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Tolstoy's dream	The agent-based approach to Economics and the Social Sciences is becoming AQ1 more and more popular among scholars interested in going beyond mainstream analyses [14]. This approach is trying to reconcile methodological individual- ism [13] with the existence of emergent phenomena in social systems [2].

Tolstoy's dream and the quest for statistical equilibrium in economics and the social sciences

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1 Tolstoy's dream

AQ1

The agent-based approach to Economics and the Social Sciences is becoming more and more popular among scholars interested in going beyond mainstream analyses [14]. This approach is trying to reconcile methodological individualism [13] with the existence of emergent phenomena in social systems [2].

Some of these concepts may appear brand new, but, at least, they can be traced back to the philosophical and scientific discussions taking place in the XIXth Century. The basic idea is that there is an analogy between human societies where many individuals interact and gases where many atoms or molecules interact. Indeed, as discussed by Hacking in *The Taming of Chance* [8], Boltzmann himself used this analogy in order to justify the atomic hypothesis. This idea was pervasive in XIXth Century thinkers. We like to think of Tolstoy's novel *War and Peace* [15] as an early agent-based simulation. The author explores the behaviour and interactions of his 580 characters during the Napoleonic invasion of Russia. More specifically, the second epilogue of the novel reveals Tolstoy's theoretical interests and his model of human history. Let Tolstoy directly speak:

> Speaking of the interaction of heat and electricity and of atoms, we cannot say why this occurs, and we say that it is so because it is inconceivable otherwise, because it must be so and that it is a law.
> The same applies to historical events. Why war and revolution occur we do not know. We only know that to produce the one or the other action, people combine in a certain formation in which they all take part, and we say that this is so because it is unthinkable otherwise, or in other words that it is a law.

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Therefore, in the XIXth Century, the analogy on which current agent-based simulations are grounded was so popolar that it found its way through literature. Unfortunately for Economics, the mechanical analogy was used in its static version and the concept of statistical equilibrium remained unknown to most economists troughout all the XXth Century and up to now.

2 Statistical equilibrium in economics

2.1 What is the common notion of equilibrium in economics?

The concept of equilibrium referred to in General Equilibrium Theory is taken from Physics. It coincides with mechanical equilibrium.

When looking for mechanical equilibrium one minimizes a potential function subject to boundary conditions, in order to find equilibrium positions; when looking for standard (micro)economic equilibrium, one maximizes a utility function subject to budget constraints (this is the consumer side, in other words, demand) and maximizes the profit subject to cost constraints (this is producer side, in other words, supply); then one equates supply and demand, and finds equilibrium quantities and prices. In both cases, the mathematical tool is optimization with constraints using the method of Lagrange multipliers.

Walras and Pareto explicitly inspired their pioneering work on General Equilibrium Theory to Physics and mechanical equilibrium. This was made clear by Ingrao and Israel [9].

2.2 What is statistical equilibrium?

Statistical equilibrium is another notion of equilibrium in Physics. It was defined by Maxwell and Boltzmann in their early work on the theory of gases, trying to reconcile mechanics and thermodynamics. In order to better understand this notion, it is useful to make use of a Markovianist approach to statistical equilibrium as discussed by Oliver Penrose (the brother of Roger Penrose) in his 1970 book [10]. By the way, a similar approach was promoted by Richard von Mises (the brother of Ludwig von Mises) in a book reprinted in 1945 (actually the book was written by R. von Mises before World War II) [16].

A finite Markov chain is a stochastic process defined as a sequence of random variables $X_1, \ldots, X_n$ on the same probability space that assume values in a finite set S, known as the state space. For a Markov chain, the predictive probability $\mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}, \ldots, X_1 = x_1)$ has the following simple form:

$$\mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}, \ldots, X_1 = x_1) = \mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}). \quad (1)$$

In other words, the predictive probability does not depend on all the past states, but on the last state occupied by the chain. As a consequence of the multiplication theorem, one gets that the finite-dimensional distribution $\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n)$ is given by:

$$\mathbb{P}(X_0 = x_0, \ldots, X_n = x_n) \\ = \mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}) \cdots \mathbb{P}(X_1 = x_1 | X_0 = x_0)\mathbb{P}(X_0 = x_0). \tag{2}$$

As a consequence of Kolmogorov's representation theorem, this means that a Markov chain is fully characterized by the knowledge of the functions $\mathbb{P}(X_m = x_m | X_{m-1} = x_{m-1})$, also known as *transition probabilities* and $\mathbb{P}(X_0 = x_0)$, also known as *initial probability distributions*. If the transition probabilities do not depend on the index m but only on the initial and on the final state, then the Markov chain is called *homogeneous*. In the following, only homogeneous Markov chains will be considered. For the sake of simplicity, it is useful to introduce the notation

$$P(x, y) = \mathbb{P}(X_m = y | X_{m-1} = x) \tag{3}$$

for the transition probability and

$$p(x) = \mathbb{P}(X_0 = x) \tag{4}$$

for the initial probability distribution. Note that $P(x, y)$ is nothing else than a matrix in the finite case under scrutiny, with the property that

$$\sum_{y \in S} P(x, y) = 1; \tag{5}$$

in other words the rows of the matrix sum up to 1 as a consequence of the addition axiom. Such matrices are called *stochastic matrices* (to be distinguished from random matrices which are matrices with random entries). Note that the initial distribution can be written as a row vector, so that one can obtain the marginal distribution of the random variable X_n as:

$$\mathbb{P}(X_n = y) = \sum_{x \in S} p(x) P^{(n)}(x, y), \tag{6}$$

where $P^{(n)}(x, y)$ represents the (x, y) entry of the n-step transition matrix.

Now, assume there is a distribution $\pi(x)$ satisfying the equation:

$$\pi(y) = \sum_{x \in S} \pi(x) P(x, y), \tag{7}$$

then $\pi(x)$ is called a *stationary distribution* or *invariant measure*. If at time step t the chain is described by $\mathbb{P}(X_t = x) = \pi(x)$, then from (7), it follows that $\mathbb{P}(X_{t+1} = x) = \pi(x) = \mathbb{P}(X_t = x)$; in other words, the distribution does not change as time goes by. Note that the states are jumping from one to another one, but the probability of finding the system in a specific state does not change. This is exactly the idea of statistical equilibrium put forward by Ludwig Boltzmann.

However, more can and should be said. First of all, the stationary distribution may not exist; secondly, the chain usually starts from a specific state, so that the initial distribution is a vector full of 0's and with a single 1 in the initial state. The latter state of affairs can be represented by a Kronecker delta $\pi(x) = \delta(x, x_0)$, where x_0 is the specific initial state. This is not a stationary distribution and the convergence of the chain to the stationary distribution is not granted at all. Fortunately, it turns out that under some rather mild conditions:

- The stationary distribution exists and it is unique;
- The chain always converges to the stationary distribution irrespective of its initial distribution.

It is indeed sufficient to consider a finite chain that is *irreducible* and *aperiodic*. A chain is irreducible if all the states are persistent; this is equivalent to claim that any state can be reached from any other state with finite probability in a finite number of steps. The chain is aperiodic if for any x, one has that $P^{(s)}(x, x) > 0$ for $s > s_0(x)$; in other words, after a possible transitory period, the probability of return is positive. All these conditions essentially mean that the s-step matrix $\mathbf{P}^{(s)}$ no more has any zero entries after a sufficient number of steps.

If the finite Markov chain is irreducible and aperiodic, then it has a unique invariant distribution $\pi(x)$ and

$$\lim_{n\to\infty} P^{(n)}(x, y) = \pi(y) \tag{8}$$

irrespective of the initial state x. This means that, after a transient period, the distribution of chain states reaches a stationary distribution, which can then be interpreted as an equilibrium distribution in the statistical sense.

2.3 Why and where statistical equilibrium may be useful in economics?

There are several possible domains of application of the concept of statistical equilibrium in Economics. Incidentally, note that many agent-based models used in Economic theory are intrinsically Markov chains (or Markovian processes). Therefore, the ideas discussed earlier naturally apply. Up to now, we have used these concepts:

- To discuss some toy models for the distribution of wealth (not of income!) as in Scalas et al. (2006) [11] and in Garibaldi et al. (2007) [6].

- To generalize a sectoral productivity model originally due to Aoki and Yoshikawa [1], in Scalas and Garibaldi (2009) [12].

In [6, 11, 12], we promote the use of a finitary approach to combinatorial stochastic processes. This approach is the subject of a forthcoming book [7] and will be illustrated by an example in the next section.

3 An example: the taxation-redistribution game

3.1 Basic descriptions

Consider a system of n coins to be divided into g agents. There are three levels of description for the system.

- (individual descriptions) Let the integers from 1 to n denote the coins and the integers from 1 to g denote the agents. Let us introduce the variables $V_1, \ldots, V_n$ whose values are given by the integers between 1 and g; by $V_i = j$, we mean the the ith coin belongs to the jth agent.
- (frequency or occupation descriptions) If the names (or labels) of the coins are irrelevant, it is possible to use the variables $Y_1, \ldots, Y_g$ where $Y_i = n_i$ is the number of coins in the pocket of the ith agent. In symbols, one can write $Y_i = \#\{V_j = i, j = 1, \ldots, n\}$. If the vector $\mathbf{Y} = \mathbf{n} = (n_1, \ldots, n_g)$ denotes a particular frequency description, one has $\sum_{i=1}^{g} n_i = n$.
- (frequency of frequencies or partitions) For $k = 1, \ldots, n$, the variables defined by $Z_k = \#\{Y_i = k, i = 1, \ldots, g\}$ give the number of agents with k coins. If the vector $\mathbf{Z} = \mathbf{z} = (z_0, \ldots, z_n)$ denotes a particular partition, it must satisfy the two constraints $\sum_{k=0}^{n} z_k = g$ and $\sum_{k=1}^{n} k z_k = n$.

Example ($n = 3$ objects (coins) into $g = 2$ categories)

- There are eight individual descriptions: $(1,1,1)$, $(1,1,2)$, $(1,2,1)$, $(2,1,1)$, $(2,2,1)$, $(2,1,2)$, $(1,2,2)$, $(2,2,2)$.
- There are four occupation vectors: $(3,0)$ corresponding to $(1,1,1)$; $(2,1)$ corresponding to $(1,1,2)$, $(1,2,1)$ and $(2,1,1)$; $(1,2)$ corresponding to $(2,2,1)$, $(2,1,2)$ and $(1,2,2)$; $(0,3)$ corresponding to $(2,2,2)$.
- There are two partition vectors: $(1,0,0,1)$ corresponding to $(3,0)$ and $(0,3)$; $(0,1,1,0)$ corresponding to $(1,2)$ and $(2,1)$.

The three basic descriptions define possible constituents of the sample space for the individual descriptions. Note that:

- For each occupation vector $\mathbf{n} = (n_1, \ldots, n_g)$ there are

$$\frac{n!}{\prod_{i=1}^{g} n_i!} \tag{9}$$

corresponding individual descriptions;

- For each partition vector $\mathbf{z} = (z_0, \ldots, z_n)$ there are

$$\frac{g!}{\prod_{i=0}^{n} z_i!} \tag{10}$$

corresponding occupation vectors;
- The total number of individual descriptions is g^n;
- The total number of occupation vectors is $(g+n-1)!/[n!(g-1)!]$;
- For the total number of partition vectors, a closed formula is not available.

3.2 Taxation (destruction) and redistribution (creation)

In this section, a stylized probabilistic model for taxation and redistribution will be introduced, based on [6]. A *taxation* is a step in which a coin is randomly taken out of n coins and a *redistribution* is a step in which the coin is given back to one of the g agents. A taxation move is equivalent to a destruction/annihilation and a redistribution move to a creation [3–5]. This model is conservative as the numbers of agents g and of coins n do not change in time. Moreover, it only includes so-called *unary* moves. If the initial state is given by $\mathbf{n} = (n_1, \ldots, n_i, \ldots, n_j, \ldots, n_g)$, the final state is $\mathbf{n}_i^j = (n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_g)$, after taxation and redistribution. Note that indebtedness is not possible. If a coin is randomly selected out of n coins, the probability of selecting a coin belonging to agent i is n_i/n. Therefore, in this model, agents are taxed proportionally to their wealth measured in terms of the number of coins in their pockets. The redistribution step is crucial as it can favour agents with many coins (a rich gets richer mechanism) or agents with few coins (a taxation scheme leading to equality). This can be done by assuming that the probability of giving the coin taken from agent i to agent j is proportional to $w_j + n_j$, where n_j is the number of coins in the pocket of agent j and w_j is a suitable weight. Depending on the choice of w_j, one can obtain different equilibrium situations. Based on the previous considerations, it is assumed that the transition probability is:

$$\mathbb{P}(\mathbf{n}_i^j|\mathbf{n}) = \frac{n_i}{n}\frac{w_j + n_j - \delta_{i,j}}{w + n - 1}, \tag{11}$$

where $w = \sum_{i=1}^{g} w_i$ and the Kronecker symbol $\delta_{i,j}$ takes into account the case $i = j$. If the condition $w_j \neq 0$ is satisfied, then also agents without coins can receive them. If all the agents are equivalent, one has $w_j = a$, uniformly and $w = ga = \theta$, so that (11) becomes

$$\mathbb{P}(\mathbf{n}_i^j|\mathbf{n}) = \frac{n_i}{n}\frac{a + n_j - \delta_{i,j}}{\theta + n - 1}. \tag{12}$$

3.3 Statistical equilibrium

From (7), one can see that the invariant distribution is the left eigenvector corresponding to eigenvalue 1 for the matrix of transition probabilities. However, the direct diagonalization of (11) is cumbersome. In this case, it is easier to use detailed balance. If a probability $p(\mathbf{n})$ can be found satisfying detailed balance, then this is an invariant distribution! In our case, if $i \neq j$, the *direct flux* is given by:

$$p(\mathbf{n})\mathbb{P}(\mathbf{n}_i^j|\mathbf{n}) = p(\mathbf{n})\frac{n_i}{n}\frac{a+n_j}{\theta+n-1} \tag{13}$$

whereas the inverse flux is given by:

$$p(\mathbf{n}_i^j)\mathbb{P}(\mathbf{n}|\mathbf{n}_i^j) = p(\mathbf{n}_i^j)\frac{n_j+1}{n}\frac{a+n_i-1}{\theta+n-1}. \tag{14}$$

Equating the two fluxes, we get

$$\frac{p(\mathbf{n})}{p(\mathbf{n}_i^j)} = \frac{n_j+1}{n_i}\frac{a+n_i-1}{a+n_j}. \tag{15}$$

The g-variate Pólya distribution discussed in the Appendix satisfies (15), so that, eventually, we get the invariant distribution for the taxation-redistribution model (it is the case $\alpha_1 = \alpha_2 = \cdots = \alpha_g = a$)

$$p(\mathbf{n}) = \frac{n!}{\theta^{[n]}}\prod_{i=1}^{g}\frac{a^{[n_i]}}{n_i!}. \tag{16}$$

Moreover, a little thought should convince the reader that the Markov chain defined by (12) is irreducible and aperiodic. Therefore, the invariant distribution (16) is unique and it is also the equilibrium distribution. Three important particular cases of (16) are:

- For $a = 1$

$$p(\mathbf{n}) = \binom{n+g-1}{n}^{-1}; \tag{17}$$

  this is the uniform distribution on all occupation vectors $\mathbf{n}$;
- For $|a| \to \infty$

$$p(\mathbf{n}) = \frac{n!}{\prod_{i=1}^{g} n_i!}\frac{1}{g^n}; \tag{18}$$

  this coincides with the multinomial distribution and corresponds to the uniform distribution on the individual descriptions;
- For $a = -1$

$$p(\mathbf{n}) = \binom{g}{n}^{-1}; \tag{19}$$

  this is again the uniform distribution on the restricted support of all occupation vectors $\mathbf{n}$ with $n_i = 0, 1$.

The case $a = 1$ coincides with the so-called *Bose-Einstein distribution*, the case $|a| \to \infty$ with the so-called *Maxwell-Boltzmann distribution*, and the case $a = -1$ leads to the so-called *Fermi-Dirac distribution*. As discussed in the

Appendix, these three remarkable cases correspond to three urn models. The Bose-Einstein distribution is related to the Pólya urn, the Maxwell-Boltzmann distribution to the Bernoullian urn and the Fermi-Dirac distribution to the hypergeometric urn. However, in this model, the parameter a needs not be confined to the three values discussed earlier and it can assume any real positive value and any negative integer value. Moreover, in our stylized model, the redistribution policy is characterized by the value of the parameter a. If a is small and positive, on has that rich agents become richer, but for $a \to \infty$ the redistribution policy becomes random: any agent has the same probability of receiving the coin. Eventually, the case $a < 0$ favours poor agents, but $|a|$ is the maximum allowed wealth for each agent.

3.4 Wealth (coin) distribution

As discussed in the Appendix, agents' exchangeability lead to a simple relationship between the joint probability distribution of partitions and the probability of a given occupation vector. One has that

$$\mathbb{P}(\mathbf{Z} = \mathbf{z}) = \frac{g!}{\prod_{i=0}^{n} z_i!}\mathbb{P}(\mathbf{Y} = \mathbf{n}) = \frac{g!}{\prod_{i=0}^{n} z_i!}\frac{n!}{\prod_{j=1}^{g} n_i!}\prod_{j=1}^{g}\frac{a^{[n_i]}}{\theta^{[n]}}$$
$$= \frac{g!n!}{\prod_{i=0}^{n} z_i!(i!)^{z_i}}\prod_{j=1}^{g}\frac{a^{[n_i]}}{\theta^{[n]}}, \qquad (20)$$

where, as discussed in Sect. 3.1, z_i is the number of agents with i coins. Now, both (16) and (20) are multivariate distributions. In order to get a univariate distribution, to be compared with empirical data, we consider the marginal distribution that describes a single agent. Given that all the agents are characterized by the same weight a, we can focus on the behaviour of the random variable $Y = Y_1$ representing the number of coins of agent 1. Starting from $Y_t = k$, one can define the following transition probabilities

$$w(k, k+1) = \mathbb{P}(Y_{t+1} = k+1|Y_t = k) = \frac{n-k}{n}\frac{a+k}{\theta+n-1}, \qquad (21)$$

meaning that a coin is randomly selected among the other $n-k$ coins belonging to the other $g-1$ agents and given to agent 1 according to the weight a and to the number of coins k,

$$w(k, k-1) = \mathbb{P}(Y_{t+1} = k-1|Y_t = k) = \frac{k}{n}\frac{\theta-a+n-k}{\theta+n-1}, \qquad (22)$$

meaning that a coin is randomly removed from agent 1 and redistributed to one of the other agents according to the weight $\theta - a$ and the number of coins $n-k$, and

$$w(k, k) = \mathbb{P}(Y_{t+1} = k|Y_t = k) = 1 - w(k, k+1) - w(k, k-1), \qquad (23)$$

meaning that agent 1 is not affected by the move taking place at step $t+1$. These equations define a birth–death Markov chain corresponding to a random walk with semi-reflecting barriers. This chain represents the wealth dynamics of a single agent interacting with a *thermal bath* consisting of the other $g-1$ agents. Indeed, the invariant (and equilibrium) distribution of the birth-death chain can be directly obtained marginalizing (16). This leads to the dichotomous Pólya distribution (see the Appendix):

$$\mathbb{P}(Y=k) = p_k = \frac{n!}{k!(n-k)!}\frac{a^{[k]}(\theta-a)^{[n-k]}}{\theta^{[n]}}. \tag{24}$$

Equation (24) can be compared with the behaviour of the agent as time goes by. As a consequence of the ergodic theorem for irreducible chains, it follows that

$$\lim_{t\to\infty}\frac{\#\{Y_s=k,\ s=0,\dots,t\}}{t} = p_k, \tag{25}$$

where p_k is given by (24). In other words, the marginal equilibrium probability is also the large-time limit of the hitting time relative frequency. These consideration are important, in order to identify the probabilistic objects to be compared to empirical (or to simulated) data.

The same procedure can be used for the wealth distribution $\mathbf{z}$. The random variable Z_k counts the number of agents with k coins. Denoting by $I_{Y_j}^{(k)} = \mathbb{I}_{Y_j=k}$ the indicator function of the event $\{Y_j=k\}$, the random variable Z_k can also be written as follows

$$Z_k = I_{Y_1}^{(k)} + I_{Y_2}^{(k)} + \ldots + I_{Y_g}^{(k)}; \tag{26}$$

Therefore, we find that

$$\mathbb{E}(Z_k) = \sum_{j=1}^{g}\mathbb{E}(I_{Y_j}^{(k)}) = \sum_{j=1}^{g}\mathbb{P}(Y_j=k), \tag{27}$$

where $\mathbb{P}(Y_j=k)$ is the marginal distributions for the jth agent. As a consequence of the equivalence of all agents, from (24) and (27), one gets that

$$\mathbb{E}(Z_k) = g\mathbb{P}(Y=k) = g\frac{n!}{k!(n-k)!}\frac{a^{[k]}(\theta-a)^{[n-k]}}{\theta^{[n]}}. \tag{28}$$

Equation (28) gives the first moment of the probability function on all possible wealth distributions (20) for the taxation-redistribution model.

The thermodynamic limit for (24) when $n \gg 1$, $g \gg 1$ and $n/g = a\chi$ leads to the negative binomial distribution as an approximation of the dichotomous Pòlya distribution (see the Appendix)

$$\mathbb{P}^{\mathrm{TL}}(Y=k) = \mathrm{NegBin}(k|a,\chi) = \frac{a^{[k]}}{k!}\left(\frac{1}{1+\chi}\right)^{a}\left(\frac{\chi}{1+\chi}\right)^{k}. \tag{29}$$

On the other side, the continuous limit for the wealth distribution is (see the Appendix)

$$f_{\mathrm{B}}(x) = \frac{\Gamma(\theta)}{\Gamma(a)\Gamma(\theta - a)} x^{a-1}(1-x)^{\theta-a-1}, \tag{30}$$

where $x = k/n$ is the continous variable corresponding to the normalized wealth of the first agent ($0 \leq x \leq 1$) and $f_{\mathrm{B}}(x)$ is its Beta probability density function. The thermodynamic limit of (30) leads to the Gamma$(x|a, u)$ density

$$p^{\mathrm{TL}}(x) = \frac{u^{-a}}{\Gamma(a)} x^{a-1} \exp\left(-\frac{x}{u}\right). \tag{31}$$

where $u = w/a$, and the meaning of w is the expected value of the wealth of the selected agent, which stays constant when the continuous thermostat becomes infinite. (See the Appendix).

3.5 Block taxation and the convergence to equilibrium

Consider the case in which taxation is made in the following way: instead of drawing a single coin from an agent at each step, $m \leq n$ coins are randomly taken from various agents and then redistributed with the mechanism described earlier, that is with a probability proportional to the actual number of coins and to an *a priori* weight. If $\mathbf{n} = (n_1, \ldots, n_g)$ is the initial occupation vector, $\mathbf{m} = (m_1, \ldots, m_g)$ (with $\sum_{i=1}^{g} m_i = m$) is the taxation vector, and $\mathbf{m}' = (m'_1, \ldots, m'_g)$ (with $\sum_{i=1}^{g} m'_i = m$) is the redistribution vector, we can also write

$$\mathbf{n}' = \mathbf{n} - \mathbf{m} + \mathbf{m}'. \tag{32}$$

The block taxation-redistribution model still has (16) as its equilibrium distribution, as the block step is equivalent to m steps of the original taxation-redistribution model. However, the convergence rate to equilibrium is faster.

The marginal analysis for the block taxation-redistribution model in terms of a birth–death Markov chain is more cumbersome than for the original model because, now, the difference $|\Delta Y|$ can vary from 0 to m. In any case, given that (24) always gives the equilibrium distribution, this means that (see the Appendix)

$$\mathbb{E}(Y) = n\frac{a}{\theta} = \frac{n}{g}, \tag{33}$$

and

$$\mathrm{Var}(Y) = n\frac{a}{\theta}\frac{\theta - a}{\theta}\frac{\theta + n}{\theta + 1} = \frac{n}{g}\frac{g-1}{g}\frac{\theta + n}{\theta + 1}. \tag{34}$$

We can write

$$Y_{t+1} = Y_t - D_{t+1} + C_{t+1}, \tag{35}$$

where D_{t+1} is the random taxation for the given agent and C_{t+1} is the random redistribution to the given agent. The expected value of D_{t+1} under the condition $Y_t = k$ is

$$\mathbb{E}(D_{t+1}|Y_t = k) = m\frac{k}{n}; \tag{36}$$

this result is valid as m coins are taken at random out of the n coins and the probability of removing a coin from the first agent is k/n under the given condition. Moreover, if $Y_t = k$ and $D_{t+1} = d$, we get that the probability of giving a coin back to agent 1 is $(a+k-d)/(\theta+n-m)$, so that, after averaging over $D_{t+1}|Y_t = k$, we have

$$\mathbb{E}(C_{t+1}|Y_t = k) = m\frac{a + k - m\dfrac{k}{n}}{\theta + n - m}. \tag{37}$$

The expected value of $Y_{t+1} - Y_t$ conditioned on $Y_t = k$ can be found from the expectation of (35) and using (36) and (37). This yields:

$$\mathbb{E}(Y_{t+1} - Y_t|Y_t = k) = -\frac{m\theta}{n(\theta + n - m)}\left(k - n\frac{a}{\theta}\right). \tag{38}$$

The following remarks on (38) are possible:

1. Equation (38) is analogous to a mean reverting equation. If, due to random fluctuations, $\mathbb{E}(Y_{t+1}|Y_t = k)$ moves away from its equilibrium expected value $na/\theta = n/g$, it will then move back towards that value;
2. If $k = na/\theta$ then the chain is first-order stationary. If one begins with n/g, then one always gets $\mathbb{E}(Y_{t+1} - Y_t|Y_t = k) = 0$;
3. $r = m\theta/(n(\theta + n - m))$ is the intensity of the restoring force. The inverse of r, gives the order of magnitude for the number of transitions needed to reach equilibrium.
4. If $m = n$, meaning that all the coins are taken and then redistributed, the new state has no memory of the previous one and statistical equilibrium is reached in a single step ($r^{-1} = 1$)!

Before concluding this section, it is interesting to discuss the case $\theta < 0$ in detail. In this case the marginal equilibrium distribution becomes the hypergeometric one:

$$\mathbb{P}(Y = k) = \frac{\dbinom{|a|}{k}\dbinom{|\theta - a|}{n - k}}{\dbinom{|\theta|}{n}}, \tag{39}$$

with $a = \theta/g$ and θ negative integers. The range of k is $(0, 1, \ldots, \min(|a|, n))$. The states with $n_i > |a|$ are transient and they do not appear any more as times goes by.

If, for instance, $|a| = 10n/g$ (ten times the average wealth), one has that $|\theta| = 10n$ and $r = 10m/(10n - n + m) \simeq (10m)/(9n)$. If $m \ll n$, this is not so far from the independent redistribution case. On the contrary, in the extreme case $|a| = n/g$, the occupation vector $\mathbf{n} = (n/g, \ldots, n/g)$ is obtained with probability 1. If an initial state containing individuals richer than $|a|$

is considered, that is if one considers (38) for $k > |a|$, then $\mathbb{E}(D_{t+1}|Y_t = k)$ is still mk/n but $E(C_{t+1}|Y_t = k, D_{t+1} = d) = 0$ unless $k - d < |a|$. More precisely, one has

$$\mathbb{E}(C_{t+1}|Y_t = k) = \begin{cases} m\dfrac{|a| - k + m\dfrac{k}{n}}{|\theta| - n + m} \text{ if } k - m\dfrac{k}{n} \leq |a| \\ 0 \text{ if } k - m\dfrac{k}{n} > |a| \end{cases} \tag{40}$$

If the average percent taxation is $f = m/n$, then one gets

$$\mathbb{E}\left(Y_{t+1} - Y_t|Y_t = k\right) = \begin{cases} -\dfrac{f\theta}{\theta - n(1-f)}\left(k_t - \dfrac{n}{g}\right) \text{ if } k(1-f) \leq |a| \\ = -k(1-f) \text{ if } k(1-f) > |a| \end{cases} \tag{41}$$

As $k(1-f)$ is the average value of Y after taxation, even if the agent is initially richer than $|a|$ he/she can participate to redistribution when the mean percentage of taxation is high enough.

Appendix: the Pólya distribution

Finite (n-step) stochastic processes

The sequence of individual random variables $V_1, \ldots, V_n$ is an n-step stochastic process. It is completely determined by the knowledge of all the finite dimensional distributions of the kind:

$$p_{V_1,\ldots,V_m}(v_1, \ldots, v_m) = \mathbb{P}(V_1 = v_1, \ldots, V_m = v_m), \tag{42}$$

where $1 \leq m \leq n$. The finite dimensional distributions are subject to Kolmogorov's compatibility conditions

$$\begin{aligned} p_{V_1,\ldots,V_m}(v_1, \ldots, v_m) &= \mathbb{P}(V_1 = v_1, \ldots, V_m = v_m) \\ = \mathbb{P}(V_{i_1} = v_{i_1}, \ldots, V_{i_m} = v_{i_m}) &= p_{V_{i_1},\ldots,V_{i_m}}(v_{i_1}, \ldots, v_{i_m}), \end{aligned} \tag{43}$$

where $i_1, \ldots, i_m$ is any of the $m!$ permutations of the indices, and

$$p_{V_1,\ldots,V_{m-1}}(v_1, \ldots, v_{m-1}) = \sum_{v_m=1}^{g} p_{V_1,\ldots,V_m}(v_1, \ldots, v_{m-1}, v_m). \tag{44}$$

Finite dimensional distributions can be conveniently characterized in terms of *predictive* probabilities. Indeed, as a consequence of the multiplication theorem (and of Bayes' theorem), one has

$$\begin{aligned} \mathbb{P}(V_1 = v_1, \ldots, V_m = v_m) &= \mathbb{P}(V_1 = v_1)\mathbb{P}(V_2 = v_2|V_1 = v_1)\cdots \\ &\cdots\mathbb{P}(V_m = v_m|V_1 = v_1, \ldots, V_{m-1} = v_{m-1}) \end{aligned} \tag{45}$$

and Kolmogorov's compatibility conditions are automatically satisfied.

Exchangeable processes

An exchangeable process is characterized by additional symmetry conditions on the finite dimensional distributions

$$p_{V_1,\ldots,V_m}(v_1,\ldots,v_m) = \mathbb{P}(V_1 = v_1,\ldots,V_m = v_m)$$
$$= \mathbb{P}(V_{i_1} = v_1,\ldots,V_{i_m} = v_m) = p_{V_{i_1},\ldots,V_{i_m}}(v_1,\ldots,v_m), \tag{46}$$

where $i_1,\ldots,i_m$ is any of the $m!$ permutations of the indices, Note that condition (46) differs from condition (43). For an exchangeable process, the probability of an individual sequence $\mathbf{V}^{(m)} = \mathbf{v}^{(m)} = (V_1 = v_1,\ldots,V_m = v_m)$ only depends on the occupation vector of the sequence $\mathbf{m} = (m_1,\ldots,m_g)$ with $\sum_{i=1}^{g} m_i = m$. This leads to:

$$\mathbb{P}(\mathbf{V}^{(m)} = \mathbf{v}^{(m)}) = \left(\frac{m!}{\prod_{i=1}^{g} m_i!}\right)^{-1} \mathbb{P}(\mathbf{Y} = \mathbf{m}) \tag{47}$$

as a consequence of (9).

The Pólya process

The Pólya process is an exchangeable process characterized by the predictive probability

$$\mathbb{P}(V_{m+1} = j | V_1 = v_1,\ldots,V_m = v_m) = \frac{\alpha_j + m_j}{\alpha + m}, \tag{48}$$

where m_j is the number of times in which category j has been observed up to step j, $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_g)$ is a vector of parameters and $\alpha = \sum_{i=1}^{g} \alpha_i$. If the new parameters $p_j = \alpha_j/\alpha$ are introduced, (48) becomes

$$\mathbb{P}(V_m = j | V_1 = v_1,\ldots,V_m = v_m) = \frac{\alpha p_j + m_j}{\alpha + m}. \tag{49}$$

$p_j = \mathbb{P}(V_1 = j)$ is the *a priori* probability of category j and (49) is nothing else than a linear mixture between initial or *a priori* probabilities and the observed frequencies. As a consequence of (48), and of exchangeability (see (47)), one gets the following finite dimensional distributions

$$\mathbb{P}(\mathbf{V}^{(m)} = \mathbf{v}^{(m)}) = \left(\frac{m!}{\prod_{i=1}^{g} m_i!}\right)^{-1} \mathrm{Polya}(\mathbf{m}|m;\boldsymbol{\alpha}), \tag{50}$$

where the multivariate generalized Pólya sampling distribution is given by:

$$\mathrm{Polya}(\mathbf{m}|m;\boldsymbol{\alpha}) = \frac{m!}{\alpha^{[m]}} \prod_{i=1}^{g} \frac{\alpha_i^{[m_i]}}{m_i!}, \tag{51}$$

where $x^{[n]} = x(x+1)\cdots(x+n-1)$ is the rising factorial.

The Pólya process encompasses the following remarkable cases:

- The multivariate hypergeometric process for integer $\alpha_j < 0, \forall j \in \{1, \ldots, g\}$ with the constraints $m_j \leq |\alpha_j|$ and $m \leq \alpha$. In this case $|\alpha_j|$ represents the initial number of marbles of colour j in an urn from which they are randomly drawn without replacement; this process is not extendible to infinity and ends after n steps;
- The multinomial process in the limit $|\alpha| \to \infty$ and $|\alpha_j| \to \infty$, with $p_j = \alpha_j/\alpha$ constant. In this case p_j represents the probability of drawing a marble of colour j with replacement from and urn; this process can be exteded to infinity;
- The Pólya urn process for integer $\alpha_j > 0, \forall j \in \{1, 2, \ldots, g\}$. In this case α_j is the initial number of marbles of colour j in an urn. They are randomly drawn and replaced with another ball of the same kind. Also this process is indefinitely extendible.

Marginal distributions

The marginal distributions for the g-variate generalized Pólya distribution can be easily derived from the predictive probability given by (48). Consider the probability $\mathbb{P}(V_{m+1} \in A | V_1 = v_1, \ldots, V_m = v_m)$, where the set A is a set of categories $A = \{j_1, \ldots, j_r\}$. This new predictive probability is given by:

$$\mathbb{P}(V_{m+1} \in A | \mathbf{V}^{(m)}) = \sum_{i=1}^{r} \mathbb{P}(V_{m+1} = j_i | \mathbf{V}^{(m)}), \tag{52}$$

where, as usual, $\mathbf{V}^{(m)} = (V_1 = v_1, \ldots, V_m = v_m)$ summarizes the evidence. In the Pólya case, $\mathbb{P}(V_{m+1} = j | \mathbf{V}^{(m)})$ is a linear function of both the weights and the occupation numbers; therefore, one gets:

$$\mathbb{P}(V_{m+1} \in A | \mathbf{V}^{(m)}) = \frac{\sum_{j \in A} \alpha_j + \sum_{j \in A} m_j}{\alpha + m} = \frac{\alpha_A + m_A}{\alpha + m}, \tag{53}$$

where $\alpha = \sum_j \alpha_j$, $\alpha_A = \sum_{j \in A} \alpha_j$ and $m_A = \sum_{j \in A} m_j$. As a direct consequence of (53), the marginal distributions of the g-variate generalized Pólya distribution are given by the dichotomous Pólya distribution of weights α_i and $\alpha - \alpha_i$, where i is the category with respect to which the marginalization is performed. In other words, one gets that

$$\sum_{m_j,\, j \neq i} \mathrm{Polya}(\mathbf{m} | m, \boldsymbol{\alpha}) = \mathrm{Polya}(m_i, m - m_i; \alpha_i, \alpha - \alpha_i)$$
$$= \frac{m!}{m_i!(m - m_i)!} \frac{\alpha_i^{[m_i]} (\alpha - \alpha_i)^{[m - m_i]}}{\alpha^{[m]}}. \tag{54}$$

Moments of the Pólya distribution

Consider the evidence vector $\mathbf{V}^{(m)} = (V_1 = v_1, \ldots, V_m = v_m)$. In the general case of g categories, it is natural to introduce the indicator function $\mathbb{I}_{X_i=j}(\omega) = I_i^{(j)}$, and define $S_m^{(j)} = \sum_{i=1}^m I_i^{(j)}$. Therefore, the random variable $S_m^{(j)}$ gives the number of successes for the jth category out of m observations or trials and $S_m^{(j)} = m_j$. One can determine $\mathbb{E}(I_i^{(k)})$ and $\mathbb{E}(I_i^{(k)} I_j^{(k)})$ and derive $\mathbb{E}(S_m^{(k)})$ as well as $\mathrm{Var}(S_m^{(k)})$. As for the expected value, one has that $\mathbb{E}(I_i^{(k)}) = 1 \cdot \mathbb{P}(I_i^{(k)} = 1) + 0 \cdot \mathbb{P}(I_i^{(k)} = 0) = \mathbb{P}(I_i^{(k)} = 1)$ coinciding with the marginal probability of success, that is the probability of observing category k at the ith step. From (48), in the absence of any evidence, one has $\mathbb{P}(I_i^{(k)} = 1) = \mathbb{P}(X_i = k) = \alpha_k/\alpha = p_k$. Therefore, the random variables $I_i^{(k)}$ are equidistributed and exchangeable, and $\mathbb{E}(S_m^{(k)}) = \sum_{i=1}^m \mathbb{E}(I_i^{(k)}) = m\mathbb{E}(I_1^{(k)})$, yielding

$$\mathbb{E}(S_m^{(k)}) = mp_k. \tag{55}$$

As for the variance $\mathrm{Var}(S_m^{(k)})$, the covariance matrix of $I_1^{(k)}, \ldots, I_m^{(k)}$ is needed. Because of the exchangeability of $I_1^{(k)}, \ldots, I_m^{(k)}$, the moment $\mathbb{E}[(I_i^{(k)})^2]$ is the same for all i, and $\mathbb{E}(I_i^{(k)} I_j^{(k)})$ is the same for all i, j, with $i \neq j$. Note that $(I_i^{(k)})^2 = I_i^{(k)}$ and this means that $\mathbb{E}[(I_i^{(k)})^2] = p_k$; it follows that

$$\mathrm{Var}(I_i^{(k)}) = \mathbb{E}[(I_i^{(k)})^2] - \mathbb{E}^2(I_i^{(k)}) = p_k(1 - p_k) \tag{56}$$

one can show that

$$\mathbb{E}(I_i^{(k)} I_j^{(k)}) = \mathbb{P}(X_i = k, X_j = k); \tag{57}$$

now, from exchangeability, from (57), and from (48), one gets

$$\begin{aligned} &\mathbb{E}(I_i^{(k)} I_j^{(k)}) = \mathbb{P}(X_i = k, X_j = k) = \mathbb{P}(X_1 = k, X_2 = k) \\ &= \mathbb{E}(I_1^{(k)} I_2^{(k)}) = \mathbb{P}(X_1 = k)\mathbb{P}(X_2 = k | X_1 = k) = p_k \frac{\alpha_k + 1}{\alpha + 1}. \end{aligned} \tag{58}$$

Therefore, the covariance matrix is given by:

$$\begin{aligned} &\mathrm{Cov}(I_i^{(k)}, I_j^{(k)}) = \mathrm{Cov}(I_1^{(k)}, I_2^{(k)}) \\ &= \mathbb{E}(I_1^{(k)} I_2^{(k)}) - \mathbb{E}(I_1^{(k)})\mathbb{E}(I_2^{(k)}) = p_k \frac{\alpha - \alpha_k}{\alpha(\alpha + 1)}. \end{aligned} \tag{59}$$

The variance of the sum $S_m^{(k)}$ follows from (56) and (59)

$$\mathrm{Var}(S_m^{(k)}) = m\mathrm{Var}(I_1^{(k)}) + m(m-1)\mathrm{Cov}(I_1^{(k)}, I_2^{(k)}) = mp_k(1-p_k)\frac{\alpha + m}{\alpha + 1}. \tag{60}$$

Thermodynamic limit

Let α_1 denote the weight of the chosen category and let $\alpha - \alpha_1$ denote the weight of the thermostat. The thermodynamic limit is $n, \alpha \gg 1$ with $\chi = n/\alpha$. Consider that

$$\frac{(\alpha-\alpha_1)^{[n-k]}}{\alpha^{[n]}} = \frac{(\alpha-\alpha_1)(\alpha-\alpha_1+1)\cdots\alpha(\alpha+1)\cdots(\alpha-\alpha_1+n-k-1)}{\alpha(\alpha+1)\cdots(\alpha+n-1)}$$

$$= \frac{(\alpha-\alpha_1)(\alpha-\alpha_1+1)\cdots(\alpha-1)}{(\alpha-\alpha_1+n-k)\cdots(\alpha+n-1)}. \quad (61)$$

The numerator contains the product $\prod_{i=1}^{\alpha_1}(\alpha - i) \simeq \alpha^{\alpha_1}$, whereas at the denominator, one has the product $\prod_{i=1}^{\alpha_1+k}(\alpha+n-i) \simeq (\alpha+n)^{\alpha_1+k}$ and the ratio is approximated by:

$$\frac{\alpha^{\alpha_1}}{(\alpha+n)^{\alpha_1+k}}; \quad (62)$$

therefore, we eventually get

$$\begin{aligned} &\mathbb{P}(k|n;\alpha_1,\alpha) \simeq \mathrm{NegBin}(k|\alpha_1,\chi) \\ &= \mathbb{P}(k|\alpha_1,\chi) = \frac{\alpha_1^{[k]}}{k!}\left(\frac{1}{1+\chi}\right)^{\alpha_1}\left(\frac{\chi}{1+\chi}\right)^{k}, k = 0,1,2,\ldots; \end{aligned} \quad (63)$$

this distribution is called *negative binomial distribution*; the geometric distribution is a particular case of (63) in which $\alpha_1 = 1$ and $\alpha = g$. If α_1 is an integer number, the usual interpretation of the negative binomial random variable is the description of the (discrete) waiting time of (i.e., the number of failures before) the first α_1th success in a Bernoullian process with parameter $p = 1/(1+\chi)$. The moments of the negative binomial distribution can be obtained from the corresponding moments of the Polya$(m_1, m-m_1; \alpha_1, \alpha-\alpha_1)$ in the limit $n, \alpha \gg 1$, with $\chi = n/\alpha$ yielding:

$$\mathbb{E}(Y_1 = k) = n\frac{\alpha_1}{\alpha} \to \alpha_1\chi, \quad (64)$$

$$\mathrm{Var}(Y_1 = k) = n\frac{\alpha_1}{\alpha}\frac{\alpha-\alpha_1}{\alpha}\frac{\alpha+n}{\alpha+1} \to \alpha_1\chi(1+\chi). \quad (65)$$

Note that if α_1 is an integer, k can be interpreted as the sum of α_1 independent geometric variables.

Continuous limit

Consider the multivariate generalized Pólya distribution given by (51). Noting that

$$\alpha^{[m]} = \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)} \quad (66)$$

(51) can be re-written as:

$$\text{Polya}(\mathbf{m}|m;\boldsymbol{\alpha}) = \frac{\Gamma(\alpha)}{\prod_{i=1}^{g}\Gamma(\alpha_i)}\frac{m!}{\Gamma(m+\alpha)}\prod_{i=1}^{g}\frac{\Gamma(m_i+\alpha_i)}{m_i!}. \tag{67}$$

The variables $x_i = m_i/m$ satisfy the following constraint

$$\sum_{i=1}^{g} x_i = \sum_{i=1}^{g}\frac{m_i}{m} = 1; \tag{68}$$

moreover, $\forall i \in \{1,\dots,g\}$, we have that $0 \leq x_i \leq 1$. If one considers the *continuous* limit in which $m \to \infty$, $m_i \to \infty$ with constant $x_i = m_i/m$ for all the categories i, one finds that

$$\frac{\Gamma(m_i+\alpha_i)}{m_i!} = \frac{\Gamma(m_i+\alpha_i)}{\Gamma(m_i+1)} \simeq m_i^{\alpha_i-1} \tag{69}$$

replacing (69) for any m_i and for m in (67) leads to

$$\begin{aligned}\text{Polya}(\mathbf{m}|m;\boldsymbol{\alpha}) &\simeq \frac{\Gamma(\alpha)}{\prod_{i=1}^{g}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{g} m_i^{\alpha_i-1}}{m^{\alpha-1}} \\ &= \frac{\Gamma\left(\sum_{i=1}^{g}\alpha_i\right)}{\prod_{i=1}^{g}\Gamma(\alpha_i)}\prod_{i=1}^{g} x_i^{\alpha_i-1}\cdot\frac{1}{m^{g-1}}.\end{aligned} \tag{70}$$

Equation (70) can be interpreted as follows; based on the exchangeability of the variables $Y_i = m_i$, the probability of the variables $X_i = Y_i/m$ of assuming values $X_1 = x_1, \dots X_n = x_n$ with $x_i = m_i/m$ is

$$\begin{aligned}\mathbb{P}(X_1 = x_1,\dots,X_n = x_n) &\simeq \frac{\Gamma\left(\sum_{i=1}^{g}\alpha_i\right)}{\prod_{i=1}^{g}\Gamma(\alpha_i)}\prod_{i=1}^{g} x_i^{\alpha_i-1}\cdot\frac{1}{m^{g-1}} \\ &\simeq \frac{\Gamma\left(\sum_{i=1}^{g}\alpha_i\right)}{\prod_{i=1}^{g}\Gamma(\alpha_i)}\prod_{i=1}^{g} x_i^{\alpha_i-1}\mathrm{d}x_1\cdots\mathrm{d}x_{g-1},\end{aligned} \tag{71}$$

where the relationship becomes exact in the continuous limit. In fact, the ratio $1/m$ can be interpreted as Δx_i because $\Delta m_i = 1$ and $x_i = m_i/m$. The function

$$p(x_1,\dots,x_g;\alpha_1,\dots\alpha_g) = p(\mathbf{x};\boldsymbol{\alpha}) = \frac{\Gamma\left(\sum_{i=1}^{g}\alpha_i\right)}{\prod_{i=1}^{g}\Gamma(\alpha_i)}\prod_{i=1}^{g} x_i^{\alpha_i-1} \tag{72}$$

defined on the simplex $\sum_{i=1}^{g} x_i = 1$ and $0 \leq x_i \leq 1$ for all the $i \in \{1,\dots g\}$ is the probability density function for the so-called *Dirichlet distribution*. Let $\mathbf{X} \sim \text{Dir}(\mathbf{x};\boldsymbol{\alpha})$ denote the fact that the random vector $\mathbf{X}$ is distributed according to the Dirichlet distribution. As a consequence of the Pólya marginalization property (53), we obtain the so-called *aggregation* property of the

Dirichlet distribution: let $X_1, \ldots, X_g$ be a sequence of random variables with values on the simplex $\sum_{i=1}^{g} x_i$ with $0 \leq x_i \leq 1$, $\forall i \in \{1, \ldots g\}$ whose distribution is $\mathrm{Dir}(x_1, \ldots, x_i, \ldots, x_{i+k}, \ldots, x_g; \alpha_1, \ldots, \alpha_i, \ldots, \alpha_{i+k}, \ldots, \alpha_g)$, then the new sequence $X_1, \ldots, X_A = \sum_{j=i}^{i+k} X_j, \ldots X_g$ is distributed according to $\mathrm{Dir}(x_1, \ldots, x_A = \sum_{j=i}^{i+k} x_j, \ldots, x_g; \alpha_1, \ldots, \alpha_A = \sum_{j=i}^{i+k} \alpha_j, \ldots, \alpha_g)$. Thanks to the aggregation property, we can find the marginal distribution of the Dirichlet distribution, whose probability density function is nothing else than the Beta distribution. If $X_1, \ldots, X_g \sim \mathrm{Dir}(x_1, \ldots, x_g; \alpha_1, \ldots, \alpha_g)$ then

$$X_i \sim \mathrm{Beta}(x_i; \alpha_i, \alpha - \alpha_i). \tag{73}$$

Starting from the probability density function $\mathrm{Beta}(x; a, b)$.

$$p(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}. \tag{74}$$

and defining $y = Ax$, then we get

$$f(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{1}{A} \left(\frac{y}{A}\right)^{a-1} \left(1 - \frac{y}{A}\right)^{b-1}, \tag{75}$$

with $y \in [0, A]$. While x is the fraction of wealth belonging to the selected agent, now y represents his absolute wealth, being A the total wealth. In the limit $A \to \infty$, $b \to \infty$, $A/b = w/a = u$ constant, the Beta density can be approximated by the $\mathrm{Gamma}(y|a, u)$ density given by:

$$g(y) = \frac{u^{-a}}{\Gamma(a)} y^{a-1} \exp\left(-\frac{y}{u}\right). \tag{76}$$

The meaning of w is the expected value of the wealth of the selected agent, which stays constant when the continuous thermostat becomes infinite.

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AQ1. Kindly provide "Summary" for this chapter.
AQ2. Kindly update the Ref. 7.