A LOW-ORDER NONCONFORMING FINITE ELEMENT FOR REISSNER-MINDLIN PLATES

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Abstract. We propose a locking-free element for plate bending problems, based on the use of nonconforming piecewise linear functions for both rotations and deflections. We prove optimal error estimates with respect to both the meshsize and the analytical solution regularity.

Key words. Nonconforming finite elements, plate bending problems, convergence analysis.

AMS subject classifications. Primary 65N30; Secondary 74S05.

1. Introduction. Nowadays, a *wide* choice of reliable finite element schemes for the approximation of Reissner-Mindlin plate problems is available in engineering and mathematical literature (see, for instance, [7] - [11], [13], [15], [24] - [28], and the references therein). However, the extension to the more complex (and more interesting) shell problems appears to be a difficult task. Indeed, only *very few and not completely satisfactory* results have been established in this direction (cf., e.g., [3], [17] - [20] and [22]).

In this paper we propose and analyze a new low-order Reissner-Mindlin plate element, some properties of which seem to be favorable for its generalization to shell problems. This triangular mixed element can be considered as a simplified variant of the one presented in [16], and it is based on the use of nonconforming piecewise linear functions for both rotations and deflections, while the shear stresses are approximated by piecewise constant functions. In actual computations the shear stress variables can be easily eliminated at the element level, and the final system to be solved involves only rotation and deflection unknowns, which *share the same nodes* (the midpoints of the edges). Compared with the element detailed in [16], the one we are going to study has the following features:

- no additional bubble functions are required;
- no additional sophisticated "reduction" operator on the shear term (other than the simple L^2-projection operator on piecewise constant functions) needs to be introduced.

In view of a possible extension to shell problems, the promising features of our element are the same as the ones met by the scheme presented in [16], i.e.

- it is a simple low-order method;
- once the shear stresses have been eliminated, all the variables into play share the same nodes;
- the element has optimal order of approximation and it is locking-free.

An outline of the paper is as follows. In Section 2 we briefly present the Reissner-Mindlin plate problem. In Section 3 we introduce the nonconforming element, together with the necessary definitions and notations. In Section 4 we develop the stability analysis, while in Section 5 we perform the error analysis. The final results (cf. Theorem 5.1 and Corollary 5.1) show that our element is locking-free and it is optimally convergent with respect to both the meshsize and the analytical solution regularity.

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Furthermore, throughout the paper we will use standard notations for Sobolev spaces and norms (cf. [14] and [23], for instance). Finally, we will denote with C a generic constant, independent of h and t, which may differ in different occurrences.

2. The Reissner-Mindlin problem. The Reissner-Mindlin equations for a clamped plate with regular and bounded midplane Ω require to find $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ such that

$$-\operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}) - \boldsymbol{\gamma} = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$-\operatorname{div} \boldsymbol{\gamma} = g \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{\gamma} = \lambda t^{-2}(\nabla w - \boldsymbol{\theta}) \quad \text{in } \Omega, \tag{2.3}$$

$$\boldsymbol{\theta} = 0, \ w = 0 \text{ on } \partial\Omega. \tag{2.4}$$

In (2.1)-(2.3), $\mathbf{C}$ is the tensor of bending moduli, $\boldsymbol{\theta}$ represents the rotations, w the transversal displacement, $\boldsymbol{\gamma}$ the scaled shear stresses and g a given transversal load. Moreover, ε is the usual symmetric gradient operator, λ is the shear modulus, and t is the thickness.

The classical variational formulation of problem (2.1)–(2.3) is

$$\begin{cases} \text{Find } (\boldsymbol{\theta}, w, \boldsymbol{\gamma}) \in \Theta \times W \times (L^2(\Omega))^2 : \\ a(\boldsymbol{\theta}, \boldsymbol{\eta}) + (\nabla v - \boldsymbol{\eta}, \boldsymbol{\gamma}) = (g, v) & (\boldsymbol{\eta}, v) \in \Theta \times W, \\ (\nabla w - \boldsymbol{\theta}, \boldsymbol{\tau}) - \lambda^{-1} t^2 (\boldsymbol{\gamma}, \boldsymbol{\tau}) = 0 & \boldsymbol{\tau} \in (L^2(\Omega))^2, \end{cases} \tag{2.5}$$

where $\Theta = (H^1_0(\Omega))^2$, $W = H^1_0(\Omega)$, $(\cdot,\cdot)$ is the inner-product in $L^2(\Omega)$ and

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) := \int_\Omega \mathbf{C}\, \varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\eta}) \mathrm{d}x.$$

It is well-known that for problem (2.5) the following *inf-sup* condition holds (cf. [14], for instance)

$$\begin{aligned} &\exists\, \beta > 0 \text{ such that:} \\ &\sup_{(\boldsymbol{\eta}, v) \in \Theta \times W} \frac{(\nabla v - \boldsymbol{\eta}, \boldsymbol{\tau})}{(||\boldsymbol{\eta}||^2_{1,\Omega} + ||v||^2_{1,\Omega})^{1/2}} \geq \beta ||\boldsymbol{\tau}||_\Gamma \qquad \forall \boldsymbol{\tau} \in \Gamma, \end{aligned} \tag{2.6}$$

where

$$\Gamma = H^{-1}(\operatorname{div}, \Omega) \quad \text{and} \quad ||\boldsymbol{\tau}||_\Gamma := (||\boldsymbol{\tau}||^2_{-1,\Omega} + ||\operatorname{div} \boldsymbol{\tau}||^2_{-1,\Omega})^{1/2}. \tag{2.7}$$

Moreover, the following regularity result is valid (cf. e.g. [7] and [21]).

PROPOSITION 2.1. *Suppose that Ω is convex and $g \in L^2(\Omega)$. Let $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ be the solution of problem* (2.5). *Then the following estimate holds*

$$||\boldsymbol{\theta}||_{2,\Omega} + ||w||_{2,\Omega} + ||\boldsymbol{\gamma}||_{H(\mathrm{div})} + t||\boldsymbol{\gamma}||_{1,\Omega} \leq C||g||_{0,\Omega}, \tag{2.8}$$

where

$$||\boldsymbol{\gamma}||^2_{H(\mathrm{div})} = ||\boldsymbol{\gamma}||^2_0 + ||\operatorname{div}\boldsymbol{\gamma}||^2_{0,\Omega}.$$

□

3. The new nonconforming element. We now introduce a nonconforming finite element approximation of problem (2.1)–(2.3) using the approach detailed in [16]. Let then $\mathcal{T}_h$ be a decomposition of Ω into triangular elements T and let us set

$$H^1(\mathcal{T}_h) := \prod_{T\in\mathcal{T}_h} H^1(T). \tag{3.1}$$

We now define suitable jump and average operators. We first denote by $\mathcal{E}_h$ the set of all the edges in $\mathcal{T}_h$, and by $\mathcal{E}_h^{\mathrm{in}}$ the set of internal edges. Let e be an internal edge of $\mathcal{T}_h$, shared by two elements T^+ and T^-, and let φ denote a function in $H^1(\mathcal{T}_h)$, or a vector in $(H^1(\mathcal{T}_h))^2$, or a tensor in $(H^1(\mathcal{T}_h))^4_s$. We define the average as usual:

$$\{\varphi\} = \frac{\varphi^+ + \varphi^-}{2} \quad \forall e \in \mathcal{E}_h^{\mathrm{in}}. \tag{3.2}$$

For a scalar function $\varphi \in H^1(\mathcal{T}_h)$ we define its jump as

$$[\varphi] = \varphi^+\mathbf{n}^+ + \varphi^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^{\mathrm{in}}, \tag{3.3}$$

while the jump of a vector $\boldsymbol{\varphi} \in (H^1(\mathcal{T}_h))^2$ is given by

$$[\boldsymbol{\varphi}] = (\boldsymbol{\varphi}^+ \otimes \mathbf{n}^+)_S + (\boldsymbol{\varphi}^- \otimes \mathbf{n}^-)_S \quad \forall e \in \mathcal{E}_h^{\mathrm{in}}, \tag{3.4}$$

where $(\boldsymbol{\varphi}\otimes\mathbf{n})_S$ denotes the symmetric part of the tensor product, and $\mathbf{n}^+$ (resp. $\mathbf{n}^-$) is the outward unit normal to ∂T^+ (resp. to ∂T^-). On the boundary edges we define jumps of scalars as $[\varphi] = \varphi\mathbf{n}$, and jumps of vectors as $[\boldsymbol{\varphi}] = (\boldsymbol{\varphi}\otimes\mathbf{n})_S$, where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. We also define averages of vectors and tensors as $\{\boldsymbol{\varphi}\} = \boldsymbol{\varphi}$. It can be easily checked that, if $\boldsymbol{\varphi}$ is a smooth tensor, and $\boldsymbol{\eta}$ a piecewise smooth vector, the following equality holds (see, e.g., [4] for a similar computation)

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \boldsymbol{\varphi}\mathbf{n}\cdot\boldsymbol{\eta}\, ds = \sum_{e\in\mathcal{E}_h} \int_e \{\boldsymbol{\varphi}\} : [\boldsymbol{\eta}]\, ds. \tag{3.5}$$

In order to introduce our scheme, we first consider the finite element spaces

$$\Theta_h = \big\{\boldsymbol{\eta} : \boldsymbol{\eta}_{|T} \in (P_1(T))^2, \ \int_e [\boldsymbol{\eta}] ds = 0 \quad \forall e \in \mathcal{E}_h\big\}, \tag{3.6}$$

$$W_h = \{v : v_{|T} \in P_1(T), \ \int_e [v] ds = 0 \quad \forall e \in \mathcal{E}_h\}, \tag{3.7}$$

$$\Gamma_h = \{\boldsymbol{\tau} : \boldsymbol{\tau}_{|T} \in (P_0(T))^2\}, \tag{3.8}$$

where $P_k(T)$ is the space of polynomials of degree at most k defined on T. We also notice that

$$\boldsymbol{\nabla}_h W_h \subset \Gamma_h, \tag{3.9}$$

where $\boldsymbol{\nabla}_h$ denotes the gradient element by element. The local degrees of freedom for the three variables are depicted in Fig. 3.1.

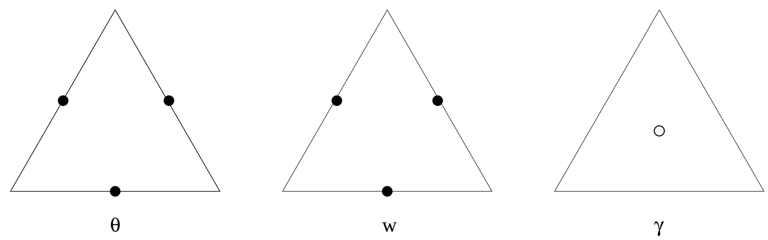

FIG. 3.1. *Local dof for the three variables*

Moreover, we introduce a penalty on the jumps of functions in Θ_h as

$$p_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}) := \sum_{e \in \mathcal{E}_h} \frac{\kappa_e}{|e|} \int_e [\boldsymbol{\theta}] : [\boldsymbol{\eta}] \, ds, \tag{3.10}$$

where $|e|$ denotes the length of the side e, and κ_e is a positive constant having the same physical dimension as $\mathbf{C}$ (for smooth $\mathbf{C}$, one could take κ_e as $|\mathbf{C}|$ evaluated at the midpoint of e).

We then define

$$a_T(\boldsymbol{\theta}, \boldsymbol{\eta}) := \int_T \mathbf{C} \, \varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\eta}) \, dx, \tag{3.11}$$

and we finally set

$$a_h(\boldsymbol{\theta}, \boldsymbol{\eta}) := \sum_{T \in \mathcal{T}_h} a_T(\boldsymbol{\theta}, \boldsymbol{\eta}) + p_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}). \tag{3.12}$$

Following the ideas of [16], the discrete problem is then

$$\begin{cases} \text{Find } (\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h) \in \Theta_h \times W_h \times \Gamma_h \\ a_h(\boldsymbol{\theta}_h, \boldsymbol{\eta}_h) + (\boldsymbol{\gamma}_h, \boldsymbol{\nabla}_h v_h - \boldsymbol{\eta}_h) = (g, v_h) \qquad (\boldsymbol{\eta}_h, v_h) \in \Theta_h \times W_h, \\ (\boldsymbol{\nabla}_h w_h - \boldsymbol{\theta}_h, \boldsymbol{\tau}_h) - \lambda^{-1} t^2 (\boldsymbol{\gamma}_h, \boldsymbol{\tau}_h) = 0 \qquad \boldsymbol{\tau}_h \in \Gamma_h. \end{cases} \tag{3.13}$$

We will use norms $||\cdot||_{\Theta_h}$ and $||\cdot||_{W_h}$ for functions in Θ_h and W_h, defined as

$$||\boldsymbol{\eta}_h||_{\Theta_h} := \left(\sum_{T\in\mathcal{T}_h} \int_T |\boldsymbol{\nabla}\boldsymbol{\eta}_h|^2 \right)^{1/2} = ||\boldsymbol{\nabla}_h\boldsymbol{\eta}_h||_{0,\Omega}, \tag{3.14}$$

$$||v_h||_{W_h} := \left(\sum_{T\in\mathcal{T}_h} \int_T |\boldsymbol{\nabla} v_h|^2 \right)^{1/2} = ||\boldsymbol{\nabla}_h v_h||_{0,\Omega}. \tag{3.15}$$

Due to the discrete Poincaré's inequality, both $||\cdot||_{\Theta_h}$ and $||\cdot||_{W_h}$ are indeed *norms* on Θ_h and W_h, not only seminorms. It has been proved in [6] (see also [12]) that there exist positive constants α and M such that

$$a_h(\boldsymbol{\eta}_h, \boldsymbol{\eta}_h) \geq \alpha ||\boldsymbol{\eta}_h||^2_{\Theta_h} \qquad \forall\, \boldsymbol{\eta}_h \in \Theta_h, \tag{3.16}$$

$$a_h(\boldsymbol{\theta}_h, \boldsymbol{\eta}_h) \leq M ||\boldsymbol{\theta}_h||_{\Theta_h} ||\boldsymbol{\eta}_h||_{\Theta_h} \qquad \forall\, \boldsymbol{\theta}_h\,, \boldsymbol{\eta}_h \in \Theta_h. \tag{3.17}$$

We remark that the coercivity property (3.16) is far from being trivial, since the bilinear form $a_h(\cdot,\cdot)$ contains only the *symmetric* gradient operator and not the *whole* gradient operator (cf. (3.11), (3.12) and (3.14)).

For functions in Γ_h we will work with the (natural) norms (cf. also (2.7))

$$||\boldsymbol{\tau}_h||_{\Gamma} \qquad \text{and} \qquad t||\boldsymbol{\tau}_h||_{0,\Omega}. \tag{3.18}$$

REMARK 3.1. *We point out that eliminating $\boldsymbol{\gamma}_h$ from system* (3.13)*, our scheme is equivalent to the following problem involving only the rotations and the vertical displacements:*

$$\begin{cases} \textit{Find } (\boldsymbol{\theta}_h, w_h) \in \Theta_h \times W_h : \\ a_h(\boldsymbol{\theta}_h, \boldsymbol{\eta}_h) + \lambda t^{-2} (\boldsymbol{\nabla}_h w_h - P_0\boldsymbol{\theta}_h, \boldsymbol{\nabla}_h v_h - P_0\boldsymbol{\eta}_h) \\ \qquad\qquad\qquad\qquad = (g, v_h) \qquad \forall (\boldsymbol{\eta}_h, v_h) \in \Theta_h \times W_h, \end{cases} \tag{3.19}$$

where P_0 denotes the L^2-projection operator on the piecewise constant functions. From (3.19) *we may notice that the method implementation turns out to be rather simple.*

4. Stability analysis. In this Section we will prove a stability result for the discretized problem 3.13, using a macroelement technique essentially developed in [26]. In what follows it will be useful to set $V := \Theta \times W$ and $V_h := \Theta_h \times W_h$, equipped with the usual product norms. We first need the following preliminary result.

PROPOSITION 4.1. *The approximation spaces defined in* (3.6)–(3.8) *satisfy the following properties:*

(P1) *There exists a linear operator* $\pi_h : W \longrightarrow W_h$ *such that*

$$||\pi_h v||_{W_h} \le c||v||_{1,\Omega} \text{ , c independent of } h$$

$$\int_\Omega \boldsymbol{\nabla}_h (v - \pi_h v) \cdot \boldsymbol{\tau}_h = 0 \qquad \forall \boldsymbol{\tau}_h \in \Gamma_h$$

(P2) *If the mesh* $\mathcal{T}_h$ *contains at least three triangles, then for* $\boldsymbol{\tau}_h \in \Gamma_h$ *condition*

$$\int_\Omega (\boldsymbol{\nabla}_h v_h - \boldsymbol{\eta}_h) \cdot \boldsymbol{\tau}_h = 0 \qquad \forall (\boldsymbol{\eta}_h, v_h) \in V_h \tag{4.1}$$

implies $\boldsymbol{\tau}_h = \mathbf{0}$.

Proof. Consider the usual nonconforming interpolating operator $\pi_h : W \longrightarrow W_h$, defined by

$$(\pi_h v)(m) = \frac{1}{|e|} \int_e v \, ds \qquad \forall e \in \mathcal{E}_h \quad \text{(with } m \text{ the midpoint of } e\text{)}.$$

It is easily seen that property **(P1)** is fulfilled.

To verify **(P2)**, for a given internal edge $e \in \mathcal{E}_h^{\text{in}}$ we first choose one of the two possible normal (resp. tangential) vectors to e, indicated in what follows as $\mathbf{n}_e$ (resp. $\mathbf{t}_e$). Let us take $\boldsymbol{\tau}_h \in \Gamma_h$ satisfying condition (4.1).

By choosing $(\mathbf{0}, v_h) \in V_h$, integrating by parts yields

$$0 = \int_\Omega \boldsymbol{\nabla}_h v_h \cdot \boldsymbol{\tau}_h = \sum_{T \in \mathcal{T}_h} \int_{\partial T} v_h \, \boldsymbol{\tau}_h \cdot \mathbf{n}_T. \tag{4.2}$$

Since equation (4.2) is true for every $(\mathbf{0}, v_h) \in V_h$, it follows that $\boldsymbol{\tau}_h \cdot \mathbf{n}_e$ is continuous across every internal edge $e \in \mathcal{E}_h^{\text{in}}$. Therefore $\boldsymbol{\tau}_h \in H(\text{div}; \Omega)$ and, obviously, $\text{div}\, \boldsymbol{\tau}_h = 0$. As a consequence, there exists φ_h (defined up to a constant) such that

$$\varphi_h \in \mathcal{L}_1^1(\Omega; \mathcal{T}_h), \qquad \boldsymbol{\tau}_h = \mathbf{curl}\, \varphi_h \tag{4.3}$$

where $\mathcal{L}_1^1(\Omega; \mathcal{T}_h)$ is the usual space of piecewise linear and continuous functions on Ω.

Fix now a generic internal edge $e \in \mathcal{E}_h^{\text{in}}$ with midpoint m, and denote with T_e^+, T_e^- the triangles sharing e as common side. Recalling that $\mathbf{curl}\, \varphi_h \cdot \mathbf{n}_e$ is constant and continuous across e, we consider $(\boldsymbol{\eta}_h, 0) \in V_h$, where $\boldsymbol{\eta}_h$ is uniquely defined by

$$\begin{cases} (\boldsymbol{\eta}_h \cdot \mathbf{t}_e)(m) = 0, \quad (\boldsymbol{\eta}_h \cdot \mathbf{n}_e)(m) = \mathbf{curl}\, \varphi_h \cdot \mathbf{n}_e \\ \boldsymbol{\eta}_h(m') = 0 \qquad \forall\, e' \in \mathcal{E}_h^{\text{in}}, \quad e' \neq e \quad \text{(with } m' \text{ the midpoint of } e'\text{)}. \end{cases} \tag{4.4}$$

Since $\boldsymbol{\tau}_h = \mathbf{curl}\, \varphi_h$ satisfies (4.1), using (4.4) we have

$$\begin{aligned} 0 &= \int_\Omega \boldsymbol{\eta}_h \cdot \mathbf{curl}\, \varphi_h = \int_{T_e^+ \cup T_e^-} \boldsymbol{\eta}_h \cdot \mathbf{curl}\, \varphi_h = \frac{|T_e^+| + |T_e^-|}{3} (\boldsymbol{\eta}_h \cdot \mathbf{curl}\, \varphi_h)(m) \\ &= \frac{|T_e^+| + |T_e^-|}{3} |\mathbf{curl}\, \varphi_h \cdot \mathbf{n}_e|^2. \end{aligned} \tag{4.5}$$

Repeating the same argument for every $e \in \mathcal{E}_h^{\text{in}}$, from (4.5) we infer that

$$\mathbf{curl}\,\varphi_h \cdot \mathbf{n}_e = \boldsymbol{\nabla}\varphi_h \cdot \mathbf{t}_e = 0 \qquad \text{for every } e \in \mathcal{E}_h^{\text{in}}. \tag{4.6}$$

Equation (4.6) implies that $\boldsymbol{\tau}_h = \mathbf{curl}\,\varphi_h$ vanishes in all the triangles $T \in \mathcal{T}_h$ having at least two sides in $\mathcal{E}_h^{\text{in}}$. Therefore, it remains to show that $\mathbf{curl}\,\varphi_h = \mathbf{0}$ also on the triangles sharing two sides with the boundary $\partial\Omega$, if there are any in the mesh $\mathcal{T}_h$. Consider then any such a triangle T, denote with e its *unique* side belonging to $\mathcal{E}_h^{\text{in}}$ and with T^{in} the triangle sharing the side e with T. Since Ω is a regular domain and $\mathcal{T}_h$ contains at least three triangles, it follows that T^{in} has at least two sides in $\mathcal{E}_h^{\text{in}}$. Hence we already know that

$$(\mathbf{curl}\,\varphi_h)_{|T^{\text{in}}} = \mathbf{0}. \tag{4.7}$$

Recalling that $\mathbf{curl}\,\varphi_h$ is constant in T, let us now take $(\boldsymbol{\eta}_h, 0) \in V_h$, where $\boldsymbol{\eta}_h$ is uniquely defined by

$$\begin{cases} \boldsymbol{\eta}_h(m) = (\mathbf{curl}\,\varphi_h)_{|T} & \text{(with } m \text{ the midpoint of } e) \\ \boldsymbol{\eta}_h(m') = 0 \qquad \forall\, e' \in \mathcal{E}_h^{\text{in}}, \quad e' \neq e & \text{(with } m' \text{ the midpoint of } e'). \end{cases} \tag{4.8}$$

Again, since $\boldsymbol{\tau}_h = \mathbf{curl}\,\varphi_h$ satisfies (4.1), by (4.7) and (4.8) we obtain

$$\begin{aligned} 0 &= \int_\Omega \boldsymbol{\eta}_h \cdot \mathbf{curl}\,\varphi_h = \int_{T\cup T^{\text{in}}} \boldsymbol{\eta}_h \cdot \mathbf{curl}\,\varphi_h = \int_T \boldsymbol{\eta}_h \cdot \mathbf{curl}\,\varphi_h \\ &= \frac{|T|}{3}\, |(\mathbf{curl}\,\varphi_h)_{|T}|^2, \end{aligned} \tag{4.9}$$

so that $\mathbf{curl}\,\varphi_h = \mathbf{0}$ also in T and the proof is complete. □

Remark 4.1. *We remark that property* **(P2)** *can be written in the following equivalent form:*

(P2') *For every* $\boldsymbol{\varphi} \in (L^2(\Omega))^2$, *the problem*

$$\begin{cases} \textit{Find } (\boldsymbol{\eta}_h, v_h) \in \Theta_h \times W_h : \\ \displaystyle\int_\Omega (\boldsymbol{\nabla}_h v_h - \boldsymbol{\eta}_h) \cdot \boldsymbol{\tau}_h = \int_\Omega \boldsymbol{\varphi} \cdot \boldsymbol{\tau}_h \qquad \forall \boldsymbol{\tau}_h \in \Gamma_h \end{cases}$$

is solvable.

4.1. Macroelement decomposition. We start by recalling some standard definitions and notations we will use in the sequel. First of all, we say that a family $\{\mathcal{T}_h\}_{h>0}$ of triangular meshes of Ω is *regular* (see [23]) if there exists a constant $\sigma > 0$ such that

$$h_T \leq \sigma \rho_T \qquad \forall\, T \in \bigcup_{h>0} \mathcal{T}_h, \tag{4.10}$$

where h_T is the diameter of the element T and ρ_T is the maximum diameter of the circles contained in T. Furthermore, a *macroelement* M is a set with connected interior part, formed by the union of a fixed number of neighboring triangles along a well-defined pattern (cf. [29]). A macroelement $M = \cup_{i=1}^m T_i$ is said to be equivalent to a reference macroelement $\widehat{M} = \cup_{i=1}^m \widehat{T}_i$ if there is a mapping $F_M : \widehat{M} \longrightarrow M$ for which the following conditions are fulfilled (cf. [29]):

1. F_M is a continuous bijection.
2. $T_i = F_M(\widehat{T}_i) \qquad \forall i \quad 1 \leq i \leq m$
3. $F_{M|\widehat{T}_i} = F_{T_i} \circ F_{\widehat{T}_i}^{-1}$, where F_{T_i} and $F_{\widehat{T}_i}$ are the usual functions mapping the standard reference triangle (of vertices $(0,0)$, $(1,0)$ and $(0,1)$) onto T_i and $\widehat{T}_i$, respectively.

From a given mesh $\mathcal{T}_h$ of Ω it is always possible to derive (obviously not in a unique manner) a “macroelement mesh” $\mathcal{M}_h$ in such a way that each $T \in \mathcal{T}_h$ is covered by some macroelement M in $\mathcal{M}_h$ and each macroelement M is equivalent to a certain reference macroelement $\widehat{M}$.

Associated with every macroelement M in $\mathcal{M}_h$, the following spaces are relevant for the stability analysis (cf. [26])

$$V_{0,M} := \big\{(\boldsymbol{\eta}_h, v_h) \in V_h : \quad (\boldsymbol{\eta}_h, v_h) = (\mathbf{0}, 0) \quad \text{in}\, \Omega \setminus M\big\}, \tag{4.11}$$

$$\Gamma_M := \big\{\boldsymbol{\tau}_h \in \Gamma_h : \quad \boldsymbol{\tau}_h = 0 \quad \text{in}\, \Omega \setminus M\big\}. \tag{4.12}$$

4.2. Fortin’s trick by macroelements. The aim of this subsection is to prove that Fortin’s trick (cf. [14]) applies to our finite element scheme, leading therefore to a suitable *inf-sup* condition with respect to the natural norms (see (2.6)). Indeed, we have the following result.

PROPOSITION 4.2. *Suppose that the family* $\{\mathcal{T}_h\}_{h>0}$ *is regular and choose a corresponding macroelement family* $\{\mathcal{M}_h\}_{h>0}$ *such that:*

1. *each macroelement M contains at least three triangles;*
2. *there is only a fixed finite number of reference macroelements* $\{\widehat{M}_1, ..., \widehat{M}_r\}$ *to which each macroelement* $M \in \cup_{h>0}\mathcal{M}_h$ *is equivalent.*

Then for the approximation spaces defined in (3.6)–(3.8) *the following* inf-sup *condition holds*

$$\begin{aligned}&\exists\, \beta > 0 \ \textit{independent of } h\textit{, such that:}\\ &\sup_{(\boldsymbol{\eta}_h, v_h)\in V_h} \frac{(\boldsymbol{\nabla}_h v_h - \boldsymbol{\eta}_h, \boldsymbol{\tau}_h)}{||(\boldsymbol{\eta}_h, v_h)||_{V_h}} \geq \beta ||\boldsymbol{\tau}_h||_\Gamma \qquad \forall \boldsymbol{\tau}_h \in \Gamma_h.\end{aligned} \tag{4.13}$$

Proof. Let $(\boldsymbol{\eta}, v) \in V$ be given. Fix an arbitrary macroelement $M \in \mathcal{M}_h$ and set

$$h_M := \max_{1\leq i\leq m} h_{T_i} \quad \text{if} \quad M = \bigcup_i^m T_i \tag{4.14}$$

Let us denote with i_M the index $1 \leq i_M \leq r$ such that M is equivalent to $\widehat{M}_{i_M}$.

Consider the problem to find $(\boldsymbol{\eta}_M, v_M) \in V_{0,M}$ solution of

$$\int_M (\boldsymbol{\nabla}_h v_M - \boldsymbol{\eta}_M) \cdot \boldsymbol{\tau}_M = \int_M (\Pi_1 \boldsymbol{\eta} - \boldsymbol{\eta}) \cdot \boldsymbol{\tau}_M \qquad \forall \boldsymbol{\tau}_M \in \Gamma_M \, , \tag{4.15}$$

where $\Pi_1 \boldsymbol{\eta}$ is the usual nonconforming interpolated of $\boldsymbol{\eta}$, defined by

$$(\Pi_1 \boldsymbol{\eta})(m) = \frac{1}{|e|} \int_e \boldsymbol{\eta} \, ds \qquad \forall e \in \mathcal{E}_h \quad \text{(with } m \text{ the midpoint of } e\text{).}$$

By property **(P2)** of Proposition 4.1, applied to the macroelement M, it follows that system (4.15) is solvable, since M contains at least three triangles (cf. also Remark 4.1). Let us take the solution of minimal V_h-norm. A scaling argument and the features of the interpolating operator Π_1 show that there exists $c(\widehat{M}_{i_M}) > 0$ such that

$$|| \, \boldsymbol{\eta}_M \, ||^2_{\Theta_h} + h_M^{-2} || \, v_M \, ||^2_{W_h} \leq c(\widehat{M}_{i_M}) || \, \boldsymbol{\eta} \, ||^2_{1,M} \tag{4.16}$$

Since h_M is obviously bounded by $|\Omega|$, inequality (4.16) implies that

$$\exists \, c_1(\widehat{M}_{i_M}) > 0 \, : \quad || \, \boldsymbol{\eta}_M \, ||^2_{\Theta_h} + || \, v_M \, ||^2_{W_h} \leq c_1(\widehat{M}_{i_M}) \, || \, \boldsymbol{\eta} \, ||^2_{1,M}. \tag{4.17}$$

Let us set

$$\boldsymbol{\eta}_F = \Pi_1 \boldsymbol{\eta} + \sum_M \boldsymbol{\eta}_M \tag{4.18}$$

$$v_F = \pi_h v + \sum_M v_M, \tag{4.19}$$

where π_h is the operator as in property **(P1)** of Proposition 4.1 (i.e. the standard nonconforming interpolation operator). We now notice that every $\boldsymbol{\tau}_h \in \Gamma_h$ can be uniquely written as $\boldsymbol{\tau}_h = \sum_M \boldsymbol{\tau}_M$, where $\boldsymbol{\tau}_M \in \Gamma_M$. Hence, recalling (4.15), from (4.18)-(4.18) we have

$$\begin{aligned}
\int_\Omega (\boldsymbol{\nabla}_h v_F - \boldsymbol{\eta}_F) \cdot \boldsymbol{\tau}_h &= \sum_M \int_M \Big[\boldsymbol{\nabla}_h (\pi_h v + v_M) - \Pi_1 \boldsymbol{\eta} - \boldsymbol{\eta}_M \Big] \cdot \boldsymbol{\tau}_M \\
&= \sum_M \Big[\int_M \boldsymbol{\nabla}_h \pi_h v \cdot \boldsymbol{\tau}_M + \int_M (\boldsymbol{\nabla}_h v_M - \Pi_1 \boldsymbol{\eta} - \boldsymbol{\eta}_M) \cdot \boldsymbol{\tau}_M \Big] \\
&= \sum_M \Big(\int_M \boldsymbol{\nabla} \, v \cdot \boldsymbol{\tau}_M - \int_M \boldsymbol{\eta} \cdot \boldsymbol{\tau}_M \Big) \\
&= \int_\Omega (\boldsymbol{\nabla} \, v - \boldsymbol{\eta}) \cdot \boldsymbol{\tau}_h
\end{aligned} \tag{4.20}$$

Therefore, for every $(\boldsymbol{\eta}, v) \in V$ we have found $\Pi_h(\boldsymbol{\eta}, v) = (\boldsymbol{\eta}_F, v_F) \in V_h$ such that

$$\int_\Omega (\boldsymbol{\nabla}_h v_F - \boldsymbol{\eta}_F)\cdot\boldsymbol{\tau}_h = \int_\Omega (\boldsymbol{\nabla}\, v - \boldsymbol{\eta})\cdot\boldsymbol{\tau}_h \qquad \forall\boldsymbol{\tau}_h\in\Gamma_h \tag{4.21}$$

Let us estimate $||\boldsymbol{\eta}_F||^2_{\Theta_h} + ||v_F||^2_{W_h}$. By using the continuity of Π_1 and π_h, and estimate (4.17), we get

$$\begin{aligned}
||\boldsymbol{\eta}_F||^2_{\Theta_h} + ||v_F||^2_{W_h} &= \Big|\Big|\,\Pi_1\boldsymbol{\eta} + \sum_M \boldsymbol{\eta}_M\Big|\Big|^2_{\Theta_h} + \Big|\Big|\,\pi_h v + \sum_M v_M\,\Big|\Big|^2_{W_h}\\
&\le 2\Big(\,||\Pi_1\boldsymbol{\eta}||^2_{\Theta_h} + ||\pi_h v||^2_{W_h} + \sum_M \big(||\,\boldsymbol{\eta}_M\,||^2_{\Theta_h} + ||\,v_M\,||^2_{W_h}\big)\Big)\\
&\le 2\Big(\,c\,\big(||v||^2_{1,\Omega} + ||\boldsymbol{\eta}||^2_{1,\Omega}\big) + \sum_M c_1(\widehat{M}_{i_M})||\boldsymbol{\eta}||^2_{1,M}\Big).
\end{aligned} \tag{4.22}$$

Since there is only a finite number of reference macroelements $\{\widehat{M}_1, .., \widehat{M}_r\}$, we obtain

$$||\boldsymbol{\eta}_F||^2_{\Theta_h} + ||v_F||^2_{W_h} \le C_1\big(||v||^2_{1,\Omega} + ||\boldsymbol{\eta}||^2_{1,\Omega}\big) \tag{4.23}$$

with $C_1 = 2\max\{c,\ c_1(\widehat{M}_1), ...,\ c_1(\widehat{M}_r)\}$. Therefore, we finally have

$$||\Pi_h(\boldsymbol{\eta}, v)||_{V_h} \le C\,\big(||\boldsymbol{\eta}||^2_{1,\Omega} + ||v||^2_{1,\Omega}\big)^{1/2} \tag{4.24}$$

with C independent of h. It is well-known (cf. [14], for instance) that (4.21) together with (4.24) implies condition (4.13) and the proof is complete. □

REMARK 4.2. *Note that it is always possible to derive, from a given regular family $\{\mathcal{T}_h\}_{h>0}$, a macroelement family $\{\mathcal{M}_h\}_{h>0}$ which fulfills the assumption of Proposition 4.2, provided in each $\mathcal{T}_h$ there are at least three triangles.*

4.3. The stability result. Once the *inf-sup* condition (4.13) has been established, suitable stability estimates can be derived using standard techniques (see, for instance, [9] and [21] for their application to Reissner-Mindlin plate problems). For the sake of completeness, we develop such a stability analysis in full details.

First, it is useful to set

$$\begin{aligned}
\mathcal{A}_h(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h; \boldsymbol{\eta}_h, v_h, \boldsymbol{\tau}_h) :=\,& a_h(\boldsymbol{\theta}_h, \boldsymbol{\eta}_h) + (\boldsymbol{\nabla}_h v_h - \boldsymbol{\eta}_h, \boldsymbol{\gamma}_h)\\
&- (\boldsymbol{\nabla}_h w_h - \boldsymbol{\theta}_h, \boldsymbol{\tau}_h) + \lambda^{-1}t^2(\boldsymbol{\gamma}_h, \boldsymbol{\tau}_h).
\end{aligned} \tag{4.25}$$

Therefore, the discrete problem (3.13) reads

$$\begin{cases}
\text{Find } (\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h) \in \Theta_h\times W_h\times\Gamma_h \text{ s.t.}\\
\mathcal{A}_h(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h; \boldsymbol{\eta}_h, v_h, \boldsymbol{\tau}_h) = (g, v_h) \qquad \forall(\boldsymbol{\eta}_h, v_h, \boldsymbol{\tau}_h)\in\Theta_h\times W_h\times\Gamma_h.
\end{cases} \tag{4.26}$$

We have the following result.

PROPOSITION 4.3. *Given $(\boldsymbol{\beta}_h, z_h, \boldsymbol{\rho}_h)\in\Theta_h\times W_h\times\Gamma_h$, there exists $(\boldsymbol{\eta}_h, v_h, \boldsymbol{\tau}_h)\in\Theta_h\times W_h\times\Gamma_h$ such that*

$$
\begin{aligned}
&||\boldsymbol{\eta}_h||_{\Theta_h} + ||v_h||_{W_h} + ||\boldsymbol{\tau}_h||_{\Gamma} + t||\boldsymbol{\tau}_h||_{0,\Omega} \\
&\qquad\qquad \leq C\Big(||\boldsymbol{\beta}_h||_{\Theta_h} + ||z_h||_{W_h} + ||\boldsymbol{\rho}_h||_{\Gamma} + t||\boldsymbol{\rho}_h||_{0,\Omega}\Big)
\end{aligned}
\tag{4.27}
$$

and

$$
\mathcal{A}_h(\boldsymbol{\beta}_h, z_h, \boldsymbol{\rho}_h; \boldsymbol{\eta}_h, v_h, \boldsymbol{\tau}_h) \geq C\Big(||\boldsymbol{\beta}_h||^2_{\Theta_h} + ||z_h||^2_{W_h} + ||\boldsymbol{\rho}_h||^2_{\Gamma} + t^2||\boldsymbol{\rho}_h||^2_{0,\Omega}\Big). \tag{4.28}
$$

Proof: Let $(\boldsymbol{\beta}_h, z_h, \boldsymbol{\rho}_h)$ be given in $\Theta_h \times W_h \times \Gamma_h$. The proof is performed in three steps.

First Step. Let us first choose $(\boldsymbol{\eta}_1, v_1, \boldsymbol{\tau}_1) \in \Theta_h \times W_h \times \Gamma_h$ as

$$
\boldsymbol{\eta}_1 = \boldsymbol{\beta}_h, \qquad v_1 = z_h, \qquad \boldsymbol{\tau}_1 = \boldsymbol{\rho}_h.
$$

It is obvious that

$$
\begin{aligned}
&||\boldsymbol{\eta}_1||_{\Theta_h} + ||v_1||_{W_h} + ||\boldsymbol{\tau}_1||_{\Gamma} + t||\boldsymbol{\tau}_1||_{0,\Omega} \\
&\qquad\qquad = ||\boldsymbol{\beta}_h||_{\Theta_h} + ||z_h||_{W_h} + ||\boldsymbol{\rho}_h||_{\Gamma} + t||\boldsymbol{\rho}_h||_{0,\Omega}.
\end{aligned}
\tag{4.29}
$$

Furthermore, it holds

$$
\mathcal{A}_h(\boldsymbol{\beta}_h, z_h, \boldsymbol{\rho}_h; \boldsymbol{\eta}_1, v_1, \boldsymbol{\tau}_1) = a_h(\boldsymbol{\beta}_h, \boldsymbol{\beta}_h) + \lambda^{-1}t^2||\gamma_h||^2_{0,\Omega}. \tag{4.30}
$$

By the coercivity of $a_h(\cdot,\cdot)$ (cf. (3.16)) it follows that

$$
\mathcal{A}_h(\boldsymbol{\beta}_h, z_h, \boldsymbol{\rho}_h; \boldsymbol{\eta}_1, v_1, \boldsymbol{\tau}_1) \geq C_1\Big(||\boldsymbol{\beta}_h||^2_{\Theta_h} + t^2||\boldsymbol{\rho}_h||^2_0\Big). \tag{4.31}
$$

Second Step. Notice that from (4.13) it follows that there exists $(\boldsymbol{\eta}_2, v_2) \in \Theta_h \times W_h$ such that

$$
||\boldsymbol{\eta}_2||_{\Theta_h} + ||v_2||_{W_h} \leq C||\boldsymbol{\rho}_h||_{\Gamma} \tag{4.32}
$$

and

$$
(\boldsymbol{\nabla}_h v_2 - \boldsymbol{\eta}_2, \boldsymbol{\rho}_h) = ||\boldsymbol{\rho}_h||^2_{\Gamma}. \tag{4.33}
$$

Choose $(\boldsymbol{\eta}_2, v_2, \boldsymbol{\tau}_2) \in \Theta_h \times W_h \times \Gamma_h$ with $\boldsymbol{\tau}_2 = 0$. We have

$$
\mathcal{A}_h(\boldsymbol{\beta}_h, z_h, \boldsymbol{\rho}_h; \boldsymbol{\eta}_2, v_2, \boldsymbol{\tau}_2) = a_h(\boldsymbol{\beta}_h, \boldsymbol{\eta}_2) + (\boldsymbol{\nabla}_h v_2 - \boldsymbol{\eta}_2, \boldsymbol{\rho}_h), \tag{4.34}
$$

so that by (4.33) it follows

$$
\mathcal{A}_h(\boldsymbol{\beta}_h, z_h, \boldsymbol{\rho}_h; \boldsymbol{\eta}_2, v_2, \boldsymbol{\tau}_2) = a_h(\boldsymbol{\beta}_h, \boldsymbol{\eta}_2) + ||\boldsymbol{\rho}_h||^2_{\Gamma}. \tag{4.35}
$$

To control the first term in the right-hand side of equation (4.35), we note that (cf. also (3.17))

$$a_h(\boldsymbol{\beta}_h, \boldsymbol{\eta}_2) \geq -\frac{M}{2\delta}||\boldsymbol{\beta}_h||^2_{\Theta_h} - \frac{\delta M}{2}||\boldsymbol{\eta}_2||^2_{\Theta_h} \geq -\frac{M}{2\delta}||\boldsymbol{\beta}_h||^2_{\Theta_h} - \frac{\delta CM}{2}||\boldsymbol{\rho}_h||^2_{\Gamma}. \tag{4.36}$$

Taking δ sufficiently small, we get

$$\mathcal{A}_h(\boldsymbol{\beta}_h, z_h, \boldsymbol{\rho}_h; \boldsymbol{\eta}_2, v_2, \boldsymbol{\tau}_2) \geq C_2||\boldsymbol{\rho}_h||^2_{\Gamma} - C_3||\boldsymbol{\beta}_h||^2_{\Theta_h}. \tag{4.37}$$

Third Step. Choose $(\boldsymbol{\eta}_3, v_3, \boldsymbol{\tau}_3) \in \Theta_h \times W_h \times \Gamma_h$ as

$$\boldsymbol{\eta}_3 = 0, \qquad v_3 = 0, \qquad \boldsymbol{\tau}_3 = -\boldsymbol{\nabla}_h z_h.$$

Notice that by (3.9) the choice above is admissible.

On one hand it is easily seen that

$$||\boldsymbol{\tau}_3||_{\Gamma} \leq C||z_h||_{\Theta_h}. \tag{4.38}$$

On the other hand it holds

$$\begin{aligned}\mathcal{A}_h(\boldsymbol{\beta}_h, z_h, \boldsymbol{\rho}_h; \boldsymbol{\eta}_3, v_3, \boldsymbol{\tau}_3) &= (\boldsymbol{\nabla}_h z_h - \boldsymbol{\beta}_h, \boldsymbol{\nabla}_h z_h) - \lambda^{-1}t^2(\boldsymbol{\rho}_h, \boldsymbol{\nabla}_h z_h)\\ &= ||z_h||^2_{W_h} - (\boldsymbol{\beta}_h, \boldsymbol{\nabla}_h z_h) - \lambda^{-1}t^2(\boldsymbol{\rho}_h, \boldsymbol{\nabla}_h z_h)\\ &\geq \left(1 - \frac{\delta}{2}\right)||z_h||^2_{W_h} - \frac{C}{2\delta}||\boldsymbol{\beta}_h||^2_{\Theta_h} - \lambda^{-1}t^2(\boldsymbol{\rho}_h, \boldsymbol{\nabla}_h z_h).\end{aligned} \tag{4.39}$$

Moreover, one has

$$-\lambda^{-1}t^2(\boldsymbol{\rho}_h, \boldsymbol{\nabla}_h z_h) \geq -t^2\left(\frac{\lambda^{-1}}{2\varepsilon}||\boldsymbol{\rho}_h||^2_{0,\Omega} + \frac{\lambda^{-1}\varepsilon}{2}||z_h||^2_{W_h}\right). \tag{4.40}$$

By (4.39)–(4.40), and taking δ and ε sufficiently small, one finally gets

$$\mathcal{A}_h(\boldsymbol{\beta}_h, z_h, \boldsymbol{\rho}_h; \boldsymbol{\eta}_3, v_3, \boldsymbol{\tau}_3) \geq C_4||z_h||^2_{W_h} - C_5||\boldsymbol{\beta}_h||^2_{\Theta_h} - C_6t^2||\boldsymbol{\rho}_h||^2_{0,\Omega}. \tag{4.41}$$

Now it only suffices to take a suitable linear combination of $\{(\boldsymbol{\eta}_i, v_i, \boldsymbol{\tau}_i)\}_{i=1}^3$ so that by (4.29), (4.31), (4.35), (4.37), (4.38) and (4.41) it follows that (4.27) and (4.28) hold. The proof is then complete. $\square$

5. Error analysis. In this Section we develop a convergence analysis for our scheme, taking advantage of Proposition 4.3.

We shall need the following result (see [1]-[2]): let T be a triangle, and let e be an edge of T. Then $\exists\, C > 0$ only depending on the minimum angle of T such that

$$||\varphi||^2_{0,e} \leq C\big(|e|^{-1}||\varphi||^2_{0,T} + |e||\varphi|^2_{1,T}\big) \qquad \varphi \in H^1(\mathcal{T}_h). \tag{5.1}$$

Clearly, (5.1) also holds for vector valued functions $\boldsymbol{\varphi} \in (H^1(\mathcal{T}_h))^2$. Moreover, we shall use the estimate (see [6])

$$\left(\sum_{e\in\mathcal{E}_h} |e|^{-1}\,||[\boldsymbol{\eta}_h]||^2_{0,e}\right)^{1/2} \leq C||\boldsymbol{\eta}_h||_{\Theta_h} \qquad \forall \boldsymbol{\eta}_h \in \Theta_h. \tag{5.2}$$

We can now prove the following Theorem.

THEOREM 5.1. *Let $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ be the solution of problem* (2.1)–(2.3). *Furthermore, let $(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h)$ be the solution of the discretized problem* (4.26). *The following error estimate holds*

$$\begin{aligned}||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_{\Theta_h}&+||w-w_h||_{W_h}+||\boldsymbol{\gamma}-\boldsymbol{\gamma}_h||_\Gamma+t||\boldsymbol{\gamma}-\boldsymbol{\gamma}_h||_{0,\Omega}\\ &\leq Ch\Big(||\boldsymbol{\theta}||_{2,\Omega}+||w||_{2,\Omega}+||\boldsymbol{\gamma}||_{H(\mathrm{div})}+t||\boldsymbol{\gamma}||_{1,\Omega}\Big).\end{aligned} \tag{5.3}$$

Proof: By Proposition 4.3, given $(\boldsymbol{\theta}_h-\boldsymbol{\theta}_I, w_h-w_I, \boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I) \in \Theta_h\times W_h\times\Gamma_h$, there exists $(\boldsymbol{\eta}_h, v_h, \boldsymbol{\tau}_h) \in \Theta_h\times W_h\times\Gamma_h$ such that

$$\begin{aligned}&||\boldsymbol{\eta}_h||_{\Theta_h}+||v_h||_{W_h}+||\boldsymbol{\tau}_h||_\Gamma+t||\boldsymbol{\tau}_h||_{0,\Omega}\\ &\leq C\Big(||\boldsymbol{\theta}_h-\boldsymbol{\theta}_I||_{\Theta_h}+||w_h-w_I||_{W_h}+||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I||_\Gamma+t||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I||_{0,\Omega}\Big)\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}&C\Big(||\boldsymbol{\theta}_h-\boldsymbol{\theta}_I||^2_{\Theta_h}+||w_h-w_I||^2_{W_h}+||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I||^2_\Gamma+t^2||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I||^2_{0,\Omega}\Big)\\ &\quad\leq \mathcal{A}_h(\boldsymbol{\theta}_h-\boldsymbol{\theta}_I, w_h-w_I, \boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I; \boldsymbol{\eta}_h, v_h, \boldsymbol{\tau}_h)\\ &\qquad= a_h(\boldsymbol{\theta}_h-\boldsymbol{\theta}_I, \boldsymbol{\eta}_h)+(\boldsymbol{\nabla}_h v_h-\boldsymbol{\eta}_h, \boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I)\\ &\qquad\quad-\big(\boldsymbol{\nabla}_h(w_h-w_I)-(\boldsymbol{\theta}_h-\boldsymbol{\theta}_I), \boldsymbol{\tau}_h\big)+\lambda^{-1}t^2(\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I, \boldsymbol{\tau}_h)\end{aligned} \tag{5.5}$$

Multiplying equation (2.1) by $\boldsymbol{\eta}_h$, integrating by parts, and using $[\boldsymbol{\theta}]=0$ we obtain

$$a_h(\boldsymbol{\theta}, \boldsymbol{\eta}_h)-(\boldsymbol{\gamma}, \boldsymbol{\eta}_h) = c_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}_h), \tag{5.6}$$

where, using (3.5),

$$c_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}_h) := \sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathbf{C}\,\varepsilon(\boldsymbol{\theta})\mathbf{n}\cdot\boldsymbol{\eta}_h\,ds = \sum_{e\in\mathcal{E}_h}\int_e\{\mathbf{C}\,\varepsilon(\boldsymbol{\theta})\}:[\boldsymbol{\eta}_h]\,ds. \tag{5.7}$$

Multiplying equation (2.2) by v_h and integrating by parts we have

$$(\boldsymbol{\gamma}, \boldsymbol{\nabla}_h v_h) = (g, v_h)+c_W(\boldsymbol{\gamma}, v_h), \tag{5.8}$$

where

$$c_W(\boldsymbol{\gamma}, v_h) := \sum_{T\in\mathcal{T}_h}\int_{\partial T}\boldsymbol{\gamma}\cdot\mathbf{n}v_h\,ds = \sum_{e\in\mathcal{E}_h}\int_e\{\boldsymbol{\gamma}\}\cdot[v_h]\,ds. \tag{5.9}$$

Multiplying equation (2.3) by $\boldsymbol{\tau}_h$ and integrating we obtain

$$(\boldsymbol{\nabla} w - \boldsymbol{\theta}, \boldsymbol{\tau}_h) - \lambda^{-1} t^2 (\boldsymbol{\gamma}, \boldsymbol{\tau}_h) = 0. \tag{5.10}$$

Therefore, from (5.6)–(5.10) we get that

$$\mathcal{A}_h(\boldsymbol{\theta}, w, \boldsymbol{\gamma}; \boldsymbol{\eta}_h, v_h, \boldsymbol{\tau}_h) = (g, v_h) + c_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}_h) + c_W(\boldsymbol{\gamma}, v_h). \tag{5.11}$$

By recalling that $(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h)$ solves (4.26), from (5.5) and (5.11) we obtain

$$\begin{aligned}
C\Big(||\boldsymbol{\theta}_h - \boldsymbol{\theta}_I||^2_{\Theta_h} + ||w_h - w_I||^2_{W_h} + ||\boldsymbol{\gamma}_h - \boldsymbol{\gamma}_I||^2_\Gamma + t^2 ||\boldsymbol{\gamma}_h - \boldsymbol{\gamma}_I||^2_{0,\Omega}\Big) \\
\le a_h(\boldsymbol{\theta} - \boldsymbol{\theta}_I, \boldsymbol{\eta}_h) - c_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}_h) + (\boldsymbol{\nabla}_h v_h - \boldsymbol{\eta}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_I) - c_W(\boldsymbol{\gamma}, v_h) \\
- \big(\boldsymbol{\nabla}_h(w - w_I) - (\boldsymbol{\theta} - \boldsymbol{\theta}_I), \boldsymbol{\tau}_h\big) + \lambda^{-1} t^2 (\boldsymbol{\gamma} - \boldsymbol{\gamma}_I, \boldsymbol{\tau}_h) \\
= T_1 + T_2 + T_3 + T_4,
\end{aligned} \tag{5.12}$$

where

$$\begin{cases}
T_1 = a_h(\boldsymbol{\theta} - \boldsymbol{\theta}_I, \boldsymbol{\eta}_h) - c_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}_h) \\
T_2 = (\boldsymbol{\nabla}_h v_h - \boldsymbol{\eta}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_I) - c_W(\boldsymbol{\gamma}, v_h) \\
T_3 = \big(\boldsymbol{\nabla}_h(w - w_I) - (\boldsymbol{\theta} - \boldsymbol{\theta}_I), \boldsymbol{\tau}_h\big) \\
T_4 = \lambda^{-1} t^2 (\boldsymbol{\gamma} - \boldsymbol{\gamma}_I, \boldsymbol{\tau}_h)
\end{cases} \tag{5.13}$$

In order to estimate the four terms above, we need to choose $\boldsymbol{\theta}_I$, w_I and $\boldsymbol{\gamma}_I$. For $\boldsymbol{\theta}_I$ and w_I we take the usual nonconforming piecewise linear interpolated of $\boldsymbol{\theta}$ and w, respectively. A suitable choice of $\boldsymbol{\gamma}_I$ is more involved and it requires the introduction of the Helmholtz decomposition for $\boldsymbol{\gamma}$ (see [14], for instance). More precisely we write

$$\boldsymbol{\gamma} = \boldsymbol{\nabla} r + \mathbf{curl}\, p \qquad r \in H^2(\Omega) \cap H^1_0(\Omega), \quad p \in H^1(\Omega)/\mathbf{R}. \tag{5.14}$$

It is easily seen that

$$\big(||r||^2_{2,\Omega} + ||p||^2_{1,\Omega}\big)^{1/2} \le C ||\boldsymbol{\gamma}||_{H(\mathrm{div})}. \tag{5.15}$$

We now take r_I as the piecewise linear and continuous Lagrange interpolated of r, and p_I as the Clemént interpolated of p. Following [21], we finally set $\boldsymbol{\gamma}_I \in \Gamma_h$ as

$$\boldsymbol{\gamma}_I = \boldsymbol{\nabla} r_I + \mathbf{curl}\, p_I. \tag{5.16}$$

We have (see [21])

$$||\boldsymbol{\gamma} - \boldsymbol{\gamma}_I||_\Gamma \le C h\, ||\boldsymbol{\gamma}||_{H(\mathrm{div})} \tag{5.17}$$

and

$$||\boldsymbol{\gamma} - \boldsymbol{\gamma}_I||_{0,\Omega} \leq Ch\,||\boldsymbol{\gamma}||_{1,\Omega}. \tag{5.18}$$

We are ready to estimate the terms in (5.13).

Estimate for T_1. Using (3.17), we have

$$a_h(\boldsymbol{\theta} - \boldsymbol{\theta}_I, \boldsymbol{\eta}_h) \leq Ch\,||\boldsymbol{\theta}||_{2,\Omega}||\boldsymbol{\eta}_h||_{\Theta_h} \tag{5.19}$$

and (cf. [16])

$$c_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}_h) \leq Ch\,||\boldsymbol{\theta}||_{2,\Omega}||\boldsymbol{\eta}_h||_{\Theta_h}. \tag{5.20}$$

Therefore

$$T_1 \leq Ch\,||\boldsymbol{\theta}||_{2,\Omega}||\boldsymbol{\eta}_h||_{\Theta_h}. \tag{5.21}$$

Estimate for T_2. Using (5.14) and (5.16) we get

$$\begin{aligned} T_2 =& \big(\boldsymbol{\nabla}_h v_h - \boldsymbol{\eta}_h, \boldsymbol{\nabla}(r - r_I) + \mathbf{curl}\,(p - p_I)\big) - c_W(\boldsymbol{\gamma}, v_h) \\ =& \big(\boldsymbol{\nabla}_h v_h, \boldsymbol{\nabla}(r - r_I)\big) + \Big\{\big(\boldsymbol{\nabla}_h v_h, \mathbf{curl}\,(p - p_I)\big) - c_W(\boldsymbol{\gamma}, v_h)\Big\} \\ &\quad - \big(\boldsymbol{\eta}_h, \boldsymbol{\nabla}(r - r_I)\big) - \big(\boldsymbol{\eta}_h, \mathbf{curl}\,(p - p_I)\big) \\ =& T_2^1 + T_2^2 + T_2^3 + T_2^4. \end{aligned} \tag{5.22}$$

• From standard approximation theory and (5.15) we have

$$T_2^1 \leq Ch\,||r||_{2,\Omega}||v_h||_{W_h} \leq Ch\,||\boldsymbol{\gamma}||_{H(\mathrm{div})}||v_h||_{W_h}. \tag{5.23}$$

• We now treat the term T_2^2: since $v_h \in W_h$ and p_I is a piecewise linear and continuous function, the discrete Helmholtz decomposition proved in [7] gives

$$(\boldsymbol{\nabla}_h v_h, \mathbf{curl}\,p_I) = 0,$$

so that, using also (5.9), we obtain

$$T_2^2 = \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\nabla}_h v_h \cdot \mathbf{curl}\,p - \sum_{e\in\mathcal{E}_h}\int_e \{\mathbf{curl}\,p\}\cdot[v_h] - \sum_{e\in\mathcal{E}_h}\int_e \{\boldsymbol{\nabla} r\}\cdot[v_h]. \tag{5.24}$$

Since

$$\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\nabla}_h v_h \cdot \mathbf{curl}\,p - \sum_{e\in\mathcal{E}_h}\int_e \{\mathbf{curl}\,p\}\cdot[v_h] = 0,$$

it follows

$$T_2^2 = -\sum_{e\in\mathcal{E}_h}\int_e \{\boldsymbol{\nabla} r\}\cdot[v_h]. \tag{5.25}$$

By a standard nonconforming approximation result and (5.15) we have

$$T_2^2 \le Ch\,||\boldsymbol{\nabla} r||_{1,\Omega}||v_h||_{W_h} \le Ch\,||\boldsymbol{\gamma}||_{H(\mathrm{div})}||v_h||_{W_h}. \tag{5.26}$$

• To bound T_2^3 we simply observe that

$$T_2^3 = -\big(\boldsymbol{\eta}_h, \boldsymbol{\nabla}(r-r_I)\big) \le Ch\,||\boldsymbol{\nabla} r||_{1,\Omega}||\boldsymbol{\eta}_h||_{0,\Omega} \le Ch\,||\boldsymbol{\gamma}||_{H(\mathrm{div})}||\boldsymbol{\eta}_h||_{\Theta_h}. \tag{5.27}$$

• Integrating by parts the term T_2^4 we get

$$\begin{aligned} T_2^4 &= -\sum_{T\in\mathcal{T}_h}\left\{\int_T \mathrm{rot}\ \boldsymbol{\eta}_h(p-p_I) + \int_{\partial T}\boldsymbol{\eta}_h\cdot\mathbf{t}_T\,(p-p_I)\right\} \\ &= -\sum_{T\in\mathcal{T}_h}\int_T \mathrm{rot}\ \boldsymbol{\eta}_h(p-p_I) - \sum_{e\in\mathcal{E}_h}\int_e \mathbf{t}_e\otimes\mathbf{n}_e : [\boldsymbol{\eta}_h]\{p-p_I\}. \end{aligned} \tag{5.28}$$

On one hand, we have

$$-\sum_{T\in\mathcal{T}_h}\int_T \mathrm{rot}\ \boldsymbol{\eta}_h(p-p_I) \le Ch\,||p||_{1,\Omega}||\boldsymbol{\eta}_h||_{\Theta_h} \le Ch\,||\boldsymbol{\gamma}||_{H(\mathrm{div})}||\boldsymbol{\eta}_h||_{\Theta_h}. \tag{5.29}$$

On the other hand, using (5.1) and (5.2) we get

$$\begin{aligned} &-\sum_{e\in\mathcal{E}_h}\int_e \mathbf{t}_e\otimes\mathbf{n}_e : [\boldsymbol{\eta}_h]\{p-p_I\} \\ &\qquad \le \left(\sum_{e\in\mathcal{E}_h}|e|\,||\{p-p_I\}||_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h}|e|^{-1}\,||[\boldsymbol{\eta}_h]||_{0,e}^2\right)^{1/2} \\ &\qquad \le C\left(\sum_{T\in\mathcal{T}_h}\big(||p-p_I||_{0,T}^2 + h_T^2|p-p_I|_{1,T}^2\big)\right)^{1/2}||\boldsymbol{\eta}_h||_{\Theta_h} \\ &\qquad \le Ch\,||p||_{1,\Omega}||\boldsymbol{\eta}_h||_{\Theta_h}. \end{aligned} \tag{5.30}$$

Therefore, from (5.28)–(5.30) and (5.15) we obtain

$$T_2^4 \le Ch\,||\boldsymbol{\gamma}||_{H(\mathrm{div})}||\boldsymbol{\eta}_h||_{\Theta_h}. \tag{5.31}$$

Collecting (5.23), (5.26), (5.27) and (5.31) we conclude that

$$T_2 \le Ch\,||\boldsymbol{\gamma}||_{H(\mathrm{div})}\Big(||\boldsymbol{\eta}_h||_{\Theta_h} + ||v_h||_{W_h}\Big). \tag{5.32}$$

Estimate for T_3. Since $\boldsymbol{\tau}_h$ is piecewise constant it follows

$$\big(\boldsymbol{\nabla}_h(w-w_I),\boldsymbol{\tau}_h\big)=0.$$

Hence

$$\begin{aligned}T_3=-(\boldsymbol{\theta}-\boldsymbol{\theta}_I,\boldsymbol{\tau}_h)&\le\left(\sum_{T\in\mathcal{T}_h}h_T^{-2}||\boldsymbol{\theta}-\boldsymbol{\theta}_I||_{0,T}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}h_T^{2}||\boldsymbol{\tau}_h||_{0,T}^2\right)^{1/2}\\&\le Ch\,||\boldsymbol{\theta}||_{2,\Omega}||\boldsymbol{\tau}_h||_{-1,\Omega}\le Ch\,||\boldsymbol{\theta}||_{2,\Omega}||\boldsymbol{\tau}_h||_{\Gamma},\end{aligned}\tag{5.33}$$

where we have used both the inverse inequality

$$\sum_{T\in\mathcal{T}_h}h_T^2||\boldsymbol{\tau}_h||_{0,T}^2\le C||\boldsymbol{\tau}_h||_{-1,\Omega}^2$$

and the definition of the Γ-norm (see (2.7)).

Estimate for T_4. We have, using (5.18)

$$T_4=\lambda^{-1}t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_I,\boldsymbol{\tau}_h)\le Ch\,t||\boldsymbol{\gamma}||_{1,\Omega}\,t||\boldsymbol{\tau}_h||_{0,\Omega}.\tag{5.34}$$

Collecting (5.21), (5.32) (5.33) and (5.34), from (5.12) we obtain

$$\begin{aligned}&\Big(||\boldsymbol{\theta}_h-\boldsymbol{\theta}_I||_{\Theta_h}^2+||w_h-w_I||_{W_h}^2+||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I||_{\Gamma}^2+t^2||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I||_{0,\Omega}^2\Big)\\&\quad\le Ch\Big(||\boldsymbol{\theta}||_{2,\Omega}+||\boldsymbol{\gamma}||_{H(\mathrm{div})}+t||\boldsymbol{\gamma}||_{1,\Omega}\Big)\\&\qquad\times\Big(||\boldsymbol{\eta}_h||_{\Theta_h}+||v_h||_{W_h}+||\boldsymbol{\tau}_h||_{\Gamma}+t||\boldsymbol{\tau}_h||_{0,\Omega}\Big).\end{aligned}\tag{5.35}$$

Using (5.4) we get

$$\begin{aligned}&||\boldsymbol{\theta}_h-\boldsymbol{\theta}_I||_{\Theta_h}+||w_h-w_I||_{W_h}+||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I||_{\Gamma}+t||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I||_{0,\Omega}\\&\quad\le Ch\Big(||\boldsymbol{\theta}||_{2,\Omega}+||\boldsymbol{\gamma}||_{H(\mathrm{div})}+t||\boldsymbol{\gamma}||_{1,\Omega}\Big),\end{aligned}\tag{5.36}$$

and estimate (5.3) follows from the triangle inequality. □

Using Proposition 2.1, from Theorem 5.1 we get an optimal error estimate with respect to h and independent of t:

Corollary 5.1. *Suppose that Ω is convex and $g\in L^2(\Omega)$. Then it holds*

$$||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_{\Theta_h}+||w-w_h||_{W_h}+||\boldsymbol{\gamma}-\boldsymbol{\gamma}_h||_{\Gamma}+t||\boldsymbol{\gamma}-\boldsymbol{\gamma}_h||_{0,\Omega}\le Ch\,||g||_{0,\Omega}.\tag{5.37}$$

□

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