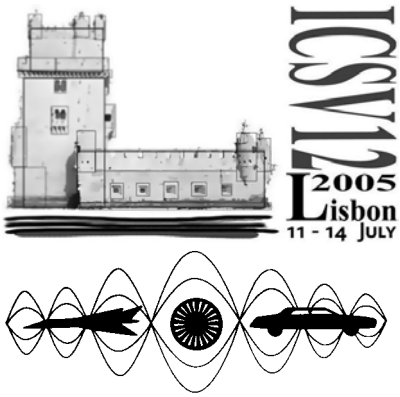

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AN INVESTIGATION ON THE ENERGY EQUIPARTITION PRINCIPLE IN STRUCTURAL DYNAMICS OF UNCERTAIN SYSTEMS

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Abstract

The question of energy sharing among complex engineering vibrating systems is still an open problem. On the basis of some recent investigations, this paper is addressed to the prediction of the long term equilibrium energies of interacting conservative resonators. More specifically, the goal would rely in a better understanding of the principle of energy equipartition, that still presents many questionable points. The analysis tries to explore both the field of linear as well as nonlinear vibrations, being the principle of equipartion obeyed in a different fashion in the two cases. Although the present work is a preliminary step in the analysis of this complex subject, some conclusions are stated and supported by the results of numerical experiments.

THE PRINCIPLE OF ENERGY EQUIPARTITION

The energy equipartion principle (EEP) was born in the context of Statistical Thermodynamics. In its simplest form the principle states that, in thermodynamic equilibrium conditions, a system of N particles with total energy E, exhibits a mean energy for each particle equal to $\bar{\varepsilon} = E / N$. A complete meaning of this statement, implies the specification of the conditions under which it holds, and the analysis of

several definitions introduced in Statistical Mechanics (SM) to specify the nature and to establish the properties of the investigated system.
The analysis of a system with a very large number of degrees of freedom, generally implies the practical impossibility of a detailed description of its motion. This consideration led historically to produce a different way of tackling dynamic problems. In the context of molecular dynamics, a statistical approach –namely Statistical Thermodynamics- is capable, in a relatively simpler way, to provide expected values of relevant quantities, such the energy. More recently, in the context of engineering, a similar problem arises in the analysis of very complicated systems, whose mathematical models have a so large number of degrees freedom to reproduce, conceptually, the same difficulties met in molecular dynamics. In this frame, the practical importance of the EEP relies in the simple prediction it provides of the energy distribution among the particles, or degrees of freedom, of the considered system. In several cases, this information would be sufficient for characterizing the response of the system for any practical purpose.
These points justify a specific analysis aimed at a better understanding of the limits of validity of the EEP in the context of engineering tools making use of an energy-statistical approach to solve vibration problems.
EEP is not an obvious result in the analysis of dynamical systems. This is clear through the consideration of the amount of assumptions that in SM are made in order to approach the EEP, assumptions often not satisfied in engineering systems. Although a more detailed discussion of similar questions is found in [1], a resume of some basic hypotheses illustrated there is given below:

● SM deals with an ensemble of systems with a constant total energy. In fact, in the light of its approach, dealing with the microscopic structure of matter, no energy dissipation exists, but only an energy transfer among the particles of the system;
● SM, under stationary conditions, uses the hypothesis of uniform probability of finding representative points of the ensemble over the equal-energy-surface in the phase space;
● SM predicts the energy partitioning between two coupled sub-systems in steady state conditions through the Gibbs-Boltzmann distribution. Besides all the previous hypotheses, the two considered sub-systems are supposed to be weakly coupled (note that ‘weak coupling’ in this context does not have the same meaning as in Statistical Energy Analysis). This means that the total energy of the whole system is just the sum of the partial energies of the two subsystems, i.e. the energy associated to the interaction forces is negligible with respect to the energy stored in each subsystem. This hypothesis is somewhere called in statistical mechanics principle of energy decomposability [2].
● The EEP is derived on the basis of the Gibbs-Boltzmann distribution using all the previously mentioned hypotheses. In fact, once the Gibbs-Boltzmann probability density function is determined, the first moment of the energy is readily calculated. The result produces the EEP in the form given at the beginning of this section.
It appears that the assumptions made to derive strictly the EEP are not obviously verified in any dynamical systems, especially in those met in structural engineering.

This way of looking at the EEP considers the energy referred to the particles of the system. However, the concept of equipartition is, sometimes, referred to the energy associated with modes (e.g. this is the case of SEA). Moreover, in other contexts the principle of equipartition is stated for dissipative systems excited by random forces (again the case often considered in SEA).
All the previous considerations show that the question of the validity of EEP is a rather complex problem.

In this paper the concept of energy equipartitioning is considered in the light of the results presented in [1, 3]. In that analysis –named Time Asymptotic Ensemble Energy Average (TAE)- the unsteady energy sharing between two coupled resonators, each consisting of a very large number of degrees of freedom (or modes) and each affected by uncertainties in its parameters, is considered. The theory allows to determine the time history of the energies of the sub-components together with the general relationship between power flow and energies of the two sub-systems.
As a particular result, the analysis also provides the long term energy response of the two sub-components, at least in the case of undamped linear systems. This result is shown to be coincident with the EEP only under particular circumstances, but its coincidence with the EEP does not seem to be a general rule.
The present analysis, starts from this last point. With the help of a systematic series of numerical experiments, an investigation about the validity of the EEP for linear as well as for nonlinear resonators, is attempted.

ASYMPTOTIC LIMITS IN ENERGY SHARING

Linear homogeneous coupled sub-systems

Some of the results found in [1, 3] and related to EEP are here shortly summarized.
A freely vibrating system *S*, isolated and conservative, satisfying given initial conditions, is considered. Two parts of *S*, S_1 and S_2, are studied such that $S \equiv S_1 \cup S_2$.
Assume that the system *S* starts to vibrate due to given initial conditions.
In [1, 3] it is demonstrated that the total energy of the *r-th* (*r=1,2*) sub-component, has the expression:

$$E^{(r)}(t) = \sum_{i,j=1}^{N} \left[a_{ij}^{(r)} \cos(\omega_i + \omega_j)t + b_{ij}^{(r)} \cos(\omega_i - \omega_j)t \right] \qquad (1)$$

where the *a*'s and *b*'s are coefficients depending on the initial conditions, on the modeshapes of *S* and on its natural frequencies ω_i.
Suppose now that inherent uncertainties affect the system *S*. As a consequence, expression (1) is not deterministic anymore, representing indeed a stochastic process. The attention is addressed to the collection of pair of interacting systems S_1 and S_2 with energies $E^{(1)}(t)$, $E^{(2)}(t)$ and more precisely to the time history of the ensemble energy average of this collection.

The way the uncertain system *S* is described is through its natural frequencies ω_i, regarded as a set of random variables, characterized by a joint probability density function $p(\omega_1,\omega_2,...,\omega_N)=p(\Omega)$. Thus, the ensemble energy average of the subsystem S_1 is:

$$\overline{E}^{(1)}(t)=\int_0^\infty \int_0^\infty ... \int_0^\infty E^{(1)}\, p(\omega_1,\omega_2,...,\omega_N)\, d\omega_1 d\omega_2 ... d\omega_N = \int_{R^N} E^{(1)}\, p(\Omega)\, d\Omega \quad (2)$$

where $d\Omega = d\omega_1 d\omega_2 ... d\omega_n$; an analogous expression is valid for the second subsystem S_2.

Substituting equation (1) into (2) and considering only the steady state energy component (the time independent contribution), i.e. the equilibrium energy $\overline{E}^{(1)}_{eq}$, of the system S_1 or S_2, yields to:

$$\begin{aligned} \overline{E}^{(1)}_{eq} &= \lim_{t\to\infty} \overline{E}^{(1)} = \sum_{i=1}^{N} \int_{\omega_i^-}^{\omega_i^+} \Pi_i(\omega_i)\, b^{(1)}(\omega_i)\, d\omega_i, \\ \overline{E}^{(2)}_{eq} &= \lim_{t\to\infty} \overline{E}^{(2)} = \sum_{i=1}^{N} \int_{\omega_i^-}^{\omega_i^+} \Pi_i(\omega_i)\, b^{(2)}(\omega_i)\, d\omega_i \end{aligned} \quad (3)$$

$$d\Omega_i = \frac{d\Omega}{d\omega_i}, \qquad \Pi_i(\omega_i) = \int_{R^{N-1}} p\,(\Omega)\; d\Omega_i$$

Equations (3) provide the expected energy approached asymptotically, i.e. when time becomes large.

It is interesting to explore the chance of simplifying these expressions. In [1] it is proven, when restricting the analysis to two coupled homogenous subsystems of the same type (cavity-cavity, beam-beam, plate-plate, etc.), that the following expressions can be obtained:

$$\overline{E}^{(1)}_{eq} = \frac{m^{(1)}}{m_0}\overline{E}_0, \qquad \overline{E}^{(2)}_{eq} = \frac{m^{(2)}}{m_0}\overline{E}_0 \quad (4)$$

where m_0, $m^{(1)}, m^{(2)}, \overline{E}_0$ are the total mass of *S*, the masses of S_1 and S_2, and the total energy of *S*, respectively.

Moreover, considering that this equilibrium conditions are determined under the hypothesis of two similar subsystems, they can be written also in an alternative form. Suppose that the modal response of the whole system includes the natural frequencies up to $\omega_{\max}$. In general, the mass of a system is related to the number *N* of modes contained in the frequency bandwidth $[0,\omega_{\max}]$. For example, for an acoustic cavity, the mode count leads to $N=\left(\frac{\omega_{\max}}{c}\right)^3 \frac{m}{6\pi^2\rho}$, where c, m, ρ are the speed of sound, the mass of the trapped gas and its mass density, respectively. For a bending plate it is

$N = \dfrac{\omega_{\max} m}{3.6 \rho c_L h^2}$, where c_L, m, ρ, h are the speed of longitudinal waves, the plate mass, the material mass density and the thickness, respectively. As a general rule $m = N f(\omega_{\max})$, where the form of the function f depends on the kind of system considered and on its properties. If the two coupled subsystems have the same form of the function f, then:

$$\frac{m^{(1)}}{m_0} = \frac{N^{(1)}}{N}, \qquad \frac{m^{(2)}}{m_0} = \frac{N^{(2)}}{N}, \qquad \frac{\bar{E}_{eq}^{(1)}}{N^{(1)}} = \frac{\bar{E}_{eq}^{(2)}}{N^{(2)}} = \frac{\bar{E}_0}{N} \tag{5}$$

where $N^{(1)}$ and $N^{(2)}$ $(N^{(1)} + N^{(2)} = N)$ are the number of modes of S_1 and S_2, respectively, contained into the frequency range $[0, \omega_{\max}]$. This expression states that, under steady conditions, the energy per mode of each subsystem is equal to the initial energy per mode of the whole system. This expresses a condition reminiscent of the EEP. However, this principle does not seem to hold in general, where equation (3) is valid, but only under the particular conditions specified above.

A first set of numerical simulations are performed to show the validity of equations (4) and (5) in the analysis of homogeneous coupled systems. The test case refers to a three dimensional acoustic cavity divided into three homogeneous cavities by two rectangular panels, as shown in figure 1. The volumes of the three cavities are approximatively in the ratio 4:1 and 5:1 with respect to the smallest one. The sides of both panels are shorter than the sides of the box, leaving rectangular openings between the subsystems. A pressure spike is generated at the initial time inside the smallest subsystems (chamber 1 in Fig. 1).

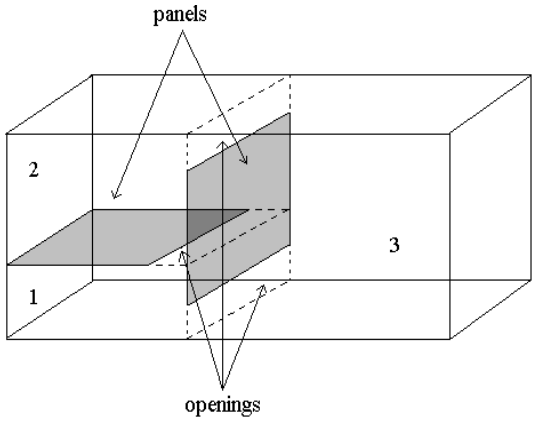

Figure 1
Scketch of the acoustic cavity

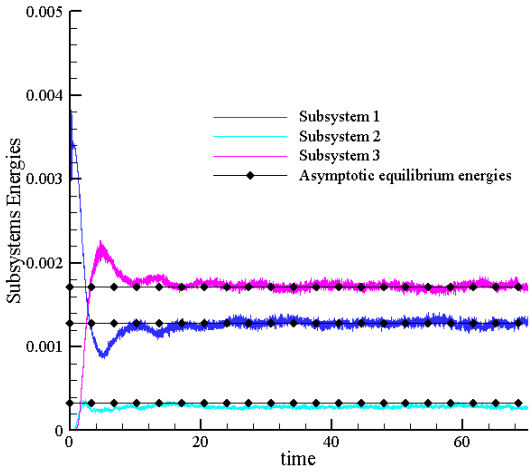

Figure 2
Comparison beetwen numerical and theoretically predicted energies

A finite difference scheme with grid 24 x 24 x 24, corresponding to 13824 degrees of

freedom, is used for the numerical solution of the acoustic wave equation.
In figure 2 the time hystories of the subsystems energies are shown and compared with the asympotic energy values predicted by eqs. (4) and (5), showing a very good agreement. Similar results are obtained for different system configurations and different pressure spike locations.

Inhomogeneous coupling

It is clearly seen how equations (4) and (5) meet difficulties when applied to the case of inhomogeneous coupled sub-systems. In fact, the two equations are not, in general, equivalent: the number of modes and the total mass of each sub-system are not proportional quantities, except in the case of coupling between homogenous sub-systems considered in the previous subsection. Thus, in general, equations (4) and (5) represent different energy equipartition principles and it is not clear which of them would be valid or even if they both fail.
A set of numerical simulations are performed on one rod, with E=1 Nm^2, A=1m^2 and L=1m, where E, A, L are the Young modulus, the mass per unit length and the total length, respectively. The rod was ideally divided into two inhomogeneous subsystems of lenght $\frac{1}{4}L$ and $\frac{3}{4}L$, respectively, and different densities: for the first rod is 2 Kg/m, for the second 1 Kg/m. Initial conditions are $u(x,0)=\sin\left(\frac{2\pi x}{L}\right)$, $\dot{u}(x,0)=0$.
The first simulation is performed using a finite difference scheme with 400 points.

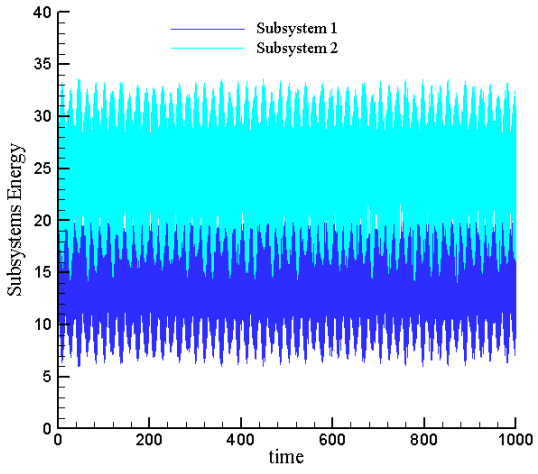

Figure 3
Rod: subsystem energies

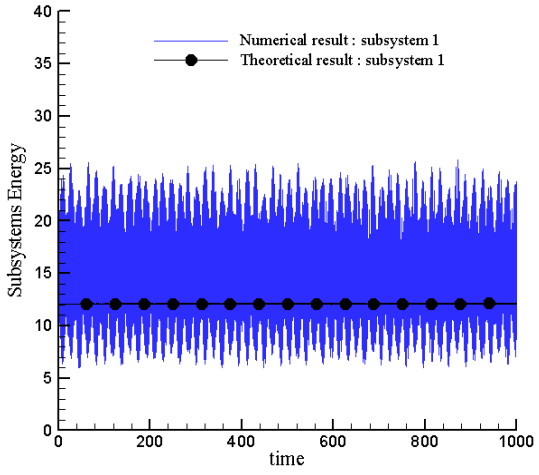

Figure 4
Rod: comparison between numerical and theoretical energy of subsystem 1

In figure 3, the time histories of the subsystems energies are shown. The energies of the two subsystems do not reach any asymptotic equilibrium conditions, but they oscillates around their mean values. Figure 4 shows a comparison between the

numerical energy trend for the first subsystem and the asymptotic equilibrium energy, evaluated by equation (4) .

Coupling between nonlinear systems

The previous numerical simulations lead to conclude that the EEP for linear systems has a limited validity. Note, in addition, that the asymptotic values of the energies of the two sub-components exhibit a certain dependency on the choice of the initial conditions for the system *S*. In fact, their selection deeply affects the set of modes involved in the system response. In particular, for linear systems, being the modes decoupled and energy independent, only those excited by the initial conditions will be included in the procees of energy sharing between the two sub-systems. Thus, none energy equipartition among the modes of the systems takes place. It is clear indeed, that a certain energy spreading of the energy among the modes is present if the two sub-systems present nonlinearities. Thus, it seems to be intuitive that nonlinearities can affect the asymptotic values of the energies influencing the process of energy sharing.
A nonlinear relationship is introduced for the rod strain-stress relationship, namely by a hardening effect $\sigma = E\varepsilon + \gamma\varepsilon^3$, where σ and ε are the axis stress and deformation respectively, E the Young modulus equal to 1 Nm^2 and γ a suitable material coefficient equal to 3.E-3 Nm^2. The systems are indeed identical for any other respects. Unlike in the linear case, the nonlinear rod approaches, after a transient, an equilibrium condition with small fluctuations around the asymptotic value, as shown in figure 5. In the same figure the asymptotic equilibrium energies evaluated by equation (5) are also plotted, showing a very good agreement. Thus, in this case eq. (4) fails and eq.(5) is correct. Finally, the standard deviations of the energy fluctuations with respect the average energy, both for the linear and the nonlinear rod, is calculated:

$$\sigma_E = \frac{\sqrt{\sum_{i=1}^{N}\left(E_i - \overline{E}\right)^2}}{N}$$

In figure 6 a comparison between the standard deviation obtained for both the linear and the nonlinear rods are plotted.
Except for the transient, the two systems show totally different time histories. In fact, while for the linear system the standard deviation does not reach an equilibrium condition, for the nonlinear system both its amplitude and fluctuations decrease with increasing time.

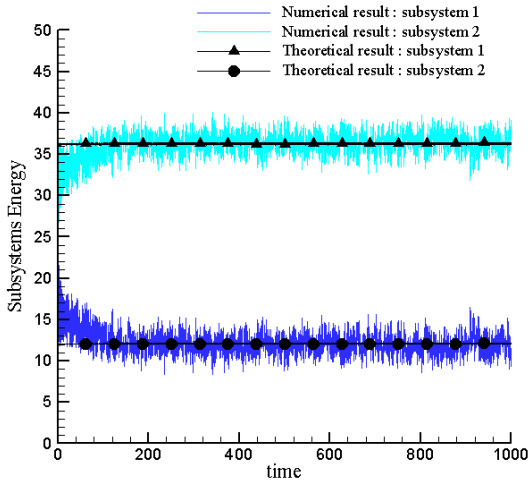

Figure 5
Nonlinear rod: comparison between numerical and theoretical energy of subsystems

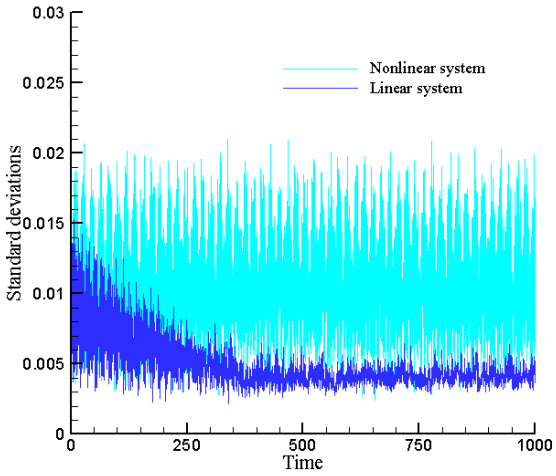

Figure 6
Comparison between the standard deviation of energy fluctuations for linear and nonlinear rod

CONCLUSIONS

In this paper the Energy Equipartition Principle (EEP) both for linear as well as for nonlinear vibrating systems is investigated. This property consists in reaching an equilibrium condition in which the energy among two (or more) sub-systems is shared in direct proportion of the number of their respective degrees of freedom. The main results are shortly summarized below:

(i) EEP does not hold in general for linear systems;
(ii) EEP holds for linear systems only when the two coupled subsystems are homogeneous;
(iii) EEP holds in general for nonlinear vibrating systems, whatever the nature of the coupled sub-systems, homogeneous or inomogeneous.

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