High-Order IMEX-RK Finite Volume Methods for Multidimensional Hyperbolic Systems.

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Abstract

In this paper we present a high-order accurate cell-centered finite volume method for the semi-implicit discretization of multidimensional hyperbolic systems in conservative form on unstructured grids. This method is based on a special splitting of the physical flux function into a convective and a non-convective part. The convective contribution to the global flux is treated implicitly by mimicking the upwinding of a scalar linear flux function while the rest of the flux is discretized in an explicit way. The spatial accuracy is ensured by allowing non-oscillatory polynomial reconstruction procedures, while the time accuracy is attained by adopting a Runge-Kutta stepping scheme. The method can be naturally considered in the framework of the **IM**plicit-**EX**plicit (IMEX) schemes and the properties of the resulting operators are analysed using the properties of M-matrices.

1 Introduction

Let us consider the multidimensional first-order hyperbolic system in divergence form

$$\frac{\partial}{\partial t}\mathbf{U} + \nabla \cdot \mathbf{F}(\mathbf{U}) = \mathbf{Q} \tag{1}$$

where the conservation of the solution vector $\mathbf{U}$ is governed by the divergence of the physical flux vector function $\mathbf{F}(\mathbf{U})$ and the right-hand-side (r.h.s.) source term $\mathbf{Q}$. This latter one can be stiff and generally takes into account the interactions among different conservative variables. Let us introduce the following assumptions on (1):

(i) according with the definitions given in Reference [5], system (1) is hyperbolic; we do not assume a-priori strict hyperbolicity;

(ii) the physical flux function is the sum of a convective and a non-convective part, denoted by $\mathbf{F}^{(c)}(\mathbf{U})$ and $\mathbf{F}^{(nc)}(\mathbf{U})$, i.e.

$$\mathbf{F}(\mathbf{U}) = \mathbf{F}^{(c)}(\mathbf{U}) + \mathbf{F}^{(nc)}(\mathbf{U}); \tag{2}$$

(iii) the convective part of the flux is the tensor product of the conservative solution vector $\mathbf{U}$ and a suitable *convective velocity field* $\mathbf{v}(\mathbf{U})$, i.e.

$$\mathbf{F}^{(c)}(\mathbf{U}) = \mathbf{U} \otimes \mathbf{v}(\mathbf{U}), \tag{3}$$

where the definition of $\mathbf{v}(\mathbf{U})$ is clearly problem dependent;

- the convection is neither faster than the fastest wave speed nor slower than the slowest one; thus the following inequality must always hold

$$\lambda_{min}(\mathbf{U}, \mathbf{n}) \leq \mathbf{v}(\mathbf{U}) \cdot \mathbf{n} \leq \lambda_{max}(\mathbf{U}, \mathbf{n}), \tag{4}$$

  where λ_{min} and λ_{max} are the minimum and maximum real eigenvalue fields of the Jacobian matrix $\dfrac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}} \cdot \mathbf{n}$, see Reference [5].

Many interesting physical models satisfy these constraints. For instance we mention the Compressible Gas Dynamic Euler Equations and the Shallow Water Equations. In the first case, $\mathbf{U}$ and $\mathbf{F}(\mathbf{U})$ are given by

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho\mathbf{v} \\ \rho E \end{pmatrix}, \qquad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho \\ \rho\mathbf{v} \\ \rho E \end{pmatrix} \otimes \mathbf{v} + p \begin{pmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{v} \end{pmatrix}, \tag{5}$$

see Reference [6]. The conservative variables are the density ρ, the momentum $\rho\mathbf{v}$, and the total energy ρE. The pressure field p is related to the conservative variables $\mathbf{U}$ by a thermodinamic relation, such as the usual polytropic state equation. In this case, the convective velocity field coincides with the fluid velocity $\mathbf{v}$ and the non-convective contribution to the physical flux, that is $\mathbf{F}^{(nc)}(\mathbf{U})$, plays the role of a pressure correction to the fluid advection of the conserved quantities. In the second case, $\mathbf{U}$ and $\mathbf{F}(\mathbf{U})$ are defined as

$$\mathbf{U} = \begin{pmatrix} h \\ h\mathbf{v} \end{pmatrix}, \qquad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} h \\ h\mathbf{v} \end{pmatrix} \otimes \mathbf{v} + \frac{1}{2} g h^2 \begin{pmatrix} \mathbf{0} \\ \mathbf{I} \end{pmatrix}, \tag{6}$$

where h is the water depth, $h\mathbf{v}$ is the momentum, and g is the gravitational constant. The non-convective part of the flux, that is $\mathbf{F}^{(nc)}(\mathbf{U})$, plays the role of the average water depth contribution to the physical flux.

Following a method-of-line approach, we numerically solve (1) by combining a shock-capturing cell-centered **F**inite **V**olume (**FV**) discretisation of the flux divergence term and an **IM**plicit-**EX**plicit **R**unge-**K**utta (**IMEX-RK**) time stepping scheme for the time derivative.

The semi-implicit FV discretisation which governs the time evolution of the i-th cell-averaged vector $\mathbf{U}_i$ reads as

$$\frac{d\mathbf{U}_i}{dt} + \frac{1}{\mathrm{Vol}_i} \sum_{j \in \sigma(i)} \mathbf{H}_{ij} = \mathbf{Q}_i, \tag{7}$$

where Vol_i is the measure (area in 2-D) of the cell i, $\mathbf{Q}_i$ is the cell-average of the source term, and $\mathbf{H}_{ij}$ is the numerical flux from cell i to cell j. The numerical flux balance of the cell i is given by summing the contributions of the numerical fluxes between the cell i and all its adjacent cells, which are listed in the set $\sigma(i)$.

2 Numerical Convective/Non-Convective Flux Splitting

In order to define completely the semi-implicit discretisation in (7), a suitable numerical flux function for $\mathbf{H}_{ij}$ has to be considered. This will be indicated

by $\mathbf{H}(\mathbf{U}, \mathbf{V}, \mathbf{n})$. Its entries are the solution states $\mathbf{U}$ and $\mathbf{V}$, and the vector $\mathbf{n}$, which defines the direction perpendicular to the cell interface where the numerical flux integral is estimated. Since the physical flux function must satisfy the convective/non-convective splitting and the constraints introduced in the previous section, it becomes natural to assume that a similar splitting holds for the numerical flux function. That is, we consider numerical flux functions that can be written as

$$\mathbf{H}(\mathbf{U}, \mathbf{V}, \mathbf{n}) = \mathbf{H}^{(c)}(\mathbf{U}, \mathbf{V}, \mathbf{n}) + \mathbf{H}^{(nc)}(\mathbf{U}, \mathbf{V}, \mathbf{n}) \tag{8}$$

where $\mathbf{H}^{(c)}$ is the numerical correspondant of $\mathbf{F}^{(c)}$ and $\mathbf{H}^{(nc)}$ the one of $\mathbf{F}^{(nc)}$. We assume the following regularity constraints on $\mathbf{H}^{(c)}$ and $\mathbf{H}^{(nc)}$, which are used in the theoretical analysis. Both the convective and the non-convective part of the numerical flux indipendently satisfy

(i) the consistency relation

$$\mathbf{H}^{(*)}(\mathbf{U}, \mathbf{U}, \mathbf{n}) = \mathbf{F}^{(*)}(\mathbf{U}) \cdot \mathbf{n} \qquad (*) = (c),\ (nc), \tag{9}$$

for every physical solution state $\mathbf{U}$ and normal vector $\mathbf{n}$;

(ii) the Lipschitz continuity condition

$$\begin{aligned} |\mathbf{H}^{(*)}(\mathbf{U}, \mathbf{V}, \mathbf{n}) - \mathbf{H}^{(*)}(\mathbf{U}', \mathbf{V}', \mathbf{n})| \leq \\ \leq L\left(|\mathbf{U} - \mathbf{U}'| + |\mathbf{V} - \mathbf{V}'|\right), \qquad (*) = (c),\ (nc), \end{aligned}$$

with respect to their arguments, that is for every couple of physical solution states $(\mathbf{U}, \mathbf{V})$, $(\mathbf{U}', \mathbf{V}')$ and normal vector $\mathbf{n}$, and with positive Lipschitz constant L.

2.1 The linear upwind flux and its properties

Some insights about the formal consequences of the previous definitions and assumptions can be obtained by considering the simplest case of the non-linear scalar advection equation

$$u_t + \nabla \cdot (\mathbf{v}(u)\, u) = 0. \tag{10}$$

In the conservation law (10) the scalar field u is advected by the non-linear convective field $\mathbf{v}(u)$. In order to apply an upwind discretisation, let us first introduce the upwind velocity along the normal direction $\mathbf{n}$, that is

$$a(u, \mathbf{n}) = (\mathbf{v}(u) \cdot \mathbf{n})^+ = \frac{\mathbf{v}(u) \cdot \mathbf{n} + |\mathbf{v}(u) \cdot \mathbf{n}|}{2}, \tag{11}$$

and which is also supposed to verify the regularity conditions of

- *non-negativity*, i.e. $a(u, \mathbf{n}) \geq 0$, for every u and $\mathbf{n}$;
- *consistency*, i.e $\mathbf{v}(u) \cdot \mathbf{n} = a(u, \mathbf{n}) - a(u, -\mathbf{n})$, for every u and $\mathbf{n}$;
- *Lipschitz continuity*, i.e. $|a(u, \mathbf{n}) - a(v, \mathbf{n})| \leq L|u - v|$, for every pair u, v and $\mathbf{n}$.

The upwind numerical flux can be written as a function of the solution states u_L and u_R given at the two sides of the cell interface where the flux has to be evaluated. Noting that

$$(\mathbf{v}(u)\cdot\mathbf{n})^- = -(\mathbf{v}(u)\cdot(-\mathbf{n}))^+ = -a(u,-\mathbf{n}), \tag{12}$$

we obtain

$$H(u_L,u_R,\mathbf{n}) = a(u_{LR},\mathbf{n})u_L - a(u_{RL},-\mathbf{n})u_R. \tag{13}$$

The symbols u_{LR} and u_{RL} denote the same intermediate solution state between u_L and u_R which is usually estimated by using a symmetric smooth function of these latters – for instance, their arithmetic mean. The subscripts L and R indicate respectively the *left* and the *right* side of the cell interface. This definition is clearly arbitrary for each cell interface, but is unique when the orientation of the vector $\mathbf{n}$ is introduced, because we assume that this latter one always points from *left* to *right*.

The FV semi-discrete scheme can be written in the matrix-like compact form

$$\mathbf{D}\frac{d\mathbf{u}}{dt} + \mathbf{A}(\mathbf{u})\mathbf{u} = \mathbf{0} \tag{14}$$

by introducing the matrix operator $\mathbf{A}(\mathbf{u})$ which takes into account the FV flux balance due to the summation term in the left-hand-side (l.h.s.) of equation (7). This approximation scheme is first order accurate in space when the cell-interface states u_L and u_R coincides with the cell-average state within the left and right adjacent cells.

If a semi-implicit Euler difference scheme is adopted for the time derivative of u in (14), we obtain the globally first-order accurate scheme

$$\left(\mathbf{I} + \Delta t\,\mathbf{D}^{-1}\mathbf{A}(\mathbf{u}^n)\right)\mathbf{u}^{n+1} = \mathbf{u}^n. \tag{15}$$

The most important issue of these two discretisations is that the spatial matrix operator $\mathbf{A}(\mathbf{u}^n)$ in (14) and in (15) is a *singular M-matrix*, and that the time evolution matrix operator $\mathbf{I} + \Delta t\,\mathbf{D}^{-1}\mathbf{A}(\mathbf{u}^n)$ is an *M-matrix*.

The accuracy in space can be increased by introducing a piecewise polynomial representation of the solution field within each cell, while the accuracy in time by adopting a multi-step Runge-Kutta method.

We shortly discuss the issue of the higher-order accuracy in space, while the similar issue in time is delayed to section 3. To achieve higher-order accuracy in space the numerical flux function is estimated by using more accurate cell-interface solution states. These ones are calculated by using a polynomial reconstruction of the solution field within each cell. This piecewise polynomial reconstruction is performed at each time step from the set of cell-averaged values which directly evolves in time. The integral of the numerical flux from the cell i to the cell j is finally given by the numerical quadrature formula

$$H_{ij}(\mathbf{u}) = |f_{ij}| \sum_{k=1}^{N_q} \omega_k H(u_i(\cdot,\mathbf{x}_{ij}^k), u_j(\cdot,\mathbf{x}_{ij}^k), \mathbf{n}), \tag{16}$$

where $|f_{ij}|$ is the measure (length in 2-D) of the face f_{ij} (edge in 2-D) shared by the cells i and j, $\mathbf{x}_{ij}^k$ is the k-th quadrature node on f_{ij} and ω_k the k-th quadrature weight. Separating the contribution to the flux balance that

contains the first-order cell-averaged terms from the contribution that contains higher-order terms, the semi-discrete scheme takes the compact form

$$\mathbf{D}\frac{d\mathbf{u}}{dt} + \mathbf{A}(\mathbf{u})\mathbf{u} = \widetilde{\mathbf{A}}(\mathbf{u})\mathbf{u}. \tag{17}$$

The matrix operator $\mathbf{A}(\mathbf{u})$ is the one already introduced in equation (14) while the matrix operator $\widetilde{\mathbf{A}}(\mathbf{u})$ takes into consideration all the effects due to the reconstruction process. This latter operator also shows an important property, because it is possible to demonstrate that when the reconstruction algorithm satisfies the following condition

$$\min\{u_i, u_j\} \leq u_i(\cdot, \mathbf{x}) \leq \max\{u_i, u_j\}, \qquad \mathbf{x} \in f_{ij}, \tag{18}$$

$\widetilde{\mathbf{A}}(\mathbf{u})$ is a *singular symmetric M-matrix*, that is a singular Stieltjes matrix [1]. We shall refer to this operator by the wording *reconstruction matrix*.

2.2 The generalized convective upwind flux

In the system case (1) the numerical convective flux $\mathbf{H}^{(c)}(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n})$ is defined first by introducing the numerical upwind convective field $\mathsf{d}(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n})$ and then by mimicking the flux form of the scalar case. Formally this yields the expression

$$\mathbf{H}^{(c)}(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n}) = \mathsf{d}(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n})\mathbf{U}_L - \mathsf{d}(\mathbf{U}_R, \mathbf{U}_L, -\mathbf{n})\mathbf{U}_R. \tag{19}$$

This definition is completed by assuming that $\mathsf{d}(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n})$ satisfies the *regularity conditions* of

(i) *non-negativity*,

$$\mathsf{d}(\mathbf{U}, \mathbf{V}, \mathbf{n}) \geq 0\,; \tag{20}$$

(ii) *Lipschitz continuity*,

$$|\mathsf{d}(\mathbf{U}, \mathbf{V}, \mathbf{n}) - \mathsf{d}(\mathbf{U}', \mathbf{V}', \mathbf{n})| \leq L\left(||\mathbf{U} - \mathbf{U}'|| + ||\mathbf{V} - \mathbf{V}'||\right), \tag{21}$$

with positive Lipschitz constant L;

(iii) and *consistency*,

$$\mathbf{v}(\mathbf{U}) \cdot \mathbf{n} = \mathsf{d}(\mathbf{U}, \mathbf{U}, \mathbf{n}) - \mathsf{d}(\mathbf{U}, \mathbf{U}, -\mathbf{n}). \tag{22}$$

Throughout the rest of the paper we shall refer to the flux form in (19) by the wording *generalized upwind flux form* or *numerical convective flux form*.

Even though (19) and the constraints on $\mathsf{d}(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n})$ can appear quite restringent, many numerical fluxes of the literature actually fit in. For instances, we can mention

- *General multi-dimensional hyperbolic systems*:
  - the Rusanov-like numerical fluxes;
  - the HLLE-like numerical fluxes [2, 3, 4, 13];

- *Compressible multi-dimensional Euler equations*:
  - the Steger & Warming flux splitting [14];
  - the Van Leer flux splitting [7, 8, 15];
  - the AUSM+ flux splitting [10, 16].

Let us consider for example the Rusanov numerical flux family. These fluxes are of the form

$$\mathbf{H}(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n}) = \frac{1}{2}(\mathbf{F}(\mathbf{U}_L) + \mathbf{F}(\mathbf{U}_R)) \cdot \mathbf{n} - \frac{1}{2}\xi(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n})(\mathbf{U}_R - \mathbf{U}_L). \quad (23)$$

In equation (23), the term $\xi(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n})$ is a scalar function, and the numerical viscosity term is a multiple of the $\mathsf{n} \times \mathsf{n}$ identity matrix $\mathbf{I}_\mathsf{n}$, that is

$$\mathcal{Q}(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n}) = \xi(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n})\mathbf{I}_\mathsf{n}. \quad (24)$$

This feature is actually common to all the numerical fluxes that can be written in the generalized upwind form, see for more details Reference [1].

The generalized upwind form of the Rusanov flux is eventually given by taking

$$\mathsf{d}(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n}) = \frac{1}{2}\left[\mathbf{v}(\mathbf{U}_L) \cdot \mathbf{n} + \xi(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n})\right], \quad (25)$$

$$\mathbf{H}^{(nc)}(\mathbf{U}_L, \mathbf{U}_R, \mathbf{n}) = \frac{1}{2}(\mathbf{F}^{(nc)}(\mathbf{U}_L) + \mathbf{F}^{(nc)}(\mathbf{U}_R)) \cdot \mathbf{n}. \quad (26)$$

3 The high-order IMEX-RK FV Scheme

The semi-discrete form of the numerical scheme developed so far can be written as

$$\frac{\partial}{\partial t}\underline{\mathbf{U}} + a(\underline{\mathbf{U}}) = b(\underline{\mathbf{U}}) \quad (27)$$

where

$$a(\underline{\mathbf{U}}) = \left(\mathbf{D}^{-1}\mathbf{A}(\underline{\mathbf{U}}) \otimes \mathbf{I}_\mathsf{n}\right)\underline{\mathbf{U}}, \quad (28)$$

and the r.h.s. term $b(\underline{\mathbf{U}})$ takes into account both the source terms and the high-order corrections arising from the non-oscillatory reconstruction process. If we apply the simplest implicit Euler discretisation method to the time derivative by taking implicitly the l.h.s. term $a(\underline{\mathbf{U}})$ and explicitly the right-hand-side term $b(\underline{\mathbf{U}})$, we have the semi-implicit discretisation

$$\left[(\mathbf{I} + \Delta t\, \mathbf{D}^{-1}\mathbf{A}(\underline{\mathbf{U}}^n)) \otimes \mathbf{I}_\mathsf{n}\right]\underline{\mathbf{U}}^{n+1} = \underline{\mathbf{U}}^n + \Delta t\, b(\underline{\mathbf{U}}^n), \quad (29)$$

which is first-order accurate in time. As in the scalar case discussed in section (2.1) the matrix $\mathbf{A}(\underline{\mathbf{U}})$ takes into account the FV flux balance contribution of the summation term in (7). Clearly, this matrix differs from the one introduced in (14), but since it refers to the same summation term of (7) we use the same symbol. As in the scalar case, it is possible to demonstrate that this matrix operators shows strong theoretical properties. That is,

- $\mathbf{A}(\underline{\mathbf{U}}^n)$ is a singular M-matrix, and
- $\mathbf{I} + \Delta t\, \mathbf{D}^{-1}\mathbf{A}(\underline{\mathbf{U}}^n)$ is an M-matrix.

The accuracy in the time discretisation can be increased by adopting an IMEX-RK advancing scheme. The *r*-**stage IMEX-RK** method is formally given by

— for each $i = 1, \ldots, r+1$ solve for $\underline{\mathbf{W}}^i$:

$$\underline{\mathbf{W}}^i + \Delta t\, \alpha'_{ii}\, a(\underline{\mathbf{W}}^i) = \underline{\mathbf{U}}^n + \Delta t \sum_{j=1}^{i-1} \left(\alpha_{ij} b(\underline{\mathbf{W}}^j) - \alpha'_{ij} a(\underline{\mathbf{W}}^j) \right) \tag{30}$$

— then compute

$$\underline{\mathbf{U}}^{n+1} = \underline{\mathbf{U}}^n + \Delta t \sum_{i=1}^{r+1} \left(\omega_i b(\underline{\mathbf{W}}^i) - \omega'_i a(\underline{\mathbf{W}}^i) \right). \tag{31}$$

The coefficients α_{ij} and α'_{ij} completely define the scheme – for a list of these latters see for example References [12, 11, 9].

The IMEX-RK scheme solves at each internal step a non-linear system of the form

$$\underline{\mathbf{W}} + \Delta t\, \alpha\, a(\underline{\mathbf{W}}) = \mathbf{r} \tag{32}$$

where α indicates the generic diagonal scheme coefficient and $\mathbf{r}$ the r.h.s. of equation (30). The following argument demonstrates the existence and uniqueness of the solution of this non-linear problem. Equation (32) defines a *fixed point* problem whose mapping is

$$\mathbf{\Phi}(\underline{\mathbf{W}}) = \left[(\mathbf{I} + \Delta t\, \alpha\, \mathbf{D}^{-1} \mathbf{A}(\underline{\mathbf{W}}))^{-1} \otimes \mathbf{I}_{\mathsf{n}} \right] \mathbf{r} \tag{33}$$

Since this mapping is a continuous function of its argument $\underline{\mathbf{W}}$ from a convex compact set into a convex compact set, the Brouwer fixed point theorem [17] implies the existence of at least one fixed point, that is one solution of (32). Furthermore, this fixed point is unique because the mapping defined by (33) is contractive under the CFL-like condition

$$\Delta t < \frac{1}{L\, \alpha\, \kappa^2 ||\mathbf{r}||_1}, \tag{34}$$

where κ is a mesh size dependent parameter.

Solving non-linear systems like (32) by using standard non-linear techniques such as the Newton method can be quite expensive from the computational cost viewpoint. A more efficient strategy consists in approximating $\underline{\mathbf{W}}$ by the solution state $\underline{\mathbf{W}}^k$ generated at the k-th step of the following IMEX-RK fixed-point iterative scheme:

$$\underline{\mathbf{W}}^0 = \underline{\mathbf{U}}^n, \tag{35}$$

$$\underline{\mathbf{W}}^i = \left(\mathbf{I} + \Delta t\, \alpha\, \mathbf{D}^{-1} \mathbf{A}(\underline{\mathbf{W}}^{i-1}) \right)^{-1} \mathbf{r}, \qquad i = 1, 2, \ldots, k. \tag{36}$$

In fact, by a straightforward Taylor development in time around t^{k+1}, there follows that $\underline{\mathbf{W}}^k$ is an $O(\Delta t^{k+1})$ approximation of the exact solution $\underline{\mathbf{W}}$ of (32).

4 Non-negativity results

Since the time evolution operator is tipically a non-singular M-matrix, its formal inverse is a non-negative matrix, that is a matrix all of whose entries are non-negative numbers. This implies almost straigthforwardly that the solution state at time t^{n+1} must be non-negative when the r.h.s. at time t^n is non-negative. Thus under some quite general assumptions non-negativity results can be drawn up. However, we emphasize that a deeper non-negativity analysis is inevitably inherent to all that implies the non-negativity of the r.h.s. of the time stepping scheme as function of the admissible solution states, which is of course problem dependent.

We just mention here the two more general results that are consequence of the M-matrix feature of the time evolution operators discussed in the previous sections. The interested reader is referred to Reference [1].

– The first order IMEX-RK (no-spatial reconstruction) is *unconditionally non-negative* i.e.

$$\mathbf{U}^n \geq \mathbf{0} \qquad \Longrightarrow \qquad \mathbf{U}^{n+1} \geq \mathbf{0} \tag{37}$$

– Higher-order IMEX-RK schemes are non-negative if

$$\left(\mathbf{I} + \Delta t \underline{\mathbf{C}}(\mathbf{U}^n)\right) \mathbf{U}^n \geq \mathbf{0}. \tag{38}$$

The non-linear matrix term $\underline{\mathbf{C}}(\mathbf{U}^n)$ depends on the reconstruction matrix $\widetilde{\mathbf{A}}(\mathbf{u})$ in the r.h.s. of equation (17).

5 Conclusions

The semi-implicit IMEX-RK scheme discussed in this paper is based on a special splitting of the physical flux vector function into a *convective* and a *non-convective* part. A cell-centered FV discretisation is then coupled to IMEX-RK time-stepping schemes. In this framework, the convective part is discretized in an implicit way, while the non-convective one in an explicit way. This coupled IMEX-RK FV integrator is strictly conservative, shock-capturing, formally n-th order accurate in space and time, and does not require the evaluation of any Jacobian matrix once re-formulated in accord with the *generalized upwind form*.

Several theoretical results have been summarized in this paper and are presented in details in Reference [1]. These results do not depend on

(i) the spatial dimension of the problem;

(ii) the numerical flux incorporated into the scheme if the flux can be re-formulated in the *generalized upwind form*;

(iii) the cell-average polynomial reconstruction and monotonizing procedure used to achieve high order accuracy in space.

Basically, this approach produces time evolution matrix operators with a peculiar block structure which is tipically given by the tensor product of M-matrices. This issue appears to be common to a wide family of numerical fluxes and of spatial FV discretisations, and is the base for the development in the

IMEX strategy context of simple and efficient resolution algorithms even for high-order discretisation schemes.

Although this approach has been detailed for unstructured meshes all of the theoretical insights that are mentioned here and proved in Reference [1] can be straightforwardly extended to structured cartesian or curvilinear meshes.

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