, given the same real number $U_{\rm i0}$, we wish to represent the ion number density (cf. Eq. (2.22b)) in the form

$$n_{\rm i}(\phi) = \int_{U_{\rm i0}}^{U_{\rm i}(\phi)} {\rm d}W'\frac{H_{\rm i}(W')}{\surd\{2[U_{\rm i}(\phi)-W']\}}. \tag{4.14}$$

Using the same terminology as for Eq. (4.14), we call the function $H_{\rm i}(W)$ the "$U_{\rm i0}$-based fractional distribution of the ion electric charge density". Lemma 3.2 (cf. Section 3) ensures that both representations of the quantity $n_{\rm i}(\phi)$ given in Eqs. (4.14) and (2.22b) are legitimate, provided the function $H_{\rm i}$ be related to the electron energy distribution function $F_{\rm i}$ according to (cf. Eq. (3.27))

$$H_{\rm i}(W/\theta+U_{\rm i0})/\surd\theta = F_{\rm i}(-[W/\theta+U_{\rm i0}])/\surd\theta + \frac{1}{\pi}\int_0^\infty {\rm d}W'\frac{\surd W'}{\surd W}\frac{F_{\rm i}(W'/\theta-U_{\rm i0})/\surd\theta}{W'+W},$$

for $W \geq 0$. (4.15)

Having performed the anticipated two pairs of transformations of the electron and ion number densities, we now use these quantities to rearrange Poisson's equation (cf. Eq. 2.24), as announced at the beginning of this section. To work out the second pair of rearrangements, we respectively use Eqs. (4.8) and (4.10) in place of Eqs. (2.21b) and (2.22b) and we introduce the function

$$H(W) = H_{\rm e}(W) - K_{\rm i}(W). \tag{4.16}$$

Then Poisson's equation (cf. Eq. (2.24)) takes the form

$$\phi_{xx}(\phi) = \int_{U_{\rm e0}}^{U_{\rm e}(\phi)} {\rm d}W' \frac{H(W')}{\surd\{2[U_{\rm e}(\phi) - W']\}}. \tag{4.17}$$

Likewise, if we respectively take the electron and ion number densities from Eqs. (4.12) and (4.14), and if we introduce the function

$$K(W) = H_{\rm i}(W) - K_{\rm e}(W), \tag{4.18}$$

then Poisson's equation takes the form

$$\phi_{xx}(\phi) = -\int_{U_{\rm i0}}^{U_{\rm i}(\phi)} {\rm d}W' \frac{K(W')}{\surd\{2[U_{\rm i}(\phi) - W']\}}. \tag{4.19}$$

Eqs. (4.17) and (4.19) are the second pair of fractional forms of Poisson's equation, announced at the beginning of this section. Given that the quantity $\phi_{xx}(\phi)$ equals the normalised charge density in the plasma (cf. Eqs. (2.18) and (2.23)), and given that the integral appearing on the right hand side of Eq. (4.17) is fractional, we call the function $H(W)$ the "$U_{\rm e0}$-based fractional distribution of the total electric charge density". Likewise, the function $-K(W)$ will be known as the "$U_{\rm i0}$-based fractional distribution of the total electric charge density".

In conclusion, the above analysis showed that, based on Poisson equation, it is always possible to write the second derivative of the electric potential ϕ_{xx} as an Abel transform of a suitable single function and that this function may be constructively related to the electron and ion energy distribution functions. The convenience of writing the quantity ϕ_{xx} in this way will be made clear in Section 5.

5 The Hilbert form of the Solutions of the Integral Poisson Equation

In this section we carry out the second part of the task, set at the beginning of Section 3, of solving Poisson's equation (cf. (2.24)) in favour of the electron and ion distribution functions. First, using the definition of the electron potential energy U_e (cf. Eq. (2.12)), we cast Eq. (4.19)) as an Abel equation (cf. e.g. [14]) for the quantity $H(W)$

$$\phi_{xx}(U_\mathrm{e}+\phi(x_\mathrm{min})) = \int_{U_{\mathrm{e}0}}^{U_\mathrm{e}} \mathrm{d}W' \frac{H(W')}{\sqrt{\{2[U_\mathrm{e}-W']\}}}, \tag{5.1}$$

whose solution we conveniently write as

$$H(W+U_{\mathrm{e}0}) = \frac{\sqrt{2}}{\pi}\frac{\mathrm{d}}{\mathrm{d}W}\int_0^W \mathrm{d}W' \frac{\phi_{xx}(W'+[\phi(x_\mathrm{min})+U_{\mathrm{e}0}])}{\sqrt{[W-W']}}, \quad \text{for } W \geq 0. \tag{5.2}$$

Following an analogous procedure, Eq. (4.19) may be solved in favour of $K(W)$, thus giving

$$-K(W/\theta + U_{\mathrm{i}0})/\sqrt{\theta} =$$

$$\frac{\sqrt{2}}{\pi}\frac{\mathrm{d}}{\mathrm{d}W}\int_0^W \mathrm{d}W' \frac{\phi_{xx}([\lim_{x\to+\infty}\phi(x) - \theta U_{\mathrm{i}0}] - W')}{\sqrt{[W - W']}},$$

$$\text{for } W \geq 0. \tag{5.3}$$

In this way, Eqs. (4.17) and (5.2) on one hand, or Eqs. (4.19) and (5.3) on the other, respectively establish a complete equivalence between the knowledge of the electric potential ϕ and that of the function $H(W)$ or $K(W)$.

Because of this equivalence, Eqs. (4.16) and (4.18) may be regarded as the first pair of equivalent forms of Poisson's equation (cf. Eq. (2.24)) announced at the beginning of this section: specifically, using Eqs. (4.9) and (4.11), we rewrite Eq. (4.16) as

$$F_{\mathrm{e}}(-[W + U_{\mathrm{e}0}]) =$$

$$-\frac{1}{\pi}\int_0^\infty \mathrm{d}W' \frac{\sqrt{W'}}{\sqrt{W}} \frac{F_{\mathrm{e}}(W' - U_{\mathrm{e}0})}{W' + W} +$$

$$\frac{1}{\pi}\mathrm{P}\int_0^\infty \mathrm{d}W' \frac{\sqrt{W'}}{\sqrt{W}} \frac{F_{\mathrm{i}}([W'/\theta - U_{\mathrm{i}0}] - U_0/\theta)/\sqrt{\theta}}{W' - W} +$$

$$H(W + U_{\mathrm{e}0}),$$

$$\text{for } W \geq 0. \tag{5.4}$$

Likewise, using Eqs. (4.13), (4.15) and (4.7), we rewrite Eq. (4.18) as

$$F_{\rm i}(-[W/\theta+U_{\rm i0}])/\sqrt{\theta} =$$

$$-\frac{1}{\pi}\int_0^\infty {\rm d}W' \frac{\sqrt{W'}}{\sqrt{W}} \frac{F_{\rm i}(W'/\theta - U_{\rm i0})/\sqrt{\theta}}{W'+W} +$$

$$\frac{1}{\pi}{\rm P}\int_0^\infty {\rm d}W' \frac{\sqrt{W'}}{\sqrt{W}} \frac{F_{\rm e}([W'-U_{\rm e0}]-U_0)}{W'-W} +$$

$$K(W/\theta+U_{\rm i0})/\sqrt{\theta},$$

$$\text{for } W \geq 0. \tag{5.5}$$

Eqs. (5.5) and Eq. (5.4) are the solutions of Poisson's equation announced at the beginning of this section. We call them the "BGK solutions". Indeed, according to the approach adopted by Bernstein, Greene and Kruskal (cf. Ref. [2]),Eq. (5.4) gives the the electron energy distribution functions for values of its argument less than $U_{\rm e0}$ (cf. the left hand side of Eq. (5.4)), provided it be given at all the other values of its argument (cf. the first term on the right hand side of Eq. (5.4)) and provided the ion energy distribution function and the electric potential be given. The same task is accomplished for the ion energy distribution function by Eq. (5.5).

We now proceed to rearranging Eqs. (5.4) and (5.5) in yet another pair of equivalent forms. First, we notice that, since the $U_{\rm e0}$-based fractional distribution H, appearing in Eq. (4.17), and the $U_{\rm i0}$-based fractional distribution K, appearing in Eq. (4.19), represent the same quantity (i.e. the total charge), they must be related. To find this relation, in the integral appearing on the right hand side of Eq. (4.17), we change the integration variable according to $W' = W'' - \phi(x_{\rm min})$, and, writing the electron potential energy $U_{\rm e}(\phi)$ as in Eq. (2.12), we get

$$\phi_{xx}(\phi) = \int_{U_{\mathrm{e}0}+\phi(x_{\min})}^{\phi} \mathrm{d}W'' \frac{G(W'' - \phi(x_{\min}))}{\sqrt{[2(\phi - W'')]}}. \tag{5.6}$$

Likewise, in the integral appearing on the right hand side of Eq. (4.19), we change the integration variable according to $W' = [\lim_{x\to+\infty} \phi(x) - W'']/\theta$, and, writing the ion potential energy $U_{\mathrm{i}}(\phi)$ as in Eq. (2.13), we get

$$\phi_{xx}(\phi) = -\int_{\phi}^{\lim_{x\to+\infty}\phi(x)-\theta U_{\mathrm{i}0}} \mathrm{d}W'' \times \\ \frac{K([\lim_{x\to+\infty}\phi(x) - W'']/\theta)/\sqrt{\theta}}{\sqrt{[2(W'' - \phi)]}}. \tag{5.7}$$

Subtracting the respective sides of Eqs. (5.6) and (5.7), an integral relation is obtained between the functions H and K, which is of the kind shown in Eq. (3.1) and in Eq. (3.20): there now, $a = U_{\mathrm{e}0} + \phi(x_{\min})$ and $b = \lim_{x\to+\infty}\phi(x) - \theta U_{\mathrm{i}0}$ and, due to Eq. (4.7), $a < b$, in such a way that the hypotheses of Lemma 3.1 and of Corollary 3.1 apply. Then, Lemma 3.1 prescribes that (cf. Eq. (3.2))

$$H(W + U_{\mathrm{e}0}) = \\ -\frac{1}{\pi}\mathrm{P}\int_0^{U_0} \mathrm{d}W' \frac{\sqrt{W'}}{\sqrt{W}} \frac{K(U_0/\theta - [W'/\theta - U_{\mathrm{i}0}])/\sqrt{\theta}}{W' - W}, \\ \text{for } W \geq 0, \tag{5.8}$$

whereas Corollary 3.1 prescribes that (cf. Eq. (3.21))

$$K(W/\theta + U_{\mathrm{i}0})/\sqrt{\theta} = \\ \frac{1}{\pi}\mathrm{P}\int_0^{U_0} \mathrm{d}W' \frac{\sqrt{W'}}{\sqrt{W}} \frac{H(U_0 - [W' - U_{\mathrm{e}0}])}{W' - W}, \\ \text{for } W \geq 0. \tag{5.9}$$

Finally, substituting Eq. (5.8) into Eq. (5.4) and Eq. (5.9) into Eq. (5.5), we respectively get

$$F_{\rm e}(-[W+U_{\rm e0}]) + \frac{1}{\pi}\int_0^\infty {\rm d}W' \frac{\sqrt{W'}}{\sqrt{W}} \frac{F_{\rm e}(W'-U_{\rm e0})}{W'+W} =$$

$$\frac{1}{\pi}{\rm P}\int_0^\infty {\rm d}W' \frac{\sqrt{W'}}{\sqrt{W}} \frac{F_{\rm i}([W'/\theta - U_{\rm i0}] - U_0/\theta)/\sqrt{\theta}}{W'-W} -$$

$$\frac{1}{\pi}{\rm P}\int_0^{U_0} {\rm d}W' \frac{\sqrt{W'}}{\sqrt{W}} \frac{K(U_0/\theta - [W'/\theta - U_{\rm i0}])/\sqrt{\theta}}{W'-W},$$

$$\text{for } W \geq 0. \tag{5.10}$$

and

$$F_{\rm i}(-[W/\theta + U_{\rm i0}])/\sqrt{\theta} + \frac{1}{\pi}\int_0^\infty {\rm d}W' \frac{\sqrt{W'}}{\sqrt{W}} \frac{F_{\rm i}(W'/\theta - U_{\rm i0})/\sqrt{\theta}}{W'+W} =$$

$$\frac{1}{\pi}{\rm P}\int_0^\infty {\rm d}W' \frac{\sqrt{W'}}{\sqrt{W}} \frac{F_{\rm e}([W'-U_{\rm e0}] - U_0)}{W'-W} +$$

$$\frac{1}{\pi}{\rm P}\int_0^{U_0} {\rm d}W' \frac{\sqrt{W'}}{\sqrt{W}} \frac{H(U_0 - [W'-U_{\rm e0}])}{W'-W},$$

$$\text{for } W \geq 0. \tag{5.11}$$

Eqs. (5.10) and Eq. (5.11) are the solutions of Poisson's equation announced at the beginning of this section. Both of these equations relate K or H, i.e. suitable transforms of ϕ_{xx} (cf. Eqs. (5.3) or (5.2)), to suitable transforms of the electron and ion energy distribution functions $F_{\rm e}$ and $F_{\rm i}$. The advantages of writing Poisson's equation in its formulation given in Eqs. (5.10) and (5.11) will be demonstrated in detail in Section 6.

6 The Sectionally Analytic Form of the Solutions of the Integral Poisson Equation

In this section we carry out the third part of the task, set at the beginning Section 3, of solving Poisson's equation in favour of the electron and ion energy distribution functions. This task consists in extending these functions in the whole complex plane of their respective arguments.

To do so, we use the forms of Poisson's equation given in Eqs. (5.10) and (5.11). We begin by introducing the complex variable z and, in the complex z-plane cut along the positive real axis, we introduce the two-valued function $z^{1/2}$. Specifically, let j denote the imaginary unit, and let $z = w \pm \mathrm{j}\epsilon$; then, if the real number w does not belong to the cut, i.e. if $w < 0$, we set

$$\lim_{\epsilon=0^+} [w \pm \mathrm{j}\epsilon]^{1/2} = \sqrt{[|w| e^{\mathrm{j}\pi}]} = +\mathrm{j}\sqrt{|w|},$$
$$\text{for } w < 0. \tag{6.1}$$

If the real number w belongs to the cut, i.e. if $w > 0$, we set

$$\lim_{\epsilon=0^+} [w \pm \mathrm{j}\epsilon]^{1/2} = \pm\sqrt{w}.$$
$$\text{for } w > 0. \tag{6.2}$$

Next, we assume that the energy distribution functions F_{e} and F_{i} are integrable over their respective domain and, in the above introduced complex z-plane, cut along the positive real axis, we define the sectionally analytic functions (cf. e.g. [11])

$$\mathsf{F}_{\rm e}(u,z)=\frac{1}{2{\rm j}\pi}\int_0^\infty {\rm d}t\frac{\sqrt{t}}{z^{1/2}}\frac{F_{\rm e}(t-u-U_{\rm e0})}{t-(-z)}, \tag{6.3}$$

$$\mathsf{F}_{\rm i}(u,z)=\frac{1}{2{\rm j}\pi}\int_0^\infty {\rm d}t\frac{\sqrt{t}}{z^{1/2}}\frac{F_{\rm i}([t-u]/\theta-U_{\rm i0})/\sqrt{\theta}}{t-(-z)}. \tag{6.4}$$

In Eqs. (6.3) and (6.4), the values of the real parameter u are such that, as the integration in t runs in the complex z-plane along the cut, and for all the appropriate values of $U_{\rm e0}$ and $U_{\rm i0}$ (cf. Eqs. (4.5) and (4.6)), the argument $t-u-U_{\rm e0}$ lies within the domain of the electron energy distribution function $F_{\rm e}$, and the argument $[t-u]/\theta-U_{\rm i0}$ lies within the domain of the ion energy distribution function $F_{\rm i}$.

Now, in the above introduced complex z-plane cut along the positive real axis, a complex number z approaching a negative real number w definitively lies outside the branch cut of the function $z^{1/2}$; also, the number $-z$, appearing in the integrals on the right hand sides of Eqs. (6.3) and (6.4), lies arbitrarily close to the integration path of those integrals, which, therefore, are singular. Then, because of Eq. (6.1) and of the Sokhotskyi-Plemelj formulæ (e.g. [11]), denoting by the symbols $\frac{\rm P}{\xi}$ and $\delta(\xi)$ respectively Cauchy's principal value and Dirac's delta distributions, the following identity may be used in those integrals, as $z=w\pm{\rm j}\epsilon$ approaches the negative real axis, provided both $[\sqrt{t}]F_{\rm e}(t-u-U_{\rm e0}))$ and $[\sqrt{t}]F_{\rm i}([t-u]/\theta-U_{\rm i0}))$ be Hölder continuous at $t=-w$:

$$\lim_{\epsilon=0^+} \frac{1}{[w \pm \mathrm{j}\epsilon]^{1/2}\{t - [-(w \pm \mathrm{j}\epsilon)]\}} =$$
$$\frac{1}{\mathrm{j}} \frac{1}{\sqrt{|w|}} \left[\frac{\mathrm{P}}{t+w} \mp \mathrm{j}\pi\delta(t+w) \right],$$
$$\text{for } t > 0 \text{ and } w < 0. \tag{6.5}$$

On the other hand, a complex number z approaching a positive real number w lies arbitrarily close to the cut; also, the number $-z$, appearing in the integrals on the right hand sides of Eqs. (6.3) and (6.4), definitively lies outside the integration path of those integrals, which, therefore, are not singular. Then, because of Eq. (6.2), the following obvious identity holds in those integrals, as $z = w \pm \mathrm{j}\epsilon$ approaches the positive real axis:

$$\lim_{\epsilon=0^+} \frac{1}{[w \pm \mathrm{j}\epsilon]^{1/2}\{t - [-(w \pm \mathrm{j}\epsilon)]\}} = \pm \frac{1}{\sqrt{w}} \frac{1}{t+w},$$
$$\text{for } t > 0 \text{ and } w > 0. \tag{6.6}$$

As an application of Eq. (6.5), its use in Eq. (6.3) and (6.4) leads to the identities

$$-\mathrm{j} \lim_{\epsilon=0^+} \{\mathsf{F}_\mathrm{e}(u, w+\mathrm{j}\epsilon) - \mathsf{F}_\mathrm{e}(u, w-\mathrm{j}\epsilon)\} =$$
$$F_\mathrm{e}(-[w+u+U_{\mathrm{e}0}]), \tag{6.7a}$$
$$\lim_{\epsilon=0^+} \{\mathsf{F}_\mathrm{e}(u, w+\mathrm{j}\epsilon) + \mathsf{F}_\mathrm{e}(u, w-\mathrm{j}\epsilon)\} =$$
$$-\frac{1}{\pi} \int_0^\infty \mathrm{d}t \frac{\sqrt{t}}{\sqrt{|w|}} \frac{F_\mathrm{e}(t-u-U_{\mathrm{e}0})}{t+w}, \tag{6.7b}$$

$$\text{for } w < 0, \tag{6.7c}$$

$$-\mathrm{j}\lim_{\epsilon=0^+}\{\mathsf{F}_\mathrm{i}(u,w+\mathrm{j}\epsilon)-\mathsf{F}_\mathrm{i}(u,w-\mathrm{j}\epsilon)\} =$$

$$F_\mathrm{i}(-\{[w+u]/\theta+U_{\mathrm{i}0}\})/\sqrt{\theta},$$

$$\text{for } w<0, \tag{6.8a}$$

whereas the application of Eq. (6.6) into the same equations immediately gives

$$-\mathrm{j}\lim_{\epsilon=0^+}\{\mathsf{F}_\mathrm{e}(u,w+\mathrm{j}\epsilon)-\mathsf{F}_\mathrm{e}(u,w-\mathrm{j}\epsilon)\} =$$

$$-\frac{1}{\pi}\int_0^\infty \mathrm{d}t\frac{\sqrt{t}}{\sqrt{w}}\frac{F_\mathrm{e}(t-u-U_{\mathrm{e}0})}{t+w},$$

$$\text{for } w>0, \tag{6.9}$$

$$-\mathrm{j}\lim_{\epsilon=0^+}\{\mathsf{F}^+_\mathrm{i}(u,w+\mathrm{j}\epsilon)-\mathsf{F}^-_\mathrm{i}(u,w-\mathrm{j}\epsilon)\} =$$

$$-\frac{1}{\pi}\int_0^\infty \mathrm{d}t\frac{\sqrt{t}}{\sqrt{w}}\frac{F_\mathrm{i}([t-u]/\theta-U_{\mathrm{i}0})/\sqrt{\theta}}{t+w},$$

$$\text{for } w>0. \tag{6.10}$$

Next, in the same complex z-plane cut along the positive real axis, as introduced above, we consider the properties of the functions defined in Eqs. (6.3) and in Eq. (6.4) evaluated at $e^{\mathrm{j}\pi}z$:

$$\mathsf{F}_\mathrm{e}(u,e^{\mathrm{j}\pi}z) = \frac{1}{2\mathrm{j}\pi}\int_0^\infty \mathrm{d}t\frac{\sqrt{t}}{e^{\mathrm{j}\pi/2}z^{1/2}}\frac{F_\mathrm{e}(t-u-U_{\mathrm{e}0})}{t-z}, \tag{6.11}$$

and

$$\mathsf{F}_\mathrm{i}(u,e^{\mathrm{j}\pi}z) = \frac{1}{2\mathrm{j}\pi}\int_0^\infty \mathrm{d}t\frac{\sqrt{t}}{e^{\mathrm{j}\pi/2}z^{1/2}}\frac{F_\mathrm{i}([t-u]/\theta-U_{\mathrm{i}0})/\sqrt{\theta}}{t-z}. \tag{6.12}$$

The situation encountered by the functions $\mathsf{F}_{\rm e}(u, e^{{\rm j}\pi} z)$ (cf. Eq. (6.11)) and $\mathsf{F}_{\rm i}(u, e^{{\rm j}\pi} z)$ (cf. Eq. (6.12)), as z approaches the real axis, is significantly different from the one encountered by the functions $\mathsf{F}_{\rm e}(u, z)$ (cf. Eq. (6.3)) and $\mathsf{F}_{\rm i}(u, z)$ (cf. Eq. (6.4)). Indeed, consider first a complex number z approaching a negative real number w: in so doing, it definitively lies outside the branch cut of the function $z^{1/2}$ and outside the integration path of the integrals on the right hand sides of Eqs. (6.11) and (6.12), which, therefore, are not singular. Then, because of Eq. (6.1), the following obvious identity holds in those integrals, as $z = w \pm {\rm j}\epsilon$ approaches the negative real axis:

$$\lim_{\epsilon=0^+} \frac{1}{[w \pm {\rm j}\epsilon]^{1/2}\{t - [w \pm {\rm j}\epsilon]\}} = \frac{1}{\rm j} \frac{1}{\sqrt{|w|}} \frac{1}{t - w},$$

for $t > 0$ and $w < 0$. (6.13)

On the other hand, a complex number z approaching a positive real number w lies arbitrarily close to both the cut and the integration path of the integrals on the right hand sides of Eqs. (6.11) and (6.12), which, therefore, are singular. Then, because of Eq. (6.2) and of the Sokhotskyi-Plemelj formulæ, the following identity may be used in those integrals, as $z = w \pm {\rm j}\epsilon$ approaches the positive real axis, provided both $[\sqrt{t}]F_{\rm e}(t - u - U_{\rm e0}))$ and $[\sqrt{t}]F_{\rm i}([t - u]/\theta - U_{\rm i0}))$ be Hölder continuous at $t = w$:

$$\lim_{\epsilon=0^+} \frac{1}{[w \pm {\rm j}\epsilon]^{1/2}\{t - [w \pm {\rm j}\epsilon]\}} =$$

$$\pm \frac{1}{\sqrt{w}} \left[\frac{\rm P}{t - w} \pm {\rm j}\pi\delta(t - w) \right],$$

for $t > 0$ and $w > 0$. (6.14)

As an application of Eq. (6.13), its use in Eq. (6.11) and (6.12) immediately gives

$$\lim_{\epsilon=0^+}\{\mathsf{F}_\mathrm{e}(u,e^{\mathrm{j}\pi}[w+\mathrm{j}\epsilon])-\mathsf{F}_\mathrm{e}(u,e^{\mathrm{j}\pi}[w-\mathrm{j}\epsilon])\}=0,$$
$$\text{for } w<0, \tag{6.15}$$

$$\lim_{\epsilon=0^+}\{\mathsf{F}_\mathrm{i}(u,e^{\mathrm{j}\pi}[w+\mathrm{j}\epsilon])-\mathsf{F}_\mathrm{i}(u,e^{\mathrm{j}\pi}[w-\mathrm{j}\epsilon])\}=0,$$
$$\text{for } w<0, \tag{6.16}$$

whereas the application of Eq. (6.14) into the same equations leads to the identities

$$\lim_{\epsilon=0^+}\{\mathsf{F}_\mathrm{e}(u,e^{\mathrm{j}\pi}[w+\mathrm{j}\epsilon])-\mathsf{F}_\mathrm{e}(u,e^{\mathrm{j}\pi}[w-\mathrm{j}\epsilon])\}=$$
$$\frac{1}{\pi}\mathrm{P}\int_0^\infty \mathrm{d}t\frac{\sqrt{t}}{\sqrt{w}}\frac{F_\mathrm{e}(t-u-U_{\mathrm{e}0})}{t-w},$$
$$\text{for } w>0, \tag{6.17}$$

$$\lim_{\epsilon=0^+}\{\mathsf{F}_\mathrm{i}(u,e^{\mathrm{j}\pi}[w+\mathrm{j}\epsilon])-\mathsf{F}_\mathrm{i}(u,e^{\mathrm{j}\pi}[w-\mathrm{j}\epsilon])\}=$$
$$\frac{1}{\pi}\mathrm{P}\int_0^\infty \mathrm{d}t\frac{\sqrt{t}}{\sqrt{w}}\frac{F_\mathrm{i}([t-u]/\theta-U_{\mathrm{i}0})/\sqrt{\theta}}{t-w},$$
$$\text{for } w>0. \tag{6.18}$$

Finally, given the positive real number U_0 (cf. Eq. (4.7)), in the same complex z-plane cut along the positive real axis, as introduced above, we consider the sectionally analytic functions

$$\mathsf{G}(z) = \frac{1}{2\mathrm{j}\pi}\int_0^{U_0} \mathrm{d}t \frac{\sqrt{t}}{z^{1/2}} \frac{H([U_0 - t] + U_{\mathrm{e}0}])}{t - z}, \tag{6.19}$$

$$\mathsf{H}(z) = \frac{1}{2\mathrm{j}\pi}\int_0^{U_0} \mathrm{d}t \frac{\sqrt{t}}{z^{1/2}} \frac{K([U_0 - t]/\theta + U_{\mathrm{i}0})/\sqrt{\theta}}{t - z}. \tag{6.20}$$

The situation encountered by the functions $\mathsf{G}(z)$ (cf. Eq. (6.19)) and $\mathsf{H}(z)$ (cf. Eq. (6.20)), as z approaches the real axis, is akin to the one encountered by the functions $\mathsf{F}_{\mathrm{e}}(u, e^{\mathrm{i}\pi}z)$ (cf. Eq. (6.11)) and $\mathsf{F}_{\mathrm{i}}(u, e^{\mathrm{i}\pi}z)$ (cf. Eq. (6.12)). In particular, in the integrals appearing on the right hand side of Eqs. (6.19) and (6.20), Eq. (6.13) holds as $z = w \pm \mathrm{j}\epsilon$ approaches the negative real axis, thus giving

$$\begin{aligned}&\mathrm{j}\lim_{\epsilon=0^+}\{\mathsf{G}(w+\mathrm{j}\epsilon) - \mathsf{G}(w-\mathrm{j}\epsilon)\} = 0,\\ &\text{for } w < 0,\end{aligned} \tag{6.21}$$

$$\begin{aligned}&\mathrm{j}\lim_{\epsilon=0^+}\{\mathsf{H}(w+\mathrm{j}\epsilon) - \mathsf{H}(w-\mathrm{j}\epsilon)\} = 0,\\ &\text{for } w < 0.\end{aligned} \tag{6.22}$$

On the other hand, in those same integrals, Eq. (6.14) holds as $z = w \pm \mathrm{j}\epsilon$ approaches the positive real axis, provided $w < U_0$ and provided both $[\sqrt{t}]H([U_0 - t] + U_{\mathrm{e}0})$ and $[\sqrt{t}]K([U_0 - t]/\theta + U_{\mathrm{i}0})$ be Hölder continuous at $t = w$, thus giving

$$\mathrm{j}\lim_{\epsilon=0^+}\{\mathsf{G}(w+\mathrm{j}\epsilon)-\mathsf{G}(w-\mathrm{j}\epsilon)\} =$$
$$\frac{1}{\pi}\mathrm{P}\int_0^{U_0}\mathrm{d}t\frac{\sqrt{t}}{\sqrt{w}}\frac{H([U_0-t]+U_{\mathrm{e}0})}{t-w},$$
$$\text{for } 0 < w < U_0, \tag{6.23}$$

$$\mathrm{j}\lim_{\epsilon=0^+}\{\mathsf{H}(w+\mathrm{j}\epsilon)-\mathsf{H}(w-\mathrm{j}\epsilon)\} =$$
$$\frac{1}{\pi}\mathrm{P}\int_0^{U_0}\mathrm{d}t\frac{\sqrt{t}}{\sqrt{w}}\frac{K([U_0-t]/\theta+U_{\mathrm{i}0}])/\sqrt{\theta}}{t-w}.$$
$$\text{for } 0 < w < U_0. \tag{6.24}$$

Now, adding the right hand sides of Eqs. (6.7a), (6.16) and (6.22), all of which apply for $w < 0$, we simply get $F_{\mathrm{e}}(-[w+U_{\mathrm{e}0}])$. Also, adding the right hand sides of Eqs. (6.9), (6.18) and (6.24), all of which apply for $w > 0$, we precisely get the right hand side of Eq. (5.10), which obviously equals the quantity $F_{\mathrm{e}}(-[w+U_{\mathrm{e}0}])$, appearing on the left hand side of Eq. (5.10) itself. These results may be conveniently summarised, *irrespective* of the sign of w, in terms of the sectionally analytic function

$$\mathfrak{F}_{\mathrm{e}}(z) = -\mathrm{j}\mathsf{F}_{\mathrm{e}}(0,z) + \mathsf{F}_{\mathrm{i}}(U_0, e^{\mathrm{j}\pi}z) - \mathrm{j}\mathsf{H}(z), \tag{6.25}$$

as follows:

$$F_{\mathrm{e}}(-[W+U_{\mathrm{e}0}]) = \lim_{\epsilon=0^+}[\mathfrak{F}_{\mathrm{e}}(W+\mathrm{j}\epsilon)-\mathfrak{F}_{\mathrm{e}}(W-\mathrm{j}\epsilon)], \tag{6.26}$$

A similar result may be established by manipulating Eqs. (6.10), (6.15), (6.21) and Eqs. (5.10), (6.10), (6.17), (6.23), and, introducing the sectionally analytic function

$$\mathfrak{F}_{\rm i}(z) = -\mathsf{j}\mathsf{F}_{\rm i}(0,z) + \mathsf{F}_{\rm e}(U_0, e^{\mathrm{j}\pi}z) + \mathsf{j}\mathsf{G}(z), \tag{6.27}$$

we have

$$F_{\rm i}(-[W+U_{\rm i0}]) = \lim_{\epsilon=0^+}[\mathfrak{F}_{\rm i}(W+\mathrm{j}\epsilon) - \mathfrak{F}_{\rm i}(W-\mathrm{j}\epsilon)], \tag{6.28}$$

In conclusion, in the above analysis, we showed that the electron energy distribution function $F_{\rm e}$ may be found as the boundary value of a suitable sectionally analytic function (cf. 6.26)): this latter function $\mathfrak{F}_{\rm e}$ (cf. Eq. (6.25)) precisely gives, through Eqs. (6.3), (6.12), (6.20) and (5.3), the continuation of the electron energy distribution function $F_{\rm e}$ in the whole complex electron energy plane. An analogous boundary value relation (cf. Eq. (6.28)) holds for the ion energy distribution function $F_{\rm i}$, whose continuation in the complex ion energy plane is provided by the sectionally analytic function $\mathfrak{F}_{\rm i}$ (cf. Eq. (6.27)), through Eqs. (6.4), (6.11), (6.19) and (5.2).

7 Conclusions

In this report, we investigated the distribution functions of the electrons and of the ions which, in stationary, collisionless conditions, self-consistently sustain an unsymmetrically distributed (skewed) electrostatic potential (cf. Fig. 1).

These distribution functions of both particle species are governed by the integral Poisson equation (cf. Eqs. (2.21b), (2.22b) and (2.24)). We determine the distribution function of one particle species (either electron or ion, according to need), once the electric potential and the distribution function of the other species are known. To do so we propose a new procedure which provides the sought distribution function directly from the distribution function of the other particle species, and from a suitable integral transform of the second derivative of the potential (cf. Eqs. (5.4) and (5.5), without the calculation of the pseudopotential used in the standard BGK approach. This procedure is based on two inversion lemmas whose proof is also an important and novel part of this report (cf. Section 3).

The main advantage of this procedure is the possibility to extend the solution of the integral Poisson equation thus found for any complex value of the particle energy (cf. Eqs. (6.25) and (6.27)). We show that such extended distributions are sectionally analytic functions of their argument, whose boundary value precisely gives the real-valued BGK solution of Poisson's equation (cf. Eqs. (6.26) and (6.26)).

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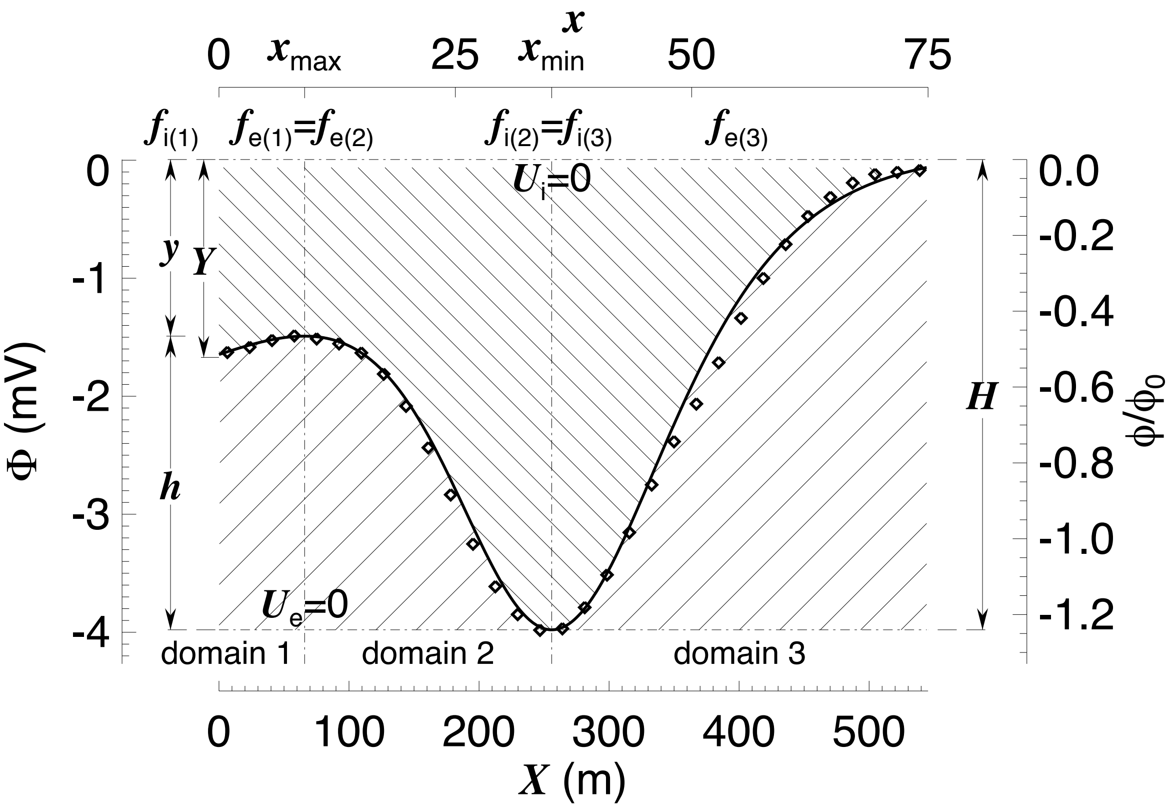
Figure caption

(1) Left and bottom axes: the electrostatic potential Φ vs. coordinate X as observed by WIND on February 2nd 1996 at 12:04:36.6 UT ($\diamond$'s). Right and top axes: the normalised potential ϕ vs. the normalised coordinate x. H, h, Y, y are the potential jumps. The horizontal dash-dotted lines $U_{\mathrm{e}} = 0$ and $U_{\mathrm{i}} = 0$ denote the reference zero values of the electron and ion potential energies. The broad- and fine-hatched areas denote the position and energy values of the negative energy electrons and ions respectively. Also shown is the scheme for for the electron and ion distributions f_{e} and f_{i}. Subscripts (1), (2) and (3) denote the domains where the potential is a monotonic function of position.



x
0
x_max
25
x_min
50
75
f_i(1)
f_e(1)=f_e(2)
f_i(2)=f_i(3)
f_e(3)
U_i=0
0
-1
-2
-3
-4
Φ (mV)
y
Y
h
H
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
φ/φ_0
U_e=0
domain 1
domain 2
domain 3
0
100
200
300
400
500
X (m)

Fig. 1.