Modelling and long-time behaviour for phase transitions with entropy balance and thermal memory conductivity

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Abstract. This paper deals with a thermomechanical model describing phase transitions with thermal memory in terms of balance and equilibrium equations for entropy and microforces, respectively. After a presentation and discussion of the model, the large time behaviour of the solutions to the related integro-differential system of PDE's is investigated.

Key words: phase transition model, entropy, thermal memory, microforces, nonlinear initial-boundary value problem, long-time behaviour of solutions.

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1. Introduction

This paper is concerned with a new model for phase transitions with thermal memory which couples a singular integro-differential equation for the entropy with a microforce balance. The resulting PDE's system has been investigated in [5] from the analytical point of view: existence and uniqueness of the solution for an associated initial and boundary value problem are proved along with continuous dependence on the data and regularity results. In the present contribution, we first detail the thermomechanical derivation of the model, then we investigate the long-time behaviour of the solutions. More precisely, we study the ω-limit set of the trajectories and characterize the ω-limit points as solutions of a suitable stationary problem.

Phase transition systems, possibly including thermal memory, have been the object of some recent studies, both for first and second order phenomena, going from the modelling aspects to mathematical results. Recalling the Ginzburg-Landau model for phase transitions of second order proposed in 1950 for superconductivity [24], let us point out that the actual development of phase field theory (see, e.g., [10, 34, 3]) led to an intense investigation of phase transitions of first order by the use of an order parameter. In this direction, we aim to cite in particular the work done by Frémond [21] and Gurtin [22, 26].

In fact, the evolution of the phase transition may be described in terms of the absolute temperature and a phase parameter characterizing the presence of one phase with respect to the other(s). The equations governing these unknowns are, in general, the first principle of thermodynamics, the so-called energy balance, and an equation written for the phase parameter.

Our specific approach in this paper follows a novel alternative way of modelling phase transitions which is devised in [6] (cf. also [7] and [4]). An approximated energy equation, which has the dimension and the structure of an entropy equation, is considered along with a balance law of microforces and micromovements, responsible for the phase transition in the system. As concerns the entropy equation, the main feature consists in the presence of a logarithmic contribution for the temperature in the evolutionary system. Then, once the related problem can be solved, the resulting singular structure of the equation turns out to be an advantage since the positivity of the unknown representing the absolute temperature is directly granted, without any application of the maximum principle, this technique being rather useless in presence of thermal memory.

The equation for the phase parameter was recovered as an equilibrium equation for the microforces, that is, the momentum balance equation for the micromovements, which are responsible for the macroscopic phase transitions in the system. To our knowledge, this fresh viewpoint to describe the dynamics of the order parameter was independently offered both by Gurtin [26] and Frémond [21]. A new direction of investigation started from here, and we may quote [8, 12, 15, 29, 30, 31, 32, 33] for related analytical results and complementary remarks.

The work done for this paper guided us to a final system, which differs from that studied in [6] for some aspects. In fact, here we introduce a different entropy flux law, and consequently modify the structure of the equation governing the temperature. More precisely, in our case the resulting entropy flux depends linearly on the temperature.

Moreover, the related equation is supplied by a suitable Dirichlet boundary condition for the temperature. All this allows us to prove more regularity on the solution than in [6], and especially to get a uniqueness result. On the other hand, the equation for the phase parameter is derived as a microforce balance following the approach proposed by Gurtin [26]. We avoid the use of the dissipative functional extensively employed by Frémond [21], the so called pseudo-potential of dissipation, to write out constitutive relations. Instead, we directly select them in a suitable way, in order to satisfy the second principle of thermodynamics, expressed by the well-known Clausius-Duhem inequality. We are detailing the derivation of the model in the following section.

The remaining part of the paper is concerned with analytical results. We focus on a PDE's system that generalizes the equations obtained in the derivation of the thermomechanical model and complement it with suitable boundary and initial conditions. Namely, we consider the Dirichlet condition for the temperature, choosing the known temperature on the boundary uniformly positive and sufficiently smooth, the homogeneous Neumann boundary conditions for the phase parameter, and initial conditions for both. Exploiting the well-posedness results obtained in [5], we study the structure of the $\omega-$limit set ω for a single trajectory of the solution.

In particular, we show that every element of ω satisfies a suitable stationary problem, which is nothing but the "natural" steady state system of our evolution problem. The main sufficient conditions for such a result are that the memory kernel fulfils some positiveness properties and proper decay conditions at infinity, and that the entropy source in the limiting stationary problem is basically nonnegative.

The results proved for the long-time behaviour can be compared with those reported in [14] and [13], and addressed to standard phase field systems with memory in the non-conserved and conserved cases, respectively. However, methods and proofs are here adapted to the specific problems we deal with. Besides, as the reader can personally check, despite the fact that in [13, 14] the authors can treat a heat flux law of Gurtin-Pipkin type [27] (while in our approach we are actually more closed to the theory of thermal memory materials proposed in [11]), it turns out that for solving our mathematical problem we face the difficulty of the logarithmic nonlinearity in the entropy equation, whereas the papers [13, 14] simply present the standard energy balance equations in their phase field systems.

2. The model

In this section, we outline the derivation of the model, for which we mainly refer to the approach proposed by Gurtin in [26] (cf. also [21] and [22]). Consider a two-phase transition occurring in a smooth domain $\Omega \subset \mathbb{R}^3$. We denote by $\vartheta \in (0, +\infty)$ the absolute temperature and by χ the order parameter, assuming that at least χ may take values in an interval containing 0. In particular, certain level sets of χ could be taken as an approximation of the interface between the two phases, provided the interface zone is sufficiently thin. From the physical point of view, to some extent χ could represent a local concentration or a rescaled proportion of one phase with respect to the other.

Balance law for microforces and thermodynamics. In the description, we restrict ourselves to phase transitions in which macroscopic deformations do not occur and microscopic accellerations are negligible (see [8] for a different position).

As already announced in the Introduction, our system comes from the coupling of the equation governing the phase parameter, derived as a balance law for microforces, and an entropy equation describing the evolution of the temperature.

First, we introduce the balance law for microforces responsible for the thermodynamical process, according to the notation in [22]. Letting $\mathbf{H}$ and B be interior microscopic forces and b stand for an external force acting at the microscopic level on the body, then the balance law reads

$$\int_{\partial S} \mathbf{H}\cdot\mathbf{n}\,d\sigma + \int_S b\,dx + \int_S B\,dx = 0 \tag{2.1}$$

for any subdomain $S \subseteq \Omega$, where $\mathbf{n}$ denotes the outward unit normal to the boundary ∂S. Then, from (2.1) we obtain the balance equation for microforces

$$\operatorname{div}\mathbf{H} + b = -B \quad \text{holding in the whole } \Omega \tag{2.2}$$

and (2.2) can be combined with a no-flux boundary condition for $\mathbf{H}$, namely

$$\mathbf{H}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{2.3}$$

An important effect of stating the balance equation (2.2) (or (2.1)) is that microforces responsible for the phase transitions have to be taken into account as mechanically induced heat sources in the first law of thermodynamics. Consequently, the new version of the first principle can be written in the form

$$\frac{d}{dt}\int_S e\,dv = -\int_{\partial S}\mathbf{q}\cdot\mathbf{n}\,d\sigma + \int_S r\,dv + \int_{\partial S}\chi_t\mathbf{H}\cdot\mathbf{n}\,d\sigma + \int_S b\chi_t\,dv \tag{2.4}$$

for any subdomain $S \subseteq \Omega$. Here, e is the internal energy, $\mathbf{q}$ the heat flux, and r the heat supply. Let us point out the meaning of (2.4): the total variation of the internal energy is given by the heat flux through the boundary, the heat supply, and the work of the external surface and distance microforces related to the phase transition process (cf. (2.1)). Hence, using (2.2), from (2.4) we obtain

$$e_t = -\operatorname{div}\mathbf{q} + r + \operatorname{div}(\mathbf{H}\chi_t) + b\chi_t = -\operatorname{div}\mathbf{q} + r - B\chi_t + \mathbf{H}\cdot\nabla\chi_t \quad \text{in } \Omega \tag{2.5}$$

and supply this equation with a boundary condition, which of course can be prescribed for the heat flux $\mathbf{q}$. Alternatively, in this paper we prefer to assume the temperature ϑ known on the boundary $\partial\Omega$ (and that naturally complies with constitutive relations), thus fixing a Dirichlet boundary condition for ϑ.

Now, we aim to discuss the thermodynamical consistence of the model we are introducing. Let us write the second principle of thermodynamics in the form of the Clausius-Duhem inequality

$$\frac{d}{dt}\int_S \eta\,dv \geq -\int_{\partial S}\frac{1}{\vartheta}\mathbf{q}\cdot\mathbf{n}\,d\sigma + \int_S \frac{r}{\vartheta}\,dv \tag{2.6}$$

for any subdomain $S \subseteq \Omega$, where η denotes the entropy. This inequality takes the local form

$$\eta_t \geq -\operatorname{div}\frac{\mathbf{q}}{\vartheta} + \frac{r}{\vartheta} = \frac{1}{\vartheta^2}\,\mathbf{q}\cdot\nabla\vartheta - \frac{1}{\vartheta}\operatorname{div}\mathbf{q} + \frac{r}{\vartheta}. \tag{2.7}$$

If we introduce the free energy ψ defined by

$$\psi = e - \vartheta\eta \tag{2.8}$$

then, with the help of (2.5) you can rewrite (2.7) as

$$\psi_t \leq -\eta\vartheta_t - \frac{1}{\vartheta}\mathbf{q}\cdot\nabla\vartheta - B\chi_t + \mathbf{H}\cdot\nabla\chi_t. \tag{2.9}$$

Now, we make precise the choice of the entropy η, the heat flux $\mathbf{q}$, and the microscopic forces B and $\mathbf{H}$. Before proceeding, we have to point out the sets of the state variables of the system and the variables defining a thermodynamical process. We assume that the state depends on the absolute temperature, the phase parameter and its gradient. Moreover, as we are allowing thermal memory in the system, we also include the summed past history of the gradient of the temperature

$$\nabla\widetilde{\vartheta}^t(s) := \int_{t-s}^{t} \nabla\vartheta(\tau)\,d\tau.$$

Here, $s \in (0, +\infty)$ denotes the time history variable and t stands for the present time. Thus, the state Σ is specified by

$$\Sigma = (\vartheta, \chi, \nabla\chi, \nabla\widetilde{\vartheta}^t). \tag{2.10}$$

For the sake of convenience we also introduce the following notation

$$\nabla\vartheta^t(s) := \nabla\vartheta(t-s)$$

and observe that

$$\frac{d}{dt}\nabla\widetilde{\vartheta}^t = \nabla\vartheta - \nabla\vartheta^t. \tag{2.11}$$

So far as concerns the thermodynamical process P, we let it be defined by the following variables

$$P = (\vartheta_t, \chi_t, \nabla\chi_t, \nabla\vartheta). \tag{2.12}$$

Our aim is to single out precise expressions of η, B, $\mathbf{H}$, and $\mathbf{q}$, which in our setting are functionals depending on the state Σ, and possibly on the process P, in such a way that the second principle of thermodynamics (2.9) is fulfilled for any possible admissible process P in (2.12).

We first discuss the relation between the free energy ψ and the entropy η, which are assumed to depend, as the internal energy e, only on the state variables. Letting $P = (\vartheta_t, 0, \mathbf{0}, \mathbf{0})$ and applying the chain rule, from (2.9) one gets

$$\frac{\partial\psi}{\partial\vartheta} = -\eta \tag{2.13}$$

so that (cf. (2.8))

$$e = \psi - \vartheta \frac{\partial \psi}{\partial \vartheta}.$$

Our choice for η is

$$\eta = c(1 + \ln \vartheta) - \lambda(\chi) \tag{2.14}$$

where $c > 0$ is the specific heat and λ is a smooth function of the order parameter that accounts for the entropy associated to the phase transition. The internal forces B and $\mathbf{H}$ are allowed to depend also on the process. Thus, letting $P = (0, \chi_t, \mathbf{0}, \mathbf{0})$ and $P = (0, 0, \nabla\chi_t, \mathbf{0})$, respectively, with the help of (2.9) we infer

$$\frac{\partial \psi}{\partial \chi} \chi_t + B\chi_t \le 0 \tag{2.15}$$

$$\left(\frac{\partial \psi}{\partial (\nabla \chi)} - \mathbf{H} \right) \cdot \nabla \chi_t \le 0 \tag{2.16}$$

which have to hold for any choice of χ_t and $\nabla\chi_t$. As it is fairly natural from the physical point of view, let the functional B and $\mathbf{H}$ linearly depend on the process, i.e.,

$$B = B_1 + B_2 \chi_t \tag{2.17}$$

$$\mathbf{H} = \mathbf{H}_1 + H_2 \nabla \chi_t \tag{2.18}$$

whence, combining (2.17) - (2.18) with (2.15) - (2.16), we derive the conditions

$$B_1 = -\frac{\partial \psi}{\partial \chi}, \quad B_2 \le 0 \tag{2.19}$$

$$\mathbf{H}_1 = \frac{\partial \psi}{\partial (\nabla \chi)}, \quad H_2 \ge 0 \tag{2.20}$$

Then, we can consider the following admissible expressions for B_1, B_2 and $\mathbf{H}_1$, H_2 (cf. (2.8), (2.13–14), and (2.5) especially)

$$B_1 = -\vartheta \lambda'(\chi) - \sigma'(\chi) - \beta(\chi), \quad B_2 = -1 \tag{2.21}$$

$$\mathbf{H}_1 = \nu \nabla \chi, \quad H_2 = 0 \tag{2.22}$$

in which σ denotes another smooth function (note the different role with respect to λ in B_1) and, instead, β stands for a monotone and maximal graph, not necessarily smooth. In fact, we let β be the subdifferential of a proper, convex, and lower semicontinuous function $\widehat{\beta}$ with values in $[0, +\infty]$. By suitably choosing λ, σ, and $\widehat{\beta}$, it turns out that the model may describe different kinds of phase transitions, as we will briefly discuss at the end of the section. Finally, let us mention the constant coefficient ν which is supposed to satisfy $\nu \ge 0$.

It remains to specify a constitutive relation for the heat flux $\mathbf{q}$ and discuss its thermodynamical consistence. We mainly refer to [26] and [6] and recall that we are dealing with thermal memory materials. The dependence of free energy ψ on the thermal gradient history $\nabla\widetilde{\vartheta}^t$ is assumed to be Fréchet-differentiable. Then, let $P = (0, 0, \mathbf{0}, \nabla\vartheta)$ and use the chain rule in (2.9). Due to (2.11), we get

$$\frac{\delta\psi}{\delta(\nabla\widetilde{\vartheta}^t)} \cdot (\nabla\vartheta - \nabla\vartheta^t) + \frac{\mathbf{q}}{\vartheta} \cdot \nabla\vartheta \le 0 \tag{2.23}$$

where $\delta\psi/\delta(\nabla\widetilde{\vartheta}^t)$ stands for the partial Fréchet derivative of ψ with respect to $\nabla\widetilde{\vartheta}^t$. In particular, we are interested to study thermodynamical potentials ψ of the form

$$\psi = \psi_1(\vartheta, \chi, \nabla\chi) + \psi_2(\nabla\widetilde{\vartheta}^t) \tag{2.24}$$

for which (2.23) reduces to

$$\frac{\delta\psi_2}{\delta(\nabla\widetilde{\vartheta}^t)} \cdot (\nabla\vartheta - \nabla\vartheta^t) + \frac{\mathbf{q}}{\vartheta} \cdot \nabla\vartheta \le 0. \tag{2.25}$$

Then, arguing similarly as in (2.18), for the heat flux $\mathbf{q}$ we state the law

$$\mathbf{q} = \vartheta(\mathbf{Q}_1 + Q_2\nabla\vartheta) \tag{2.26}$$

which has the additional factor ϑ to be simplified in (2.25). Basically, the simplest choices for $\mathbf{Q}_1$ and Q_2 lead to

$$\mathbf{Q}_1 = \int_0^{+\infty} k'(s)\nabla\widetilde{\vartheta}^t(s)\, ds, \qquad Q_2 = -k_0 \tag{2.27}$$

where k_0 denotes a positive coefficient and k is a suitable memory kernel such that $k' \in L^1(0, +\infty) \cap H^1(0, +\infty)$ and (cf. [1, 23])

$$k_0 - \int_0^{+\infty} k'(s) \cos\omega s\, ds > 0 \qquad \text{for all } \omega \in \mathbb{R}. \tag{2.28}$$

Let us point out that for insulating materials, such as ice and water, the above assumptions on the heat conductivity are in agreement with the physical evidence.

Remarks on thermal free energies. A comparison between (2.25) and (2.26–27) yields the inequality

$$\frac{\delta\psi_2}{\delta(\nabla\widetilde{\vartheta}^t)} \cdot (\nabla\vartheta - \nabla\vartheta^t) \le k_0\, |\nabla\vartheta|^2 - \int_0^{+\infty} k'(s)\nabla\widetilde{\vartheta}^t(s) \cdot \nabla\vartheta\, ds. \tag{2.29}$$

If we consider only function ψ_2 for which

$$\psi_2(\mathbf{0}) = 0 \tag{2.30}$$

anyway one can check that there are many functionals Ψ satisfy inequality (2.29) and condition (2.30). In particular all the functionals which fulfil

$$\frac{d}{dt}\, \Psi(\nabla\widetilde{\vartheta}^t) \le -\int_0^{\infty} \widehat{k}'(s)\nabla\widetilde{\vartheta}^t(s)\, ds \cdot \nabla\vartheta(t)$$

comply with our requirements. As in [19] (see also [20, 18]), one may consider the maximum free energy

$$\Psi_M(\nabla\widetilde{\vartheta}^t) = \frac{1}{2}\int_0^\infty \int_0^\infty h''(|s_1 - s_2|)\nabla\widetilde{\vartheta}^t(s_1)\cdot\nabla\widetilde{\vartheta}^t(s_2)\,ds_1 ds_2$$

defined on the set

$$D_M = \left\{\widetilde{\mathbf{g}}^t : (0,+\infty)\to\mathbb{R}^3 \ : \ \int_0^\infty \int_0^\infty h''(|s_1 - s_2|)^t(s_1)\widetilde{\mathbf{g}}^t(s_1)\cdot\widetilde{\mathbf{g}}^t(s_2)\,ds_1 ds_2 < \infty\right\}$$

where h is a sufficiently smooth kernel satisfying the thermodynamic condition (2.28) with $k_0 = 0$ as well. The domain D_M is such that if D is the domain of any other free energy $\Psi : D \to (0,+\infty)$, there holds $D_M \subset D$. Moreover the maximum free energy has the property that

$$\Psi_M(\widetilde{\mathbf{g}}^t) \ge \psi_2(\widetilde{\mathbf{g}}^t) \quad \text{for all } \widetilde{\mathbf{g}}^t \in D_M.$$

Another free energy we would like to mention is the Graffi-Volterra free energy Ψ_G specified by

$$\Psi_G(\widetilde{\mathbf{g}}^t) = -\frac{1}{2}\int_0^\infty k'(s)\widetilde{\mathbf{g}}^t(s)\cdot\widetilde{\mathbf{g}}^t(s)\,ds \tag{2.31}$$

where the kernel $k \in C^2(0,+\infty)$ is such that

$$k'(s) < 0, \quad k''(s) \ge 0 \quad \text{ for all } s > 0. \tag{2.32}$$

The domain of definition of the functional in (2.31) is given by

$$D_G = \left\{\widetilde{\mathbf{g}}^t : (0,+\infty)\to\mathbb{R}^3 \ : \ \int_0^\infty k'(s)\widetilde{\mathbf{g}}^t(s)\cdot\widetilde{\mathbf{g}}^t(s)\,ds > -\infty\right\}.$$

We also note that in [1] the minimum free energy $\Psi_m : D_m \to (0,+\infty)$ is considered, using the Golden rappresentation given in [25] for a viscoelastic material. Moreover, as proved in [17], for any kernel $k \in C^2(0,+\infty)$ such that

$$\alpha k'(s) + k''(s) \ge 0 \quad \text{ for some } \alpha \in (0,+\infty)$$

and the related Graffi-Volterra free energy we have

$$\frac{d}{dt}\Psi_G(\nabla\widetilde{\vartheta}^t) \le -\alpha\Psi_G(\nabla\widetilde{\vartheta}^t) + \mathbf{q}(\nabla\widetilde{\vartheta}^t)\cdot\nabla\vartheta(t).$$

The PDE's system. Now, we are in the position of recovering the partial differential equations we will deal with. Before proceeding, let us rewrite (2.5) on account of (2.17–22) and (2.26). Since (cf. (2.8), (2.13), and (2.11))

$$\begin{aligned} e_t = &\ \psi_t + \vartheta_t\eta + \vartheta\eta_t \\ = &- \eta\vartheta_t + \frac{\partial\psi}{\partial\chi}\chi_t + \frac{\partial\psi}{\partial(\nabla\chi)}\cdot\nabla\chi_t + \frac{\delta\psi}{\delta(\nabla\widetilde{\vartheta}^t)}\cdot(\nabla\vartheta - \nabla\vartheta^t) + \vartheta_t\eta + \vartheta\eta_t \\ = &- B_1\chi_t + \mathbf{H}_1\cdot\nabla\chi_t + \frac{\delta\psi}{\delta(\nabla\widetilde{\vartheta}^t)}\cdot(\nabla\vartheta - \nabla\vartheta^t) + \vartheta\eta_t \end{aligned}$$

and, on the other hand,

$$\begin{aligned} e_t &= -\operatorname{div}\mathbf{q} + r - B_1\chi_t + \chi_t^2 + \mathbf{H}_1\cdot\nabla\chi_t \\ &= -\vartheta\operatorname{div}\frac{\mathbf{q}}{\vartheta} - \frac{1}{\vartheta}\mathbf{q}\cdot\nabla\vartheta + r - B_1\chi_t + \chi_t^2 + \mathbf{H}_1\cdot\nabla\chi_t \end{aligned}$$

we get the equation

$$\vartheta\eta_t = -\vartheta\operatorname{div}(\mathbf{Q}_1 + Q_2\nabla\vartheta) - \frac{\mathbf{q}}{\vartheta}\cdot\nabla\vartheta + r + \chi_t^2 - \frac{\delta\psi}{\delta(\nabla\widetilde{\vartheta}^t)}\cdot(\nabla\vartheta - \nabla\vartheta^t). \tag{2.33}$$

Now, thanks to the small perturbation assumptions, we neglect the high order nonlinearities

$$\chi_t^2 \quad \text{and} \quad -\frac{\mathbf{q}}{\vartheta}\cdot\nabla\vartheta - \frac{\delta\psi}{\delta(\nabla\widetilde{\vartheta}^t)}\cdot(\nabla\vartheta - \nabla\vartheta^t)$$

which are also non-negative due to (2.25). We refer to [6] for some more details and the exact form of the nonlinearities (see, in particular, [6, Remark 2.3]). Then, (2.33) reduces to the particular approximation $\vartheta\eta_t + \vartheta\operatorname{div}(\mathbf{Q}_1 + Q_2\nabla\vartheta) = r$ which, dividing by $\vartheta > 0$, yields

$$\eta_t + \operatorname{div}(\mathbf{Q}_1 + Q_2\nabla\vartheta) = R \tag{2.34}$$

where $R = r/\vartheta$. This is the reason for which in the following the equation (2.34) will be called entropy equation. Therefore, substituting the expressions of η and $\mathbf{Q}_1$, Q_2 given by (2.14) and (2.27), with the help of some integration by parts in time (one may argue as in [6, formula (2.55)]) we obtain

$$c(\log\vartheta - \lambda(\chi))_t - \operatorname{div}(k_0\nabla\vartheta + k*\nabla\vartheta) = R(t) + \operatorname{div}\int_{-\infty}^{0} k(t-s)\nabla\vartheta(s)\,ds \tag{2.35}$$

where $*$ stands for the usual time convolution product over the interval $(0,t)$, i.e., if a and c are summable time functions, then

$$(a*c)(t) := \int_0^t a(t-s)c(s)\,ds, \quad t > 0.$$

In the following, by abuse of notation we denote by R the entropy source, including external sources and the summed past history of the gradient of the temperature $\int_{-\infty}^t k(\,\cdot\, - s)\nabla\vartheta(s)\,ds$ which is assumed to be known.

Next, in view of (2.21–22) and considering the simple situation in which $b = 0$ (no microscopic external action on the body), the balance law for microforces (2.2) becomes

$$\chi_t - \Delta\chi + \beta(\chi) + \sigma'(\chi) \ni -\vartheta\lambda'(\chi) \tag{2.36}$$

where the symbol $\ni$ (instead of $=$) is due to the presence of the possibly multivalued graph β. The attentive reader may note that we have set $\nu = 1$, this for the sake of simplicity.

The above equations are combined with suitable boundary and initial conditions. In particular, concerning boundary conditions and recalling (2.3) and (2.22), it turns out that we fixed a homogeneous Neumann condition for χ

$$\partial_n \chi = 0 \quad \text{on the boundary } \Gamma := \partial\Omega$$

which is rather usual in this kind of problems. On the contrary, for the equation involving heat flux and temperature, we supply it with the already envisaged (cf. the comments below (2.5)) Dirichlet boundary condition for ϑ

$$\vartheta = \vartheta_\Gamma \quad \text{on } \Gamma$$

(absolute temperature known on the boundary), with the datum ϑ_Γ being strictly positive and sufficiently smooth. In addition, we prescribe Cauchy conditions for $\ln\vartheta$ and χ, i.e.,

$$(\ln\vartheta)(0) = \ln\vartheta_0, \quad \chi(0) = \chi_0.$$

The resulting system is highly nonlinear, and the main difficulties lie in the treatment of nonlinearities coupled with the presence of the convolution product involving the temperature gradient. However, existence and uniqueness of a global solution, as well as some regularity properties, for the related initial boundary value problem have been proved in [5]. The long-time behaviour of the solution is rather investigated in the remaining sections of this paper, under reasonable and appropriate assumptions on the trajectories of the data $R(t)$ and $\vartheta_\Gamma(t)$ as t goes to $+\infty$.

Examples of possible nonlinearities. Finally, we discuss some possible choices for β (i.e., for $\widehat{\beta}$), σ, and λ. Concerning β, let us restrict ourselves to the (significant) case when $\widehat{\beta}$ is the indicator function of the interval $[0,1]$, defined by $I_{[0,1]}(\chi) = 0$ if $\chi \in [0,1]$, and $I_{[0,1]}(\chi) = +\infty$ otherwise. Thus, the order parameter χ is constrained to assume value only in the interval $[0,1]$ and the subdifferential $\beta(\chi) = \partial I_{[0,1]}$ is actually a multivalued maximal monotone graph, with $\xi \in \partial I_{[0,1]}(\chi)$ if and only if

$$\xi \begin{cases} \le 0 & \text{if } \chi = 0 \\ = 0 & \text{if } 0 < \chi < 1 \\ \ge 0 & \text{if } \chi = 1 \end{cases}.$$

A first modelling situation we aim to consider regards solid-liquid phase transitions. Then, typical nonlinearities λ and σ in this case are given by

$$\lambda(\chi) = -\int_{1/2}^{\chi} l(\xi)\, d\xi, \quad \sigma(\chi) = -\lambda(\chi)\vartheta_c + 4a\chi(1-\chi) \tag{2.37}$$

where $l(\chi) > 0$ represents the (possibly constant) latent heat of the phase transition, ϑ_c denotes the critical phase transition temperature, and $a > 0$ is the maximum value of the function σ, attained at the midpoint $\chi = 1/2$ and measuring the depth of the potential wells corresponding to the different phases. If one examines the form the free energy ψ (more precisely, the component ψ_1 in (2.24)), by referring for instance to [16, Introduction] and owing to (2.37) it results that if $\vartheta \neq \vartheta_c$ one of the two points $\chi = 0$ and $\chi = 1$ is always preferred for equilibrium, according whether $\vartheta < \vartheta_c$ or $\vartheta > \vartheta_c$ respectively (so that the high temperature phase corresponds to the value $\chi = 1$).

Another interesting situation that can be included in our modelling approach is the Ising model of ferromagnetism. In this case, we take

$$\lambda(\chi) = -4b\chi(1-\chi), \quad \sigma(\chi) = -b\vartheta_c(1-2\chi)^2 \tag{2.38}$$

where the parameter b is analogous with a in the previous framework, but here the value ϑ_c corresponds to the so-called Curie temperature. This occurrence is rather different from the other situation, since now we have (cf. equation (2.36)) $\sigma'(\chi) + \vartheta\lambda'(\chi) = 4b(\vartheta_c - \vartheta)(1-2\chi)$ and the free energy may assume either two absolute minima at $\chi = 0, 1$ with the same value (two symmetric phase variants) if $\vartheta < \vartheta_c$ (at low temperatures) or just one absolute minimum in the midpoint $\chi = 1/2$ if $\vartheta > \vartheta_c$ (only one cristalline phase at high temperatures). This behaviour is proper of martensitic transformations (see, e.g., [21, Chapter 13]).

3. Mathematical results

In this section, we describe the mathematical problem more carefully. Moreover, we list the assumptions and state our results. In the sequel, we let Ω be a bounded open set in $\mathbb{R}^3$ whose boundary Γ is assumed to be smooth. It is convenient to set

$$H := L^2(\Omega), \qquad V := H^1(\Omega), \qquad V_0 := H_0^1(\Omega) \tag{3.1}$$

$$W := \{v \in H^2(\Omega) : \partial_n v = 0\} \tag{3.2}$$

where ∂_n denotes the normal derivative. We endow H, V, and W with their usual scalar products and norms, and use a self-explaining notation, like $\|\cdot\|_V$. For the sake of simplicity, the same symbol will be used both for a space and for any power of it. We note that the norms $\|v\|_V$ and $\|\nabla v\|_H$ are equivalent for $v \in V_0$, and recall that V_0' coincides with the Sobolev space $H^{-1}(\Omega)$.

As far as the structure of the system is concerned, we are given four functions $\widehat{\beta}$, λ, σ, k, and two constants k_0, α satisfying the conditions listed below.

$$\widehat{\beta} : \mathbb{R} \to [0, +\infty] \text{ is convex, proper, lower semicontinuous, and } \widehat{\beta}(0) = 0 \tag{3.3}$$

$$\lim_{|r| \to +\infty} |r|^{-2}\,\widehat{\beta}(r) = +\infty \tag{3.4}$$

$$\lambda, \sigma \in C^1(\mathbb{R}) \text{ and } \lambda', \sigma' \text{ are Lipschitz continuous} \tag{3.5}$$

$$k \in W^{1,1}(0,+\infty) \cap L^2(0,+\infty) \quad \text{and} \quad k_0, \alpha > 0 \tag{3.6}$$

$$\int_0^{+\infty} \bigl(k_0 v(t) + (k * v)(t)\bigr) v(t)\, dt \geq \alpha \int_0^{+\infty} |v(t)|^2\, dt \quad \text{for every } v \in L^2(0,+\infty) \tag{3.7}$$

$$\widehat{k} \in L^2(0,+\infty) \tag{3.8}$$

where we have set

$$\widehat{k}(t) := \int_t^{+\infty} k(s)\, ds \quad \text{for } t \geq 0\,. \tag{3.9}$$

We define the graph β in $\mathbb{R}\times\mathbb{R}$ by

$$\beta := \partial\widehat{\beta} \tag{3.10}$$

and note that β is maximal monotone and that $\beta(0) \ni 0$. The same symbol β will be used for the maximal monotone operators induced on L^2 spaces.

Remark 3.1. As already pointed out in the previous section, assumption (3.7) is well established from the physical point of view and complies with the second law of thermodynamics (see also [23, Remark 3.3]). Of course, a sufficient condition for it is that k is a kernel of positive type, which means (3.7) itself with $\alpha = k_0$. Such a property is fulfilled if k is a positive, decreasing, and convex function of class C^2, in addition to (3.6) (see, e.g., [2, Subsection IV.4.1]). ∎

Note that assumption (3.6) allows us to define

$$k_\infty := \int_0^{+\infty} k(s)\, ds \tag{3.11}$$

and that (3.7) ensures that

$$k_0 + k_\infty \geq \alpha \tag{3.12}$$

as one can see taking the characteristic function of $(0,T)$ as v in (3.7) and letting T tend to $+\infty$.

The data of our problem are four functions R, ϑ_Γ, ϑ_0, and χ_0. For the sake of convenience, we split the assumptions on such data into two sets, starting with the requirements ensuring well-posedness for any fixed final time (see [5]). We assume that two constants ϑ_* and ϑ^* are given such that

$$0 < \vartheta_* \leq \vartheta^* < +\infty \tag{3.13}$$

and that the conditions listed below hold.

$$R \in L^2(0,T;H) \quad \text{for every } T \in (0,+\infty) \tag{3.14}$$

$$\vartheta_\Gamma \in C^0([0,T];H^{1/2}(\Gamma)) \cap W^{1,1}(0,T;L^\infty(\Gamma)) \cap H^1(0,T;H^{-1/2}(\Gamma)) \quad \text{for every } T \in (0,+\infty) \tag{3.15}$$

$$\vartheta_* \leq \vartheta_\Gamma \leq \vartheta^* \quad \text{a.e. on } \Gamma\times(0,+\infty) \tag{3.16}$$

$$\vartheta_0 \in L^\infty(\Omega), \quad \vartheta_* \leq \vartheta_0 \leq \vartheta^* \quad \text{a.e. in } \Omega \tag{3.17}$$

$$\chi_0 \in V \quad \text{and} \quad \widehat{\beta}(\chi_0) \in L^1(\Omega). \tag{3.18}$$

The function ϑ_Γ is the boundary datum for the temperature and we introduce its harmonic extension $\vartheta_{\mathcal{H}}$, namely,

$$\vartheta_{\mathcal{H}}(t) \in V, \quad \Delta\vartheta_{\mathcal{H}}(t) = 0, \quad \text{and} \quad \vartheta_{\mathcal{H}}(t)|_\Gamma = \vartheta_\Gamma(t) \quad \text{for a.a. } t \in (0,+\infty)\,. \tag{3.19}$$

In [5] the final time T is fixed and it is studied the problem of finding a triplet (ϑ, χ, ξ) which satisfies the regularity conditions

$$\vartheta \in L^2(0,T;V) \quad \text{and} \quad u := \vartheta - \vartheta_{\mathcal{H}} \in L^2(0,T;V_0) \tag{3.20}$$
$$\vartheta > 0 \quad \text{a.e. in } \Omega \times (0,T) \quad \text{and} \quad \ln\vartheta \in L^\infty(0,T;H) \cap H^1(0,T;V_0') \tag{3.21}$$
$$\chi \in L^2(0,T;W) \cap H^1(0,T;H) \tag{3.22}$$
$$\xi \in L^2(0,T;H) \tag{3.23}$$

and fulfils the initial–boundary value problem

$$\partial_t\bigl(\ln\vartheta(t) - \lambda(\chi(t))\bigr) - \Delta(k_0 u + k * u)(t) = R(t) \quad \text{in } V_0', \text{ for a.a. } t \in (0,T) \tag{3.24}$$
$$\partial_t\chi - \Delta\chi + \xi + \sigma'(\chi) = -\lambda'(\chi)\vartheta \qquad \text{a.e. in } \Omega \times (0,T) \tag{3.25}$$
$$\xi \in \beta(\chi) \quad \text{a.e. in } \Omega \times (0,T) \tag{3.26}$$
$$(\ln\vartheta)(0) = \ln\vartheta_0 \quad \text{and} \quad \chi(0) = \chi_0. \tag{3.27}$$

In [5], the above problem is studied carefully from the mathematical point of view, and existence, uniqueness, regularity, and continuous dependence results are proved. From [5, Thm. 2.1], we deduce the following

Theorem 3.2. *Assume* (3.3–10) *and* (3.13–19) *with the notation* (3.1–2). *Then, there exists a unique triplet* (ϑ, χ, ξ) *satisfying* (3.20–23) *and solving problem* (3.24–27) *for every* $T \in (0,+\infty)$. ∎

In this paper, we study the long time behaviour of such a solution. To this aim, we assume that the data R and ϑ_Γ suitably reach asymptotic values R_∞ and $\vartheta_{\Gamma,\infty}$, respectively, in the following sense (where R_∞ and $\vartheta_{\Gamma,\infty}$ can be meant also as functions on $(0,+\infty)$ constant with respect to time)

$$R_\infty \in H \quad \text{and} \quad \vartheta_{\Gamma,\infty} \in H^{1/2}(\Gamma) \tag{3.28}$$
$$R - R_\infty \in L^2(0,+\infty;H) \cap L^\infty(0,+\infty;H) \tag{3.29}$$
$$\vartheta_\Gamma - \vartheta_{\Gamma,\infty} \in L^2(0,+\infty;H^{-1/2}(\Gamma)). \tag{3.30}$$

Moreover, we need some more regularity of R and ϑ_Γ, namely

$$\partial_t R \in L^2(0,+\infty;H) \tag{3.31}$$
$$\partial_t\vartheta_\Gamma \in L^p(0,+\infty;L^\infty(\Gamma)) \quad \text{for every } p \in [1,+\infty] \tag{3.32}$$
$$\partial_t^2\vartheta_\Gamma \in L^1(0,+\infty;L^\infty(\Gamma)). \tag{3.33}$$

Remark 3.3. Our assumptions on ϑ_Γ and the general theory of harmonic functions ensure a number of properties and estimates for its harmonic extension $\vartheta_{\mathcal{H}}$ (see (3.19)) and for the harmonic extension $\vartheta_{\mathcal{H},\infty}$ of $\vartheta_{\Gamma,\infty}$ defined by

$$\vartheta_{\mathcal{H},\infty} \in V, \quad \Delta\vartheta_{\mathcal{H},\infty} = 0, \quad \text{and} \quad \vartheta_{\mathcal{H},\infty}|_\Gamma = \vartheta_{\Gamma,\infty}. \tag{3.34}$$

Some of them, namely

$$
\begin{aligned}
&\vartheta_{\mathcal{H}} - \vartheta_{\mathcal{H},\infty} \in L^2(0,+\infty;H) \\
&\partial_t \vartheta_{\mathcal{H}} \in L^p(0,+\infty;L^\infty(\Omega)) \quad \text{for every } p \in [1,+\infty] \\
&\partial_t^2 \vartheta_{\mathcal{H}} \in L^1(0,+\infty;L^\infty(\Omega)) \\
&\vartheta_* \le \vartheta_{\mathcal{H}}(t) \le \vartheta^* \quad \text{and} \quad \vartheta_* \le \vartheta_{\mathcal{H},\infty} \le \vartheta^* \quad \text{a.e. in } \Omega, \text{ for a.a. } t \in (0,+\infty)
\end{aligned}
\tag{3.35}
$$

will be used in the sequel. ■

Now, we introduce the $\omega-$limit related to the trajectory of the pair (ϑ, χ). We set

$$
\begin{aligned}
\omega = \bigl\{(\vartheta_\infty, \chi_\infty) \in H \times V : \quad &(\vartheta(t_n), \chi(t_n)) \to (\vartheta_\infty, \chi_\infty) \\
&\text{strongly in } H \times V \text{ for some sequence } \{t_n\} \nearrow +\infty\bigr\}.
\end{aligned}
\tag{3.36}
$$

Such an $\omega-$limit depends on the data, of course. However, we do not stress it in the notation. Besides, we consider the stationary problem of finding $(\vartheta_{\mathcal{S}}, \chi_{\mathcal{S}}, \xi_{\mathcal{S}})$ such that

$$
\vartheta_{\mathcal{S}} \in V, \quad \chi_{\mathcal{S}} \in W, \quad \text{and} \quad \xi_{\mathcal{S}} \in H \tag{3.37}
$$

$$
-(k_0 + k_\infty)\Delta\vartheta_{\mathcal{S}} = R_\infty \quad \text{a.e. in } \Omega, \qquad \vartheta_{\mathcal{S}}|_\Gamma = \vartheta_{\Gamma,\infty} \tag{3.38}
$$

$$
-\Delta\chi_{\mathcal{S}} + \xi_{\mathcal{S}} + \sigma'(\chi_{\mathcal{S}}) = -\lambda'(\chi_{\mathcal{S}})\vartheta_{\mathcal{S}} \qquad \text{a.e. in } \Omega \tag{3.39}
$$

$$
\xi_{\mathcal{S}} \in \beta(\chi_{\mathcal{S}}) \quad \text{a.e. in } \Omega. \tag{3.40}
$$

For the sake of convenience we also set

$$
u_{\mathcal{S}} := \vartheta_{\mathcal{S}} - \vartheta_{\mathcal{H},\infty} \tag{3.41}
$$

where $\vartheta_{\mathcal{H},\infty}$ is defined by (3.34). Then, we have

$$
u_{\mathcal{S}} \in V_0 \quad \text{and} \quad -(k_0 + k_\infty)\Delta u_{\mathcal{S}} = R_\infty \quad \text{a.e. in } \Omega\,. \tag{3.42}
$$

As a final assumption, we require that

$$
\inf_\Omega \vartheta_{\mathcal{S}} > 0. \tag{3.43}
$$

Remark 3.4. The above assumption makes sense since (3.38) is a proper definition of $\vartheta_{\mathcal{S}}$. Indeed, $k_0 + k_\infty > 0$, due to (3.12) and the last of (3.6). Note that a sufficient condition for (3.43) is that $R_\infty \ge 0$, thanks to the maximum principle. Moreover, note that the second of (3.37) contains the Neumann boundary condition $\partial_n \chi_{\mathcal{S}} = 0$ on Γ (cf. (3.2)). Finally, observe that $u_{\mathcal{S}} \in H^2(\Omega) \subset L^\infty(\Omega)$. Hence, accounting also for (3.43), we can assume that

$$
\vartheta_* \le \vartheta_{\mathcal{S}} \le \vartheta^* \quad \text{and} \quad |u_{\mathcal{S}}| \le \vartheta^* \quad \text{a.e. in } \Omega \tag{3.44}
$$

since such inequalities can be achieved just by changing the meaning of ϑ_* and ϑ^*. On the contrary, we point out that no positive lower bound is known for ϑ. ■

Our result is the theorem stated below, which gives some properties of ω and a relationship between ω and the above stationary problem.

Theorem 3.5. *Let* (3.3–10) *and* (3.13–19) *hold with the notation* (3.1–2). *Moreover, assume* (3.28–30), (3.31–33), *and* (3.43) *and recall definition* (3.36). *Then, the set* ω *is nonempty, compact, and connected with respect to the strong topology of* $H \times V$. *Moreover,* ω *is contained in* $V \times W$ *and for every* $(\vartheta_\infty, \chi_\infty) \in \omega$ *there exists* $\xi_{\mathcal{S}} \in H$ *such that* $(\vartheta_\infty, \chi_\infty, \xi_{\mathcal{S}})$ *solves the stationary problem* (3.37–40). ∎

Remark 3.6. Equation (3.39) with homogeneous Neumann boundary condition is the Euler-Lagrange equation of the functional

$$\mathcal{F}(z) := \frac{1}{2}\int_\Omega |\nabla z|^2 + \int_\Omega \bigl((\widehat{\beta} + \sigma)(z) + \lambda(z)\vartheta_{\mathcal{S}}\bigr), \quad z \in V.$$

Using (3.3–5), it is easy to see that $\mathcal{F}$ has an absolute minimum. On the other hand, as our assumption does not imply any convexity of $\mathcal{F}$, one cannot expect uniqueness for its Euler-Lagrange equation. Therefore, just the component $\vartheta_{\mathcal{S}}$ of the solution to problem (3.37–40) is unique, in general, and the whole trajectory $\{\vartheta(t),\ t \geq 0\}$ tends to $\vartheta_{\mathcal{S}} = \vartheta_\infty$ weakly in V and strongly in H as t tends to $+\infty$, while nothing can be concluded as far as the component χ is concerned. ∎

4. Proof of Theorem 3.5

In this section we prove Theorem 3.5. Our procedure is the following. First we perform a number of a priori estimates that provide some compactness and ensure, in particular, that the $\omega-$limit is nonempty and fulfills the basic properties stated in Theorem 3.5. Then, we pick any element $(\vartheta_\infty, \chi_\infty) \in \omega$ and prove its relationship with the limit problem (3.37–40). Our argument is the following. We choose a sequence $\{t_n\} \nearrow +\infty$ according to definition (3.36) and introduce the auxiliary functions

$$\vartheta_n(t) = \vartheta(t + t_n) \quad \text{and} \quad \chi_n(t) = \chi(t + t_n), \quad t \in [0, +\infty) \tag{4.1}$$

which solve problems close to (3.24–27). We show that the a priori estimates derived in the previous steps yield a number of estimates for such functions which allow us to take a weak limit point $(\vartheta^\infty, \chi^\infty)$ of the sequence $\{(\vartheta_n, \chi_n)\}$. We infer that $(\vartheta^\infty, \chi^\infty)$ solves a system close to (3.37–40), and the last step of the proof is to show that $(\vartheta^\infty, \chi^\infty)$ does not depend on time and coincides with the original pair $(\vartheta_\infty, \chi_\infty)$ of the $\omega-$limit.

Before starting to prove Theorem 3.5, let us recall some tools. As far as convolutions are concerned, we remark the identity

$$\partial_t(a * b) = a(0)b + (\partial_t a) * b \tag{4.2}$$

which holds whenever it makes sense, and the Young theorem

$$\|a * b\|_{L^r(0,T;X)} \leq \|a\|_{L^p(0,T)} \|b\|_{L^q(0,T;X)} \tag{4.3}$$

where X is a Banach space, $p, q, r \in [1, \infty]$ satisfy $1/r = (1/p) + (1/q) - 1$, and $T \in (0, +\infty]$. Next, we recall the Poincaré and Sobolev inequalities

$$\|v\|_H \le M_\Omega \|\nabla v\|_H \quad \text{for every } v \in V_0 \tag{4.4}$$

$$\|v\|_{L^p(\Omega)} \le M_\Omega \|v\|_V \quad \text{for every } v \in V \text{ and for } 1 \le p \le 6 \tag{4.5}$$

where M_Ω is a constant depending on Ω, only. Finally, we exploit the well-known Gronwall lemma in the following form (see, e.g., [9, Lemma A.4, p. 156]). Let $a \in [0, +\infty)$ and let $b, \varphi : [0, +\infty) \to \mathbb{R}$ be measurable nonnegative functions. Then

$$\varphi(t) \le a + \int_0^t b(s)\, \varphi(s)\, ds \quad \text{for a.a. } t \in (0, +\infty) \qquad \text{implies}$$

$$\varphi(t) \le a \exp(\|b\|_{L^1(0,+\infty)}) \quad \text{for a.a. } t \in (0, +\infty). \tag{4.6}$$

As far as constants are concerned, we use a general rule. In the sequel, δ denotes an arbitrary positive parameter, whose value is chosen whenever it is convenient to do it, while the symbol c stands for different constants which depend only on Ω and on the constants and the norms of the functions involved in the assumptions of our statements. A notation like c_δ or $c(T)$ allows the constant to depend on the specified parameter, in addition. Hence, the meaning of such symbols might change from line to line and even in the same chain of inequalities. On the contrary, we use a different notation for precise constants (cf. (4.4–5)) which we could refer to. Finally, we set

$$Q_t := \Omega \times (0, t) \quad \text{for } 0 < t \le +\infty\,. \tag{4.7}$$

As said at the beginning of the present section, our proof of Theorem 3.5 relies on a number a priori estimates. However, the regularity of the solution is not sufficient to completely justify the calculation we would like to perform. Therefore, we should come back to the procedure used in [5], where problem (3.24–27) has been solved by passing to the limit as $\varepsilon \searrow 0$ in an approximating problem depending on the positive parameter ε, and prove a priori estimates which are uniform with respect to ε. However, in order not to make the exposition too heavy, we prefer to proceed formally on the solution of problem (3.24–27). Of course, we think of a more regular structure and of smoother initial data for a while, but it is understood that we cannot use constants related to such a further regularity.

We remark that the source term R and the boundary datum are smooth enough by assumption and that no new property of the initial data ϑ_0 and χ_0 is needed (i.e., more regularity is assumed just for the approximating initial data) since we use weighted test functions, if necessary.

Now, we recall the main feature of the approximating problem, which has the following form

$$\partial_t\big(\varepsilon\vartheta_\varepsilon + \ln_\varepsilon(\vartheta_\varepsilon) - \lambda_\varepsilon(\chi_\varepsilon)\big) - \Delta(k_0 u_\varepsilon + k * u_\varepsilon) = R \qquad \text{a.e. in } \Omega \times (0, T) \tag{4.8}$$

$$\partial_t \chi_\varepsilon - \Delta\chi_\varepsilon + \xi_\varepsilon + \sigma'(\chi_\varepsilon) = -\lambda_\varepsilon'(\chi_\varepsilon)\vartheta_\varepsilon \qquad \text{a.e. in } \Omega \times (0, T) \tag{4.9}$$

$$\text{where} \quad u_\varepsilon := \vartheta_\varepsilon - \vartheta_{\mathcal{H}} \quad \text{and} \quad \xi_\varepsilon := \beta_\varepsilon(\chi_\varepsilon) \tag{4.10}$$

$$u_\varepsilon = \partial_t \chi_\varepsilon = 0 \quad \text{on } \Gamma \times (0, T) \tag{4.11}$$

$$\vartheta_\varepsilon(0) = \vartheta_{0\varepsilon} \quad \text{and} \quad \chi_\varepsilon(0) = \chi_0. \tag{4.12}$$

In equations (4.8–12), the functions $\ln_\varepsilon$, β_ε , and λ_ε are suitable Lipschitz continuous approximations of $\ln$, β , and λ , respectively, while $\vartheta_{0\varepsilon}$ is a regularization of ϑ_0 . It has been proved that problem (4.8–12) has a solution $(\vartheta_\varepsilon, \chi_\varepsilon, \xi_\varepsilon)$ and that such a solution tends to (ϑ, χ, ξ) in some appropriate topology as $\varepsilon \searrow 0$, at least for a subsequence. Hence, the solution of problem (3.24–27) will satisfy new a priori estimates, provided that corresponding uniform estimates are fulfilled by the solution of problem (4.8–12) and that just norms related either to reflexive Banach spaces or to dual spaces of separable Banach spaces are involved.

So, as it will be clear in a moment, the estimates in $L^\infty(0, +\infty; L^1(\Omega))$ we formally derive directly on the solution (ϑ, χ, ξ) actually do not hold in the limit. On the other hand, such estimates are used just as tools in the sequel. Hence, everything would work if we were dealing with the approximating problem. In order to clarify this point, we write a remark after each formal estimate.

First a priori estimate. We write the difference between (3.24) and the equality in (3.42) and test it by $u - u_{\mathcal{S}}$. Then, we multiply (3.25) by $\partial_t \chi$ and formally integrate by parts over Ω . Finally, we sum the obtained equality to each other and integrate over $(0, t)$. We obtain

$$
\begin{aligned}
&\int_0^t \langle \partial_t \ln \vartheta(s), u(s) - u_{\mathcal{S}} \rangle \, ds \\
&\quad + k_0 \int_{Q_t} |\nabla (u - u_{\mathcal{S}})|^2 + \int_{Q_t} (k * \nabla u - k_\infty \nabla u_{\mathcal{S}}) \cdot \nabla (u - u_{\mathcal{S}}) \\
&\quad + \int_{Q_t} |\partial_t \chi|^2 + \frac{1}{2} \int_\Omega |\nabla \chi(t)|^2 + \int_\Omega \widehat{\beta}(\chi(t)) \\
&= \int_{Q_t} (R - R_\infty)(u - u_{\mathcal{S}}) + \frac{1}{2} \int_\Omega |\nabla \chi_0|^2 + \int_\Omega (\widehat{\beta} + \sigma)(\chi_0) - \int_\Omega \sigma(\chi(t)) \\
&\quad + \int_{Q_t} \partial_t \lambda(\chi) \, (u - u_{\mathcal{S}} - \vartheta)
\end{aligned}
\tag{4.13}
$$

and treat each term that need some manipulation, separately. Using (3.44), we have

$$
\begin{aligned}
&\int_0^t \langle \partial_t \ln \vartheta(s), u(s) - u_{\mathcal{S}} \rangle \, ds = \int_0^t \langle \partial_t \ln \vartheta(s), \vartheta(s) - \vartheta_{\mathcal{H}}(s) - u_{\mathcal{S}} \rangle \, ds \\
&= \int_{Q_t} \partial_t (\vartheta - u_{\mathcal{S}} \ln \vartheta) - \int_{Q_t} (\partial_t \ln \vartheta) \vartheta_{\mathcal{H}} \\
&= \int_\Omega (\vartheta(t) - u_{\mathcal{S}} \ln \vartheta(t)) - \int_\Omega (\vartheta_0 - u_{\mathcal{S}} \ln \vartheta_0) \\
&\quad - \int_\Omega \ln \vartheta(t) \, \vartheta_{\mathcal{H}}(t) + \int_\Omega \ln \vartheta_0 \, \vartheta_{\mathcal{H}}(0) + \int_{Q_t} \ln \vartheta \, \partial_t \vartheta_{\mathcal{H}} \\
&\geq \int_\Omega (\vartheta(t) - \vartheta_{\mathcal{S}} \ln \vartheta(t)) - \int_\Omega (\vartheta_{\mathcal{H}}(t) - \vartheta_{\mathcal{H},\infty}) \ln \vartheta(t) + \int_{Q_t} \ln \vartheta \, \partial_t \vartheta_{\mathcal{H}} - c \\
&= \int_\Omega (\vartheta(t) + \vartheta_{\mathcal{S}} \ln^- \vartheta(t)) - \int_\Omega \vartheta_{\mathcal{S}} \ln^+ \vartheta(t)
\end{aligned}
$$

$$
\begin{aligned}
&- \int_\Omega (\vartheta_{\mathcal{H}}(t) - \vartheta_{\mathcal{H},\infty}) \ln \vartheta(t) + \int_{Q_t} \ln \vartheta \, \partial_t \vartheta_{\mathcal{H}} - c \\
&\geq \int_\Omega (\vartheta(t) + \vartheta_* \ln^- \vartheta(t)) - \vartheta^* \int_\Omega \ln^+ \vartheta(t) \\
&\quad - (\vartheta^* - \vartheta_*) \int_\Omega |\ln \vartheta(t)| - \int_0^t \|\ln \vartheta(s)\|_{L^1(\Omega)} \|\partial_t \vartheta_{\mathcal{H}}(s)\|_{L^\infty(\Omega)} \, ds - c. \qquad (4.14)
\end{aligned}
$$

Now, we set

$$
\mathcal{L}(r) := r + \vartheta_* \ln^- r \quad \text{for } r > 0 \qquad (4.15)
$$

and note that $\mathcal{L}(r) \geq \delta_0$ for some $\delta_0 > 0$ and every $r > 0$, and that $|\ln r| \leq \delta \mathcal{L}(r) + c_\delta$. Hence, the intergals over Ω on the right hand side of (4.14) are dominated by any fraction of the first term, and we conclude that

$$
\int_0^t \langle \partial_t \ln \vartheta(s), u(s) - u_{\mathcal{S}} \rangle \, ds \geq \frac{1}{2} \int_\Omega \mathcal{L}(\vartheta(t)) - c - \int_0^t \|\partial_t \vartheta_{\mathcal{H}}(s)\|_{L^\infty(\Omega)} \mathcal{L}(\vartheta(s)) \, ds.
$$

Now, we deal with the second and third term of (4.13) and use assumption (3.7). We have

$$
\begin{aligned}
&k_0 \int_{Q_t} |\nabla(u - u_{\mathcal{S}})|^2 + \int_{Q_t} (k * \nabla u - k_\infty \nabla u_{\mathcal{S}}) \cdot \nabla(u - u_{\mathcal{S}}) \\
&= k_0 \int_{Q_t} |\nabla(u - u_{\mathcal{S}})|^2 + \int_{Q_t} (k * \nabla(u - u_{\mathcal{S}})) \cdot \nabla(u - u_{\mathcal{S}}) \\
&\quad + \int_{Q_t} \nabla(k * u_{\mathcal{S}} - k_\infty u_{\mathcal{S}}) \cdot \nabla(u - u_{\mathcal{S}}) \\
&\geq \alpha \int_{Q_t} |\nabla(u - u_{\mathcal{S}})|^2 - \delta \int_{Q_t} |\nabla(u - u_{\mathcal{S}})|^2 - c_\delta \int_{Q_t} |\nabla(k * u_{\mathcal{S}} - k_\infty u_{\mathcal{S}})|^2.
\end{aligned}
$$

On the other hand, we see that (3.8–9) imply

$$
\begin{aligned}
&\int_{Q_t} |\nabla(k * u_{\mathcal{S}} - k_\infty u_{\mathcal{S}})|^2 = \int_{Q_t} |k * \nabla u_{\mathcal{S}} - k_\infty \nabla u_{\mathcal{S}}|^2 \\
&= \int_{Q_t} \left| \int_0^s k(\tau) \nabla u_{\mathcal{S}} \, d\tau - \int_0^{+\infty} k(\tau) \nabla u_{\mathcal{S}} \, d\tau \right|^2 \\
&= \int_{Q_t} |\widehat{k} \, \nabla u_{\mathcal{S}}|^2 = \int_\Omega |\nabla u_{\mathcal{S}}|^2 \int_0^t |\widehat{k}(s)|^2 \, ds \leq c.
\end{aligned}
$$

Finally, the remaining terms on the left hand side are nonnegative (cf. (3.3)). Hence, let us consider the right hand side. The first term is easy to handle using the Poincaré inequality (4.4) and condition (3.29). We have indeed

$$
\begin{aligned}
&\int_{Q_t} (R - R_\infty)(u - u_{\mathcal{S}}) \leq \delta \int_{Q_t} |u - u_{\mathcal{S}}|^2 + c_\delta \int_{Q_t} |R - R_\infty|^2 \\
&\leq \delta M_\Omega \int_{Q_t} |\nabla(u - u_{\mathcal{S}})|^2 + c_\delta.
\end{aligned}
$$

As the next terms of (4.13) are given quantities (cf. (3.18)), we deal with the σ term. It is easy to see that (3.3–5) imply that

$$r^2 \leq \delta \widehat{\beta}(r) + c_\delta \quad \text{and} \quad |\lambda(r)| + |\sigma(r)| \leq M_{\lambda,\sigma}(r^2+1) \quad \text{for every } r \in \mathbb{R} \tag{4.16}$$

for some constant $M_{\lambda,\sigma}$. Hence, we deduce that

$$-\int_\Omega \sigma(\chi(t)) \leq \delta \int_\Omega \widehat{\beta}(\chi(t)) + c_\delta.$$

Next, we treat the last integral of (4.13). Using (4.16) once more and owing to (3.35), we have

$$\begin{aligned}
&\int_{Q_t} \partial_t \lambda(\chi)\,(u - u_{\mathcal{S}} - \vartheta) = -\int_{Q_t} \partial_t \lambda(\chi)\,(\vartheta_{\mathcal{H}} + u_{\mathcal{S}}) \\
&= -\int_\Omega \lambda(\chi(t))(\vartheta_{\mathcal{H}}(t) + u_{\mathcal{S}}) + \int_\Omega \lambda(\chi_0)(\vartheta_{\mathcal{H}}(0) + u_{\mathcal{S}}) + \int_{Q_t} \lambda(\chi)\,\partial_t \vartheta_{\mathcal{H}} \\
&= -\int_\Omega \lambda(\chi(t))\vartheta_{\mathcal{S}} - \int_\Omega \lambda(\chi(t))(\vartheta_{\mathcal{H}}(t) - \vartheta_{\mathcal{H},\infty}) \\
&\quad + \int_\Omega \lambda(\chi_0)(\vartheta_{\mathcal{H}}(0) + u_{\mathcal{S}}) + \int_{Q_t} \lambda(\chi)\,\partial_t \vartheta_{\mathcal{H}} \\
&\leq \vartheta^* M_{\lambda,\sigma} \int_\Omega (|\chi(t)|^2 + 1) + M_{\lambda,\sigma} \|\vartheta_{\mathcal{H}}(t) - \vartheta_{\mathcal{H},\infty}\|_{L^\infty(\Omega)} \int_\Omega (|\chi(t)|^2 + 1) \\
&\quad + \int_\Omega \lambda(\chi_0)(\vartheta_{\mathcal{H}}(0) + u_{\mathcal{S}}) + M_{\lambda,\sigma} \int_0^t \|\partial_t \vartheta_{\mathcal{H}}(s)\|_{L^\infty(\Omega)} (\|\chi(s)\|_H^2 + 1) \\
&\leq \delta \int_\Omega \widehat{\beta}(\chi(t)) + \int_0^t \|\partial_t \vartheta_{\mathcal{H}}(s)\|_{L^\infty(\Omega)} \|\widehat{\beta}(\chi(s))\|_{L^1(\Omega)}\, ds + c_\delta.
\end{aligned}$$

At this point, we collect (4.13) and all the inequalities we have obtained and choose δ small enough. Then, owing to (3.35), we apply the Gronwall lemma (4.6) and conclude that the following a priori estimate holds

$$\begin{aligned}
&\|\mathcal{L}(\vartheta)\|_{L^\infty(0,+\infty;L^1(\Omega))} + \|u - u_{\mathcal{S}}\|_{L^2(0,+\infty;V)} \\
&\quad + \|\partial_t \chi\|_{L^2(0,+\infty;H)} + \|\nabla \chi\|_{L^\infty(0,+\infty;H)} + \|\widehat{\beta}(\chi)\|_{L^\infty(0,+\infty;L^1(\Omega))} \leq c.
\end{aligned} \tag{4.17}$$

Consequences. In view of (4.15) and (4.16), we infer that

$$\|\vartheta\|_{L^\infty(0,+\infty;L^1(\Omega))} + \|\ln \vartheta\|_{L^\infty(0,+\infty;L^1(\Omega))} + \|\chi\|_{L^\infty(0,+\infty;V)} \leq c. \tag{4.18}$$

Next, we deduce an estimate involving the convolution term. We have

$$-\big(k * \Delta u - k_\infty \Delta u_{\mathcal{S}}\big) = -k * \Delta(u - u_{\mathcal{S}}) + \widehat{k}\,\Delta u_{\mathcal{S}}.$$

On the other hand, $\Delta u_{\mathcal{S}} = -(k_0 + k_\infty)^{-1} R_\infty$ is a known element of H. Hence, using the Young theorem, (3.8), and (4.17), we deduce

$$\begin{aligned}
&\|-(k * \Delta u - k_\infty \Delta u_{\mathcal{S}})\|_{L^2(0,+\infty;V_0')} \le \|-k * \Delta(u - u_{\mathcal{S}})\|_{L^2(0,+\infty;V_0')} + c\|\widehat{k}\|_{L^2(0,+\infty)} \\
&\le \|k\|_{L^1(0,+\infty)} \|\Delta(u - u_{\mathcal{S}})\|_{L^2(0,+\infty;V_0')} + c \le c\|u - u_{\mathcal{S}}\|_{L^2(0,+\infty;V)} + c \le c.
\end{aligned}$$

By comparison in the difference between (3.24) and (3.42), by (3.29) we get

$$\|\partial_t(\ln\vartheta - \lambda(\chi))\|_{L^2(0,+\infty;V_0')} \le c. \tag{4.19}$$

Remark 4.1. As said before, the above estimates (4.17) and (4.18) should be performed on the approximating problems. Doing that, we would obtain a uniform bound for both ϑ_ε and $\ln_\varepsilon(\vartheta_\varepsilon)$ in the space $L^\infty(0,+\infty;L^1(\Omega))$. Moreover, we note that the main trouble in our formal procedure is the derivation of (4.13), since the time derivative $\partial_t \ln\vartheta$ belongs just to $L^2(0,T;V_0')$. On the contrary, as the graph of the logarithm is replaced by a bi-Lipschitz relation (see (4.8)), the corresponding term of the approximating problem is a function. ∎

Second a priori estimate. We set for convenience $\zeta(t) = \tanh t$ for $t \ge 0$ and note that both ζ and ζ' are bounded by 1. Now, we take the difference between (3.24) and (3.42) and test it by $\zeta\partial_t u = \zeta\partial_t(u - u_{\mathcal{S}})$. Next, we differentiate (3.25) with respect to time and obtain a second order equation. Then, we test it by $\zeta\partial_t\chi$. Finally, we add the equalities we get to each other and integrate over $(0,t)$. We can write

$$\begin{aligned}
&\int_{Q_t} (\partial_t \ln\vartheta)\zeta\,\partial_t u + k_0 \int_{Q_t} \zeta\,\nabla(u - u_{\mathcal{S}}) \cdot \nabla\partial_t(u - u_{\mathcal{S}}) \\
&\quad + \int_{Q_t} \zeta\,\partial_t^2\chi\,\partial_t\chi + \int_{Q_t} \zeta|\nabla\partial_t\chi|^2 + \int_{Q_t} \zeta\,\beta'(\chi)|\partial_t\chi|^2 \\
&= -\int_{Q_t} \zeta\big(k * \nabla u - k_\infty \nabla u_{\mathcal{S}}\big) \cdot \nabla\partial_t(u - u_{\mathcal{S}}) \\
&\quad + \int_{Q_t} \zeta(R - R_\infty)\,\partial_t(u - u_{\mathcal{S}}) - \int_{Q_t} \zeta\,\sigma''(\chi)|\partial_t\chi|^2 \\
&+ \int_{Q_t} \zeta\,\lambda'(\chi)\,\partial_t\chi\,\partial_t(u - \vartheta) - \int_{Q_t} \zeta\,\lambda''(\chi)\,\vartheta\,|\partial_t\chi|^2
\end{aligned} \tag{4.20}$$

and treat each term separately. We have

$$\int_{Q_t} (\partial_t \ln\vartheta)\zeta\,\partial_t u = \int_{Q_t} \zeta\,\frac{|\partial_t\vartheta|^2}{\vartheta} - \int_{Q_t} (\partial_t \ln\vartheta)\,\zeta\,\partial_t\vartheta_{\mathcal{H}}.$$

The first term is nonnegative. We move the second one to the right hand side and estimate it by integrating by parts as follows

$$\begin{aligned}
&\int_{Q_t} (\partial_t \ln\vartheta)\,\zeta\,\partial_t\vartheta_{\mathcal{H}} = \zeta(t) \int_\Omega \ln\vartheta(t)\,\partial_t\vartheta_{\mathcal{H}}(t) - \int_{Q_t} \ln\vartheta\,\partial_t(\zeta\partial_t\vartheta_{\mathcal{H}}) \\
&\le \|\partial_t\vartheta_{\mathcal{H}}\|_{L^\infty(Q_\infty)} \|\ln\vartheta(t)\|_{L^1(\Omega)} \\
&\quad + \big(\|\partial_t\vartheta_{\mathcal{H}}\|_{L^1(0,+\infty;L^\infty(\Omega))} + \|\partial_t^2\vartheta_{\mathcal{H}}\|_{L^1(0,+\infty;L^\infty(\Omega))}\big)\, \|\ln\vartheta\|_{L^\infty(0,+\infty;L^1(\Omega))} \le c
\end{aligned}$$

where we have used (3.35) (and Remark 4.1 if we are thinking of performing the estimate on the approximating problem). The next term is easy to handle. In view of (4.17), we have indeed

$$\int_{Q_t} \zeta \nabla(u-u_{\mathcal{S}}) \cdot \nabla \partial_t (u-u_{\mathcal{S}}) = \frac{\zeta(t)}{2} \int_\Omega |\nabla(u(t)-u_{\mathcal{S}})|^2 - \frac{1}{2} \int_{Q_t} \zeta' |\nabla(u-u_{\mathcal{S}})|^2$$
$$\geq \frac{\zeta(t)}{2} \int_\Omega |\nabla(u(t)-u_{\mathcal{S}})|^2 - c.$$

We obtain similarly

$$\int_{Q_t} \zeta\, \partial_t^2 \chi\, \partial_t \chi \geq \frac{\zeta(t)}{2} \int_\Omega |\partial_t \chi(t)|^2 - c$$

and we can deal with the right hand side, since the next integrals are nonnegative. The convolution term needs much more work, as usual. We have

$$\begin{aligned} &- \int_{Q_t} \zeta \big(k * \nabla u - k_\infty \nabla u_{\mathcal{S}}\big) \cdot \nabla \partial_t (u-u_{\mathcal{S}}) \\ &= - \int_{Q_t} \zeta (k * \nabla (u-u_{\mathcal{S}})) \cdot \nabla \partial_t (u-u_{\mathcal{S}}) \\ &\quad - \int_{Q_t} \zeta \big(k * \nabla u_{\mathcal{S}} - k_\infty \nabla u_{\mathcal{S}}\big) \cdot \nabla \partial_t (u-u_{\mathcal{S}}) \end{aligned} \tag{4.21}$$

and deal with each term separately. We estimate the first one integrating by parts. Owing to (4.2–3) and once more to (4.17), we infer

$$\begin{aligned} &- \int_{Q_t} \zeta (k * \nabla (u-u_{\mathcal{S}})) \cdot \nabla \partial_t (u-u_{\mathcal{S}}) \\ &= -\zeta(t) \int_\Omega (k * \nabla(u-u_{\mathcal{S}}))(t) \cdot \nabla(u(t)-u_{\mathcal{S}}) + \int_{Q_t} \zeta' (k * \nabla(u-u_{\mathcal{S}})) \cdot \nabla(u-u_{\mathcal{S}}) \\ &\quad + k(0) \int_{Q_t} \zeta |\nabla(u-u_{\mathcal{S}})|^2 + \int_{Q_t} \zeta (k' * \nabla(u-u_{\mathcal{S}})) \cdot \nabla(u-u_{\mathcal{S}}) \\ &\leq \delta \zeta(t) \int_\Omega |\nabla(u(t)-u_{\mathcal{S}})|^2 \\ &\quad + \Big(c_\delta \|k\|^2_{L^2(0,+\infty)} + \|k\|_{L^1(0,+\infty)} + |k(0)| + \|k'\|_{L^1(0,+\infty)} \Big) \int_{Q_t} |\nabla(u-u_{\mathcal{S}})|^2 \\ &\leq \delta \zeta(t) \int_\Omega |\nabla(u(t)-u_{\mathcal{S}})|^2 + c_\delta. \end{aligned}$$

Consider the second term of (4.21). Noting that $\widehat{k}$ is bounded due to (3.6), owing to (3.6) itself and to (3.8), and using (4.17) again, we obtain

$$- \int_{Q_t} \zeta \big(k * \nabla u_{\mathcal{S}} - k_\infty \nabla u_{\mathcal{S}}\big) \cdot \nabla \partial_t (u-u_{\mathcal{S}}) = \int_{Q_t} \zeta \widehat{k}\, \nabla u_{\mathcal{S}} \cdot \nabla \partial_t (u-u_{\mathcal{S}})$$

$$
\begin{aligned}
&= \zeta(t)\,\widehat{k}(t)\int_\Omega \nabla u_{\mathcal S}\cdot\nabla(u(t)-u_{\mathcal S})\\
&\quad - \int_{Q_t}\zeta'\,\widehat{k}\;\nabla u_{\mathcal S}\cdot\nabla(u-u_{\mathcal S}) + \int_{Q_t}\zeta k\,\nabla u_{\mathcal S}\cdot\nabla(u-u_{\mathcal S})\\
&\leq \delta\zeta(t)\int_\Omega |\nabla(u(t)-u_{\mathcal S})|^2 + c_\delta|\widehat{k}(t)|^2\int_\Omega|\nabla u_{\mathcal S}|^2\\
&\quad + \frac12\left(\int_0^t|\widehat{k}(s)|^2\,ds + \int_0^t|k(s)|^2\,ds\right)\int_{Q_t}|\nabla u_{\mathcal S}|^2 + \int_{Q_t}|\nabla(u-u_{\mathcal S})|^2\\
&\leq \delta\zeta(t)\int_\Omega|\nabla(u(t)-u_{\mathcal S})|^2 + c_\delta.
\end{aligned}
$$

As both term of (4.21) are estimated, we come back to (4.20) and consider the next integral. Owing to the Poincaré inequality and to (3.29), (3.31), we easily have

$$
\begin{aligned}
&\int_{Q_t}\zeta(R-R_\infty)\,\partial_t(u-u_{\mathcal S}) = \zeta(t)\int_\Omega (R(t)-R_\infty)(u(t)-u_{\mathcal S})\\
&\quad - \int_{Q_t}\zeta'\,(R-R_\infty)(u-u_{\mathcal S}) - \int_{Q_t}\zeta\,(\partial_t R)(u-u_{\mathcal S})\\
&\leq \delta\zeta\int_\Omega|\nabla(u(t)-u_{\mathcal S})|^2 + c_\delta\int_\Omega|R(t)-R_\infty|^2\\
&\quad + \int_{Q_t}|\nabla(u-u_{\mathcal S})|^2 + c\int_{Q_t}\left(|R-R_\infty|^2+|\partial_t R|^2\right)\\
&\leq \delta\zeta\int_\Omega|\nabla(u(t)-u_{\mathcal S})|^2 + c_\delta.
\end{aligned}
$$

The σ'' term of (4.20) is bounded thanks to (3.5) and to (4.17). Hence, let us come to the second last integral. Accounting for (3.5), (3.35) and (4.17), we obtain

$$
\begin{aligned}
&\int_{Q_t}\zeta\,\lambda'(\chi)\,\partial_t\chi\,\partial_t(u-\vartheta) = -\int_{Q_t}\zeta\,\lambda'(\chi)\,\partial_t\chi\,\partial_t\vartheta_{\mathcal H} \leq c\int_{Q_t}(1+|\chi|)\,|\partial_t\chi|\,|\partial_t\vartheta_{\mathcal H}|\\
&\leq \int_{Q_t}|\partial_t\chi|^2 + \int_{Q_t}|\partial_t\vartheta_{\mathcal H}|^2 + \int_{Q_t}|\chi|\,|\partial_t\chi|\,|\partial_t\vartheta_{\mathcal H}| \leq c + \int_{Q_t}|\chi|\,|\partial_t\chi|\,|\partial_t\vartheta_{\mathcal H}|
\end{aligned}
$$

and we have to estimate the last integral. We do that using (4.18) and noting that (3.35) imply $\partial_t\vartheta_{\mathcal H}\in L^2(0,+\infty;L^\infty(\Omega))$.

$$
\begin{aligned}
&\int_{Q_t}|\chi|\,|\partial_t\chi|\,|\partial_t\vartheta_{\mathcal H}| \leq \int_0^t \|\chi(s)\|_H\|\partial_t\chi\|_H\|\partial_t\vartheta_{\mathcal H}(s)\|_{L^\infty(\Omega)}\,ds\\
&\leq \int_{Q_t}|\partial_t\chi|^2 + \int_0^t\|\chi(s)\|_H^2\|\partial_t\vartheta_{\mathcal H}(s)\|_{L^\infty(\Omega)}^2\,ds\\
&\leq c + \|\chi\|_{L^\infty(0,+\infty;H)}^2\|\partial_t\vartheta_{\mathcal H}(s)\|_{L^2(0,+\infty;L^\infty(\Omega))}^2\,ds \leq c.
\end{aligned}
$$

Finally, we deal with the last term of (4.20). By (3.5), we have

$$
\begin{aligned}
-\int_{Q_t} \zeta\, \lambda''(\chi)\, \vartheta\, |\partial_t \chi|^2 &\le c \int_{Q_t} \zeta\, \vartheta\, |\partial_t \chi|^2 \\
&= c \int_{Q_t} \zeta(\vartheta_{\mathcal{H}} + u_{\mathcal{S}}) |\partial_t \chi|^2 + c \int_{Q_t} \zeta(u - u_{\mathcal{S}}) |\partial_t \chi|^2 .
\end{aligned}
$$

The first integral on the right hand side is bounded by (3.35), (3.44), and (4.17), and the second one can be treated owing the Hölder, Sobolev, and Poincaré inequalities and using (4.17). We have indeed

$$
\begin{aligned}
&\int_{Q_t} \zeta(u - u_{\mathcal{S}}) |\partial_t \chi|^2 \le \int_0^t \zeta(s) \|u(s) - u_{\mathcal{S}}\|_{L^4(\Omega)} \|\partial_t \chi(s)\|_{L^4(\Omega)} \|\partial_t \chi(s)\|_{L^2(\Omega)}\, ds \\
&\le M_\Omega^2 \int_0^t \zeta(s) \|u(s) - u_{\mathcal{S}}\|_V \|\partial_t \chi(s)\|_V \|\partial_t \chi(s)\|_H\, ds \\
&\le c \int_0^t \zeta(s) \|u(s) - u_{\mathcal{S}}\|_V \|\partial_t \chi(s)\|_H^2\, ds \\
&\quad + c \int_0^t \zeta(s) \|u(s) - u_{\mathcal{S}}\|_V \|\nabla \partial_t \chi(s)\|_H \|\partial_t \chi(s)\|_H\, ds \\
&\le c \int_0^t \zeta(s) \|\partial_t \chi(s)\|_H^2 \left(1 + \|u(s) - u_{\mathcal{S}}\|_V^2\right) \\
&\quad + \delta \int_{Q_t} \zeta |\nabla \partial_t \chi|^2 + c_\delta \int_0^t \zeta(s) \|u(s) - u_{\mathcal{S}}\|_V^2 \|\partial_t \chi(s)\|_H^2\, ds \\
&\le c + \delta \int_{Q_t} \zeta |\nabla \partial_t \chi|^2 + c_\delta \int_0^t \zeta(s) \|\nabla(u(s) - u_{\mathcal{S}})\|_H^2 \|\partial_t \chi(s)\|_H^2\, ds
\end{aligned}
$$

and we point out that $\|\partial_t \chi(\,\cdot\,)\|_H^2 \in L^1(0, +\infty)$ by (4.17). Therefore, we collect (4.20) and all the inequalities we have derived. Then, we choose δ small enough and apply the Gronwall lemma (4.6). We obtain

$$
\begin{aligned}
&\int_{Q_\infty} \zeta \, \frac{|\partial_t \vartheta|^2}{\vartheta} + \sup_{t \ge 0} \zeta(t) \int_\Omega |\nabla(u(t) - u_{\mathcal{S}})|^2 \\
&\quad + \sup_{t \ge 0} \zeta(t) \int_\Omega |\partial_t \chi(t)|^2 + \int_{Q_\infty} \zeta |\nabla \partial_t \chi|^2 \le c
\end{aligned}
\tag{4.22}
$$

whence, in particular,

$$
\begin{aligned}
&\|\partial_t \sqrt{\vartheta}\|_{L^2(1,+\infty;H)} + \|u - u_{\mathcal{S}}\|_{L^\infty(1,+\infty;V_0)} + \|\vartheta\|_{L^\infty(1,+\infty;V)} \\
&\quad + \|\partial_t \chi\|_{L^\infty(1,+\infty;H)} + \|\partial_t \chi\|_{L^2(1,+\infty;V)} \le c.
\end{aligned}
\tag{4.23}
$$

Remark 4.2. In the above argument, we have differentiated (3.25). Such a procedure would be correct when dealing with the approximating problem (4.8–12), provided that its solution is smooth enough. Now, one could go through the proofs of [5] and see that the approximating solution is smoother provided that the data, the functions ln, λ, σ, and the graph β are approximated with some more care. On the other hand, the passage to the limit as $\varepsilon \searrow 0$ uses just very general properties and does not rely on a precise approximation. For instance, as far as β is concerned, one can see that the monotonicity of β_ε and the Mosco convergence of its primitive $\widehat{\beta}_\varepsilon$ to $\widehat{\beta}$ are sufficient to handle the ξ term. ∎

Third a priori estimate. Using (3.5) and the Sobolev inequality, we immediately see that (4.17) implies

$$\|\lambda'(\chi)\|_{L^\infty(0,+\infty;L^6(\Omega))} + \|\sigma'(\chi)\|_{L^\infty(0,+\infty;L^6(\Omega))} \leq c.$$

Hence, from (4.23) we also deduce

$$\|\lambda'(\chi)\vartheta\|_{L^\infty(1,+\infty;H)} \leq c.$$

Accounting for the estimate (4.23) of $\partial_t\chi$, we infer that

$$\|-\Delta\chi + \xi\|_{L^\infty(1,+\infty;H)} \leq c$$

by comparison in (3.25). Therefore, a standard argument shows that both $\Delta\chi$ and ξ are estimated in the same space. Due to the Neumann boundary condition for χ and the elliptic regularity theory, we conclude that

$$\|\chi\|_{L^\infty(1,+\infty;W)} + \|\xi\|_{L^\infty(1,+\infty;H)} \leq c. \tag{4.24}$$

Fourth a priori estimate. We derive a bound for $\partial_t\vartheta$. We have $\partial_t\vartheta = 2\sqrt{\vartheta}\,\partial_t\sqrt{\vartheta}$. On the other hand, (4.23) and the Sobolev inequality (4.5) imply that

$$\|\sqrt{\vartheta}\|_{L^\infty(1,+\infty;L^{12}(\Omega))} = \|\vartheta\|^{1/2}_{L^\infty(1,+\infty;L^6(\Omega))} \leq c\|\vartheta\|^{1/2}_{L^\infty(1,+\infty;V)} \leq c.$$

Therefore, combining with (4.23) through the Hölder inequality, we conclude that

$$\|\partial_t\vartheta\|_{L^2(1,+\infty;L^{12/7}(\Omega))} \leq 2\|\sqrt{\vartheta}\|_{L^\infty(1,+\infty;L^{12}(\Omega))}\|\partial_t\sqrt{\vartheta}\|_{L^2(1,+\infty;L^2(\Omega))} \leq c. \tag{4.25}$$

Consequence. Using (4.2), we obtain

$$\partial_t(k*u) = ku(0) + k*\partial_t u = k(\vartheta_0 - \vartheta_{\mathcal{H}}(0)) + k*\partial_t\vartheta - k*\partial_t\vartheta_{\mathcal{H}}.$$

Combining it with (3.6), (4.25), and (3.35), we infer that

$$\|\partial_t(k*u)\|_{L^2(1,+\infty;L^{12/7}(\Omega))} \leq c. \tag{4.26}$$

Study of the $\omega-$limit. First of all, we observe that ϑ is an $L^{12/7}(\Omega)$ valued continuous function on $[1,+\infty)$ and χ is a V valued continuous function on the same interval, thanks to (4.25) and (4.23), respectively. Accounting for the estimates of $\|\vartheta\|_{L^\infty(1,+\infty;V)}$ and $\|\chi\|_{L^\infty(1,+\infty;W)}$ given by (4.23) and (4.24), we deduce that (the continuous representatives of) ϑ and χ are continuous also with respect to the weak topologies of V and W, respectively. Hence, we have

$$\|\vartheta(t)\|_V + \|\chi(t)\|_W \leq c \quad \text{for every } t \geq 1 \tag{4.27}$$

and the $\omega-$limit ω given by (3.36) is nonempty and contained in $V \times W$. Next, using the compact embeddings $W \subset V \subset H$, we immediately see that ω is relatively compact in $H \times V$. Moreover, general results (see, e.g., [28]) imply that it is compact and connected with respect to the strong topology of $H \times V$. Hence, to prove Theorem 3.5, it remains to show that for every element $(\vartheta_\infty, \chi_\infty) \in \omega$ the pair $(\vartheta_{\mathcal{S}}, \chi_{\mathcal{S}}) := (\vartheta_\infty, \chi_\infty)$ yields the first two components of a solution $(\vartheta_{\mathcal{S}}, \chi_{\mathcal{S}}, \xi_{\mathcal{S}})$ to problem (3.37–40). Therefore, we pick $(\vartheta_\infty, \chi_\infty) \in \omega$ and a sequence $\{t_n\} \nearrow +\infty$ such that

$$(\vartheta(t_n), \chi(t_n)) \to (\vartheta_\infty, \chi_\infty) \quad \text{strongly in } H \times V . \tag{4.28}$$

We can assume $t_n \geq 1$ for every n. Moreover, as any subsequence of $\{t_n\}$ enjoys the same properties of the original sequence, we do not change the notation when passing to a subsequence. We introduce the functions ϑ_n and χ_n given by (4.1) and the functions u_n, ξ_n, η_n, and R_n defined similarly, i.e.,

$$u_n(t) := u(t{+}t_n),\ \xi_n(t) := \xi(t{+}t_n),\ \eta_n(t) := (k{*}u)(t{+}t_n),\ R_n(t) := R(t{+}t_n) \tag{4.29}$$

for $t \geq 0$. Note that $\eta_n \neq k * u_n$. Then, $(\vartheta_n, \chi_n, \xi_n, \eta_n)$ solves the system

$$\partial_t\big(\ln\vartheta_n(t) - \lambda(\chi_n(t))\big) - k_0\Delta u_n(t) - \Delta\eta_n(t) = R_n(t)$$
$$\text{in } V_0', \text{ for a.a. } t \in (0,+\infty) \tag{4.30}$$
$$\partial_t\chi_n - \Delta\chi_n + \xi_n + \sigma'(\chi_n) = -\lambda'(\chi_n)\vartheta_n \qquad \text{a.e. in } Q_\infty \tag{4.31}$$
$$\xi_n \in \beta(\chi_n) \quad \text{a.e. in } Q_\infty \tag{4.32}$$
$$\vartheta_n(0) = \vartheta(t_n) \quad \text{and} \quad \chi_n(0) = \chi(t_n). \tag{4.33}$$

Now, we would pass to the limit in such a system as $n \nearrow \infty$. More precisely, we look at all functions involved in (4.30–33) and take their weak limits. Then, we identify such limits in term of the element $(\vartheta_\infty, \chi_\infty)$ of ω we have fixed in (4.28).

Weak limits. Collecting all the estimates we have obtained in the previous steps, we see that

$$\begin{aligned}
&\|\vartheta_n\|_{L^\infty(0,T;V)\cap H^1(0,T;L^{12/7}(\Omega))} + \|\chi_n\|_{L^\infty(0,T;W)\cap H^1(0,T;V)\cap W^{1,\infty}(0,T;H)} \\
&\quad + \|\xi_n\|_{L^\infty(0,T;H)} + \|\eta_n\|_{L^\infty(0,T;V)\cap H^1(0,T;L^{12/7}(\Omega))} \\
&\quad + \|\partial_t(\ln\vartheta_n - \lambda(\chi_n))\|_{L^2(0,T;V_0')} \leq c(T)
\end{aligned}$$

where $T \in (0,+\infty)$ is arbitrary. Then, taking a sequence $\{T_k\} \nearrow +\infty$ and using a diagonal procedure, we deduce that there exist functions ϑ^∞, χ^∞, ξ^∞, and η^∞

on $(0,+\infty)$ such that the following weak star convergences hold (for a subsequence) for every $T \in (0,+\infty)$

$$\vartheta_n \to \vartheta^\infty \quad \text{in } L^\infty(0,T;V) \cap H^1(0,T;L^{12/7}(\Omega)) \tag{4.34}$$
$$\chi_n \to \chi^\infty \quad \text{in } L^\infty(0,T;W) \cap H^1(0,T;V) \cap W^{1,\infty}(0,T;H) \tag{4.35}$$
$$\xi_n \to \xi^\infty \quad \text{in } L^\infty(0,T;H) \tag{4.36}$$
$$\eta_n \to \eta^\infty \quad \text{in } L^\infty(0,T;V) \cap H^1(0,T;L^{12/7}(\Omega)). \tag{4.37}$$

Using [35, Cor. 4, Sec. 8], we deduce the strong convergences

$$\vartheta_n \to \vartheta^\infty \quad \text{in } C^0([0,T];H) \tag{4.38}$$
$$\chi_n \to \chi^\infty \quad \text{in } C^0([0,T];V) \tag{4.39}$$

whence also

$$u_n \to u^\infty := \vartheta^\infty - \vartheta_{\Gamma,\infty}$$
$$\text{strongly in } C^0([0,T];H) \text{ and weakly star in } L^\infty(0,T;V_0). \tag{4.40}$$

Moreover, the convergence we have obtained are sufficient to identify the limits of the nonlinear terms. For instance, we have $\xi^\infty \in \beta(\chi^\infty)$ since we can apply, e.g., [2, p. 42]. On the other hand, just by direct computation, one sees that a bound of the form $\|v\|_{L^p(1,+\infty;X)} \le c$, where $p < +\infty$ and X a Banach space, implies $v_n \to 0$ strongly in $L^p(0,+\infty;X)$, where $v_n(t) := v(t+t_n)$. Hence, the a priori estimates (4.19), (4.17), (4.25), and (4.26) yield the strong convergences

$$\partial_t(\ln\vartheta_n - \lambda(\chi_n)) \to 0 \quad \text{in } L^2(0,+\infty;V_0') \tag{4.41}$$
$$\partial_t\chi_n \to 0 \quad \text{in } L^2(0,+\infty;H) \tag{4.42}$$
$$\partial_t\vartheta_n \to 0 \quad \text{in } L^2(0,+\infty;L^{12/7}(\Omega)) \tag{4.43}$$
$$\partial_t\eta_n \to 0 \quad \text{in } L^2(0,+\infty;L^{12/7}(\Omega)) \tag{4.44}$$

respectively. In particular, we can take $n \nearrow \infty$ in (4.30–31) and get

$$-k_0\Delta u^\infty - \Delta\eta^\infty = R_\infty \qquad \text{in } V_0', \quad \text{a.e. in } (0,+\infty) \tag{4.45}$$
$$-\Delta\chi^\infty + \xi^\infty + \sigma'(\chi^\infty) = -\lambda'(\chi^\infty)\,\vartheta^\infty \quad \text{and} \quad \xi^\infty \in \beta(\chi^\infty) \quad \text{a.e. in } Q_\infty. \tag{4.46}$$

Moreover, the boundary conditions are fulfilled as well, since u^∞ and χ^∞ take values in V_0 and in W, respectively.

Conclusion. The proof of Theorem 3.5 is complete whenever we show that

$$\vartheta^\infty(t) = \vartheta_\infty, \quad u^\infty(t) = u_{\mathcal S}, \quad \chi^\infty(t) = \chi_\infty, \quad \text{and} \quad \eta^\infty(t) = k_\infty u_{\mathcal S}$$
$$\text{for every } t \in [0,+\infty). \tag{4.47}$$

Indeed, in such a case, we have $\partial_t\xi^\infty = 0$ by comparison in (4.46), and we can take the constant value of ξ^∞ as $\xi_{\mathcal S}$. We prove that $\vartheta^\infty = \vartheta_\infty$. For every t we have

$$\vartheta^\infty(t) = \vartheta^\infty(0) + \int_0^t \partial_t\vartheta^\infty(s)\,ds = \lim_{n\to\infty}\vartheta_n(0) + \lim_{n\to\infty}\int_0^t \partial_t\vartheta_n(s)\,ds = \vartheta_\infty$$

the limits being understood in $L^{12/7}(\Omega)$, by (4.43). It follows that $u^\infty = u_{\mathcal{S}}$. The same argument used for ϑ^∞ , applied to (4.42), leads to $\chi^\infty = \chi_\infty$. It remains to prove that $\eta^\infty = k_\infty u_{\mathcal{S}}$, or $\eta^\infty = k_\infty u^\infty$, as we already know that $u^\infty = u_{\mathcal{S}}$. First of all, we can combine (4.37) and (4.44) and derive that $\partial_t \eta^\infty = 0$, i.e., η^∞ is a constant. On the other hand, we can present $\eta^\infty - k_\infty u^\infty$ as follows

$$\eta^\infty - k_\infty u^\infty = (\eta^\infty - \eta_n) + k * (u_n - u^\infty) + (\eta_n - k * u_n) + (k * u^\infty - k_\infty u^\infty) \tag{4.48}$$

and we argue this way. Recalling the Ascoli compactness theorem (cf., e.g., [35, Thm. 3, Sec. 6]), we see that

$$\begin{gathered} \eta^\infty - \eta_n \to 0 \quad \text{and} \quad k * (u_n - u^\infty) \to 0 \\ \text{strongly in } C^0([0,T]; L^1(\Omega)) \text{ for every } T < +\infty \end{gathered} \tag{4.49}$$

the former following also from (4.37) and (4.44), the latter being a consequence of (4.40) and the Young theorem. As far as the third term of (4.48) is concerned, we have for $t \geq 0$

$$\begin{aligned} \eta_n(t) - (k * u_n)(t) &= \int_0^{t+t_n} k(s) u(t + t_n - s)\, ds - \int_0^t k(s) u(t + t_n - s)\, ds \\ &= \int_t^{t+t_n} k(s) u(t + t_n - s)\, ds \end{aligned}$$

whence

$$\begin{aligned} &\|\eta_n(t) - (k * u_n)(t)\|_{L^1(\Omega)} \\ &\leq \|u\|_{L^\infty(0,+\infty;L^1(\Omega))} \int_t^{+\infty} |k(s)|\, ds \leq c \int_t^{+\infty} |k(s)|\, ds \end{aligned} \tag{4.50}$$

since $u \in L^\infty(0, +\infty; L^1(\Omega))$ due to (4.18) and (3.35). Finally,

$$(k * u^\infty - k_\infty u^\infty)(t) = \int_0^t k(s)\, u^\infty\, ds - k_\infty u^\infty = -u^\infty \int_t^{+\infty} k(s)\, ds$$

so that

$$\|(k * u^\infty - k_\infty u^\infty)(t)\|_{L^1(\Omega)} \leq c \int_t^{+\infty} |k(s)|\, ds. \tag{4.51}$$

At this point, we can easily conclude. Fix any $\varepsilon > 0$. Then, choose T such that the right hand sides of (4.50) and (4.51) are $\leq \varepsilon$ for every $t \geq T$. Next, using (4.49), choose n in order that

$$\|\eta^\infty - \eta_n\|_{C^0([0,T];L^1(\Omega))} \leq \varepsilon \quad \text{and} \quad \|k * (u_n - u^\infty)\|_{C^0([0,T];L^1(\Omega))} \leq \varepsilon.$$

Hence, the left hand side of (4.48) satisfies $\|(\eta^\infty - k_\infty u^\infty)(T)\|_{L^1(\Omega)} \leq 4\varepsilon$. We infer that $\eta^\infty - k_\infty u^\infty = 0$ since both η^∞ and u^∞ are independent of t and ε is arbitrary. This completes the proof.

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