Computation of sparse circulant permanents via determinants

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Abstract

We consider the problem of computing the permanent of circulant matrices. We apply some recent results on the computation of the number of perfect matchings of small genus graphs in order to show that the permanent of a circulant matrix with three nonzero entries per row is the linear combination of just four determinants (of circulant matrices with the same structure as the original matrix). We also show that the same result holds true for a class of circulant matrices with four nonzero entries per row related to the dimer problem with periodic boundary conditions. Conversely, we give hints at the fact that more general circulants do not share similar properties, and thus should be dealt with by means of radically different approaches.

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1. Introduction

The computation of the permanent seems to be a very hard task, even for sparse $(0,1)$ matrices. A number of results show that it is extremely unlikely that there is a polynomial time algorithm for computing the permanent (see [22,23], and also [3,4]). The best known algorithm is due to Ryser [20] and takes $O(n^{2^n})$ operations, where $n$ is the matrix size.
It is a striking fact that the computation of the permanent essentially keeps its general difficulty when restricted to (0, 1) matrices with only three nonzero entries per row and column [3]. It is thus natural to impose on the matrices additional constraints which could make the permanent easier to compute.

An interesting and extensively studied special case is provided by circulant matrices. Some structure of permanents of circulants has been exposed by the investigation on the linear recurrences they satisfy [15] and by the employment of sophisticated algebraic techniques, based on the notion of permanental compound, which yield interesting asymptotic estimates [16]. Other structural and computational properties of these permanents have been analyzed in [1, 2].

Although the above results suggest that there could be room for tractability, they fail to provide polynomial time algorithms, even for circulants with three nonzero entries per row.

In this paper we show that the permanent of a circulant matrix with three nonzero entries per row is the linear combination of just four determinants of circulant matrices with the same structure as the original matrix. We also show that the same result holds true for a class of circulant matrices with four nonzero entries per row, whose permanents are related to the dimer problem with periodic boundary conditions.

Such an efficient computational approach is based upon recent results by Galluccio and Loebl [5, 6], Tesler [21], and Galluccio et al. [7, 8], which show how to express the permanent of a matrix associated with a small genus graph as the linear combination of a few determinants. This approach finds its historical roots in the work of Kac and Ward [9] and Kasteleyn [10–12] on the Ising model and the dimer problem. In particular, Kasteleyn was the first to show the crucial role of the genus of the graph in determining the feasibility of computing permanents via determinants. Other related counting problems on regular lattices have been addressed in [18, 19].

The rest of this paper is organized as follows. In Section 2 we introduce the main notation used throughout the paper. In Section 3 we recall some results from [21], which will be instrumental to prove our results. Building upon this, in Section 4 we prove properties of permanents of very sparse circulants which will lead to an extremely efficient computational scheme; such permanents will indeed be expressed as the linear combination of four determinants.

In Section 5 we give an algebraic counterpart to the combinatorial derivation of our results; in particular we show the crucial role of the cancellation which occurs during the expansion of the determinant in establishing the feasibility of the computation the permanent as a linear combination of a small number of determinants. In Section 6 we contrast our result with the asymptotic estimates by Minc, and we show that the matrices whose permanents can be expressed as a linear combination of four determinants coincide with those for which Minc can provide accurate asymptotic estimates. Finally, in Section 7 we give some concluding remarks.
2. Preliminaries

Let \( \mathcal{S} \) be the set of all permutations of the first \( n \) integers. The permanent of an \( n \times n \) matrix \( A \equiv (a_{i,j}) \) is defined as

\[
\text{per}(A) = \sum_{\sigma \in \mathcal{S}} \prod_{i=1}^{n} a_{i,\sigma_i},
\]

where \( \sigma = (\sigma_1, \ldots, \sigma_n) \), while the determinant is defined as

\[
\text{det}(A) = \sum_{\sigma \in \mathcal{S}} (-1)^{|\sigma|} \prod_{i=1}^{n} a_{i,\sigma_i},
\]

where \( |\sigma| \) denotes the sign of the permutation \( \sigma = (\sigma_1, \ldots, \sigma_n) \).

A \((0, 1)\)-matrix \( A \) is said to be convertible if there exists a \((-1, 1)\)-matrix \( X \equiv (x_{i,j}) \) such that \( \text{per}(A) = \text{det}(A \ast X) \), where \( \ast \) denotes the elementwise product, i.e., the \((i, j)\)th entry of the matrix \( A \ast X \) is \( a_{i,j}x_{i,j} \).

We now introduce the notion of Pfaffian of a skew symmetric matrix of even size. Let us consider all the partitions \( P \) into pairs of the set \( \{1, 2, \ldots, 2n-1, 2n\} \), i.e., \( \{(i_1, j_1), \ldots, (i_n, j_n)\} \). Let \( \sigma_{p} \) denote the sign of the permutation \( (i_1, j_1, i_2, \ldots, i_n, j_n) \). The Pfaffian of a skew symmetric \( 2n \times 2n \) matrix \( B \) can be defined as

\[
\text{Pf}(B) = \sum_{p \in P} \sigma_{p} b_{i_1,j_1} b_{i_2,j_2} \cdots b_{i_n,j_n}.
\]

The permanent of a \((0, 1)\) matrix has an interpretation in terms of both the digraph and the bipartite graph that can be associated with the matrix. More precisely, if \( A \) is an \( n \times n \) \((0, 1)\) matrix, we denote by \( D(A) \) the digraph whose adjacency matrix is \( A \) and by \( G[A] \) the \( 2n \)-node bipartite graph associated with \( A \) in the natural way. Then the permanent of \( A \) is equal to the number of cycle covers of \( D(A) \) as well as to the number of perfect matchings of \( G[A] \). Recall that a cycle cover of \( D(A) \) is a node disjoint covering of all the nodes of \( D(A) \) in terms of its cycles, whereas a matching of \( G[A] \) is a set of pairwise node disjoint edges, and a perfect matching (or 1-factor) of \( G[A] \) is a matching such that each node of \( G[A] \) is incident to exactly one of the edges forming the matching.

It is also useful to introduce the signed adjacency matrix of a digraph, which is defined as \( A - A^T \), where \( A \) is the ordinary adjacency matrix.

Let \( P \) denote the \((0, 1)\) \( n \times n \) matrix with 1’s only in positions \((i, i+1), i = 1, 2, \ldots, n-1, \) and \((n, 1)\). Any \((0, 1)\) circulant matrix can be written in the form

\[
P^{t_1} + P^{t_2} + \cdots + P^{t_k},
\]

where \( 0 \leq t_1 < t_2 < \cdots < t_k < n \).

3. Counting perfect matchings in small genus graphs

Kasteleyn \[10–12\] showed that it is possible to change the signs to some elements of the adjacency matrix \( A \) of a given planar graph \( G \) in such a way that the Pfaffian of the signed adjacency matrix of \( A \) is equal to the number of perfect matchings of \( G \).

This immediately translates into the well known fact that, given a matrix \( A \) whose associated bipartite graph \( G[A] \) is planar, then there exists a matrix \( A' \), with \(|A'| = A\) elementwise, such that \( \text{per}(A) = |\text{det}(A')| \). In turn, this means that all the terms of \( \text{det}(A') \) have the same sign and thus there is no cancellation among them.

This property does not hold for general graphs, bipartite or not. However, Kasteleyn also stated, without a proof, that the number of perfect matchings in a graph
embeddable in a surface of genus\(^1\) \(g\) is given by a linear combination of \(4^g\) Pfaffians of matrices obtained by changing the sign to some of the entries of the adjacency matrix of the graph. Recently this result has been proved and generalized by Galluccio and Loebl \([5,6]\), and, independently, by Tesler \([21]\). As a straightforward consequence, the permanent of a matrix associated with a graph of genus \(g\) can be computed as a linear combination of \(4^g\) determinants.

Tesler’s construction seems to be at the same time simpler and more general than the one proposed by Galluccio and Loebl, and thus in this paper we have found it convenient to use the former. Incidentally, this will give us the freedom to choose between embeddings into orientable and non-orientable surfaces.

Tesler proved the following property of any graph drawing, which generalizes the concept of “admissible orientation” introduced by Kasteleyn.

**Theorem 1** \([21]\). Once drawn in the plane, any graph may be oriented so that every perfect matching has sign \(\epsilon_m = \epsilon_0 \cdot (-1)^{\kappa(m)}\), where \(\epsilon_0\) is a constant (equal to either 1 or \(-1\)), and \(\kappa(m)\) denotes the number of crossings in the matching \(m\). Such an orientation is called crossing orientation.

In order to determine and exploit a crossing orientation, Tesler used a standard way to draw a graph on a surface of genus \(g\); in the following we summarize his approach.

A compact boundaryless 2-dimensional surface \(S\) of arbitrary genus can be represented in the plane by the following **plane model**.

Consider a set of \(n\) labels \(a_1, \ldots, a_n\), and a polygon \(P\) with \(2n\) sides coupled in \(n\) pairs \(p_j, p'_j, j = 1, \ldots, n\). For each pair, paste together the sides \(p_j\) and \(p'_j\) and label both of them with \(a_j\) if, following clockwise the boundary of \(P\), the pasted sides \(p_j\) and \(p'_j\) are traversed in the same direction. In this case we say that \(S\) is \(j\)-non-orientable. If the two pasted sides do not have the same orientation, label with \(a_j\) the side traversed clockwise and with \(a_j^{-1}\) the other one. We then say that \(S\) is \(j\)-orientable.

Starting at any side of \(P\) and proceeding clockwise, we form a word \(\sigma\) reading off the \(2n\) labels. If the occurrences of \(a_j\) or \(a_j^{-1}\) are interleaved with those of \(a_k\) or \(a_k^{-1}\), we say that \(\sigma\) is \(j, k\)-alternating.

Given a graph \(G\) we embed it in a suitable plane model as follows. Edges which are fully contained inside \(P\) do not cross and are called 0-edges. Edges that go through sides \(p_j\) and \(p'_j\) are called \(j\)-edges. An edge can be both a \(j\)-edge and a \(k\)-edge if it goes through sides \(p_j, p'_j, p_k, p'_k\).

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\(^1\) We say that a graph embeds in a surface if it can be drawn on it without any edge crossing. The genus \(g\) of a surface in \(\mathbb{R}^3\) is the maximum number of nonintersecting closed curves which one can draw on the surface without disconnecting it. (Equivalently, we can say that a surface has genus \(g\) if it is topologically equivalent to a sphere with \(g\) “handles”.) For instance, a sphere and a plane have genus 0, while a torus has genus 1.
Fig. 1. Two embeddings of two graphs sharing the same planar portion. On the left we show a surface with word $\sigma = a_1a_2a_1^{-1}a_2$ (Klein bottle); on the right a surface with word $\sigma = a_1a_2a_1^{-1}a_2^{-1}$ (surface of genus 1).

The main property of this embedding is that, if $\sigma$ is, for instance, $j$, $k$-alternating, then each $j$-edge crosses each $k$-edge exactly once. An example is illustrated in Fig. 1.

Tesler proved that, once a graph has been embedded in a plane model, a crossing orientation can be easily obtained according to the following scheme:

- Add some auxiliary edges (which will disappear later) so that the planar part of the graph is connected and all the vertices belong to a face.\(^2\) (This will be instrumental to the execution of the next step.)
- Determine an admissible orientation of the planar part, i.e., an orientation such that all the faces are clockwise odd, i.e., the number of edges of each face which are oriented clockwise is odd.
- For each $j$-edge, consider its union with the planar part and orient it in such a way that the face it forms is clockwise odd.

Once a crossing orientation of a graph embedded in a plane model is obtained, it is simple to determine the matrices whose Pfaffians, combined linearly, will count the number of perfect matchings.

Since we are mainly interested in the embedding of bipartite graphs in orientable surfaces, we state the following restricted version of Tesler’s main result.

**Theorem 2** [21]. Let $G$ be a bipartite graph with $2n$ vertices ($n$ for each side of the bipartition), embedded in a surface of genus 1 represented as a planar map with word $\sigma = a_1a_2a_1^{-1}a_2^{-1}$ (see Fig. 1), and let $A$ be an $n \times n$ matrix representing the adjacencies of $G$. Let $A'$ be the ‘signed’ version of $A$, representing a crossing orientation of $G$. In general the embedding will contain $k$-edges, for $k = 0, 1, 2$, and edges which are both 1- and 2-edges.

Let $B_{x,y}$ be a matrix defined as follows. If $(i, j)$ is a 0- (resp. a 1-, 2-) edge, then $B_{x,y}(i, j) = A'(i, j)$ (resp. $B_{x,y}(i, j) = xA'(i, j)$, $B_{x,y}(i, j) = yA(i, j)$). If $(i, j)$

\(^2\) A face is a cycle whose interior does not contain any node.
is both a 1- and a 2-edge, then $B_{x,y}(i,j) = xyA'(i,j)$. Finally, if $(i,j)$ is not an edge of $G$, then $B_{x,y}(i,j) = 0$. Then

$$2 \per(A) = \det(B_{1,1}) + \det(B_{1,-1}) + \det(B_{-1,1}) - \det(B_{-1,-1}).$$

4. Permanent of very sparse circulants

Let $Q^k$ denote the $n \times n \{0,1\}$ Toeplitz matrix whose entries are all zeros, except for the diagonal that starts from position $(1,k+1)$, and let $\hat{Q}^k = (Q^{n-k})^T$. Recalling that $P$ denotes the $\{0,1\}$ circulant matrix whose first row is all zero except for the entry $(1,2)$, we see that $P^k = Q^k + \hat{Q}^k$. Let us finally denote with $\hat{P}^k$ the skew-circulant matrix $Q^k - \hat{Q}^k$.

The application of Tesler’s construction to circulant matrices of the form $I + P + P^j$ leads to the following result.

**Lemma 3.** The following equalities hold

$$2 \per(I + P + P^j) = \det(I + P + P^j) + \det(I + P - P^j) + \det(I + \hat{P} + (-1)^j\hat{P}^j) - \det(I + \hat{P} - (-1)^j\hat{P}^j)$$

if $n$ is odd, and

$$2 \per(I + P + P^j) = \det(I + \hat{P} + \hat{P}^j) + \det(I + \hat{P} - \hat{P}^j) + \det(I + P + (-1)^jP^j) - \det(I + P - (-1)^jP^j)$$

if $n$ is even.

**Proof.** The proof consists of applying Theorem 2 to an appropriate embedding of the graph corresponding to $A = I_n + P_n + P_n^j$. In the following we assume that $n$ and $j$ are odd. The other cases can be handled in the same way. $A'$ will denote the signed adjacency matrix of the digraph obtained by orienting the edges of $G$.

We use labels $1, 3, \ldots, 2n-1$ for vertices of $G[A]$ indexing the rows of $A$, and labels $2, 4, \ldots, 2n$ for those indexing the columns of $A$, in such a way that if $A'(i,j) = 1$, then there is the edge $(2i-1, 2j)$.

The edges corresponding to the matrix $I + P$ form a cycle of length $2n$ which contains all the vertices (see Fig. 2). Let us delete edge $(2, 2n-1)$ from this cycle, and identify the chain so obtained with the planar part of the embedding (see Fig. 3).

We then add the auxiliary edges $(i, i+2)$, for $i = 1, \ldots, 2n-2$ (dashed lines in Fig. 3). These edges disrupt the bipartite structure, but are only needed to establish a crossing orientation of the graph, and do not appear in $A'$.

By analyzing the simple structure of the planar part of the graph, it is clear that an admissible orientation of $G$ is given by $(2i-1) \rightarrow (2i), i = 1, \ldots, n$, and $(2i-1) \rightarrow (2i+2), i = 1, \ldots, n-1$, for the edges on the chain, which thus have a posi-
Fig. 2. Two drawings of the graph corresponding to $I_7 + P_7 + P_3$.

tive sign in $A'$, and $(2i - 1) \to (2i + 1)$ and $(2i + 2) \to (2i)$, $i = 1, \ldots, n - 1$, for the auxiliary edges.

We dispose of the remaining edges as follows. Edges corresponding to $P^j$ are divided in two sets: the first one is given by $(1, 2 + 2j), (3, 4 + 2j), \ldots, (k, k + 1 + 2j)$, for $k + 1 + 2j = 2n$, i.e., $k = 2(n - j) - 1$. Since the indices do not overlap, these edges can be drawn as 1-edges without crossing, as shown in Fig. 3. The second set of edges is given by $(k + 2, 2), (k + 4, 4), \ldots, (k + j, 2j)$, which can be drawn as 2-edges, and thus treated similarly.

The crossing orientation of these edges can be easily obtained by means of parity arguments, and taking advantage of the high regularity of the chain. In particular all the edges in the first set have orientation from even-labeled vertices to odd-labeled vertices, i.e., negative sign in $A'$, while all the edges in the second set have the opposite orientation.

The last edge to be considered is $(2, 2n - 1)$, which closes the cycle corresponding to $I + P$. It can be drawn as a self-crossing 1,-2-edge as shown in Fig. 3. Its crossing orientation, for technical reasons due to the self-crossing property, is given by $2 \to (2n - 1)$.

Summarizing we have $A' = I + Q - \hat{Q} - Q^j + \hat{Q}^j$, and thus

$$B_{x,y} = I + Q - xy\hat{Q} - xQ^j + y\hat{Q}^j.$$  

Recalling formula (2), we finally obtain

$$2\text{per}(A) = \det(I + Q - \hat{Q} - Q^j + \hat{Q}^j) + \det(I + Q + \hat{Q} + Q^j + \hat{Q}^j)$$

$$+ \det(I + Q + \hat{Q} - Q^j - \hat{Q}^j) - \det(I + Q - \hat{Q} + Q^j - \hat{Q}^j),$$

which, using $P^k = Q^k + \hat{Q}^k$ and $\hat{P}^k = Q^k - \hat{Q}^k$, can be rewritten as

$$2\text{per}(A) = \det(I + \hat{P} - \hat{P}^j) + \det(I + P + P^j)$$

$$+ \det(I + P - P^j) - \det(I + \hat{P} - \hat{P}^j).$$
Fig. 3. Embedding of the graph corresponding to $I_7 + P_7 + P_3^3$.

Proceeding in the same way in the other three cases ($n$ odd, $j$ even and $n$ even, $j$ odd/even) we prove the claim. □

We now show that there is an even simpler expression for the permanent of $I + P + P^j$, which involves determinants of circulant matrices.

**Corollary 4.** Let $A$ be an $n \times n$ matrix of the form $I + P + P^j$, with $n = 2^t n_0$ and $n_0$ odd, and let $\omega$ be a primitive $(2^{t+1})$-root of the unity. In particular, for $n = 2n_0$, $\omega = i$. Then $\text{per}(A)$ can be expressed as the linear combination of four determinants of circulant matrices, i.e.,

$$2 \text{per}(A) = \det(I + P + P^j) + \det(I + P - P^j)$$
$$+ \det(I - P + P^j) + \det(-I + P + P^j),$$

(1)

for $n$ odd, and

$$2 \text{per}(A) = \det(I + \omega P + \omega^j P^j) + \det(I + \omega P - \omega^j P^j)$$
$$+ \det(I + P + (-1)^j P^j) - \det(I + P - (-1)^j P^j),$$

(2)

for $n$ even.
Proof. Let $D_\epsilon$ be the diagonal matrix whose $(i,i)$th entry is $\epsilon^{i-1}$.

The claim easily follows from the equalities:

$D_\omega^{-1} \hat{P}^j D_\omega = D_{\omega_n}^{-1} (Q^j - \hat{Q}^j) D_\omega = \omega^j Q^j - \omega^{j-n} \hat{Q}^j$

$= \omega^j Q^j + \omega^j \hat{Q}^j = \omega^j P^j$,

where we have used the fact that $\omega^j = -\omega^{j-n}$. Indeed multiplying both sides by $\omega_n^j$ we obtain $\omega_n^j = -1$ which follows from $\omega_n = (\omega^2)^{\frac{n}{2}} = (-1)^{\frac{n}{2}} = -1$. Then we have, for example,

$$\det(I + \hat{P} + \hat{P}^j) = \det(D_\omega^{-1}(I + \hat{P} + \hat{P}^j)D_\omega)$$

$$= \det(I + \omega P + \omega^j P^j). \quad \Box$$

The following corollary generalizes formulas (1) and (2) to circulants of the form $A = I + P^i + P^j$. The proof takes advantage of the following simple lemma.

**Lemma 5.** Let $A$ and $B$ be $(0,1)$ matrices such that the bipartite graphs $G[A]$ and $G[B]$ are isomorphic. Then there exists a permutation matrix $\Pi$ for which $\Pi A \Pi^{-1} = B$.

**Proof.** By definition of graph isomorphism, the claim is equivalent to saying that if $G[A]$ and $G[B]$ are isomorphic, then $D(A)$ and $D(B)$ are also isomorphic. This is easily seen by direct inspection. □

**Corollary 6.** Let $(a,b)$ denote the greatest common divisor between $a$ and $b$. Let $A = I + P^i + P^j$, $j > i$, be of size $n$, where at least one of the values $(n,j-i)$, $(n,i)$, $(n,j)$ is equal to 1. Let $\omega$ be defined as in Corollary 4. We have

$$2 \text{per}(A) = \det(I + P^i + P^j) + \det(I - P^i + P^j)$$

$$+ \det(I + P^i - P^j) + \det(-I + P^i + P^j)$$

for $n$ odd, and

$$2 \text{per}(A) = \det(I - (\omega)^i P^i + \omega^j P^j) + \det(I + \omega^i P^i - \omega^j P^j)$$

$$+ \det(I + P^i - (-1)^i P^j) - \det(I - (-1)^i P^i - (-1)^j P^j)$$

for $n$ even.

**Proof.** We prove the claim for $n$ odd, the other case being analogous. Let us first assume that $(n,j-i) = 1$. This implies that the edges corresponding to $P^i + P^j$ in $G[A]$ form a cycle of length $2n$, while the edges corresponding to $I$ can be drawn as chords of length $d$, where $d$ depends only on $n$, $i$, $j$. It is easy to prove that there is a value $k$ such that $B = I + P + P^k$ has a representation with the same chord length $d$ (see [2]). Thus $G[B]$ is isomorphic to $G[A]$, and $\text{per}(A) = \text{per}(B)$. Since $G[A]$ and $G[B]$ are isomorphic, by Lemma 5 there exists a permutation matrix $\Pi$ such that $\Pi A \Pi^{-1} = B$. 


Moreover, the permutation $\Pi$ maps cycles into cycles. Indeed, it clearly maps edges corresponding to $I_n^i$ in $A$ into edges corresponding to $I_n^j$ in $B$. Then the remaining edges ($P^i + P^j$ and $P^+P^k$) form two $2n$ cycles in $A$ and $B$, and since the corresponding graphs are bipartite, once two vertices have been identified by the isomorphism, e.g., one in $P^i$ and the other one in $P^k$, it implies that all the vertices in $P^i$ are mapped into vertices of $P^k$; similarly for the vertices of $P^j$ and $P^k$.

Since $\det(\Pi)\det(\Pi^{-1}) = 1$, we have that either

$$\det(I + \alpha P^i + \beta P^j) = \det(I + \alpha P^j + \beta P^k),$$

(3)

or

$$\det(I + \alpha P^j + \beta P^i) = \det(I + \beta P^j + \alpha P^k),$$

(4)

for two arbitrary constants $\alpha$ and $\beta$.

The symmetry of (1) allows us to use either (3) or (4) in order to obtain

$$2 \text{per}(I + P^i + P^j) = 2 \text{per}(I + P^j + P^k)
= \det(I + P^i + P^j) + \det(I + P^j + P^k)
+ \det(I - P^i + P^k) - \det(I - P^j + P^k)
= \det(I + P^i + P^j) + \det(I + P^j + P^k)
+ \det(I - P^i - P^j) - \det(I - P^j - P^k).$$

To deal with the case $(n, j - i) > 1$ we have to use the assumption that at least one of the values $(n, j - i), (n, i), (n, j)$ is equal to 1. This implies that either $(n, i) = 1$ or $(n, j) = 1$, so that we can proceed as before taking into account either the matrix $A' = P^{n-i}A = I + P^{i-j} + P^{n-j}$ or $A'' = P^{n-j}A = I + P^{n-j} + P^{n+i-j}$. Indeed $P^t$ is a permutation matrix for every $t$, and so $\text{per}(A') = \text{per}(A'') = \text{per}(A)$. □

The approach used above cannot be extended to handle circulant matrices with more than three nonzero entries per row, unless they have a very special structure. Indeed the well known lower bound $g \geq \lceil 1 - (n/2) + (nd/4)(1 - (2/\gamma)) \rceil$ (where $d$ and $\gamma$ denote degree and girth, respectively) implies that the genus of a regular graph of degree at least 4 grows linearly with $n$, whenever $\gamma > 4$.

For instance, among the circulant matrices with 4 ones per row only those with two consecutive ones (up to isomorphisms) are associated with graphs of genus 1. Indeed it is easy to check that the graph corresponding to the matrix $A_j = I + P + P^j + P^{j+1}$ has genus 1 (see Fig. 4) and its actual embedding in the torus can be derived from that of $I + P + P^j$.

Combining these observations with Corollary 6, we obtain the following lemma.

**Lemma 7.** Let $A$ be an $n \times n$ matrix of the form $I + P + P^j + P^{j+1}$. Then $\text{per}(A)$ can be expressed as

$$\text{per}(A) = \text{per}(A') = \text{per}(A'').$$
Fig. 4. Embedding the graphs $G[I + P + P^3]$ (left) and $G[I + P + P^3 + P^4]$ (right) in the torus. We show two fragments (without wrap-around) of a square-section torus. Along the horizontal direction, we embed all the vertices together with the edges that connect them in a cycle. (Note that these edges correspond to two of the three “diagonals” of the matrix, e.g., $I + P$.) The other edges, corresponding to $P^3$, can be arranged in order to form a path which crosses the torus in a spiral-like fashion. On the right, the edges related to $P^4$ find their place besides the edges corresponding to $P^3$. To improve readability, edges (3,10) and (5,12) are the only ones related to $P^4$ actually shown.

$$2\text{per}(A) = \det(I + \hat{P} + (-1)^{j} \hat{P}^{j} - (-1)^{j} \hat{P}^{j+1})$$  \hfill (5)
$$+ \det(I + P - P^{j} + P^{j+1})$$  \hfill (6)
$$+ \det(I + P + P^{j} - P^{j+1})$$  \hfill (7)
$$- \det(I + \hat{P} - (-1)^{j} \hat{P}^{j} + (-1)^{j} \hat{P}^{j+1}),$$  \hfill (8)

for $n$ odd, and

$$2\text{per}(A) = \det(I + P + (-1)^{j} P^{j} - (-1)^{j} P^{j+1})$$
$$+ \det(I + \hat{P} - \hat{P}^{j} + \hat{P}^{j+1})$$
$$+ \det(I + \hat{P} + \hat{P}^{j} - \hat{P}^{j+1})$$
$$- \det(I + P - (-1)^{j} P^{j} + (-1)^{j} P^{j+1}),$$

for $n$ even.

**Proof.** We omit the proof, which is similar to that of Lemma 3. □

An interesting special case is provided by the $2k \times 2k$ matrix $I + P + P^k + P^{k+1}$, whose permanent turns out to be equal to $2^{2k} + 2^{k+1}$.

Noticing that the terms (5)–(8) above can be coupled in pairs with the same absolute value, we can also state the following result.

**Corollary 8.** Let $A$ be an $n \times n$ matrix of the form $I + P + P^j + P^{j+1}$. If $n$ is odd, then

$$\text{per}(A) = \det(I + P - P^{j} + P^{j+1}) + \det(I + P + P^{j} - P^{j+1}).$$

An immediate consequence of the results of this section is that both $\text{per}(I + P^j + P^l)$ and $\text{per}(I + P + P^j + P^{j+1})$ can be computed in linear time, because of the obvious linear time computability of the corresponding determinants. Examples of
values of these permanents together with the related determinants are shown in Tables 1 and 2.

5. An algebraic viewpoint

A close look at the formulas of Corollary 6 reveals that the equality between the sum of the four determinants and twice the permanent holds term by term, i.e., that each permutation is captured correctly, as shown in the following example.

Example 1. Let us consider the $9 \times 9$ matrix $A = I + P + P^4$, and four permutations $\pi_1, \pi_2, \pi_3,$ and $\pi_4$, where
• \( \pi_1 \) corresponds to \( A_{\pi_1} \equiv a_{1,2}a_{2,3}a_{3,7}a_{4,4}a_{5,5}a_{6,6}a_{7,8}a_{8,9}a_{9,1} \), as shown below

\[
\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,2} & a_{2,3} & a_{2,4} \\
a_{3,3} & a_{3,4} & a_{3,5} \\
a_{4,4} & a_{4,5} & a_{4,6} \\
a_{5,5} & a_{5,6} & a_{5,7} \\
a_{6,6} & a_{6,7} & a_{6,8} \\
a_{7,7} & a_{7,8} & a_{7,9} \\
a_{8,8} & a_{8,9} & a_{8,9} \\
a_{9,1} & a_{9,2} & a_{9,9}
\end{pmatrix}
\]

• \( \pi_2 \) to \( A_{\pi_2} \equiv a_{1,2}a_{2,6}a_{3,3}a_{4,8}a_{5,5}a_{6,1}a_{7,7}a_{8,9}a_{9,4} \),

• \( \pi_3 \) to \( A_{\pi_3} \equiv a_{1,2}a_{2,3}a_{3,7}a_{4,4}a_{5,5}a_{6,6}a_{7,8}a_{8,9}a_{9,4} \),

• \( \pi_4 \) to \( A_{\pi_4} \equiv a_{1,5}a_{2,2}a_{3,3}a_{4,4}a_{5,5}a_{6,6}a_{7,7}a_{8,8}a_{9,9} \).

Since \( \pi_1 \) is odd and \( \pi_2, \pi_3, \pi_4 \) are even, we obtain \( \det(A) = \cdots + A_{\pi_1} + A_{\pi_2} + A_{\pi_3} + A_{\pi_4} + \cdots \).

Considering how the signs of the entries in \( I \pm P \pm P^4 \) impact on the sign of \( A_{\pi_4} \), we must have, in the determinant expansion

\[
\det(I + P + P^4) = \cdots + A_{\pi_1} + A_{\pi_2} + A_{\pi_3} + A_{\pi_4} + \cdots \\
\det(I + P - P^4) = \cdots + A_{\pi_1} + A_{\pi_2} - A_{\pi_3} + A_{\pi_4} + \cdots \\
\det(I - P + P^4) = \cdots + A_{\pi_1} + A_{\pi_2} + A_{\pi_3} - A_{\pi_4} + \cdots \\
\det(-I + P + P^4) = \cdots + A_{\pi_1} - A_{\pi_2} + A_{\pi_3} + A_{\pi_4} + \cdots .
\]

This implies that the expression \( \det(I + P + P^4) + \det(I + P - P^4) + \det(I - P + P^4) + \det(-I + P + P^4) \) (which is equal to \( 2 \det(A) \)) sums up to \( 2A_{\pi_1} + 2A_{\pi_2} + 2A_{\pi_3} + 2A_{\pi_4} + \cdots \), as expected.

The property illustrated by Example 1 has the following implication on the cycle structure of the digraphs associated with the matrices of Corollary 6.

**Corollary 9.** Let \( P_A, P_B, \) and \( P_C \) be three different cyclic permutations which act on the integers \( 0, 1, \ldots, n - 1 \), for \( n \) prime. For \( i = A, B, C \), let us denote with \( P_i(x) \) the action of \( P_i \) on \( x \), i.e., \( P_i(x) = x + k_i \mod n \), where \( k_A, k_B, \) and \( k_C \) are distinct. Let us define a function \( \mathbb{P}() \) representing a permutation, such that \( P(x) \) is either \( P_A(x) \), \( P_B(x) \), or \( P_C(x) \). Let us finally denote with \( \mathbb{P}P_i \) the number of values \( 0 \leq x < n \) such that \( P(x) = P_i(x) \), for \( i = A, B, C \). Then the permutation corresponding to \( P \) is odd if and only if \( \mathbb{P}P_A, \mathbb{P}P_B, \) and \( \mathbb{P}P_C \) are all odd.
Proof. The proof follows from the application of Corollary 6 to the matrix \( I + P^{k_A} + P^{k_C} \), which has the same permanent as \( P^{k_A} + P^{k_B} + P^{k_C} \). Indeed, according to Corollary 6, \( n \) being odd, we have 

\[
2\text{per}(M) = \det(M) + \det(M_A) + \det(M_B) + \det(M_C),
\]

where \( A, B, \) and \( C \) describe the powers of \( P \) corresponding to the permutations \( P_A, P_B, \) and \( P_C \), \( M = A + B + C \), and \( M_X = M - 2X \), for \( X = A, B, C \). For every fixed permutation \( P \), the parity of \( \#P_A, \#P_B \) and \( \#P_C \) impacts on the sign with which it appears in the determinants of \( \det(M), \det(M_A), \det(M_B), \) and \( \det(M_C) \) as summarized in the table below.

<table>
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<th>( #P_C )</th>
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Since all the terms of the permanent expansion have positive sign, then the sign of the permutation \( P \) (which is the same in the determinants of \( \det(M), \det(M_A), \det(M_B), \) and \( \det(M_C) \)) must be negative whenever \( \#P_A, \#P_B \) and \( \#P_C \) are odd (see last row of the table above).

As in Corollary 6, this argument can be generalized to the case of matrices of composite size, for suitable values of \( n, k_A, k_B, \) and \( k_C \). □

Corollary 9 implies that different permutations which contain the same number of elements from each of the diagonals of \( I + P^i + P^j \) must have the same parity, so that, in the expansion of the determinant, they will appear with the same sign, and so will not cancel each other. This property will be instrumental to the analysis of the conditions under which the expansion of the determinant can be used to recover the value of the permanent.

It is in fact possible to find a correspondence between perfect matchings and terms in the expansion of the determinant, by avoiding the phenomenon of cancellation. In general this can be achieved by employing the method of variables (see [13, p. 315]). Given a graph \( G[A] \), we introduce a matrix \( A(x) \) whose \( (i, j) \)th entry is a formal variable \( x_{ij} \) if the \( (i, j) \)th entry of \( A \), the adjacency matrix of \( G \), is equal to 1, and 0 otherwise. Thus \( \det(A(x)) \) is a polynomial in the variables \( x_{ij} \), and there is a one to one correspondence between every term in the expansion of \( \det(A(x)) \) and a perfect matching of \( G \). It is also clear that different terms in the expansion are products of different variables, so that they do not cancel each other.

An important result which establishes a surprising connection between the number of perfect matchings and the determinant is shown in [13, Theorem 8.4.1, p. 330]. It states that if \( G^* \) is a random orientation of \( G \) (where each edge is oriented, independently of the other edges, with probability \( 1/2 \) in either direction), then the
expected value of the determinant of the signed adjacency matrix of \( G^* \) is equal to the number of perfect matchings of \( G \).

Combined with the method of variables, this fact implies that the permanent can be computed as the average of all the determinants obtained by setting each of the nonzero entries to 1 and \(-1\). In general, this approach is not practical, because it involves the computation of a number of determinants equal to \( 2|E| \), where \( E \) denotes the edge set of \( G \). However it can become practical under special circumstances, such as the case of small genus graphs, where, as we are going to see, cancellation occurs in a very restricted way.

The key observation is that in these special cases it is not necessary to provide a distinct variable for each edge in the graph, but rather the same variable can correspond to many edges, still giving a guarantee that no cancellation will occur.

Indeed the algebraic counterpart of the formulas for the permanent of circulants with three ones per row consists of the fact that if one expands the determinant of the matrix \( I + aP + bP^k \), then no cancellation occurs (see the arguments at the beginning of this section), so that the determinant and the permanent contain the same terms, each with the right coefficient, except perhaps for the sign. This is the algebraic counterpart of the fact that entries from the same diagonal of the circulant matrix \( I + P + P^k \) correspond to edges of \( G[I + P + P^k] \) which can be oriented in the same way (see the results of the previous section).

Summarizing, we can say that in matrices corresponding to bipartite graphs of genus 1 the edges using the two “bridges” form two classes, and this can be interpreted, in algebraic terms, as the fact that a single variable can be substituted to the entries corresponding to edges in one class, so that two variables are enough to avoid cancellation.

**Example 2.** Let \( A(x, y, z) \) be the \( 9 \times 9 \) matrix \( xI + yP + zP^3 \). Then we have

\[
\det(A(x, y, z)) = x^9 + y^9 + 9x^2y^6z + 18x^4y^3z^2 + 3x^6z^3 - 9xy^3z^5 + 3x^3z^6 + z^9,
\]

and

\[
\per(A(x, y, z)) = x^9 + y^9 + 9x^2y^6z + 18x^4y^3z^2 + 3x^6z^3 + 9xy^3z^5 + 3x^3z^6 + z^9.
\]

Notice that \( \det(A(x, y, z)) \) and \( \per(A(x, y, z)) \) contain exactly the same monomials, and only differ for the sign of one coefficient.

The following example shows that for very simple circulant matrices not of the type studied in the previous section cancellation occurs if a single variable is used to represent all the entries in each diagonal.

**Example 3.** Let \( A(x, y, z, q) \) be the \( 6 \times 6 \) matrix \( xI + yP + zP^2 + qP^4 \). Then we have
\[ \det(A(x, y, z, q)) = q^6 + 2q^3 x^3 + x^6 - 3q^4 y^2 + 6q^3 y^2 + 3q^2 y^4 - y^6 \]
\[ - 6q^4 xz - 6q x^4 z + 6xy^4 z + 9q^2 x^2 z^2 - 9x^2 y^2 z^2 \]
\[ + 2q^3 z^3 + 2x^3 z^3 + 6qy^2 z^3 - 6q x z^4 + z^6, \]

while
\[ \per(A(x, y, z, q)) = q^6 + 2q^3 x^3 + x^6 + 3q^4 y^2 + 6q^3 y^2 + 3q^2 y^4 + y^6 \]
\[ + 6q^4 xz + 6q x^4 z + 12q^2 xy^2 z + 6xy^4 z + 9q^2 x^2 z^2 \]
\[ + 9x^2 y^2 z^2 + 2q^3 z^3 + 2x^3 z^3 + 6qy^2 z^3 + 6q x z^4 + z^6. \]

Except for signs, there is just a single different term, which is the coefficient of the monomial \( q^2 xy^2 z \), whose value is 12 in the permanent expansion, and which is 0 in the determinant. However this is enough to prevent any hope of computing \( \per(I + P + P^2 + P^4) \) as the linear combination of determinants of matrices of the form \( xI + yP + zP^2 + qP^4 \).

It is easy to verify that, in order to avoid cancellation in the matrix of Example 3 it is necessary to use distinct variables for all the entries in one of the four diagonals, so that, if the matrix size is \( n \), \( n + 3 \) variables are enough, as opposed to \( 4n \).

In Section 4, we have applied Tesler’s results in the case of orientable surfaces. The following example shows that further savings in the number of determinants are sometimes possible by employing complex coefficients; for instance the permanent of the matrix \( I + P + P^2 \) of odd size (which is nonconvertible) can be written as the real part of a single determinant of a complex matrix. This construction turns out to correspond to embeddings into non-orientable surfaces.

**Example 4.** Let us consider the matrix \( A(x) = I + xP + P^2 \) for \( n = 13 \), and let \( p(x) = \det(A(x)) \).

We have
\[ p(x) = 2 + 13 x - 91 x^3 + 182 x^5 - 156 x^7 + 65 x^9 - 13 x^{11} + x^{13}, \]
\[ \per(A(x)) = 2 + 13 x + 91 x^3 + 182 x^5 + 156 x^7 + 65 x^9 + 13 x^{11} + x^{13}. \]

Evaluating \( p(\cdot) \) at \( ix \), where \( i \) denotes the imaginary unity, we obtain
\[ p(ix) = 2 + i(13 x + 91 x^3 + 182 x^5 + 156 x^7 + 65 x^9 + 13 x^{11} + x^{13}), \]
from which it is easy to derive \( \per(A) = \text{Re}[(1 - i)p(\cdot)] \), where \( \text{Re}(\cdot) \) denotes the real part of the complex number \( x \).

Example above can be explained as an application of Tesler’s results for non-orientable surfaces. Indeed, the graph corresponding to \( I + P + P^2 \) for \( n \) odd can be embedded in a non-orientable surface of word \( a_1 a_1 \). This is easily accomplished
Fig. 5. Embedding the graph corresponding to the matrix $I + P + P^2$ in a non-orientable surface (case $n = 5$).

using the edges corresponding to $I$ and $P^2$ to build a cycle of length $2n$, and noticing that the edges corresponding to $P$ connect opposite vertices in the cycle. Hence it is possible to draw all these edges in a non-orientable fashion as shown in Fig. 5 for $n = 5$.

In general, when the matrix size $n$ is odd, applying Tesler’s results we have

$$\text{per}(I + P + P^2) = \text{Re}\left[\left(1 + (-1)^{(n+1)/2}\right)\det(I + iP + P^2)\right].$$

6. Comparisons with Minc’s approach

Minc adopted in [16] a technique based on permanental compounds which did not lead to substantial computational consequences, but allowed him to obtain asymptotic estimates of the limit, for $n$ approaching infinity, of the $n$th root of the permanent of some special circulants. It is remarkable to notice that the class of matrices for which Minc could actually carry out his analysis (the class characterized by convertible permanental compounds) coincides with the class of circulants associated with graphs of genus 1 studied here.

Our work thus complements Minc’s, and makes it feasible to verify experimentally the quality of his estimates (see Tables 3 and 4).

7. Further results and conclusions

It is important to mention some natural extensions of the results of this paper, which, for the sake of a simpler presentation, have been stated for $(0, 1)$ circulant matrices, despite their more general validity.

Indeed the fact that the equality between the permanent and the sum of four determinants holds term by term implies that the formulas hold true not only for

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3 This is because Tesler’s construction is applied to weighted graphs.
Table 3

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circulants matrices with three arbitrary nonzero entries per row and column, but also for matrices whose nonzero pattern reflects the circulant structure, although they are not circulant elementwise.

Another class of matrices to which the same results can be applied is that of Toeplitz matrices with five nonzero diagonals, provided that the diagonals below the main diagonal do not involve rows touched by the diagonals located above it. In this case the associated graph has still genus 1.

As observed by Minc in [16], the circulant matrix \(I + P + P^j + P^{j+1}\) is associated with the dimer problem with periodic boundary conditions; more precisely it models the dimer problem on a 2-dimensional toroidal brick with suitable side lengths (see [16, pp. 40–41]). Our approach can thus be used to count very efficiently the number of ways in which such brick can be dissected into dimers.

In this paper we have analyzed a nontrivial class of matrices with three or four nonzero entries per row and column, and introduced the first efficient algorithm for the computation of its permanents.

The general framework to which this paper belongs is that of the feasibility of computing permanents via determinants, a question of major importance, first raised by Polya in 1913 [17], and answered in 1997 by a computationally sound characterization of convertibility [14].

Further work to be done include studying Minc’s conjectures (see [16, pp. 36–37, and 39–40]) at the light of the tools made available by the results of this paper; in particular, it is now feasible to make large scale experiments which could provide counterexamples or guide us towards proofs. Furthermore it will be interesting to apply some randomized techniques to less sparse circulants, taking advantage of the arguments of Section 5.
Table 4

The three central columns contain the value of \( \frac{\text{per}(L_n + P + P^{t-1} + P^t)^{1/n}}{n} \), for several values of \( n \) and \( t \). Last column reports the corresponding estimates obtained by Minc for \( \lim_{n \to \infty} \frac{\text{per}(L_n + P + P^{t-1} + P^t)^{1/n}}{n} \). Notice that there is a discrepancy in the quality of the estimates for odd vs even size, justified by the way the approximations are computed (see [16, p. 36]).

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Acknowledgment

The authors wish to thank Gianfranco Bilardi for several fruitful discussions, and for conjecturing the formula of Corollary 6.

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[16] H. Minc, Permanental compounds and permanents of (0, 1) circulants, Linear Algebra Appl. 86 (1987) 11–42.