The method of mothers for non-overlapping non-matching DDM

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Abstract

In this paper we introduce a variant of the three-field formulation where we use only two sets of variables. Considering, to fix the ideas, the homogeneous Dirichlet problem for $-\Delta u = g$ in Ω, our variables are *i)* the approximations u_h^s of u in each sub-domain Ω^s (each on its own grid), and *ii)* an approximation ψ_h of u on the *skeleton* (the union of the interfaces of the sub-domains) on an independent grid (that could often be uniform). The novelty is in the way to derive, from ψ_h, the values of each trace of u_h^s on the boundary of each Ω^s. We do it by solving an auxiliary problem on each $\partial\Omega^s$ that resembles the mortar method but is more flexible. Under suitable assumptions, quasi-optimal error estimates are proved, uniformly with respect to the number and size of the subdomains. A preliminary version of the method and of its theoretical analysis has been presented in [7].

1 Introduction

Assume, for simplicity, that we have to solve the model problem

$$\text{find } u \in H_0^1(\Omega) \text{ such that } -\Delta u = g \text{ in } \Omega, \tag{1}$$

on a polygonal or polyhedral domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, where g is a given function sufficiently regular in Ω. In order to apply a Domain Decomposition technique we split Ω into polygonal or polyhedral sub-domains Ω^s $(s = 1, 2, \ldots, S)$ and we consider the *skeleton*

$$\Sigma := \cup_{s=1}^S \Gamma^s, \tag{2}$$

where $\Gamma^s := \partial\Omega^s$ (while $\Gamma = \partial\Omega$ will denote the boundary of Ω).

For the sake of simplicity from now on we shall use a three-dimensional notation, and speak therefore of *faces, edges* and *vertexes*. The change of terminology in the polygonal case is obvious and left to the reader. Then, for each $s = 1, 2, \ldots, S$, we denote by Γ_r^s, for $r = 1, 2, \ldots, R^s$, each of the R^s polygonal *faces* of the polyhedron Ω^s.

On Σ we consider

$$\Phi := \{\phi \in L^2(\Sigma) : \exists v \in H_0^1(\Omega) \text{ with } \phi = v_{|\Sigma}\} \equiv H_0^1(\Omega)_{|\Sigma}, \tag{3}$$

while on each Ω^s, for $s = 1, 2, \ldots, S$, we consider the space

$$V^s := H^1(\Omega^s). \tag{4}$$

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In its turn, $H^1_0(\Omega)$ can be identified with a subspace of

$$V := \{v \in L^2(\Omega),\ v_{|\Omega^s} \in V^s,\ \forall s = 1, \dots, S\}, \tag{5}$$

and in particular, setting $v^s := v_{|\Omega^s}$ and introducing the notation

$$v_{|\Sigma} = \phi \qquad \Leftrightarrow \qquad v^s_{|\Gamma^s} = \phi_{|\Gamma^s}, \forall s = 1, \dots, S, \tag{6}$$

we can write

$$H^1_0(\Omega) = \{v \in V \text{ such that } \exists \phi \in \Phi \text{ with } v_{|\Sigma} = \phi\}. \tag{7}$$

When discretizing the problem, we assume to be given a decomposition $\mathcal{T}^\Sigma_\delta$ of Σ and a corresponding space Φ_δ of piecewise polynomials. A key difference between this numerical discretization and the previous one proposed in [7] is that now we consider a non-conforming approximation Φ_δ of Φ. In particular, we allow the elements of Φ_δ to be discontinuous across two adjacent faces of Σ, which is quite convenient for the practical implementation of the method.

We also assume that in each Ω^s we are given a decomposition $\mathcal{T}^s_h \equiv \mathcal{T}^{\Omega^s}_h$, with a corresponding space $V^s_h \subset V^s$ of piecewise polynomials, and we set

$$V_h := \{v \in V \text{ such that } v_{|\Omega^s} \in V^s_h,\ \forall s = 1, \dots, S\}. \tag{8}$$

It is clear that each decomposition $\mathcal{T}^s_h$ will induce a decomposition $\mathcal{T}^{\Gamma^s}_h$ on Γ^s and a corresponding space of traces $V^s_{h\,|\Gamma^s} \subset V^s{}_{|\Gamma^s}$. On the other hand the restriction of $\mathcal{T}^\Sigma_\delta$ to each Γ^s also induces a decomposition $\mathcal{T}^{\Gamma^s}_\delta$ of Γ^s and another space of piecewise polynomials $\Phi^s_\delta := \Phi_{\delta|\Gamma^s}$ made by the restrictions of the functions in Φ_δ to Γ^s. Hence, on each Γ^s we have *two* decompositions (one coming from $\mathcal{T}^\Sigma_\delta$ and one from $\mathcal{T}^s_h$) and two spaces of piecewise polynomial functions (one from Φ_δ and one from V^s_h). Note, incidentally, that on each face belonging to two different sub-domains we shall have *three* decompositions and three spaces: one from Σ and the other two from the two sub-domains.

The first basic idea of our method is to design for every sub-domain Ω^s a linear operator $\mathcal{G}^s$ (the *generation operator*) that maps every *mother* $\phi_\delta \in \Phi_\delta$ into an element (*daughter*) $v^s_{h|\Gamma^s} = \mathcal{G}^s(\phi_\delta) \in V^s_{h\,|\Gamma^s}$. Together with the individual $\mathcal{G}^s$ we consider a *global* operator $\mathcal{G}$ defined as

$$\mathcal{G}(\phi_\delta) = (\mathcal{G}^1(\phi_\delta), \dots, \mathcal{G}^S(\phi_\delta)), \tag{9}$$

and, similarly to (6), we use the notation

$$v_{h|\Sigma} = \mathcal{G}(\phi_\delta) \qquad \Leftrightarrow \qquad v^s_{h|\Gamma^s} = \mathcal{G}^s(\phi_\delta),\ \forall s = 1, \dots, S.$$

The way to construct the operators $\mathcal{G}^s$ constitutes the second basic idea of this paper, and will be described in a while.

Once we have the operators $\mathcal{G}^s$ we can consider the subspace $\mathcal{S}_h$ of V_h made of *sisters* (that is, daughters of the same mother):

$$\mathcal{S}_h := \{v_h \in V_h \text{ such that } \exists \phi_\delta \in \Phi_\delta \text{ with } v_{h|\Sigma} = \mathcal{G}(\phi_\delta)\} \subset V_h. \tag{10}$$

We point out that in our previous definitions we consider as *daughter*, at the same time, a trace $v^s_{h|\Gamma^s}$, and any function $v^s_h \in V^s_h$ having that same trace. It is clear, comparing (10) with (7), that $\mathcal{S}_h$ can be seen as a nonconforming approximation of $H^1_0(\Omega)$. This allows us to consider the following discrete formulation. We set

$$a_s(u, v) := \int_{\Omega^s} \nabla u \cdot \nabla v \mathrm{d}x \qquad \text{and} \qquad a(u, v) := \sum_{s=1}^{S} a_s(u^s, v^s) \tag{11}$$

and we look for $u_h \in \mathcal{S}_h$ such that

$$a(u_h, v_h) = \int_\Omega g\, v_h \mathrm{d}x \quad \forall v_h \in \mathcal{S}_h. \tag{12}$$

It is clear that, under reasonable assumptions on the subspaces Φ_δ and V_h^s and on the generation operators $\mathcal{G}^s$, problem (12) will have good stability and accuracy properties.

The idea of introducing the space Φ_δ, and defining a nonconforming approximation of $H_0^1(\Omega)$ by taking the subset of V_h whose elements take (in some weak sense) value $\phi_\delta \in \Phi_\delta$ on the skeleton Σ, is one of the main ideas of the three field formulation (see [11]). Following that approach, for each sub-domain Ω^s we could take a space M_h^s of Lagrange multipliers, and, for every $\phi_\delta \in \Phi_\delta$, we could define $\mathcal{G}^s(\phi_\delta) \in V^s_{h\,|\Gamma^s}$ by

$$\int_{\Gamma^s} (\phi_\delta - \mathcal{G}^s(\phi_\delta))\mu_h^s \,\mathrm{d}x = 0 \qquad \forall \mu_h^s \in M_h^s. \tag{13}$$

In general, however, equation (13) can fail to have a solution, or the solution can fail to be unique, unless the spaces M_h^s and $V^s_{h\,|\Gamma^s}$ have the same dimension *and* satisfy a suitable *inf-sup* condition. In fact, the three field approach fits in the present framework only if we allow the generator $\mathcal{G}$ to be a set-valued operator and, accordingly, if we change the condition $v_{h|\Sigma} = \mathcal{G}(\phi_\delta)$ into $v_{h|\Sigma} \in \mathcal{G}(\phi_\delta)$ in the definition of *sister* space $\mathcal{S}_h$. This is not a problem in the definition and in the analysis of the three fields formulation. However, a method where the trace of the elements v_h^s on Γ^s is uniquely determined by an element of Φ_δ would have clear advantages. In particular it would allow to use standard Dirichlet solvers (which can easily be found already implemented and whose optimization is well understood) as a brick for treating the equation in the subdomain. As we said, in order for $\mathcal{G}^s(\phi_\delta)$ to be uniquely determined by (13) the spaces M_h^s and $V^s_{h\,|\Gamma^s}$ must have the same dimension. A simple minded choice is $M_h^s \equiv V^s_{h\,|\Gamma^s}$, that guarantees existence and uniqueness of the solution of (13) together with optimal stability and accuracy properties of the projector $\mathcal{G}^s$. This choice however is not the optimal one: in fact, in the estimate of the error for problem (12), there seems to be no way to get rid of a term like

$$\sum_{s=1}^{S} \int_{\Gamma^s} \frac{\partial u}{\partial \mathbf{n}^s} (\phi_\delta - \mathcal{G}^s(\phi_\delta))\,\mathrm{d}x. \tag{14}$$

An obvious way to treat the term in (14) is to use the fact that $\phi_\delta - \mathcal{G}^s(\phi_\delta)$ is orthogonal to all elements in M_h^s, so that we can subtract from $\partial u/\partial \mathbf{n}^s$ any element of M_h^s. In particular we are interested in subtracting a suitable approximation $\mu_I^s \simeq \partial u/\partial \mathbf{n}^s$. It is then crucial to be able to find in M_h^s a μ_I^s that approximates $\partial u/\partial \mathbf{n}^s$ with the needed order. However, $\partial u/\partial \mathbf{n}^s$ is *discontinuous* passing from one face to another of the same subdomain. And if the space M_h^s is made of *continuous* functions (as it would be with the choice $M_h^s \equiv V^s_{h\,|\Gamma^s}$), then the order of approximation (say, in $H^{-1/2}(\partial\Omega^s)$) cannot be better than $O(h)$. Hence, we do need an M_h^s made of functions that can be discontinuous when passing from one face to another of the same Ω^s. The requirement to contain a suitable amount of discontinuities and the one to have the same dimension of $V^s_{h\,|\Gamma^s}$ seem very difficult to conciliate.

A quite similar difficulty is met in the *mortar method*, (see e.g. [5], [4], [17], [21]), in particular in three dimensions. There, the requirement that M_h^s have the same dimension as $V^s_{h\,|\Gamma^s}$ is relaxed as little as possible. The values of a "weakly continuous" function v_h^s at nodes which are interior to the faces of Γ^s on the slave sides are uniquely determined by the weak continuity equation, while the degrees of freedom corresponding to nodes on the edges of Γ^s (whose union forms the so called wire-basket) are free. Remark that this difficulty can be overcome (see [1]) if one uses a mixed formulation for the Laplace problem, since in such an approach the multiplier needs to approximate u, which is continuous, rather than its normal derivative. We point out that the mortar method can be described in the framework given here, again by allowing $\mathcal{G}$ to be a set-valued operator: Φ_δ would correspond to the traces of v_h on the "master sides" (or "mortars") and $\mathcal{G}^s$ would be defined as *the identity* on master sides and as *one of the available mortar projections* on "slave sides".

The idea, here, is to give up the equality of the dimensions (but still obtain a well defined operator $\mathcal{G}^s$) by changing (13) into a slightly more complicated formulation, involving an additional Lagrange multiplier. Let us see the main features of this path.

We choose first a space M_h^s having in mind the fact that we must be able to use it for approximating $\partial u/\partial \mathbf{n}^s$ with the right order. We also need its dimension to be smaller than (or equal to) that of $V^s_{h\,|\Gamma^s}$. Then we change (13) in the following way. For every $\phi \in L^2(\Sigma)$ we look for *a pair* $(\widetilde{v}_h^s, \widetilde{\mu}_h^s)$ in $V^s_{h\,|\Gamma^s} \times M_h^s$ such that

$$\int_{\Gamma^s} (\phi - \widetilde{v}_h^s)\, \mu_h^s \,\mathrm{d}x = 0 \qquad \forall \mu_h^s \in M_h^s \tag{15}$$

and

$$\sum_{T\in\mathcal{T}_h^{\Gamma^s}} h_T^{-1} \int_T (\phi - \widetilde{v}_h^s)\, v_h^s \,\mathrm{d}x + \int_{\Gamma^s} \widetilde{\mu}_h^s\, v_h^s \,\mathrm{d}x = 0 \qquad \forall v_h^s \in V^s_{h\,|\Gamma^s}. \tag{16}$$

Then we set

$$\mathcal{G}^s(\phi) := \widetilde{v}_h^s. \tag{17}$$

It is clear that in (15)-(16) the number of equations will always be equal to the number of unknowns. It is also clear that if (by shear luck) we have $\phi_{|\Gamma^s} \in V^s_{h\,|\Gamma^s}$, then

$$\phi_{|\Gamma^s} \in V^s_{h\,|\Gamma^s} \quad\Rightarrow\quad \mathcal{G}^s(\phi) = \phi_{|\Gamma^s} \quad \textit{and} \quad \widetilde{\mu}_h^s = 0. \tag{18}$$

This will, in the end, provide for the new approach (15)-(17) an optimal order of accuracy (as we had for the previous simple-minded (13)). It is, finally, also obvious that some sort of *inf-sup* condition will be needed in order to ensure existence and uniqueness of the solution of (15)-(16), unless some suitable additional stabilization is introduced. However, the possibility of escaping the cage of the equal dimensionality of M_h^s and $V^s_{h\,|\Gamma^s}$ opens a whole lot of interesting possibilities. We shall see two examples in §4. In the first example, we shall take as $V^s_{h\,|\Gamma^s}$ the space of quadratic and globally continuous finite elements and as M_h^s the space of linear finite elements, continuous within each face but discontinuous across the faces. In the second example $V^s_{h\,|\Gamma^s}$ will be the space of continuous finite elements of degree k enriched by suitable *bubble* functions, while M_h^s will be formed by fully discontinuous finite elements of degree $k-1$.

In this paper we shall follow the path indicated above. In the next section we shall make precise all the necessary assumptions and definitions, and in §3 we shall derive abstract error bounds for problem (12) when the operators $\mathcal{G}^s$ are constructed as in (15)-(17). In §4 we shall present some possible choices for the finite element spaces and discuss their stability and accuracy properties. A step of our analysis is based on properties of some Besov spaces and on abstract tools of interpolation of function spaces; the Appendices will be devoted to those rather technical topics.

2 Preliminaries

Throughout the analysis, we shall make use of the classical Lebesgue spaces $L^p(\omega)$, endowed with the norm $\|\cdot\|_{L^p(\omega)}$, where $\omega \subseteq \Omega$ is a manifold of dimension n or $n-1$, and $1 \le p \le \infty$. we shall also need the notion of L^2-seminorm, defined as

$$|v|_{L^2(\omega)} := \|v - \bar{v}\|_{L^2(\omega)}, \tag{19}$$

$\bar{v}$ denoting the mean value of v on ω.

Moreover, we shall make use of the Sobolev spaces $H^\alpha(\omega) \equiv W^{\alpha,2}(\omega)$, for $\alpha \in \mathbb{R}$, endowed with the usual norms $\|\cdot\|_{H^\alpha(\omega)}$ and seminorms $|\cdot|_{H^\alpha(\omega)}$. In particular, Sobolev spaces of fractional order for $-1 < \alpha < 1$ will be needed (see Appendix A or [18] for more details). Finally, part of our analysis (Lemmata 8 and 9 in Appendix B) will need some Besov spaces and some tools of interpolation theory between function spaces, which are briefly recalled in Appendix A.

The rest of this section is devoted to the presentation of the notation and assumptions to be made on the decomposition and on the discretizations.

2.1 Assumptions on Ω and on the domain decomposition

We assume that Ω is an open polyhedron of diameter L, and that each Ω^s, for $s = 1, \dots, S$, is also an open polyhedron, of diameter H_{Ω^s}; we assume that the intersection of two different Ω^s is empty, and that the union of the closures of all Ω^s is the closure of Ω. As in (2) the *skeleton* Σ will be the union of the boundaries $\partial\Omega^s$. The diameter of each face Γ^s_r will be denoted by $H_{\Gamma^s_r}$. We *do not* assume that this decomposition is *compatible*. This means that we *do not* assume that the intersection of the closure of two different Ω^s is either a common face, or a common edge, or a common vertex.

Furthermore, we assume that each Ω^s is the image of a reference polyhedron $\widehat{\Omega}^{\widehat{s}}$ (of unitary diameter) in a set $\{\widehat{\Omega}^1, \dots, \widehat{\Omega}^{\widehat{S}}\}$, through a bounded map B_{Ω^s} with bounded Jacobian ∇B_{Ω^s}. We also assume that B_{Ω^s} maps each face, edge or vertex of $\widehat{\Omega}^{\widehat{s}}$ onto a face, edge or vertex of Ω^s. Clearly $\widehat{S}$ is the (possibly small) number of different kind of polyhedra that form the partition in subdomains. For example, if the subdomains are either tetrahedra or hexahedra, then $\widehat{S} = 2$ and $\widehat{\Omega}^1$ and $\widehat{\Omega}^2$ are the reference tetrahedron and hexahedron, respectively. The shape regularity of each Ω^s is measured by $\|\nabla B_{\Omega^s}\|_{L^\infty(\widehat{\Omega}^{\widehat{s}})}\|\nabla(B^{-1}_{\Omega^s})\|_{L^\infty(\Omega^s)}$, and our estimates will depend on

$$\kappa_0 := \sup_{s=1,\dots,S} \|\nabla B_{\Omega^s}\|_{L^\infty(\widehat{\Omega}^{\widehat{s}})}\|\nabla(B^{-1}_{\Omega^s})\|_{L^\infty(\Omega^s)},$$

as well as on the set of reference polyhedra $\{\widehat{\Omega}^1, \dots, \widehat{\Omega}^{\widehat{S}}\}$, but will be uniform with respect to the actual number S or sizes H_{Ω^s}, $s = 1, \dots, S$, of the subdomains.

2.2 Assumptions on the decomposition $\mathcal{T}^\Sigma_\delta$

We assume that we are given a family $\{\mathcal{T}^\Sigma_\delta\}_\delta$ of decompositions of Σ. Each decomposition $\mathcal{T}^\Sigma_\delta$ is made of open triangles, in such a way that the intersection of two different triangles is empty, and the union of the closures of all triangles is Σ. We denote by $\mathcal{T}^{\Gamma^s}_\delta$ and $\mathcal{T}^{\Gamma^s_r}_\delta$ the restrictions of $\mathcal{T}^\Sigma_\delta$ to Γ^s and Γ^s_r, respectively. *Within each face* Γ^s_r, we assume *compatibility*, that is, we assume that the intersection of the closures of two different triangles lying on each Γ_r^s is either empty, a common edge or a common vertex. We assume, as usual, *shape regularity*, for instance by assuming that the ratio between the diameter of each triangle and the radius of its biggest inscribed circle is smaller than κ_1, with κ_1 independent of δ. Furthermore, we assume that each mesh $\mathcal{T}_\delta^{\Gamma_r^s}$ is *quasi-uniform*: there exists a constant κ_2, independent of the family index δ, such that, if $\delta_{\Gamma_r^s}^{\min}$ and $\delta_{\Gamma_r^s}^{\max}$ are the minimum and the maximum diameters (respectively) of the triangles in $\mathcal{T}_\delta^{\Gamma_r^s}$, then $\delta_{\Gamma_r^s}^{\min} \geq \kappa_2 \delta_{\Gamma_r^s}^{\max}$, for all $r = 1, \ldots, R^s$ and $s = 1, \ldots, S$.

2.3 Assumptions on the decompositions $\mathcal{T}_h^s$ (and $\mathcal{T}_h^{\Gamma^s}$)

We assume that we are given, for each $s = 1, \ldots, S$, a family $\{\mathcal{T}_h^s\}_h$ of decompositions of Ω^s. Each decomposition is made of open tetrahedrons in such a way that the intersection of two different tetrahedrons is empty, and the union of the closures of all tetrahedrons is Ω^s. We also assume *compatibility*: the intersection of the closures of two different tetrahedrons is either empty, a common face, a common edge, or a common vertex. Finally we assume *shape regularity*, for instance by assuming that the ratio between the diameter of each tetrahedron and the radius of its biggest inscribed sphere is smaller than κ_3, with κ_3 independent of the family index h. We point out that we *do not* assume quasi-uniformity for the meshes $\mathcal{T}_h^s$. We denote by h_K the diameter of an element $K \in \mathcal{T}_h^s$; the parameter $h_{\Omega^s}^{\max}$ denotes the maximum diameter of the elements in $\mathcal{T}_h^s$. We recall that the triangulation $\mathcal{T}_h^{\Gamma^s}$ is the restriction to Γ^s of $\mathcal{T}_h^s$; h_T denotes the diameter of an element $T \in \mathcal{T}_h^{\Gamma^s}$; we also introduce the notation $h_{\Gamma^s}^{\min}$ and $h_{\Gamma^s}^{\max}$ for denoting the minimum and the maximum diameter, respectively, of the elements $T \in \mathcal{T}_h^{\Gamma^s}$.

2.4 Definitions of the spaces V, Φ, $\Phi^\star$, and M_r^s

The space V is defined in (5) and it is endowed with the seminorm and norm:

$$|v|_V^2 := \sum_{s=1}^{S} \|\nabla v\|_{L^2(\Omega^s)}^2, \qquad \forall v \in V, \tag{20}$$

$$\|v\|_V^2 := L^{-2} \|v\|_{L^2(\Omega)}^2 + |v|_V^2, \qquad \forall v \in V. \tag{21}$$

The natural norm induced by $|\cdot|_V$ on the space of continuous mothers Φ by the definition (3) is

$$\|\phi\|_\Phi^2 := \sum_{s=1}^{S} |\phi|_{H^{1/2}(\Gamma^s)}^2, \qquad \forall \phi \in \Phi.$$

In view of a non-conforming approximation Φ_δ of Φ, we introduce the space

$$\Phi^\star := \left\{\phi \in L^2(\Sigma) : \phi_{|\Gamma} = 0,\ \phi_{|\Gamma_r^s} \in H^{1/2}(\Gamma^s), \forall r = 1, \ldots, R^s, s = 1, \ldots, S\right\} \tag{22}$$

endowed with the *norm*

$$\|\phi\|_{\Phi^\star}^2 := \sum_{s=1}^{S} \left(H_{\Omega^s}^{-1} |\phi|_{L^2(\Gamma^s)}^2 + \sum_{r=1}^{R^s} |\phi|_{H^{1/2}(\Gamma_r^s)}^2 \right). \tag{23}$$

It is not difficult to realize that $\|\cdot\|_*$ is indeed a norm. In fact, $\|\phi\|_* = 0$ implies that $\phi_{|\Gamma^s}$ is a constant c^s and since ϕ is single-valued, all such constants are necessarily equals.

Since since $\phi = 0$ on Γ, we easily conclude that $\phi = 0$ on Σ. In this respect, see also Lemma 1.

On each face Γ_r^s we define $M_r^s := H^{-1/2}(\Gamma_r^s)$, endowed with the norm

$$\|\mu\|_{M_r^s} := \sup_{v \in H^{1/2}(\Gamma_r^s)} \frac{\langle \mu, v \rangle}{\left(H_{\Gamma_r^s}^{-1} \|v\|^2_{L^2(\Gamma_r^s)} + |v|^2_{H^{1/2}(\Gamma_r^s)} \right)^{1/2}}. \tag{24}$$

2.5 Assumptions on the discretizations Φ_δ, V_h^s, and M_h^s

We denote by $\mathbb{P}_\kappa(\omega)$ the space of polynomials of degree at most κ on ω. Our assumptions on the discrete spaces Φ_δ, V_h^s and M_h^s are:

$$\Phi_\delta \subseteq \left\{ \phi \in \Phi^\star : \phi_{|T} \in \mathbb{P}_\kappa(T),\, T \in \mathcal{T}_\delta^\Sigma \right\}, \tag{25}$$

$$V_h^s \subseteq \{ v^s \in V^s \text{ such that } v^s_{|K} \in \mathbb{P}_\kappa(K),\, K \in \mathcal{T}_h^s \}, \tag{26}$$

and

$$M_h^s \subseteq \{ \mu \in L^2(\Gamma^s) \text{ such that } \mu_{|T} \in \mathbb{P}_\kappa(T),\, T \in \mathcal{T}_h^{\Gamma^s} \}. \tag{27}$$

Using the notation of [8] for the usual Lagrange finite element spaces, we have

$$V_h^s \subseteq \mathcal{L}^1_\kappa(\mathcal{T}_h^s), \quad M_h^s \subseteq \mathcal{L}^0_\kappa(\mathcal{T}_h^{\Gamma^s}), \quad \Phi_\delta \subseteq \mathcal{L}^0_\kappa(\mathcal{T}_\delta^\Sigma),$$

with the additional assumption that the functions $\phi_\delta \in \Phi_\delta$ are null on Γ and continuous on the faces Γ_r^s; note that in the case of a non-compatible subdivision into subdomains, the continuity is required on the union of partially overlapped faces.

We assume that there exist bounded lifting operators from $V^s_{h\,|\Gamma^s}$ to V_h^s. More precisely, for all $s = 1, \ldots, S$ and for all $v_h^s \in V^s_{h\,|\Gamma^s}$, there exists an extension $v_h^s \in V_h^s$ such that

$$|v_h^s|_{H^1(\Omega^s)} \leq C |v_h^s|_{H^{1/2}(\Gamma^s)}, \tag{28}$$

with a constant C which only depends on the shape regularity of the mesh. This property is actually true for almost all reasonable finite element spaces, as we shall see later on. Finally, we make the following minimal assumptions on V_h^s and M_h^s:

$$\begin{gathered} V_h^s \text{ contains the constants on } \Omega^s,\ \forall s = 1, \ldots, S; \\ M_h^s \text{ and } \Phi_\delta \text{ contain the constants on } \Gamma_r^s,\ \forall s = 1, \ldots, S,\ \forall r = 1, \ldots, R^s. \end{gathered} \tag{29}$$

2.6 The operators $\mathcal{G}^s$ and the compatibility assumptions among the discretizations

Having defined the spaces V_h^s (and therefore $V^s_{h\,|\Gamma^s}$) and M_h^s, we can now consider the operators $\mathcal{G}^s$ (that will always be given by (15)-(17)) together with the global operator $\mathcal{G}$ (still given by (9)), and then we can define the space of sisters $\mathcal{S}_h$, always as in (10).

We can now turn to the more important assumptions, that will require some compatibility conditions among the spaces Φ_δ^s, $V^s_{h\,|\Gamma^s}$ and M_h^s.

Our first assumption will deal with the well-posedness of problem (15)-(17). As this is a problem in classical mixed form, we have no real escape but assuming an *inf-sup* condition on the spaces $V^s_{h\,|\Gamma^s}$ and M_h^s. In particular we define, for any real α, the norm

$$\|\nu\|^2_{h,\alpha,\Gamma^s} := \sum_{T \in \mathcal{T}_h^{\Gamma^s}} h_T^{-2\alpha} \|\nu\|^2_{0,T}, \qquad \forall \nu \in L^2(\Gamma^s), \tag{30}$$

and we make the following assumption:

$\exists\gamma_0 > 0$ such that $\forall s = 1, \ldots, S$ and $\forall h > 0$

$$\inf_{\mu_h^s \in M_h^s \setminus \{0\}} \sup_{v_h^s \in V^s_{h\,|\Gamma^s} \setminus \{0\}} \frac{\int_{\Gamma^s} v_h^s \, \mu_h^s \, \mathrm{d}x}{\|v_h^s\|_{h,1/2,\Gamma^s} \; \|\mu_h^s\|_{h,-1/2,\Gamma^s}} \geq \gamma_0. \tag{31}$$

Condition (31) will be, in a sense, the only nontrivial assumption that we have to take into account in the definition of our spaces V_h^s and M_h^s. However, in §4, we are going to see some families of elements where (31) can be checked rather easily.

Our last assumption will deal with the *bound on the mother*. We point out that, so far, we did not assume that an element of the space of sisters $\mathcal{S}_h$ had a unique mother. Indeed, we do not need it. we shall simply ask that

$$\exists\gamma_1 > 0 \text{ such that } \forall v_h \in \mathcal{S}_h, \exists\phi_\delta \in \Phi_\delta \text{ with } \mathcal{G}(\phi_\delta) = v_{h|\Sigma} \text{ and } \gamma_1 \|\phi_\delta\|_{\Phi^\star} \leq |v_h|_V . \tag{32}$$

We point out that a sufficient condition for (32) to hold is that Φ_δ and the M_h^s's are chosen in such a way that they satisfy an *inf-sup* condition. More precisely, we have the following proposition.

Proposition 1. *Let the assumptions of §2.1 hold. Assume that there exists a γ_1', independent of the meshes and of s and r, such that: for each $s = 1, \ldots, S$, and each $r = 1, \ldots, R^s$, if Γ_r^s is an internal face we have*

$$\inf_{\phi_\delta \in \Phi_{\delta|\Gamma_r^s} \setminus \{0\}} \sup_{\mu_h^s \in M_h^s \setminus \{0\}} \frac{\int_{\Gamma_r^s} \phi_\delta \, \mu_h^s \, \mathrm{d}x}{\|\mu_h^s\|_{M_r^s} \left(H_{\Gamma_r^s}^{-1} \|\phi_\delta\|^2_{L^2(\Gamma_r^s)} + |\phi_\delta|^2_{H^{1/2}(\Gamma_r^s)} \right)^{1/2}} \geq \gamma_1' > 0. \tag{33}$$

Then (32) holds.

Proof. Let v_h be a daughter of a given mother ϕ_δ, i.e., $v_{h|\Sigma} = \mathcal{G}(\phi_\delta)$. Consider a single subdomain Ω^s with boundary Γ^s. We recall that, for the assumptions (15) and (29), $v_h - \bar{v}_h = \mathcal{G}^s(\phi_\delta - \bar{\phi}_\delta)$, where $\bar{v}_h$ and $\bar{\phi}_\delta$ are the the mean values of v_h and ϕ_δ, respectively, on Γ^s. If we use the inf-sup condition (33) for $\phi_\delta - \bar{\phi}_\delta$, we get, on each single face Γ_r^s of Ω^s

$$\begin{aligned}
&\gamma_1' \left(H_{\Gamma_r^s}^{-1} \|\phi_\delta - \bar{\phi}_\delta\|^2_{L^2(\Gamma_r^s)} + |\phi_\delta - \bar{\phi}_\delta|^2_{H^{1/2}(\Gamma_r^s)} \right)^{1/2} \\
&\qquad \leq \sup_{\mu_h^s \in M_h^s \setminus \{0\}} \frac{\int_{\Gamma^s} (\phi_\delta - \bar{\phi}_\delta) \, \mu_h^s \, \mathrm{d}x}{\|\mu_h^s\|_{M_r^s}} \\
&\qquad = \sup_{\mu_h^s \in M_h^s \setminus \{0\}} \frac{\int_{\Gamma^s} (v_h - \bar{v}_h) \, \mu_h^s \, \mathrm{d}x}{\|\mu_h^s\|_{M_r^s}} \\
&\qquad \leq \left(H_{\Gamma_r^s}^{-1} \|v_h - \bar{v}_h\|^2_{L^2(\Gamma_r^s)} + |v_h - \bar{v}_h|^2_{H^{1/2}(\Gamma_r^s)} \right)^{1/2}.
\end{aligned} \tag{34}$$

Thanks to the trace inequality, we have

$$\begin{aligned}
\sum_{r=1}^{R^s} \left(H_{\Gamma_r^s}^{-1} \|v_h - \bar{v}_h\|^2_{L^2(\Gamma_r^s)} + |v_h - \bar{v}_h|^2_{H^{1/2}(\Gamma_r^s)} \right) &\leq C \left(H_{\Omega^s}^{-1} |v_h|^2_{L^2(\Gamma^s)} + |v_h|^2_{H^{1/2}(\Gamma^s)} \right) \\
&\leq C |v_h|^2_{H^1(\Omega^s)}.
\end{aligned}$$

Then we get, after summing over $r = 1, \dots, R^s$,

$$\gamma_1 \left(H_{\Omega^s}^{-1} |\phi_\delta|^2_{L^2(\Gamma^s)} + \sum_{r=1}^{R^s} |\phi_\delta|^2_{H^{1/2}(\Gamma_r^s)} \right)^{1/2} \leq |v_h|_{H^1(\Omega^s)}, \tag{35}$$

for a suitable choice of $\gamma_1 > 0$. Squaring (35) and summing for $s = 1, \dots, S$ eventually yields (32). □

Observe that Proposition 1 can be weakened: indeed, we ask an inf-sup condition (33) on both sides of each internal face which is shared between two subdomains, while one of the two conditions is enough. Anyway, as it is clear from the proof, the assumptions of Proposition 1 are stronger than (32). In particular they imply the uniqueness of the mother, which, as already remarked, is not strictly needed in the following.

Assumptions (33) (or the milder (32)) will have to be verified case by case. We recall that, by a well known argument, a way for an inf-sup condition of such kind to be satisfied is that on each face Γ_r^s the mesh $\mathcal{T}_\delta^{\Gamma_r^s}$ is coarser than the mesh induced by $\mathcal{T}_h^s$ (see §4).

3 Basic Error Estimates

The goal of this section is to show an optimal bound for the error $\|u - u_h\|_V$, in terms of the approximation properties of the discrete spaces. Under the assumptions of §2, the error estimates will be independent of the number or the size of the subdomains. From now on, C and C_i will denote strictly positive constants, possibly different at each occurrence, which may depend only on the set of reference polyhedra $\{\widehat{\Omega}^1, \dots, \widehat{\Omega}^{\widehat{S}}\}$, the polynomial degree κ, the constants κ_0, κ_1, κ_2, κ_3, appearing in the shape regularity and quasi-uniformity assumptions on the meshes, and the stability constants γ_0, γ_1 of §2.

3.1 Preliminary results

We need some preliminary lemma. The first lemma introduces a sort of Poincaré inequality for the space of mothers.

Lemma 1. *Under the assumptions of §2.1, we have*

$$\sum_{s=1}^{S} H_{\Omega^s} \|\phi\|^2_{L^2(\Gamma^s)} \leq CL^2 \sum_{s=1}^{S} H_{\Omega^s}^{-1} |\phi|^2_{L^2(\Gamma^s)}, \qquad \forall \phi \in \Phi^\star. \tag{36}$$

Proof. Let $\bar{\phi}$ be the piecewise constant function which, in each subdomain Ω^s, is equal to the mean value of ϕ on $\Gamma^s \equiv \partial\Omega^s$. By the triangle inequality, since $H_{\Omega^s}^{1/2} \leq LH_{\Omega^s}^{-1/2}$, we have

$$\begin{aligned} H_{\Omega^s}^{1/2} \|\phi\|_{L^2(\Gamma^s)} &\leq H_{\Omega^s}^{1/2} \|\phi - \bar{\phi}\|_{L^2(\Gamma^s)} + H_{\Omega^s}^{1/2} \|\bar{\phi}\|_{L^2(\Gamma^s)} \\ &\leq LH_{\Omega^s}^{-1/2} |\phi|_{L^2(\Gamma^s)} + H_{\Omega^s}^{1/2} \|\bar{\phi}\|_{L^2(\Gamma^s)}. \end{aligned} \tag{37}$$

Then, we only have to show that

$$\sum_{s=1}^{S} H_{\Omega^s} \|\bar{\phi}\|^2_{L^2(\Gamma^s)} \leq CL^2 \sum_{s=1}^{S} H_{\Omega^s}^{-1} |\phi|^2_{L^2(\Gamma^s)}, \qquad \forall \phi \in \Phi^\star. \tag{38}$$

Let x_1 denote the first coordinate of a point $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{n}^s$ be the outward normal direction on Γ^s, and n_1^s be the first component of $\mathbf{n}^s$. Assume for simplicity that Ω contains the origin of $\mathbb{R}^n$, so that $\|x_1\|_{L^\infty(\Omega)} \leq L$ (otherwise we would take, instead of x_1, a polynomial $p = x_1 - c$, vanishing at some point of Ω). Thanks to the assumptions of §2.1 we have, integrating by parts,

$$\sum_{s=1}^{S} H_{\Omega^s} \|\bar{\phi}\|^2_{L^2(\Gamma^s)} \leq C \sum_{s=1}^{S} \|\bar{\phi}\|^2_{L^2(\Omega^s)}$$
$$= C \sum_{s=1}^{S} \int_{\Omega^s} \bar{\phi}^2 \frac{\partial x_1}{\partial x_1} = C \sum_{s=1}^{S} \int_{\Gamma^s} \bar{\phi}^2 x_1 n_1^s. \tag{39}$$

On the other hand, since ϕ is single-valued on Σ and vanishes on Γ, we have

$$\sum_{s=1}^{S} \int_{\Gamma^s} \phi^2 x_1 n_1^s = 0.$$

Using this in (39), then the Cauchy-Schwarz and the Young inequality we get

$$\begin{aligned}
\sum_{s=1}^{S} H_{\Omega^s} \|\bar{\phi}\|^2_{L^2(\Gamma^s)} &\leq C \sum_{s=1}^{S} \int_{\Gamma^s} \left(\bar{\phi}^2 - \phi^2\right) x_1 n_1^s \\
&= C \sum_{s=1}^{S} \int_{\Gamma^s} \left(2\left(\bar{\phi} - \phi\right)\bar{\phi} - \left(\bar{\phi} - \phi\right)^2\right) x_1 n_1^s \\
&\leq C \left(\sum_{s=1}^{S} L \|\phi - \bar{\phi}\|_{L^2(\Gamma^s)} \|\bar{\phi}\|_{L^2(\Gamma^s)} + \sum_{s=1}^{S} L \|\phi - \bar{\phi}\|^2_{L^2(\Gamma^s)} \right) \\
&\leq \frac{1}{2} \sum_{s=1}^{S} H_{\Omega^s} \|\bar{\phi}\|^2_{L^2(\Gamma^s)} + C \sum_{s=1}^{S} H^{-1}_{\Omega^s} L^2 \|\phi - \bar{\phi}\|^2_{L^2(\Gamma^s)}.
\end{aligned}$$

which gives (38), and eventually (36). □

The next result is a Poincaré inequality for functions in V.

Lemma 2. *Under the assumptions of §2.1, let $v \in V$; then*

$$\|v\|_{L^2(\Omega)} \leq C \left(L^2 |v|^2_V + \sum_{s=1}^{S} H_{\Omega^s} \|\bar{v}^s\|^2_{L^2(\Gamma^s)} \right)^{1/2}, \tag{40}$$

where $\bar{v}^s$ denotes the mean value of v^s on Γ^s.

Proof. By the triangle inequality, we have

$$\|v\|_{L^2(\Omega^s)} \leq \|v - \bar{v}\|_{L^2(\Omega^s)} + \|\bar{v}\|_{L^2(\Omega^s)} =: I + II.$$

Thanks to the Poincaré inequality, we have

$$I \leq C H_{\Omega^s} |v|_{H^1(\Omega^s)}.$$

Considering now II we have

$$II \leq C H^{1/2}_{\Omega^s} \|\bar{v}\|_{L^2(\Gamma^s)};$$

squaring and summing over all the subdomains, we get

$$\|v\|_{L^2(\Omega)} \leq C \left[\sum_{s=1}^{S} \left(H_{\Omega^s}^2 |v^s|_{H^1(\Omega^s)}^2 + H_{\Omega^s} \|\bar{v}^s\|_{L^2(\Gamma^s)}^2 \right) \right]^{1/2}. \tag{41}$$

Since $H_{\Omega^s} \leq L$, (41) yields (40). □

Assume now that $v_h \in \mathcal{S}_h$, so that $v_{h|\Sigma} = \mathcal{G}(\phi_\delta)$ for some mother $\phi_\delta \in \Phi_\delta$; then from (15) and (29), we have $\int_{\Gamma^s} v_h - \phi_\delta = 0$, and therefore, using (36) and (32), we get the following Poincaré-like estimates for $\mathcal{S}_h$.

Corollary 1. *Under the assumptions of §2, we have*

$$\|v_h\|_V \leq CL|v_h|_V, \qquad \forall v_h \in \mathcal{S}_h. \tag{42}$$

We shall now prove a sort of inverse inequality in the space $V^s_{h\,|\Gamma^s}$.

Lemma 3. *Under the assumptions of §2, the following inverse inequality holds:*

$$|v_h^s|_{H^{1/2}(\Gamma^s)} \leq C\|v_h^s\|_{h,1/2,\Gamma^s}, \qquad \forall v_h^s \in V^s_{h\,|\Gamma^s}. \tag{43}$$

Proof. We shall actually prove that (43) holds for all $v_h^s \in \mathcal{L}^1_\kappa(\mathcal{T}_h^{\Gamma^s})$. It is well known that a function in $\mathcal{L}^1_\kappa(\mathcal{T}_h^s)$ is uniquely identified by its values at a set $\{x_i\}_i$ of nodes corresponding to the canonical Lagrange basis. With an abuse of notation, we extend v_h^s to a function of $\mathcal{L}^1_\kappa(\mathcal{T}_h^s)$ setting $v_h^s(x_i) = 0$ at all the internal nodes (i.e., the nodes of $\mathcal{T}_h^s$ not lying on Γ^s). From the classical trace theorem, we have $|v_h^s|_{H^{1/2}(\Gamma^s)} \leq C|v_h^s|_{H^1(\Omega^s)}$. Let us then bound the $H^1(\Omega^s)$ seminorm of v_h^s. By definition v_h^s is non zero only on those tetrahedrons $T \in \mathcal{T}_h^s$ which are adjacent to the boundary. Let K be one of such tetrahedrons, with $m \geq 1$ faces lying on Γ^s, and let $T_i \in \mathcal{T}_h^{\Gamma^s}$, $i = 1, \ldots, m$ be those faces. Thanks to standard arguments, we can write:

$$|v_h^s|_{H^1(K)}^2 \leq Ch_K^{-1}\|v_h^s\|_{L^2(\partial K)}^2 \leq C\sum_{i=1}^{m} h_{T_i}^{-1}\|v_h^s\|_{L^2(T_i)}^2. \tag{44}$$

For each element K' that share only an edge or a vertex with Γ^s, there is an element K which own a face with that edge or vertex. In this case, by standard arguments, we get

$$|v_h^s|_{H^1(K')}^2 \leq C|v_h^s|_{H^1(K')}^2,$$

and then we can still use (44) to bound $|v_h^s|_{H^1(K')}^2$. Finally, adding the contributions of all elements adjacent to Γ^s, we obtain that

$$|v_h^s|_{H^1(\Omega^s)} \leq C\|v_h^s\|_{h,1/2,\Gamma^s},$$

which implies (43). □

3.2 Estimates on the operator $\mathcal{G}^s$

We now look in more detail at the operator $\mathcal{G}^s$. Thanks to the classical theory of mixed finite elements (see [8]) it is immediate to see that the following lemma holds true.

Lemma 4. *Let $s = 1, \dots, S$; assume that the inf-sup condition* (31) *is satisfied; then*

$$\|\mathcal{G}^s(\phi)\|_{h,1/2,\Gamma^s} \leq C\|\phi\|_{h,1/2,\Gamma^s}, \qquad \forall \phi \in L^2(\Gamma^s), \tag{45}$$

where the norms are the ones defined in (30).

We point out that the norm $\|\cdot\|_{h,1/2,\Gamma^s}$, which is induced by the bilinear form $(u, v) \mapsto \sum_{T\in\mathcal{T}_h^{\Gamma^s}} \int_T h_T^{-1} u\, v\, \mathrm{d}x$, plays the role of a discrete $H^{1/2}(\Gamma^s)$ norm. See also the result of Lemma 3 above.

We observe that Lemmata 4 and 3 trivially imply the continuity of $\mathcal{G}^s$ from $L^2(\Gamma^s)$ (endowed with the norm $\|\cdot\|_{h,1/2,\Gamma^s}$), to $H^{1/2}(\Gamma^s)$. However a stronger result holds, stated in the following theorem.

Theorem 1. *Under the assumptions of §2, $\mathcal{G}^s$ is continuous from $H^{1/2}(\Gamma^s)$ to $H^{1/2}(\Gamma^s)$:*

$$|\mathcal{G}^s(\phi)|_{H^{1/2}(\Gamma^s)} \leq C|\phi|_{H^{1/2}(\Gamma^s)}, \qquad \forall \phi \in H^{1/2}(\Gamma^s). \tag{46}$$

Proof. First, we introduce the Clément interpolant $\phi_I \in V^s_{h\,|\Gamma^s}$ of ϕ (see [14]), for which the same arguments as in [14] give

$$\|\phi - \phi_I\|_{h,1/2,\Gamma^s} \leq C|\phi|_{H^{1/2}(\Gamma^s)}. \tag{47}$$

Moreover, we have the stability property (see Lemma 8 in Appendix B)

$$|\phi_I|_{H^{1/2}(\Gamma^s)} \leq C|\phi|_{H^{1/2}(\Gamma^s)}. \tag{48}$$

Since $\mathcal{G}^s$ is linear and using the triangle inequality, we have

$$|\mathcal{G}^s(\phi)|_{H^{1/2}(\Gamma^s)} \leq |\mathcal{G}^s(\phi - \phi_I)|_{H^{1/2}(\Gamma^s)} + |\mathcal{G}^s(\phi_I)|_{H^{1/2}(\Gamma^s)} = I + II.$$

Making use of Lemma 3, Lemma 4 and (47), we get

$$\begin{aligned} I = |\mathcal{G}^s(\phi - \phi_I)|_{H^{1/2}(\Gamma^s)} &\leq C\|\mathcal{G}^s(\phi - \phi_I)\|_{h,1/2,\Gamma^s} \\ &\leq C\|\phi - \phi_I\|_{h,1/2,\Gamma^s} \\ &\leq C|\phi|_{H^{1/2}(\Gamma^s)}. \end{aligned}$$

Moreover, recalling (18) (that is, $\mathcal{G}^s(\phi_I) = \phi_I$) and then using (48), we have

$$II = |\mathcal{G}^s(\phi_I)|_{H^{1/2}(\Gamma^s)} \leq C|\phi|_{H^{1/2}(\Gamma^s)},$$

which eventually gives (46). □

We are also interested to the case of ϕ that are discontinuous across the faces. In this case $\phi_{|\Gamma^s} \notin H^{1/2}(\Gamma^s)$, and Theorem 1 is useless. However, we have the following result, whose proof is based on the Lemmata of Appendix B.

Theorem 2. *Under the assumptions of §2, for $1 \leq s \leq S$, it holds that*

$$|\mathcal{G}^s(\phi)|^2_{H^{1/2}(\Gamma^s)} \leq C\left(1 + \log\left(\frac{H_{\Omega^s}}{h^{\min}_{\Gamma^s}}\right)\right)\left(H^{-1}_{\Omega^s}|\phi|^2_{L^2(\Gamma^s)} + |\phi|_{L^2(\Gamma^s)}\sum_{r=1}^{R^s}|\phi|_{H^1(\Gamma^s_r)}\right), \tag{49}$$

for any $\phi \in L^2(\Gamma^s)$ such that $\phi_{|\Gamma^s_r} \in H^1(\Gamma^s_r)$ for each $r = 1, \dots, R^s$.

Proof. We set

$$\epsilon := \frac{1}{\log\left(\frac{H_{\Omega^s}}{h_{\Gamma^s}^{\min}}\right)}. \tag{50}$$

If $\epsilon \geq 1/2$ (that is if $e^2 h_{\Gamma^s}^{\min} \geq H_{\Omega^s}$) then (48) easily follows from (43) and (45). Then we only need to consider the case $\epsilon < 1/2$. Still denoting by $\phi_I \in V^s_{h\,|\Gamma^s}$ the Clément interpolant of ϕ, we have the error estimate

$$\|\phi - \phi_I\|_{h,1/2-\epsilon,\Gamma^s} \leq C|\phi|_{H^{1/2-\epsilon}(\Gamma^s)}. \tag{51}$$

Furthermore, Lemma 8 in Appendix B gives

$$|\phi_I|_{H^{1/2}(\Gamma^s)} \leq C(h_{\Gamma^s}^{\min})^{-\epsilon}|\phi|_{H^{1/2-\epsilon}(\Gamma^s)}. \tag{52}$$

Therefore, reasoning as in the proof of Theorem 1

$$|\mathcal{G}^s(\phi)|_{H^{1/2}(\Gamma^s)} \leq |\mathcal{G}^s(\phi - \phi_I)|_{H^{1/2}(\Gamma^s)} + |\mathcal{G}^s(\phi_I)|_{H^{1/2}(\Gamma^s)} = I + II.$$

Making use of Lemma 3, Lemma 4, and (51), we get

$$\begin{aligned} I = |\mathcal{G}^s(\phi - \phi_I)|_{H^{1/2}(\Gamma^s)} &\leq C\|\mathcal{G}^s(\phi - \phi_I)\|_{h,1/2,\Gamma^s} \\ &\leq C\|\phi - \phi_I\|_{h,1/2,\Gamma^s} \\ &\leq C(h_{\Gamma^s}^{\min})^{-\epsilon}\|\phi - \phi_I\|_{h,1/2-\epsilon,\Gamma^s} \\ &\leq C(h_{\Gamma^s}^{\min})^{-\epsilon}|\phi|_{H^{1/2-\epsilon}(\Gamma^s)}. \end{aligned}$$

Moreover, using (52), we have

$$II = |\mathcal{G}^s(\phi_I)|_{H^{1/2}(\Gamma^s)} = |\phi_I|_{H^{1/2}(\Gamma^s)} \leq C(h_{\Gamma^s}^{\min})^{-\epsilon}|\phi|_{H^{1/2-\epsilon}(\Gamma^s)}.$$

Invoking Lemma 9 in Appendix B, we get then

$$\begin{aligned} |\mathcal{G}^s(\phi)|_{H^{1/2}(\Gamma^s)} &\leq I + II \\ &\leq C(H_{\Omega^s})^{\epsilon}(h_{\Gamma^s}^{\min})^{-\epsilon}\epsilon^{-1/2}\left(H_{\Omega^s}^{-1}|\phi|^2_{L^2(\Gamma^s)} + |\phi|_{L^2(\Gamma^s)}\sum_{r=1}^{R^s}|\phi|_{H^1(\Gamma^s_r)}\right)^{1/2}; \end{aligned} \tag{53}$$

recalling (50) we have $(H_{\Omega^s})^{\epsilon}(h_{\Gamma^s}^{\min})^{-\epsilon}\epsilon^{-1/2} = e\left(\log\left(\frac{H_{\Omega^s}}{h_{\Gamma^s}^{\min}}\right)\right)^{1/2}$, which eventually gives (49). □

3.3 Interpolation estimates

Let now $\psi_I \in \Phi_\delta$ be an interpolant of the exact solution $\psi := u_{|\Sigma}$. For every Ω^s ($s = 1, \dots, S$), let $\tilde{u}_I^s \in V_h^s$ be defined as the unique solution of

$$\begin{cases} \tilde{u}_I^s = \mathcal{G}^s(\psi_I) \text{ on } \Gamma^s \\ a_s\left(\tilde{u}_I^s, v_h^s\right) = \int_{\Omega^s} g\, v_h^s \,\mathrm{d}x \qquad \forall v_h^s \in V_h^s \cap H_0^1(\Omega^s). \end{cases} \tag{54}$$

Let $\tilde{u}_I$ be equal to $\tilde{u}_I^s$ in each Ω^s, $s = 1, \dots, S$. It is clear that $\tilde{u}_I \in \mathcal{S}_h$. We are now going to estimate the distance between u and $\tilde{u}_I$.

Using the definition (54) of $\tilde{u}_I^s$ and thanks to the assumption (28), we can apply the usual theory for estimating the error for each Dirichlet problem in Ω^s:

$$|u-\tilde{u}_I^s|_{H^1(\Omega^s)} \le C\left(\inf_{v_h^s\in V_h^s}|u-v_h^s|_{H^1(\Omega^s)} + |u-\tilde{u}_I^s|_{H^{1/2}(\Gamma^s)}\right). \tag{55}$$

It is clear now that the crucial step is to estimate $|u-\tilde{u}_I^s|_{H^{1/2}(\Gamma^s)}$. To this aim we consider a generic $v_h \in V^s_{h\,|\Gamma^s}$ and we write

$$\begin{aligned} |u-\tilde{u}_I^s|_{H^{1/2}(\Gamma^s)} &= |u-\mathcal{G}^s(\psi_I)|_{H^{1/2}(\Gamma^s)} \\ &\le |u-v_h|_{H^{1/2}(\Gamma^s)} + |v_h-\mathcal{G}^s(u)|_{H^{1/2}(\Gamma^s)} \\ &\quad +|\mathcal{G}^s(u)-\mathcal{G}^s(\psi_I)|_{H^{1/2}(\Gamma^s)} \end{aligned} \tag{56}$$

Recalling (18), , we have $v_h = \mathcal{G}^s(v_h)$ and, using Theorem 1, we easily get

$$|v_h-\mathcal{G}^s(u)|_{H^{1/2}(\Gamma^s)} = |\mathcal{G}^s(v_h-u)|_{H^{1/2}(\Gamma^s)} \le C|u-v_h|_{H^{1/2}(\Gamma^s)}.$$

On the other hand, using Theorem 2, we obtain

$$\begin{aligned} &|\mathcal{G}^s(u)-\mathcal{G}^s(\psi_I)|_{H^{1/2}(\Gamma^s)} = |\mathcal{G}^s(u-\psi_I)|_{H^{1/2}(\Gamma^s)} \\ &\quad\le C\left(1+\log\left(\frac{H_{\Omega^s}}{h^{\min}_{\Gamma^s}}\right)\right)^{1/2} \\ &\qquad\cdot\left(H^{-1}_{\Omega^s}|u-\psi_I|^2_{L^2(\Gamma^s)} + |u-\psi_I|_{L^2(\Gamma^s)}\sum_{r=1}^{R^s}|u-\psi_I|_{H^1(\Gamma^s_r)}\right)^{1/2}. \end{aligned}$$

Substituting this back in (56), and then in (55), we get

$$\begin{aligned} |u-\tilde{u}_I^s|_{H^1(\Omega^s)} \le C\Bigg[&\inf_{v_h^s\in V_h^s}|u-v_h^s|_{H^1(\Omega^s)} + \left(1+\log\left(\frac{H_{\Omega^s}}{h^{\min}_{\Gamma^s}}\right)\right)^{1/2} \\ &\cdot\left(H^{-1}_{\Omega^s}|u-\psi_I|^2_{L^2(\Gamma^s)} + |u-\psi_I|_{L^2(\Gamma^s)}\sum_{r=1}^{R^s}|u-\psi_I|_{H^1(\Gamma^s_r)}\right)^{1/2}\Bigg]. \end{aligned} \tag{57}$$

We now remark that $u \notin \mathcal{S}_h$ while $\tilde{u}_I \notin H^1_0(\Omega)$. Therefore, in order to estimate $u-\tilde{u}_I$ in $L^2(\Omega)$, we can neither use the usual Poincaré inequality nor apply (42), but we have to do it directly. Using (40) we have immediately

$$\|u-\tilde{u}_I\|^2_{L^2(\Omega)} \le CL^2\sum_{s=1}^{S}|u-\tilde{u}_I^s|^2_{H^1(\Omega^s)} + \sum_{s=1}^{S}H_{\Omega^s}\left\|\frac{1}{|\Gamma^s|}\int_{\Gamma^s}u-\tilde{u}_I^s\right\|^2_{L^2(\Gamma^s)}. \tag{58}$$

Now, recalling that $\tilde{u}_I^s = \mathcal{G}^s(\psi_I)$ on Γ^s and $\int_{\Gamma^s}\mathcal{G}^s(\psi_I) = \int_{\Gamma^s}\psi_I$ (thanks to (15) and (29)), we get

$$\left\|\frac{1}{|\Gamma^s|}\int_{\Gamma^s}u-\tilde{u}_I^s\right\|^2_{L^2(\Gamma^s)} = \left\|\frac{1}{|\Gamma^s|}\int_{\Gamma^s}u-\psi_I\right\|^2_{L^2(\Gamma^s)} \le \|u-\psi_I\|^2_{L^2(\Gamma^s)}. \tag{59}$$

Furthermore, thanks to (36), that is,

$$\sum_{s=1}^{S}H_{\Omega^s}\|u-\psi_I\|^2_{L^2(\Gamma^s)} \le CL^2\sum_{s=1}^{S}H^{-1}_{\Omega^s}|u-\psi_I|^2_{L^2(\Gamma^s)},$$

we have

$$\sum_{s=1}^{S} H_{\Omega^s} \left\| \frac{1}{|\Gamma^s|} \int_{\Gamma^s} u - \tilde{u}_I^s \right\|^2_{L^2(\Gamma^s)} \leq C L^2 \sum_{s=1}^{S} H_{\Omega^s}^{-1} |u - \psi_I|^2_{L^2(\Gamma^s)} . \tag{60}$$

Collecting (57), (58), and (60) and recalling that $\psi_I \in \Phi_\delta$ is an arbitrary approximation of $u_{|\Sigma}$, we finally get the following approximation estimate.

Lemma 5. *Under the assumptions of §2, let u be the exact solution of* (1) *and let $\tilde{u}_I$ be constructed as in (54). Then we have*

$$\begin{aligned} \|u - \tilde{u}_I\|_V^2 \leq C \sum_{s=1}^{S} \left(\inf_{v_h^s \in V_h^s} |u - v_h^s|^2_{H^1(\Omega^s)} \right) + C \inf_{\phi_\delta \in \Phi_\delta} \left\{ \sum_{s=1}^{S} \left(1 + \log\left(\frac{H_{\Omega^s}}{h_{\Gamma^s}^{\min}} \right) \right) \right. \\ \left. \cdot \left(H_{\Omega^s}^{-1} |u - \phi_\delta|^2_{L^2(\Gamma^s)} + |u - \phi_\delta|_{L^2(\Gamma^s)} \sum_{r=1}^{R^s} |u - \phi_\delta|_{H^1(\Gamma_r^s)} \right) \right\}. \end{aligned} \tag{61}$$

3.4 Error estimates

We are now ready to analyze problem (12) and derive the abstract error estimate.

Theorem 3. *Under the assumptions of §2, let u be the exact solution of* (1) *and u_h be the solution of* (12)*. Then we have*

$$\begin{aligned} \|u - u_h\|_V^2 \leq C \inf_{v_h \in V_h} \left\{ |u - v_h|_V^2 \right\} + C \sum_{s=1}^{S} \inf_{\mu_h^s \in M_h^s} \left\{ \sum_{r=1}^{R^s} \left\| \frac{\partial u}{\partial \mathbf{n}^s} - \mu_h^s \right\|^2_{M_r^s} \right\} \\ + C \inf_{\phi_\delta \in \Phi_\delta} \left\{ \sum_{s=1}^{S} \left(1 + \log\left(\frac{H_{\Omega^s}}{h_{\Gamma^s}^{\min}} \right) \right) \right. \\ \left. \cdot \left(H_{\Omega^s}^{-1} |u - \phi_\delta|^2_{L^2(\Gamma^s)} + |u - \phi_\delta|_{L^2(\Gamma^s)} \sum_{r=1}^{R^s} |u - \phi_\delta|_{H^1(\Gamma_r^s)} \right) \right\}. \end{aligned} \tag{62}$$

Proof. With our assumptions, it is easy to see that problem (12) has a unique solution. We now set $e_h := \tilde{u}_I - u_h \in \mathcal{S}_h$, where $\tilde{u}_I$ is defined in (54). Using the definition (11) and adding and subtracting u we have:

$$|e_h|_V^2 = a(e_h, e_h) = a(\tilde{u}_I - u, e_h) + a(u - u_h, e_h) =: I + II. \tag{63}$$

Using (12) and integrating $a(u, e_h)$ by parts in each Ω^s we obtain

$$\begin{aligned} II = a(u - u_h, e_h) &= \sum_{s=1}^{S} \int_{\Omega^s} g\, e_h^s \,\mathrm{d}x + \sum_{s=1}^{S} \int_{\Gamma^s} \frac{\partial u}{\partial \mathbf{n}^s} e_h^s \,\mathrm{d}x - \sum_{s=1}^{S} \int_{\Omega^s} g\, e_h^s \,\mathrm{d}x \\ &= \sum_{s=1}^{S} \int_{\Gamma^s} \frac{\partial u}{\partial \mathbf{n}^s} e_h^s \,\mathrm{d}x. \end{aligned} \tag{64}$$

As $e_h \in \mathcal{S}_h$, using assumption (32) there will be a mother $\eta_\delta \in \Phi_\delta$ with $\gamma_1 \|\eta_\delta\|_{\Phi^\star} \leq |e_h|_V$, such that $\mathcal{G}(\eta_\delta) = e_h|_\Sigma$. Hence the continuity of $\partial u / \partial \mathbf{n}^s$ on the interfaces between subdomains, and the fact that η_δ is single-valued on the skeleton Σ yield

$$II = \sum_{s=1}^{S} \int_{\Gamma^s} \frac{\partial u}{\partial \mathbf{n}^s} (e_h^s - \eta_\delta) \,\mathrm{d}x = \sum_{s=1}^{S} \int_{\Gamma^s} \frac{\partial u}{\partial \mathbf{n}^s} (\mathcal{G}^s(\eta_\delta) - \eta_\delta) \,\mathrm{d}x. \tag{65}$$

We can now use the definition of $\mathcal{G}^s$ (see (16)) and subtract from $\partial u/\partial \mathbf{n}^s$ any function $\mu_h^s \in M_h^s$. In particular, to fix the ideas, we take for instance the best approximation of $\partial u/\partial \mathbf{n}^s$ in M_h^s, and denote it by ν_I^s. In this way we obtain

$$\begin{aligned} II &= \sum_{s=1}^{S} \int_{\Gamma^s} \big(\frac{\partial u}{\partial \mathbf{n}^s} - \nu_I^s\big)(\mathcal{G}^s(\eta_\delta) - \eta_\delta))\,\mathrm{d}x \\ &= \sum_{s=1}^{S} \sum_{r=1}^{R^s} \int_{\Gamma_r^s} \big(\frac{\partial u}{\partial \mathbf{n}^s} - \nu_I^s\big)(\mathcal{G}^s(\eta_\delta) - \eta_\delta))\,\mathrm{d}x. \end{aligned} \tag{66}$$

We recall that $\mathcal{G}^s(\eta_\delta) = e_h^s$ on Γ^s. We also point out that (thanks to (29)) we can assume that the mean value of $\partial u/\partial \mathbf{n}^s - \nu_I^s$ on each face Γ_r^s is zero, so that we can use the $H^{1/2}$-seminorm of e_h^s and η_δ instead of the norm in the estimate. Then, recalling (32) for η_δ, that is.

$$\sum_{s=1}^{S} \sum_{r=1}^{R^s} |\eta_\delta|^2_{H^{1/2}(\Gamma_r^s)} \le \gamma_1^{-2} \, |e_h|_V^2 \,,$$

and using the Cauchy-Schwarz inequality, as well as the standard trace inequality in each Ω^s for e_h^s, we obtain

$$\begin{aligned} II &\le \sum_{s=1}^{S} \sum_{r=1}^{R^s} \left\| \frac{\partial u}{\partial \mathbf{n}^s} - \nu_I^s \right\|_{M_r^s} \Big(|e_h^s|_{H^{1/2}(\Gamma_r^s)} + |\eta_\delta|_{H^{1/2}(\Gamma_r^s)} \Big) \\ &\le C \Big(\sum_{s=1}^{S} \sum_{r=1}^{R^s} \left\| \frac{\partial u}{\partial \mathbf{n}^s} - \nu_I^s \right\|^2_{M_r^s} \Big)^{1/2} |e_h|_V \,. \end{aligned} \tag{67}$$

Since

$$I = a\,(\tilde{u}_I - u, e_h) \le |\tilde{u}_I - u|_V \; |e_h|_V \,, \tag{68}$$

we obtain from (63), (67), and (68)

$$|e_h|_V \le C \left(\sum_{s=1}^{S} \sum_{r=1}^{R^s} \left\| \frac{\partial u}{\partial \mathbf{n}^s} - \nu_I^s \right\|^2_{M_r^s} \right)^{1/2} + C \, |u - \tilde{u}_I|_V \,. \tag{69}$$

Using the triangle inequality and Corollary 1 (since $e_h \in \mathcal{S}_h$) we get

$$\|u - u_h\|_V \le \|u - \tilde{u}_I\|_V + \|e_h\|_V \le \|u - \tilde{u}_I\|_V + C\,|e_h|_V, \tag{70}$$

that, together with (69) and (61), eventually gives (62). □

4 Examples and Remarks

In this section we want to show two examples of finite element discretizations that satisfy the abstract assumptions of §2, and derive the corresponding error bounds in terms of suitable powers of the mesh-sizes. For each example, we shall show that the two inf-sup conditions (31) and (33) hold. The former inf-sup will be proved by a constructive argument, while, for the latter, we shall assume that the mesh $\mathcal{T}_\delta^\Sigma$ is coarser than the mesh induced on Σ by $\mathcal{T}_h^s$, and then we shall make use of the classical argument of [2].

4.1 First example

Under the assumptions on the subdomain subdivision and on the triangulations made in §2, we consider here the following choice of the finite element spaces Φ_δ, V_h^s and M_h^s, $s = 1, \ldots, S$:

$$\Phi_\delta := \left\{\phi \in \Phi^\star : \phi_{|T} \in \mathbb{P}_2(T),\, T \in \mathcal{T}_\delta^\Sigma\right\}, \tag{71}$$

$$V_h^s := \{v^s \in V^s \text{ such that } v^s_{|K} \in \mathbb{P}_2(K),\, K \in \mathcal{T}_h^s\}, \tag{72}$$

and

$$M_h^s := \left\{\begin{array}{l} \mu \in L^2(\Gamma^s) \text{ such that } \mu_{|T} \in \mathbb{P}_1(T),\, T \in \mathcal{T}_h^{\Gamma^s} \text{ and} \\ \quad \mu_{|\Gamma_r^s} \text{ is continuous, } \forall r = 1, \ldots, R^s, s = 1, \ldots, S \end{array}\right\}. \tag{73}$$

We point out that Φ_δ and M_h^s are made of functions that are continuous on the faces Γ_r^s, but discontinuous across two adjacent faces, while the V_h^s are made of continuous functions (within each subdomain Ω^s).

We can now discuss the various abstract assumptions that have been made in §2. To start with, condition (29) is obviously satisfied. Similarly, (28) holds as shown, for instance, in [6].

In order for the *inf-sup* condition (31) to hold, we assume a very weak condition on each mesh $\mathcal{T}_h^{\Gamma^s}$. We denote by $\mathcal{T}_h^{\Gamma_r^s} \subset \mathcal{T}_h^{\Gamma^s}$ the set of the elements of $\mathcal{T}_h^{\Gamma^s}$ lying on the face Γ_r^s, and by $\check{\mathcal{T}}_h^{\Gamma_r^s}$ the set of those elements $T \in \mathcal{T}_h^{\Gamma_r^s}$ that have their three vertices on $\partial\Gamma_r^s$ (for example, the set of the gray elements in Figure 1). We say that two triangles are *adjacent* when they share an edge. We assume that

For each triangle $T \in \check{\mathcal{T}}_h^{\Gamma_r^s}$ there exists an adjacent triangle not belonging to $\check{\mathcal{T}}_h^{\Gamma_r^s}$. (74)

The assumption above is always satisfied unless the decomposition is *absurdly coarse*.

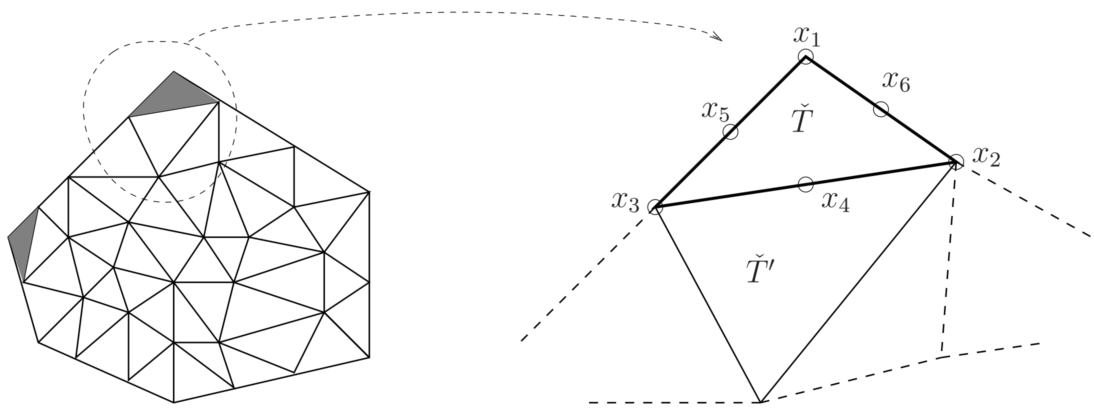

Figure 1: Example of a mesh on a face Γ_r^s: on the left, the two elements in gray belong to $\check{\mathcal{T}}_h^{\Gamma_r^s}$; one of them is shown on the right, and a local numbering of the degrees of freedom is introduced.

Lemma 6. *Let* M_h^s *and* V_h^s *be constructed as in* (72) *and* (73)*, respectively. Under the assumption (74), the inf-sup condition* (31) *holds.*

Proof. We are going to check the inf-sup condition *face by face*. Then, for $r = 1, \ldots, R^s$, and $s = 1, \ldots, S$, given $\mu_h \in M^s_{h|\Gamma_r^s}$, we are going to construct a $v_h = v_h(\mu_h) \in V^s_{h|\Gamma_r^s}$ which is null on $\partial\Gamma_r^s$ and such that

$$\int_{\Gamma_r^s} v_h\mu_h \geq C_1\|\mu_h\|^2_{-1/2,h,\Gamma_r^s}, \tag{75}$$

$$\|v_h\|_{1/2,h,\Gamma_r^s} \leq C_2\|\mu_h\|_{-1/2,h,\Gamma_r^s}. \tag{76}$$

Let h_I be the piecewise linear, continuous, null on $\partial\Gamma_r^s$, Clément interpolant of the piecewise constant mesh-size function assuming on each triangle T the value h_T. Given $T \in \mathcal{T}_h^{\Gamma_r^s}$ we denote by ω_T the patch of the neighboring elements on Γ_r^s (i.e., the union of those elements belonging to $\mathcal{T}_h^{\Gamma_r^s}$, sharing a vertex or an edge with T). We recall (see [14]) that, since $h_I \geq 0$ we have

$$C_1 \min_{T'\in\omega_T} h_{T'} \leq \|h_I\|_{L^\infty(T)} \leq C_2 \max_{T'\in\omega_T} h_{T'}, \qquad \forall T \in \mathcal{T}_h^{\Gamma_r^s}\backslash\check{\mathcal{T}}_h^{\Gamma_r^s},$$

while h_I is null on $T \in \check{\mathcal{T}}_h^{\Gamma_r^s}$; moreover, for the shape regularity of the mesh $\mathcal{T}_h^{\Gamma_r^s}$,

$$C_1 \max_{T'\in\omega_T} h_{T'} \leq h_T \leq C_2 \min_{T'\in\omega_T} h_{T'}\ , \qquad \forall T \in \mathcal{T}_h^{\Gamma_r^s}$$

whence

$$C_1 h_T \leq \|h_I\|_{L^\infty(T)} \leq C_2 h_T, \qquad \forall T \in \mathcal{T}_h^{\Gamma_r^s}\backslash\check{\mathcal{T}}_h^{\Gamma_r^s}. \tag{77}$$

Define $\dot{v}_h := h_I\mu_h$; clearly, $\dot{v}_h$ is piecewise quadratic, continuous and null on $\partial\Gamma_r^s$. Moreover, thanks to (77),

$$\|\dot{v}_h\|_{1/2,h,\Gamma_r^s} \leq \left(\sum_{T\in\mathcal{T}_h^{\Gamma_r^s}} h_T^{-1}\|h_I\|^2_{L^\infty(T)}\|\mu_h\|^2_{L^2(T)}\right)^{1/2} \leq C\|\mu_h\|_{-1/2,h,\Gamma_r^s}; \tag{78}$$

and, invoking a standard scaling argument,

$$\int_T \dot{v}_h\mu_h = \int_T h_I\mu_h^2 \geq Ch_T\|\mu_h\|^2_{L^2(T)}, \quad \forall T \in \mathcal{T}_h^{\Gamma_r^s}\backslash\check{\mathcal{T}}_h^{\Gamma_r^s}, \tag{79}$$

while $\dot{v}_h$ is null on the elements belonging to $\check{\mathcal{T}}_h^{\Gamma_r^s}$.

We consider now an element $\check{T} \in \check{\mathcal{T}}_h^{\Gamma_r^s}$ and use the notation of Figure 1 (right part). Recall the assumption (74), which implies the existence of an adjacent element (denoted $\check{T}'$) not belonging to $\check{\mathcal{T}}_h^{\Gamma_r^s}$. Hence we can define $\check{v}_{\check{T}} \in V^s_{h|\Gamma_r^s}$ null outside the two elements $\check{T}$ and $\check{T}'$ (and hence, by continuity, null on $\left(\partial\check{T}\cup\partial\check{T}'\right)\setminus\left(\partial\check{T}\cap\partial\check{T}'\right)$), and such that $\check{v}_{\check{T}}(x_4) = h_{\check{T}}\mu_h(x_1)$. With the usual scaling arguments

$$h_{\check{T}}^{-1/2}\|\check{v}_{\check{T}}\|_{L^2(\check{T}\cup\check{T}')} \leq Ch_{\check{T}}^{1/2}\|\mu_h\|_{L^2(\check{T})}, \tag{80}$$

and

$$\begin{aligned}\int_{\check{T}\cup\check{T}'} \check{v}_{\check{T}}\mu_h &\geq C_1 h_{\check{T}} h_{\check{T}}^{n-1}\mu_h^2(x_1) - C_2 h_{\check{T}} h_{\check{T}}^{n/2}\mu_h(x_1)\|\mu_h\|_{L^2(\check{T}')}\\ &\geq C_3 h_{\check{T}}\|\mu_h\|^2_{L^2(\check{T})} - C_4 h_{\check{T}}\|\mu_h\|^2_{L^2(\check{T}')}.\end{aligned} \tag{81}$$

We can repeat the same construction for all $\check{T} \in \check{\mathcal{T}}_h^{\Gamma_r^s}$. Observe that the element $\check{T}'$ in the estimate above can be the same for different elements of $\check{\mathcal{T}}_h^{\Gamma_r^s}$, but their number is uniformly bounded because of the shape regularity assumptions in § 2.3. Our assumption is that

$$v_h := \dot{v}_h + \beta \sum_{\check{T}\in\check{\mathcal{T}}_h^{\Gamma_r^s}} \check{v}_{\check{T}}.$$

From (78) and (80) we get

$$\|v_h\|_{1/2,h,\Gamma_r^s} \le (C_1 + C_2\beta)\|\mu_h\|_{-1/2,h,\Gamma_r^s},$$

while from (79) and (81) we get

$$\int_{\Gamma_r^s} v_h\mu_h \ge \min\{C_1 - C_2\beta, C_3\beta\}\|\mu_h\|^2_{-1/2,h,\Gamma_r^s}.$$

For a suitable value of β we obtain therefore (75)–(76). □

We consider now the bound on the mother (32). We shall apply Proposition 1. Consider a single internal face Γ_r^s. We recall that, for the definition (24), we have

$$\inf_{\phi\in H^{1/2}(\Gamma_r^s)\backslash\{0\}} \sup_{\mu\in H^{-1/2}(\Gamma_r^s)\backslash\{0\}} \frac{\langle\mu,\phi\rangle}{\|\mu\|_{M_r^s}\left(H_{\Gamma_r^s}^{-1}\|\phi\|^2_{L^2(\Gamma_r^s)} + |\phi|^2_{H^{1/2}(\Gamma_r^s)}\right)^{1/2}} = 1.$$

Therefore, assuming the mesh $\mathcal{T}_\delta^{\Gamma_r^s}$ "coarse enough" compared with $\mathcal{T}_h^{\Gamma_r^s}$, and applying the technique of [2], one gets

$$\inf_{\phi_\delta\in\mathcal{L}_2^1(\mathcal{T}_\delta^{\Gamma_r^s})\backslash\{0\}} \sup_{\mu_h^s\in M_h^s\backslash\{0\}} \frac{\int_{\Gamma_r^s}\phi_\delta\,\mu_h^s\,\mathrm{d}x}{\|\mu_h^s\|_{M_r^s}\left(H_{\Gamma_r^s}^{-1}\|\phi_\delta\|^2_{L^2(\Gamma_r^s)} + |\phi_\delta|^2_{H^{1/2}(\Gamma_r^s)}\right)^{1/2}} \ge \gamma_1' > 0, \tag{82}$$

with γ_1' independent of the meshes and the face Γ_r^s, under the assumptions of §2. In particular, recalling that $\mathcal{T}_\delta^{\Gamma_r^s}$ is supposed to be quasi-uniform with mesh-size $\delta_{\Gamma_r^s}$, there exists a constant $\rho > 1$ such that if $\delta_{\Gamma_r^s} \ge \rho \max_{T\in\mathcal{T}_h^{\Gamma_r^s}} h_T$, then (82) holds; ρ is in fact independent of the face Γ_r^s, still under the assumptions of §2.

We can collect the previous results, together with the abstract error estimates of the previous section, in the following theorem.

Theorem 4. *Under the assumptions of §2, if the discrete spaces Φ_δ, M_h^s and V_h^s are defined in* (71), (73) *and* (72), *and if (82) holds, then*

$$\|u-u_h\|_V^2 \le C\sum_{s=1}^S\left(\sum_{K\in\mathcal{T}_h^s}\left(h_K^4|u|^2_{H^3(K)}\right) + \left(1+\log\left(\frac{H_{\Omega^s}}{h_{\Gamma^s}^{\min}}\right)\right)\sum_{r=1}^{R^s}\delta_{\Gamma_r^s}^5|u|^2_{H^3(\Gamma_r^s)}\right) \tag{83}$$

Proof. For the assumptions and results of this section, we can use Theorem 3: then, we have to estimate the right hand side of (62) in terms of suitable powers of the mesh-size. If we introduce the nodal interpolant $u_I \in V_h^s$ of u and $\nu_I \in M_h^s$ of $\partial u/\partial\mathbf{n}^s$, by the usual approximation estimates we get

$$|u-u_I|^2_{H^1(\Omega^s)} + \sum_{r=1}^{R^s}\left\|\frac{\partial u}{\partial\mathbf{n}^s} - \nu_I\right\|^2_{M_r^s} \le C\sum_{K\in\mathcal{T}_h^s} h_K^4|u|_{H^3(K)};$$

similarly denoting by $\psi_I \in \Phi_\delta$ the nodal interpolant of $u_{|\Sigma}$, we have

$$H_{\Omega^s}^{-1}|u-\psi_I|^2_{L^2(\Gamma^s)} + |u-\psi_I|_{L^2(\Gamma^s)} \sum_{r=1}^{R^s} |u-\psi_I|_{H^1(\Gamma^s_r)} \le \sum_{r=1}^{R^s} \delta^5_{\Gamma^s_r} |u|^2_{H^3(\Gamma^s_r)}.$$

□

Remark 1. *We observe that a* $\log\left(\frac{H_{\Omega^s}}{h^{\min}_{\Gamma^s}}\right)$ *appears as a factor in the term involving the approximation properties of* Φ_δ*, though it is, roughly speaking, compensated by the higher order accuracy we have for that term, when the exact solution* u *is smooth enough. Moreover, the higher order accuracy in the approximation of* Φ_δ *allows to actually take the mesh* $\mathcal{T}^\Sigma_\delta$ *coarser, face by face, than the mesh* $\mathcal{T}^s_h$*, and therefore satisfy the* inf-sup *condition (82) without losing in accuracy.*

Remark 2. *This framework can be extended by taking* k*-degree elements for* V^s_h $(k \ge 2)$ *and* $(k-1)$*-degree, discontinuous across two different faces, elements for the multiplier spaces* M^s_h*; for* Φ_δ*,* k'*-elements on a coarse-enough mesh (quasi-uniform on each face) are allowed, for any* $k' \ge 1$*. The error analysis of this section can be adjusted in a straightforward way. The case of* Φ_δ *made of global polynomial within each face (similarly to what proposed in [16]), which may give advantages in the evaluation of (15) and (16), can be considered as well.*

4.2 Second example

We discuss a now a second possible choice of the finite element spaces. Given an integer $k \ge 1$, we consider $(k-1)$-degree fully discontinuous multiplier spaces M^s_h. Counting the degrees of freedom, it is clear that now k-degree continuous elements as V^s_h are too poor in view of the inf-sup condition (31); therefore we have to enrich such a V^s_h. We shall show that a simple stabilization of the problem, made by adding suitable *boundary bubbles* to V^s_h, leads to optimal convergence properties and, at the same time, provides a very easy implementation. This is reminiscent of what has been done for instance in [3], [9], [12], and [10], but simpler and more effective.

More in details, our choice here is

$$\Phi_\delta := \left\{\phi \in \Phi^\star : \phi_{|T} \in \mathbb{P}_k(T),\, T \in \mathcal{T}^\Sigma_\delta\right\}, \tag{84}$$

and

$$M^s_h := \left\{\mu \in L^2(\Gamma^s) \text{ such that } \mu_{|T} \in \mathbb{P}_{k-1}(T),\, T \in \mathcal{T}^{\Gamma^s}_h\right\}; \tag{85}$$

We point out that Φ_δ is made of functions continuous inside the faces Γ^s_r, while M^s_h is made of functions that are, a priori, totally discontinuous from one element to another.

The choice of each V^s_h will be slightly more elaborate. We set

$$V^s_h := \{v^s \in V^s \text{ such that } v^s_{|K} \in \mathcal{P}_K,\, \forall K \in \mathcal{T}^s_h\}, \tag{86}$$

with $\mathcal{P}_K$ to be chosen. For each tetrahedron $K \in \mathcal{T}^s_h$ with no faces belonging to Γ^s we take $\mathcal{P}_K := \mathbb{P}_k$. If instead K has a face f on Γ^s we consider the cubic function b_f on K that vanishes on the three remaining internal faces of K, and we augment the space $\mathbb{P}_k$ with the bubble space B^f_{k+2} obtained multiplying b_f times the functions in $\mathcal{Q}_f \equiv \mathbb{P}_{k-1}(f)$ (that is the space of polynomials of degree $\le k-1$ on f: remember that the face f will be one of the triangles $T \in \mathcal{T}_h^{\Gamma^s}$). If K has another face on Γ^s we repeat the operation, further augmenting the space $\mathbb{P}_k$. In summary

$$\mathcal{P}_K := \mathbb{P}_k(K) + \{\bigoplus_{f\subset\Gamma^s} B^f_{k+2}\} \equiv \mathbb{P}_k + \{\bigoplus_{f\subset\Gamma^s} b_f\mathbb{P}_{k-1}(f)\}. \tag{87}$$

We note that $\bigoplus b_f\mathbb{P}_{k-1}(f)$ is a direct sum, but its sum with $\mathbb{P}_k(K)$ is not direct whenever $k \geq 3$. This however will not be a problem for the following developments.

We can now turn to the various abstract assumptions that have been made in §2. As before, (29) is obviously satisfied, and (28) is proved, for instance, in [6]. We consider then the *inf-sup* condition (31).

Lemma 7. *Let M_h^s and V_h^s be constructed as in* (84) *and in* (86)*, respectively. Then the inf-sup condition* (31) *holds.*

Proof. For every $\mu_h^s \in M_h^s$ we construct $v_h^s \in V_h^s$ as

$$v_h^s = \sum_{T\in\mathcal{T}_h^{\Gamma^s}} h_T\, b_T\, \mu_h^s \tag{88}$$

where as before b_T is the cubic function on K (the tetrahedron having T as one of its faces) vanishing on the other three faces of K and having mean value 1 on T. It is not too difficult to check that

$$\|\mu_h^s\|_{h,-1/2,\Gamma^s}\ \|v_h^s\|_{h,1/2,\Gamma^s} \leq C \int_{\mathcal{T}_h^{\Gamma^s}} v_h^s\mu_h^s \tag{89}$$

that is precisely the *inf-sup* condition (31) that we need. □

For what concern the *inf-sup* on the mother, it is clear that if $\mathcal{T}_\delta^\Sigma$ is "coarse enough" on each face, compared with the meshes of the two sub-domains having that face in common, then we can reason as in §4.2, getting (33) and then (32). Finally we have, still reasoning as in §4.2, the error estimate stated below.

Theorem 5. *Under the assumptions of §2, for the discrete spaces Φ_δ, M_h^s and V_h^s defined in* (84)*,* (85) *and* (86) *with* (87)*, and if* (32) *holds, then*

$$\begin{aligned}\|u-u_h\|_V^2 \leq C\sum_{s=1}^{S}\Bigg(&\sum_{K\in\mathcal{T}_h^s}\left(h_K^{2k}|u|^2_{H^{k+1}(K)}\right)\\ &+\left(1+\log\left(\frac{H_{\Omega^s}}{h_{\Gamma^s}^{\min}}\right)\right)\sum_{r=1}^{R^s}\delta_{\Gamma_r^s}^{2k+1}|u|^2_{H^{k+1}(\Gamma_r^s)}\Bigg)\end{aligned} \tag{90}$$

We end this section with some observations on the actual implementation of the method when the bubble stabilization (87) is used.

Indeed, let us see how the computation of the generation operators $\mathcal{G}^s$ can be performed in practice. Assume that we are given a function ϕ in, say, $L^2(\Gamma^s)$. We recall that, to compute $\mathcal{G}^s(\phi) := \widetilde{v_h^s}$, we have to find the pair $(\widetilde{v_h^s}, \widetilde{\mu_h^s}) \in V^s_{h\,|\Gamma^s} \times M_h^s$ such that

$$\int_{\Gamma^s} (\phi - \widetilde{v_h^s})\,\mu_h^s\,\mathrm{d}x = 0 \qquad \forall \mu_h^s \in M_h^s, \tag{91}$$

$$\sum_{T\in\mathcal{T}_h^{\Gamma^s}} \int_T h_T^{-1}(\phi - \widetilde{v_h^s})\, v_h^s \, \mathrm{d}x + \int_{\Gamma^s} \widetilde{\mu_h^s}\, v_h^s \, \mathrm{d}x = 0 \qquad \forall v_h^s \in V^s_{h\,|\Gamma^s}. \tag{92}$$

We also recall that, with the choice (87), the space $V^s_{h\,|\Gamma^s}$ can be written as $V^s_{h\,|\Gamma^s} = \mathcal{L}^1_k(\mathcal{T}_h^{\Gamma^s}) + B_{k+2}(\mathcal{T}_h^{\Gamma^s})$ where $\mathcal{L}^1_k(\mathcal{T}_h^{\Gamma^s})$ is, as before, the space of continuous piecewise polynomials of degree k on the mesh $\mathcal{T}_h^{\Gamma^s}$, and $B_{k+2}(\mathcal{T}_h^{\Gamma^s})$ is the space of bubbles of degree $k+2$, always on $\mathcal{T}_h^{\Gamma^s}$. In order to write it as a *direct sum* we introduce the space

$$W^s = \{v_h^s \in V^s_{h\,|\Gamma^s} \text{ such that } \int_{\Gamma^s} v_h^s\, \mu_h^s \mathrm{d}x = 0 \;\forall \mu_h^s \in M_h^s\}. \tag{93}$$

We can decompose $V^s_{h\,|\Gamma^s}$ as $W^s \oplus B_{k+2}(\mathcal{T}_h^{\Gamma^s})$. Since $V^s_{h\,|\Gamma^s}$ is spanned by the usual finite element basis functions and $B_{k+2}(\mathcal{T}_h^{\Gamma^s})$ is formed by bubbles, then W^s admits a local basis. We can then split *in a unique way* $\widetilde{v_h^s} = \widetilde{w} + \widetilde{b}$ with $\widetilde{w} \in W^s$ and $\widetilde{b}$ in $B_{k+2}(\mathcal{T}_h^{\Gamma^s})$. It is now clear that $\widetilde{b}$ can be computed immediately from (91) that becomes:

$$\int_{\Gamma^s} (\phi - \widetilde{b})\, \mu_h^s \, \mathrm{d}x = 0 \qquad \forall \mu_h^s \in M_h^s. \tag{94}$$

Once $\widetilde{b}$ is known, one can compute $\widetilde{w}$ from (92) that easily implies

$$\sum_{T\in\mathcal{T}_h^{\Gamma^s}} \int_T h_T^{-1}(\phi - \widetilde{w})\, w, \mathrm{d}x = \sum_{T\in\mathcal{T}_h^{\Gamma^s}} \int_T h_T^{-1}\widetilde{b}\, w, \mathrm{d}x \qquad \forall w \in W^s. \tag{95}$$

In this way the saddle point problem (91)-(92) splits into two smaller subproblems, each with a symmetric and positive definite matrix. In particular (94) can be solved element by element, so that (95) is the only system of a relevant size that has to be solved.

A Fractional order Sobolev and Besov spaces.

Spaces of fractional order are required for dealing with the mothers, with the traces of functions on Σ and with the multipliers appearing in the definition of the operators $\mathcal{G}^s$. For example, the space of *continuous* mothers Φ, defined in (3), is naturally endowed with an $H^{1/2}$ topology. Roughly speaking, we can think of Φ as a sort of $H^{1/2}(\Sigma)$. However, this topology is too strong for our discrete space Φ_δ, which is a non-conforming approximation of Φ. In order to carry out a sharp error analysis, we make use of Besov spaces; in particular, the space $B^{1/2}_{2,\infty}$ will be the weaker replacement of $H^{1/2}$. The space $B^{1/2}_{2,\infty}$ is of order 1/2 (as is $H^{1/2}$ itself) but it contains discontinuous functions (as the ones in Φ_δ). This space is in between $H^{1/2-\epsilon}$ and $H^{1/2}$, for all positive ϵ.

In this appendix we give the definitions and some properties of (fractional order) Sobolev and Besov spaces, and some notions of interpolation theory between function spaces that will be needed in Appendix B.

Let $\omega \subset \mathbb{R}^n$ be a regular manifold of dimension $d \le n$ ($d = n-1$ or $d = n$ are the two cases of interest). Given α such that $0 < \alpha < 1$, the Sobolev space $H^\alpha(\omega)$ is endowed with the seminorm and norm

$$\begin{aligned} |v|^2_{H^\alpha(\omega)} &:= \iint_{\omega\times\omega} \frac{|v(x)-v(y)|^2}{|x-y|^{d+2\alpha}}\, dxdy \\ \|v\|^2_{H^\alpha(\omega)} &:= \|v\|^2_{L^2(\omega)} + |v|^2_{H^\alpha(\omega)}; \end{aligned} \tag{96}$$

see, for more details, [15, (1.3.3.3)] or [20, (4.4.1/8)]. As we shall see later, an equivalent definition can be given making use of the interpolation theory. On $H^0(\omega) \equiv L^2(\omega)$, we define the seminorm

$$|v|_{L^2(\omega)} := \|v - \bar{v}\|_{L^2(\omega)},$$

$\bar{v}$ denoting the mean value of v on ω, while on $H^1(\omega)$ the usual seminorm is

$$|v|_{H^1(\omega)} := \|\nabla v\|_{L^2(\omega)}.$$

Spaces of negative order are defined by duality (see [18]). In particular, in our analysis we make use of $H^{-1/2}$, which is the dual of $H^{1/2}$.

Let ω be the image of a reference manifold $\widehat{\omega}$ through a one-to-one map B_ω (bounded with bounded Jacobian), and let $\widehat{v}$ be the pullback on $\widehat{\omega}$ of a function $v : \omega \to \mathbb{R}$; passing from one domain to the other, the seminorms $|\cdot|_{H^\alpha}$ defined above scale as

$$\begin{aligned} |v|_{H^\alpha(\omega)} &\leq \|\nabla(B_\omega^{-1})\|^\alpha_{L^\infty(\omega)} \|\det(\nabla B_\omega)\|^{1/2}_{L^\infty(\widehat{\omega})} |\widehat{v}|_{H^\alpha(\widehat{\omega})} \\ |\widehat{v}|_{H^\alpha(\widehat{\omega})} &\leq \|\nabla(B_\omega)\|^\alpha_{L^\infty(\widehat{\omega})} \|\det(\nabla(B_\omega^{-1}))\|^{1/2}_{L^\infty(\omega)} |v|_{H^\alpha(\omega)}, \end{aligned} \tag{97}$$

for $0 \leq \alpha \leq 1$. Inequalities (97) are an extension of the classical ones for integer order Sobolev spaces (see, e.g., [13]), and easily follow from the change of variable rule for integrals and from the Cauchy-Schwarz inequality.

We now recall the definition of *interpolation spaces*, according to the so called *K-method* (see [20, §1.3]). Given two Banach spaces A_0 and A_1, assuming $A_1 \subset A_0$, and given two parameters $0 < \theta < 1$ and $1 \leq p \leq \infty$, we define a norm $\|\cdot\|_{(A_0,A_1)_{\theta,p}}$ as

$$\|v\|_{(A_0,A_1)_{\theta,p}} := \left[\int_0^{+\infty} \inf_{v_1 \in A_1} \Big(t^{-\theta} \|v - v_1\|_{A_0} + t^{1-\theta} \|v_1\|_{A_1} \Big)^p \frac{dt}{t} \right]^{\frac{1}{p}} \tag{98}$$

if $1 \leq p < \infty$, while, when $p = +\infty$,

$$\|v\|_{(A_0,A_1)_{\theta,+\infty}} := \sup_{t>0} \inf_{v_1 \in A_1} \Big(t^{-\theta} \|v - v_1\|_{A_0} + t^{1-\theta} \|v_1\|_{A_1} \Big). \tag{99}$$

We then define $(A_0, A_1)_{\theta,p} \subset A_0$ as the space of functions v in A_0 such that $\|v\|_{(A_0,A_1)_{\theta,p}}$ is finite.

An important property of the interpolated norm is that [20, Theorem 1.3.3.g]

$$\|v\|_{(A_0,A_1)_{\theta,p}} \leq C_{\theta,p} \|v\|^{1-\theta}_{A_0} \|v\|^\theta_{A_1}, \qquad \forall v \in A_1. \tag{100}$$

The most important result of interpolation theory states that a linear operator $\mathcal{L}$ which is continuous from A_0 into B_0 and from A_1 into B_1 is also continuous from $(A_0, A_1)_{\theta,p}$ into $(B_0, B_1)_{\theta,p}$.Precisely

$$\left. \begin{aligned} \|\mathcal{L}v\|_{B_0} \leq C_0 \|v\|_{A_0}, \forall v \in A_0 \\ \|\mathcal{L}v\|_{B_1} \leq C_1 \|v\|_{A_1}, \forall v \in A_1 \end{aligned} \right\} \quad \Rightarrow \quad \|\mathcal{L}v\|_{(B_0,B_1)_{\theta,p}} \leq C_\theta \|v\|_{(A_0,A_1)_{\theta,p}}, \forall v \in (A_0, A_1)_{\theta,p}, \tag{101}$$

where $C_\theta = C_0^{1-\theta} C_1^\theta$. Fractional order Sobolev spaces can be characterized by interpolation of integer order spaces: for example,

$$H^\alpha(\omega) = (L^2(\omega), H^1(\omega))_{\alpha,2}, \qquad \forall \alpha \text{ such that } 0 < \alpha < 1, \tag{102}$$

see [20] for more details.

Besov spaces, a more general class of spaces, can be defined by interpolation as well; in particular, we shall need in Appendix B the following spaces

$$\begin{aligned} B^{1/2}_{2,1}(\omega) &= (L^2(\omega), H^1(\omega))_{1/2,1}, \\ B^{1/2}_{2,\infty}(\omega) &= (L^2(\omega), H^1(\omega))_{1/2,\infty}, \\ B^{1/2}_{2,\infty,0}(\omega) &= (L^2(\omega), H^1_0(\omega))_{1/2,\infty}, \end{aligned} \tag{103}$$

where $H^1_0(\omega)$ is the space of functions of $H^1(\omega)$ vanishing on the boundary $\partial\omega$. The following continuous inclusion holds (see [19])

$$B^{1/2}_{2,1}(\omega) \hookrightarrow B^{1/2}_{2,\infty,0}(\omega). \tag{104}$$

Another useful inclusion (due to [20, Theorem 1.3.3.e]) is $B^{1/2}_{2,\infty}(\omega) \hookrightarrow H^\alpha(\omega)$, which holds for all $\alpha < 1/2$; however, the constant in the norm inequality depends on α. In particular, we have the following theorem.

Theorem 6. *Let $0 < \epsilon \le 1/2$; then*

$$\|\phi\|_{H^{1/2-\epsilon}(\omega)} \le C_\omega \epsilon^{-1/2} \|\phi\|_{B^{1/2}_{2,\infty}(\omega)}, \qquad \forall \phi \in B^{1/2}_{2,\infty}(\omega), \tag{105}$$

where the constant C_ω depends on ω but is independent of ϵ.

Proof. When ϵ is far from 0 (e.g., $\epsilon > 1/4$) the result is well known, and is a consequence of [20, Theorem 1.3.3.g]). Consider therefore the case $\epsilon \le 1/4$. Thanks to (102), we have

$$\|\phi\|_{H^{1/2-\epsilon}(\omega)} \le C_\omega \|\phi\|_{(L^2(\omega),H^1(\omega))_{1/2-\epsilon,2}};$$

then, using the definition (98), for any $\phi_0 = \phi_0(t)$ and $\phi_1 = \phi_1(t)$ such that $\phi = \phi_0(t) + \phi_1(t)$, with $\phi_0(t) \in L^2(\omega)$ and $\phi_1(t) \in H^1(\omega)$, for all $t > 0$

$$\begin{aligned} \|\phi\|^2_{(L^2(\omega),H^1(\omega))_{1/2-\epsilon,2}} &\le \int_0^{+\infty} \left(t^{-1/2+\epsilon}\|\phi_0\|_{L^2(\omega)} + t^{1/2+\epsilon}\|\phi_1\|_{H^1(\omega)}\right)^2 \frac{dt}{t} \\ &= \int_0^{1} \left(t^{-1/2+\epsilon}\|\phi_0\|_{L^2(\omega)} + t^{1/2+\epsilon}\|\phi_1\|_{H^1(\omega)}\right)^2 \frac{dt}{t} \\ &\quad + \int_1^{+\infty} \left(t^{-1/2+\epsilon}\|\phi_0\|_{L^2(\omega)} + t^{1/2+\epsilon}\|\phi_1\|_{H^1(\omega)}\right)^2 \frac{dt}{t} \\ &= I + II. \end{aligned}$$

Taking $\phi_0 = \phi$ and $\phi_1 = 0$ when $t \ge 1$, we have

$$II \le \|\phi\|^2_{L^2(\omega)} \int_1^{+\infty} t^{-2+2\epsilon}\, dt = (1-2\epsilon)^{-1}\|\phi\|^2_{L^2(\omega)};$$

since $\|\phi\|_{L^2(\omega)} \le \|\phi\|_{B^{1/2}_{2,\infty}(\omega)}$, we get

$$II \le 2\|\phi\|^2_{B^{1/2}_{2,\infty}(\omega)}.$$

On the other hand

$$I \le \left(\sup_{0<t<1} \left\{ t^{-1/2} \|\phi_0\|_{L^2(\omega)} + t^{1/2} \|\phi_1\|_{H^1(\omega)} \right\} \right)^2 \int_0^1 t^{2\epsilon-1}\, dt$$
$$\le (2\epsilon)^{-1} \left(\sup_{0<t<1} \left\{ t^{-1/2} \|\phi_0\|_{L^2(\omega)} + t^{1/2} \|\phi_1\|_{H^1(\omega)} \right\} \right)^2 ;$$

taking the infimum over all the admissible ϕ_0 and ϕ_1 we get

$$I \le (2\epsilon)^{-1} \|\phi\|^2_{B^{1/2}_{2,\infty}(\omega)},$$

from which we obtain (105). □

B Two results involving fractional order spaces

In this Appendix, we report the proof of the estimates (48), (52), and (53), making use of the notions and tools of Appendix A.

We still denote by C a constant, possibly different at each occurrence, which may depend only on the set of reference polyhedra $\{\widehat{\Omega}^1, \ldots, \widehat{\Omega}^{\widehat{S}}\}$, the polynomial degree κ, and the constants κ_0, κ_1, κ_2, κ_3, appearing in the shape regularity and quasi-uniformity assumptions on the meshes (see §2).

The first result states two fractional order stability estimates for the Clément interpolant.

Lemma 8. *Under the assumptions of §2.1, let $1 \le s \le S$, and, for all $v \in L^2(\Gamma^s)$, let $v_I \in V^s_{h\,|\Gamma^s}$ denote the Clément interpolant of v (see [14]). If $v \in H^{1/2}(\Gamma^s)$, we have*

$$|v_I|_{H^{1/2}(\Gamma^s)} \le C |v|_{H^{1/2}(\Gamma^s)}. \tag{106}$$

Moreover, given $0 < \epsilon \le 1/2$, if $v \in H^{1/2-\epsilon}(\Gamma^s)$, we have

$$|v_I|_{H^{1/2}(\Gamma^s)} \le C (h^{\min}_{\Gamma^s})^{-\epsilon} |v|_{H^{1/2-\epsilon}(\Gamma^s)}. \tag{107}$$

Proof. The bound (43) implies, in particular, the inverse estimate

$$|v_I|_{H^{1/2}(\Gamma^s)} \le C h^{\min\,-1/2}_{\Gamma^s} |v_I|_{L^2(\Gamma^s)}. \tag{108}$$

Then the estimates (106) and (107) can be obtained from the usual stability estimates

$$\|v_I\|_{L^2(\Gamma^s)} \le C \|v\|_{L^2(\Gamma^s)}, \tag{109}$$

$$|v_I|_{H^1(\Gamma^s)} \le C |v|_{H^1(\Gamma^s)}. \tag{110}$$

making use of the interpolation theorem (101). In order to guarantee that the constants C of (106) and (107) do not depend on H_{Ω^s}, we reason on the reference polyhedron $\widehat{\Omega}^{\widehat{s}}$, which is mapped onto Ω^s by the map B_{Ω^s}, as assumed in §2.1. Therefore, using (97)we shift inequalities (109), (110) and (108) on $\widehat{\Gamma}^{\widehat{s}} := \partial\widehat{\Omega}^{\widehat{s}}$, obtaining

$$\|\widehat{v}_I\|_{L^2(\widehat{\Gamma}^{\widehat{s}})} \le \widehat{C} \|\widehat{v}\|_{L^2(\widehat{\Gamma}^{\widehat{s}})}, \tag{111}$$

$$|\widehat{v}_I|_{H^1(\widehat{\Gamma}^{\widehat{s}})} \le \widehat{C} |\widehat{v}|_{H^1(\widehat{\Gamma}^{\widehat{s}})}, \tag{112}$$

$$|\widehat{v}_I|_{H^{1/2}(\widehat{\Gamma}^{\widehat{s}})} \le \widehat{C} (H_{\Omega^s}/h^{\min}_{\Gamma^s})^{1/2} \|\widehat{v}_I\|_{L^2(\widehat{\Gamma}^{\widehat{s}})}, \tag{113}$$

where $\widehat{v_I} = v_I \circ B_{\Gamma^s}$ and $\widehat{v} = v \circ B_{\Gamma^s}$ with $B_{\Gamma^s} := B_{\Omega^s|\widehat{\Gamma^s}}$. Observe that the constant $\widehat{C}$ in (111) is related to the constant C in (109) by the inequality

$$\widehat{C} \leq \|det\nabla B_{\Gamma^s}\|^{1/2}_{L^\infty(\widehat{\Gamma^s})}\|det\nabla(B^{-1}_{\Gamma^s})\|^{1/2}_{L^\infty(\Gamma^s)}C \leq \|\nabla B_{\Omega^s}\|^{n/2}_{L^\infty(\widehat{\Omega^s})}\|\nabla(B^{-1}_{\Omega^s})\|^{n/2}_{L^\infty(\Omega^s)}C \leq \kappa_0^{n/2}C,$$

where κ_0 is defined in §2.1. A similar argument holds when comparing the constant $\widehat{C}$ in (112), and in (113), with the corresponding C in (110) and (108), respectively. For (113), we also made use of $\|\nabla B^{-1}_{\Omega^s}\|^{-1}_{L^\infty(\Omega^s)} \leq H_{\Omega^s}$, due to the fact that the diameter of $\widehat{\Omega^s}$ is unitary.

We can now apply the interpolation theorem (101) (to the operator $\mathcal{L}\widehat{v} = \widehat{v_I}$, that is, $\mathcal{L}$ is the pull-back, after the Clément interpolation, after the push-forward) and (102): from (111) and (112) we get

$$\|\widehat{v_I}\|_{H^{1/2}(\widehat{\Gamma^s})} \leq \widehat{C}\|\widehat{v}\|_{H^{1/2}(\widehat{\Gamma^s})}, \tag{114}$$

for any $\widehat{v}$ in $H^{1/2}(\widehat{\Gamma^s})$, and therefore

$$|\widehat{v_I}|_{H^{1/2}(\widehat{\Gamma^s})} \leq \widehat{C}|\widehat{v}|_{H^{1/2}(\widehat{\Gamma^s})}, \tag{115}$$

taking the quotient with respect to the constants. Still by interpolation, now starting from (111) with (113) and from (114) we obtain

$$\|\widehat{v_I}\|_{H^{1/2}(\widehat{\Gamma^s})} \leq \widehat{C}(H_{\Omega^s}/h^{\min}_{\Gamma^s})^{-\epsilon}\|\widehat{v}\|_{H^{1/2-\epsilon}(\widehat{\Gamma^s})}, \tag{116}$$

giving

$$|\widehat{v_I}|_{H^{1/2}(\widehat{\Gamma^s})} \leq \widehat{C}(H_{\Omega^s}/h^{\min}_{\Gamma^s})^{-\epsilon}|\widehat{v}|_{H^{1/2-\epsilon}(\widehat{\Gamma^s})}. \tag{117}$$

Note that the norm $\|\cdot\|_{H^{1/2-\epsilon}}$ in the right hand side of (116) has been obtained by interpolation from $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^{1/2}}$. In this particular case, the interpolation process is uniform with respect to ϵ.

We can now shift (115) and (117) to Γ^s. As before, this will introduce factors in the estimates which depend on $\|\nabla B_{\Omega^s}\|_{L^\infty(\widehat{\Omega^s})}\|\nabla B^{-1}_{\Omega^s}\|_{L^\infty(\Omega^s)}$ (and which are therefore bounded in terms of κ_0). This gives (106) and (107). □

Below is our second lemma.

Lemma 9. *Under the assumptions of §2.1, let* $0 < \epsilon \leq 1/2$ *and* $1 \leq s \leq S$*; then*

$$|\phi|^2_{H^{1/2-\epsilon}(\Gamma^s)} \leq C\frac{H^{2\epsilon}_{\Omega^s}}{\epsilon}\left(H^{-1}_{\Omega^s}|\phi|^2_{L^2(\Gamma^s)} + |\phi|_{L^2(\Gamma^s)}\sum_{r=1}^{R^s}|\phi|_{H^1(\Gamma^s_r)}\right), \tag{118}$$

for any $\phi \in L^2(\Gamma^s)$ *with* $\phi_{|\Gamma^s_r} \in H^1(\Gamma^s_r)$*, for each* $r = 1, \ldots, R^s$*.*

Proof. We use a scaling argument as in the proof of Lemma 8, and make use of the same notation. Since $H_{\Omega^s} \leq \|\nabla B_{\Omega^s}\|_{L^\infty(\widehat{\Omega^s})}$ and $\|\nabla B^{-1}_{\Omega^s}\|^{-1}_{L^\infty(\Omega^s)} \leq H_{\Omega^s}$ (recall that the diameter of $\widehat{\Omega^s}$ is unitary) and for the scaling properties (97), the estimate (118) is in fact equivalent to

$$|\widehat{\phi}|^2_{H^{1/2-\epsilon}(\widehat{\Gamma}^s)} \leq \frac{\widehat{C}}{\epsilon}\left(|\widehat{\phi}|^2_{L^2(\widehat{\Gamma^s})} + |\widehat{\phi}|_{L^2(\widehat{\Gamma^s})}\sum_{r=1}^{R^s}|\widehat{\phi}|_{H^1(\widehat{\Gamma}^s_r)}\right), \tag{119}$$

where $\widehat{\phi} = \phi \circ B_{\Gamma^s}$, $B_{\Omega^s}(\widehat{\Gamma}^{\widehat{s}}) = \Gamma^s$ and $B_{\Omega^s}(\widehat{\Gamma}^{\widehat{s}}_r) = \Gamma^s_r$, for each $r = 1, \dots, R^s$. In particular, the constant C of (118) can be easily related to the constant $\widehat{C}$ of (119) by the inequality

$$\begin{aligned} C &\le \|det\nabla B_{\Gamma^s}\|_{L^\infty(\widehat{\Gamma}^{\widehat{s}})}\|det\nabla(B_{\Gamma^s}^{-1})\|_{L^\infty(\Gamma^s)}\|\nabla B_{\Gamma^s}\|_{L^\infty(\widehat{\Gamma}^{\widehat{s}})}\|\nabla(B_{\Gamma^s}^{-1})\|_{L^\infty(\Gamma^s)}\widehat{C} \\ &\le \|\nabla B_{\Omega^s}\|^{n+1}_{L^\infty(\widehat{\Omega}^{\widehat{s}})}\|\nabla(B_{\Omega^s}^{-1})\|^{n+1}_{L^\infty(\Omega^s)}\widehat{C} \\ &\le \kappa_0^{n+1}\widehat{C}, \end{aligned}$$

where κ_0 is defined in §2.1. Moreover, defining

$$\tilde{\phi} := \widehat{\phi} - \frac{1}{|\widehat{\Gamma}^{\widehat{s}}|}\int_{\widehat{\Gamma}^{\widehat{s}}} \widehat{\phi},$$

it is clear that (119) will follow if we prove that

$$\|\tilde{\phi}\|^2_{H^{1/2-\epsilon}(\widehat{\Gamma}^{\widehat{s}})} \le \frac{\widehat{C}}{\epsilon}\sum_{r=1}^{R^s}\|\tilde{\phi}\|_{L^2(\widehat{\Gamma}^{\widehat{s}}_r)}\|\tilde{\phi}\|_{H^1(\widehat{\Gamma}^{\widehat{s}}_r)}. \tag{120}$$

To prove (120) we first remark that

$$\prod_{r=1}^{R^s} L^2(\widehat{\Gamma}^{\widehat{s}}_r) \subset L^2(\widehat{\Gamma}^{\widehat{s}})$$

and

$$\prod_{r=1}^{R^s} H^1_0(\widehat{\Gamma}^{\widehat{s}}_r) \subset H^1(\widehat{\Gamma}^{\widehat{s}}).$$

Hence, by interpolation,

$$\prod_{r=1}^{R^s} B^{1/2}_{2,\infty,0}(\widehat{\Gamma}^{\widehat{s}}_r) \subset B^{1/2}_{2,\infty}(\widehat{\Gamma}^{\widehat{s}}).$$

Recalling (104) (i.e., $B^{1/2}_{2,1}(\widehat{\Gamma}^{\widehat{s}}_r) \subset B^{1/2}_{2,\infty,0}(\widehat{\Gamma}^{\widehat{s}}_r)$), we are led to

$$\|\tilde{\phi}\|^2_{B^{1/2}_{2,\infty}(\widehat{\Gamma}^{\widehat{s}})} \le C\sum_{r=1}^{R^s}\|\tilde{\phi}\|^2_{B^{1/2}_{2,1}(\widehat{\Gamma}^{\widehat{s}}_r)}. \tag{121}$$

Using Theorem 6, then (121) and (100), (120) easily follows.

Moreover it is easy to see that the constant C in (118) depends on κ_0 and on the reference domains $\widehat{\Omega}_{\widehat{s}}$. □

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