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Computation of Market Equilibria via the Excess Demand Function*

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Abstract

We consider the computation of equilibria for exchange economies. The general problem is unlikely to admit efficient algorithms. We develop and adapt a number of tools which allow us to take advantage of the structure of equilibria, when the market satisfies a property, called weak gross substitutability, which guarantees that the equilibria form a convex set. Using these tools we derive two polynomial time algorithms: the first one is a simple and efficient discrete version of the tâtonnement process, while the second one is based on the Ellipsoid method, and achieves a better dependence on the approximation parameter. Our approach does not make use of the specific form of the utility functions of the individual traders, and it is thus more general than previous work.

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1 Introduction

The market equilibrium problem consists of finding a set of prices and allocations of goods to economic agents such that each agent maximizes her utility, subject to her budget constraints, and the market clears. The equilibrium equations, which are satisfied under mild assumptions [1], express a static condition characterized by the fact that the market demand for each good equals its market supply. This notion fails to predict any kind of dynamics leading to an equilibrium, although it conveys the intuition that, in any process leading to a stable state where demand equals supply, a disequilibrium price of a good will have to increase if the demand for such a good exceeds its supply, and viceversa [3, 4].

The proofs of existence of equilibria [12] use general fixed point theorems and therefore do not tell us how an equilibrium can be efficiently computed. An important question that theoretical computer scientists have begun to address is whether there are efficient algorithms for computing equilibria. The general problem seems to be computationally hard (see [13], p.526), and the research has been focusing on markets which satisfy some economically meaningful restrictions.

It has been shown that equilibria can be computed in polynomial time in various special cases, the most important of which are *exchange economies*[1] when traders have utility functions that are linear [8, 11], Cobb-Douglas [9], or a range of CES functions [7]. These important special cases are all instances where the market satisfies a property called *weak gross substitutability* (WGS). Classical results in economics - see below - show that the price equilibria in such markets are characterized by an infinite number of linear inequalities and therefore form a convex set.

The combinatorial algorithm presented in [8] takes advantage of the special structure of the demand when the traders have linear utility functions and proportional initial endowments. The results in [11, 7] are based on explicit descriptions of the convex sets which characterize the equilibria for linear and a range of CES functions.

In this paper, we show that under fairly general assumptions, there are polynomial-time algorithms to compute equilibria in all markets that satisfy weak gross substitutability. To show our results, we need to build on the proofs that characterize the equilibria as a convex set using the right assumptions and ideas.

We present and analyze two algorithms:

1. A discrete version of the price-adjustment mechanism, known as tâtonnement, which computes an approximation to the equilibrium in polynomial time, and with a polynomial dependence on the approximation parameter. This algorithm is particularly simple. A preliminary version of the result was presented in [6].

2. An algorithm based on the Ellipsoid algorithm, which approximates the equilibrium in polynomial time, and which turns out to be exponentially better than algorithm 1. in terms of the dependence on the approximation parameter. A preliminary version of the result was presented in [5].

[1]Exchange economies are markets where the focus is on trading goods, while their actual production is ignored.

As a consequence, we obtain alternative polynomial time algorithms for computing equilibria for exchange economies with linear, Cobb-Douglas, and a range of CES utility functions that satisfy weak gross substitutability. Unlike previously known polynomial time algorithms, our approach does not make use of the specific form of these utility functions and is in this sense more general.

Our results are built upon Lemma 1.1 below, which has been proven by Arrow, Block, and Hurwicz [2] (the related definitions are in the next section).

Lemma 1.1 *If an equilibrium price vector $\hat{\pi}$ satisfies $\hat{\pi}_j > 0$, for each good j, if the market satisfies gross substitutability (GS), positive homogeneity, and Walras' law, then for any non-equilibrium price vector π such that $\pi_j > 0$ for each j, we have $\hat{\pi}^T Z(\pi) > 0$, where $Z(\pi)$ is the market excess demand at price π.*

The Lemma generalizes to the case where there is only WGS [3, 4], and immediately implies that the set of equilibrium prices form a convex set. It also gives, for any positive price vector π that is not an equilibrium price vector, a *separating hyperplane* [10], that is, a hyperplane that separates π from the convex set of equilibrium prices. Indeed we have $\sum_j \hat{\pi}_j Z_j(\pi) > 0$, but $\sum_j \pi_j Z_j(\pi) = 0$, by Walras' law.

The Lemma can be used to show the convergence under WGS of the tâtonnement process governed by the differential equation system:

$$\frac{d\pi_k}{dt} = G_k(Z_k(\pi)), \quad k = 1, 2, \ldots, n, \tag{1}$$

where $G_k()$ is some continuous and differentiable, sign-preserving function.

The rest of this paper is organized as follows. In Section 2 we introduce the exchange market model and provide some basic definitions. In Section 3 we describe a transformation of the input market into another one with certain desirable properties. In Section 4 we describe a computationally useful separation inequality.

In Section 5 we present and analyze a discrete version of the tâtonnement process that runs in polynomial time. In Section 6 we present a different algorithm with a better dependance on the approximation parameter, which uses the Ellipsoid algorithm.

Finally, in Section 7 we provide some concluding remarks.

2 Preliminaries

We first describe the exchange market model and provide some basic definitions.

Let us consider a market M with m economic agents who represent traders of n goods. Let $\mathbf{R}^n_+$ denote the subset of $\mathbf{R}^n$ with all nonnegative coordinates. The j-th coordinate in $\mathbf{R}^n$ will stand for good j.

Each trader i has a concave utility function $u_i : \mathbf{R}^n_+ \to \mathbf{R}_+$, which represents her preferences for the different bundles of goods, and an initial endowment of goods $w_i = (w_{i1}, \ldots, w_{in}) \in \mathbf{R}^n_+$.

The utility function u_i is nonsatiable if for any $x \in \mathbf{R}^n_+$ there is a $y \in \mathbf{R}^n_+$ such that $u_i(y) > u_i(x)$. Nonsatiation is considered, in the theory of equilibrium, a standard and extremely mild assumption (see [12], p. 42). We will assume that each trader is initially endowed with a strictly positive amount of at least one good, that is, $w_i \neq \mathbf{0}$. Let $W_j = \sum_i w_{ij}$ denote the total amount of good j in the market.

The input size of M is defined to be the number of traders plus the number of goods plus the number of bits needed for encoding the rational numbers that describe the utility functions and initial endowments.

An equilibrium is defined to be a vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}^n_+$ at which there is a bundle $\bar{x}_i = (\bar{x}_{i1}, \ldots, \bar{x}_{in}) \in \mathbf{R}^n_+$ of goods for each trader i such that the following two conditions hold:

(i) For each trader i, the vector $\bar{x}_i$ maximizes $u_i(x)$ subject to the constraints $\pi \cdot x \leq \pi \cdot w_i$ and $x \in \mathbf{R}^n_+$, and

(ii) For each good j, $\sum_i \bar{x}_{ij} \leq W_j$.

Note that the constraint[2] $\pi \cdot x \leq \pi \cdot w_i$ in (i) says that the bundle x should cost no more than the *income* $\pi \cdot w_i$ of trader i. Thus an equilibrium is a price vector at which the market clears when traders exchange their initial endowments for a bundle of goods in an optimal way.

For any price vector π, the vector $x_i(\pi)$ that maximizes $u_i(x)$ subject to the constraints $\pi \cdot x \leq \pi \cdot w_i$ and $x \in \mathbf{R}^n_+$ is called the *demand* of trader i at prices π.

In this paper, we assume that the maximizing vector is unique if it exists. This is the case with many of the commonly used utility functions. Notice that for any $\pi \in \mathbf{R}^n_+$, the set $\{x \in \mathbf{R}^n_+ | \pi \cdot x \leq \pi \cdot w_i\}$ of feasible bundles for trader i is non-empty, since $\mathbf{0}$ is contained in it. However, it can be unbounded if some of the prices in π are zero. In such a situation, $u_i(x)$ may not attain its maximum, and if this happens we say that the demand $x_i(\pi)$ is undefined. We stress that $x_i(\pi)$ can be undefined only if one or more components of π is zero.

The *excess demand* of trader i is $z_i(\pi) = x_i(\pi) - w_i$. Then $X_k(\pi) = \sum_i x_{ik}(\pi)$ denotes the *market demand* (or aggregate demand) of good k at prices π, and $Z_k(\pi) = X_k(\pi) - W_k = \sum_i z_{ik}(\pi)$ the *market excess demand* of good k at prices π. The vectors $X(\pi) = (X_1(\pi), \ldots, X_n(\pi))$ and $Z(\pi) = (Z_1(\pi), \ldots, Z_n(\pi))$ are called *market demand* (or aggregate demand) and *market excess demand*, respectively. Naturally, market demand and excess demand are undefined if some individual demand is undefined. The nonsatiability of the utility functions implies that at any price π for which the demand is well-defined, *Walras' Law* holds: $\pi^T Z(\pi) = 0$.

In terms of the excess demand function, the equilibrium is defined as a vector of prices $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{R}^n_+$ such that $Z_j(\pi) \leq 0$, for each j.

In this article we assume that (the excess demand of) the market M satisfies weak gross substitutability (WGS). That is, for any two sets of prices π and π' such that $0 < \pi_j \leq \pi'_j$, for each j, and $\pi_j < \pi'_j$ for some j, we have that $\pi_k = \pi'_k$ for any good k implies $Z_k(\pi) \leq Z_k(\pi')$. That is, increasing the prices for some of the goods while keeping some others fixed cannot cause a decrease in the aggregate demand for the goods whose price is fixed. Clearly, a market satisfies WGS if the excess demand of each individual trader does.

[2]Given two vectors x and y, we use $x \cdot y$ or $x^T y$ to denote their inner product.

2.1 Approximate Equilibria

The algorithms we develop here compute approximate equilibria. To keep the definitions of approximate equilibria simple, we assume that all the utility functions $u()$ discussed in this paper satisfy $u(0) = 0$.

Definition 2.1 *A bundle $x_i \in \mathbf{R}^n_+$ is a μ-*approximate demand *(for $\mu \geq 1$) of trader i at prices π if $u_i(x_i) \geq \frac{1}{\mu}u^*$ and $\pi^T x_i \leq \mu\pi^T w_i$, where $u^* = max\{u_i(x) | x \in \mathbf{R}^n_+, \pi^T x \leq \pi^T w_i\}$.*

A price vector $\pi \in \mathbf{R}^n_+$ is a weak μ-approximate equilibrium *($\mu \geq 1$) if there is a bundle x_i for each i such that (1) for each trader i, x_i is a μ-approximate demand of trader i at prices π, and (2) $\sum_i x_{ij} \leq \mu \sum_i w_{ij}$ for each good j.*

Observe that a 1-approximate equilibrium is just an equilibrium. For $\mu > 1$, a μ-approximate equilibrium relaxes the requirement that traders obtain optimal bundles to the requirement that traders obtain nearly optimal bundles. The relaxation on other fronts – approximate satisfaction of budget constraints, and approximate market clearance– are only included for reasons of convenience. This is established in the following lemma.

Lemma 2.2 *Let prices π be a weak μ-approximate equilibrium, and x_i's be the corresponding allocations. Let $y_i = \frac{1}{\mu}x_i$. Then y_i is a μ^2-approximate demand at π, satisfying (i) $\pi \cdot y_i \leq \pi \cdot w_i$ for each trader i, and (ii) $\sum_i y_{ij} \leq \sum_i w_{ij}$ for each good j.*

Proof: We have $\pi \cdot y_i = \frac{1}{\mu}\pi \cdot x_i \leq \pi \cdot w_i$, since x_i is a μ-approximate demand at prices π. This establishes (i). Similarly, we have $\sum_i y_{ij} = \frac{1}{\mu}x_{ij} \leq \sum_i w_{ij}$, and thus we have (ii). Finally, by the concavity of the u_i's it follows that

$$\begin{aligned} u_i(y_i) &\geq \frac{1}{\mu}u_i(x_i) + \Big(1 - \frac{1}{\mu}\Big)u_i(0) \\ &\geq \frac{1}{\mu}u_i(x_i) \\ &\geq \frac{1}{\mu^2}u_i(x_i(\pi)), \end{aligned}$$

where $x_i(\pi)$ denotes the actual demand of trader i. ■

We also need a simple property of approximate demands that concerns their resilience to small price changes.

Lemma 2.3 *Let π and π' be two sets of prices in $\mathbf{R}^n_+$, and $\varepsilon > 0$ be such that for each j we have (1) $\pi'_j \leq (1+\varepsilon) \cdot \pi_j$, and (2) $\pi_j \leq (1+\varepsilon) \cdot \pi'_j$. Let x_i be a μ-approximate demand for trader i at prices π. Then x_i is a $(1+\varepsilon)^2\mu$-approximate demand for trader i at prices π'.*

Proof:

We will first show that x_i approximately satisfies the budget constraint, to within a factor of $(1+\varepsilon)^2\mu$, at prices π'. Using (1) and the fact that $\pi \cdot x_i \leq \mu\pi \cdot w_i$, we get

$$\pi' \cdot x_i = \sum_j \pi'_j x_{ij} \leq (1+\varepsilon)\sum_j \pi_j x_{ij} \leq (1+\varepsilon)\mu\pi \cdot w_i. \quad (2)$$

Using (2) we get

$$\pi \cdot w_i \;=\; \sum_j \pi_j w_{ij} \;\le\; (1+\varepsilon)\sum_j \pi'_j w_{ij} \;=\; (1+\varepsilon)\pi' \cdot w_i. \tag{3}$$

Substituting (3) in (2), we get

$$\pi' \cdot x_i \le (1+\varepsilon)^2 \mu \pi' \cdot w_i.$$

We will now show that x_i approximately maximizes trader i's utility function at π'. Let $x^* = x_i(\pi)$ and $y^* = x_i(\pi')$.

Set $z = \frac{y^*}{(1+\varepsilon)^2}$. By definition of y^*, we have

$$\pi' \cdot y^* \le \pi' \cdot w_i.$$

Using (1) and (2), we transform this into

$$\frac{1}{(1+\varepsilon)}\pi \cdot y^* \le (1+\varepsilon)\pi \cdot w_i.$$

This implies that $\pi \cdot z \le \pi \cdot w_i$. Since $z \in \mathbf{R}^n_+$ and $\pi \cdot z \le \pi \cdot w_i$, $u_i(x^*) \ge u_i(z)$.

Therefore,

$$\begin{aligned}
u_i(x^*) &\ge u_i(z) \\
&= u_i\left(\frac{y^*}{(1+\varepsilon)^2}\right) = u_i\left(\frac{y^*}{(1+\varepsilon)^2} + (1 - \frac{1}{(1+\varepsilon)^2}) \cdot \mathbf{0}\right) \\
&\ge \frac{1}{(1+\varepsilon)^2} u_i(y^*) + \left(1 - \frac{1}{(1+\varepsilon)^2}\right) u_i(\mathbf{0}) \qquad \text{(by concavity of } u_i\text{)} \\
&= \frac{1}{(1+\varepsilon)^2} u_i(y^*) \qquad \text{(since } u_i(\mathbf{0}) = 0\text{)}.
\end{aligned}$$

Since $u_i(x_i) \ge \frac{1}{\mu} u_i(x^*)$, it follows that

$$u_i(x_i) \ge \frac{1}{\mu(1+\varepsilon)^2} u_i(y^*).$$

This completes the proof.

■

2.2 Demand Oracle

Our algorithms will assume a subroutine to compute the demand approximately at a given price vector. This is a reasonable assumption, since the demand is computed by solving a convex program for each trader.

Definition 2.4 *An exchange market M is said to be equipped with a* demand oracle *if there is an algorithm that takes as input a price vector $\pi \in \mathbf{Q}^n_+$ and a positive rational $\sigma < 1$, and returns a vector $Y = (Y_1, Y_2, \ldots, Y_n)$ such that $|Y_j - Z_j(\pi)| \le \sigma$ for all j. The algorithm is required to run in polynomial time in the input size and in* $\log(1/\sigma)$.

We assume henceforth that the market M is equipped with a demand oracle.

3 A Useful Transformation

We will now describe a transformation of the input market M (the one for which we wish to compute an equilibrium) into a market $\hat{M}$ whose excess demand function has some desirable properties. Furthermore, the ratio of the maximum to minimum price at any equilibrium price vector for $\hat{M}$ is nicely bounded. The market $\hat{M}$ can be thought of as a "perturbation" of M, and readily inherits WGS as well as a demand oracle.

It is convenient to describe the transformation in two steps. In the first, we obtain an intermediate market M' in which the total amount of each good is 1. The new utility function of the i-th trader is given by $u'_i(x_1, \ldots, x_n) = u_i(W_1 x_1, \ldots, W_n x_n)$. It can be verified that, if $u_i()$ is concave, then $u'_i()$ is concave. The new initial endowment of the j-th good held by the i-th trader is $w'_{ij} = w_{ij}/W_j$. Let w'_i denote $(w'_{i1}, \ldots, w'_{in}) \in \mathbf{R}^n_+$. Clearly, $W'_j = \sum_i w'_{ij} = 1$.

The following lemma summarizes some key properties of the transformation.

Lemma 3.1 *1. For any $\mu \geq 1$, $(x_{i1}, \ldots, x_{in})$ is a μ-approximate demand at prices $(\pi_1, \ldots, \pi_n)$ for trader i in M' if and only if $(W_1 x_{i1}, \ldots, W_n x_{in})$ is a μ-approximate demand at prices $(\frac{\pi_1}{W_1}, \ldots, \frac{\pi_n}{W_n})$ for trader i in M.*

2. For any $\mu \geq 1$, $(\pi_1, \ldots, \pi_n)$ is a weak μ-approximate equilibrium for M' if and only if $(\frac{\pi_1}{W_1}, \ldots, \frac{\pi_n}{W_n})$ is a weak μ-approximate equilibrium for M.

3. M' has a demand oracle if M does. The excess demand of M' satisfies WGS if the excess demand of M does.

We transform M' into the market $\hat{M}$ as follows. Let $0 < \eta \leq 1$ be a parameter. For each trader i, the new utility function and initial endowments are the same, that is, $\hat{u}_i() = u'_i()$, and $\hat{w}_i = w'_i$. The new market $\hat{M}$ has one extra trader, whose initial endowment is given by $\hat{w}_{m+1} = (\eta, \ldots, \eta)$, and whose utility function is the Cobb-Douglas[3] function $u_{m+1}(x_{m+1}) = \prod_j x_{m+1,j}^{1/n}$. A trader with this Cobb-Douglas utility function spends $\frac{1}{n}$-th of her budget on each good. Stated precisely, $\pi_j x_{m+1,j}(\pi) = \pi \cdot \hat{w}_{m+1}/n$. The extra trader allows us to show that at any equilibrium for $\hat{M}$ the ratio between the largest price and the smallest price is bounded above.

Note that the total amount of good j in the market $\hat{M}$ is $\hat{W}_j = \sum_{i=1}^{m+1} \hat{w}_{ij} = 1 + \eta$.

Lemma 3.2 *(1) The market $\hat{M}$ has an equilibrium. (2) Every equilibrium π of $\hat{M}$ satisfies the condition $\frac{\max_j \pi_j}{\min_j \pi_j} \leq 2n/\eta$. (3) For any $\mu \geq 1$, a weak μ-approx equilibrium for $\hat{M}$ is a weak $\mu(1+\eta)$-approx equilibrium for M'. (4) $\hat{M}$ satisfies WGS if M' does. (5) $\hat{M}$ has a demand oracle if M' does.*

Proof: (1) follows from standard arguments. Briefly, a quasi-equilibrium $\pi \in \mathbf{R}^n_+$ with $\pi_j \neq \mathbf{0}$ always exists ([12], Chapter 17, Proposition 17.BB.2). A quasi-equilibrium π is defined as a price vector at which there are allocations $x_i \in \mathbf{R}^n_+$ for each trader i so that (a) $\sum_i x_{ij} \leq \sum_i \hat{w}_{ij}$ for each good j, and (b) for each trader i, any bundle y such that $u_i(y) > u_i(x_i)$ should satisfy $\pi \cdot y \geq \pi \cdot \hat{w}_i$. Observe that (b) is a weakening of the requirement

[3]The Cobb-Douglas utility function has the general form $u_i(x) = \prod_j (x_{ij})^{a_{ij}}$, where $a_{ij} \geq 0$ and $\sum_j a_{ij} = 1$.

that x_i be a utility-maximizing bundle at price π; in particular, a trader whose income is 0 at price π is allowed to have an x_i that is not utility maximizing.

However, at price π the income $\pi \cdot \hat{w}_{m+1}$ of the $(m+1)$'th trader is strictly positive. This ensures that that $\pi_j > 0$ for each good j. Since each trader is assumed to have a strictly positive amount of each good, this means that the income $\pi \cdot \hat{w}_i$ of each trader i is strictly positive. From this it follows that each x_i is actually utility maximizing (see [12], Chapter 17, Proposition 17.BB.1), and so π is an equilibrium.

For (2), assume that the equilibrium price vector π is scaled so that $\max_j \pi_j = 1$. At price π, the income of the $(m+1)$'th trader is $\pi \cdot \hat{w}_{m+1} \geq \eta$. Since the $(m+1)$'th trader has the Cobb-Douglas utility function described above, she spends exactly a fraction $1/n$ of her income on each good. For any good k, her demand for the good is therefore at least $\frac{\eta}{n\pi_k}$. We must have $\frac{\eta}{n\pi_k} \leq \hat{W}_k = (1+\eta) \leq 2$, which implies that $\pi_k \geq \frac{\eta}{2n}$.

For (3), assume that π is a weak μ-approximate equilibrium for $\hat{M}$, and, for $1 \leq i \leq m+1$, let x_i be the corresponding bundles. Evidently, for each $1 \leq i \leq m$, x_i is a μ-approximate demand for i in the market M', and thus also a $\mu(1+\eta)$-approximate demand. For each good k, we have $\sum_{i=1}^{m+1} x_{ik} \leq \mu\hat{W}_k$. Since $x_{m+1,k} \geq 0$, this implies that $\sum_{i=1}^{m} x_{ik} \leq \mu\hat{W}_k = \mu(1+\eta)W'_k$. Thus π is a weak $\mu(1+\eta)$-approx equilibrium for M'.

For (4), note that the individual excess demand of the $(m+1)$'th trader satisfies WGS. The claim follows because the aggregate excess demand of $\hat{M}$ is the sum of the aggregate excess demand of M' and the individual excess demand of the $(m+1)$'th trader.

(5) follows for the same reason. ■

4 Computationally Useful Separation Inequalities

Our strategy is to compute a $(1+\varepsilon)$-approximate equilibrium for $\hat{M}$. From Lemma 3.1 and Lemma 3.2 (applied with $\eta = \varepsilon$), this $(1+\varepsilon)$-approximate equilibrium will then be a $(1+O(\varepsilon))$-approximate equilibrium for M.

We define $\Delta = \{\pi \in \mathbf{R}^n_+ | \eta/2n \leq \pi_j \leq 1 \text{ for each } j\}$. Note that Lemma 3.2 implies that $\hat{M}$ has an equilibrium price in Δ. We define $\Delta^+ = \{\pi \in \mathbf{R}^n_+ | \eta/2n - \eta/4n \leq \pi_j \leq 1 + \eta/4n \text{ for each } j\}$.

Our algorithms aim to find an equilibrium for $\hat{M}$ in Δ. In this section, we describe the main tool that lets our algorithms make progress from an arbitrary candidate vector $\pi \in \Delta^+$. Assuming that π is not a weak $(1+\varepsilon)$-approx equilibrium for $\hat{M}$, the following two lemmas show that the hyperplane normal to $Z(\pi)$ and passing through π separates π from all points within a distance δ of any equilibrium of $\hat{M}$ in Δ. We henceforth let $Z(\pi)$ and $X(\pi)$ denote, respectively, the excess demand vector and the aggregate demand vector in the market $\hat{M}$.

Lemma 4.1 *For any $\pi \in \Delta^+$, $||Z(\pi)||_2 \leq 8n^2/\eta$.*

Proof: In the following sequence, the third inequality follows from Walras' Law using simple manipulations, the fourth inequality holds because $\pi \in \Delta^+$, and the fifth inequality holds because $\hat{W}_j \leq 2$ for each j.

$$\begin{aligned}
||Z(\pi)||_2 &\leq \sum_j |Z_j(\pi)| \\
&\leq \sum_j X_j(\pi) + \sum_j \hat{W}_j \\
&\leq \frac{\max_k \pi_k}{\min_k \pi_k} \sum_j \hat{W}_j + \sum_j \hat{W}_j \\
&\leq \frac{2n}{\eta} \sum_j \hat{W}_j + \sum_j \hat{W}_j \\
&\leq \frac{4n^2}{\eta} + 2n \\
&\leq \frac{8n^2}{\eta}.
\end{aligned}$$

■

The following lemma and its proof build upon classic work of Arrow et al. [2], and, in particular, on Lemma 1.1 stated in the Introduction of this paper.

Lemma 4.2 *Let $\pi \in \Delta^+$ be a price vector that is not a weak $(1+\varepsilon)$-approximate equilibrium for $\hat{M}$, for some $\varepsilon > 0$. Then for any equilibrium $\hat{\pi} \in \Delta$, we have $\hat{\pi} \cdot Z(\pi) \geq \delta > 0$, where $\delta \leq 1$ and $1/\delta$ is bounded by a polynomial in n, $\frac{1}{\varepsilon}$, and $\frac{1}{\eta}$.*

Proof:
Let us assume that the goods are indexed so that

$$\frac{\pi_1}{\hat{\pi}_1} \leq \frac{\pi_2}{\hat{\pi}_2} \leq \cdots \leq \frac{\pi_n}{\hat{\pi}_n}.$$

Let $t_i = \frac{\pi_i}{\hat{\pi}_i}$; we have $t_1 \leq t_2 \leq \cdots \leq t_n$.
We are going to define a sequence of price vectors $\pi^1, \ldots, \pi^n$. Let

$$\pi^s = (\pi_1, \pi_2, \ldots, \pi_{s-1}, \pi_s = t_s\hat{\pi}_s, t_s\hat{\pi}_{s+1}, t_s\hat{\pi}_{s+2}, \ldots, t_s\hat{\pi}_n).$$

Thus π^s is the component-wise minimum of the two vectors π and $t_s\hat{\pi}$. Note that π^1 is the equilibrium vector $t_1\hat{\pi}$ and π^n is the vector π.

It is helpful to imagine that the vector $t_1\hat{\pi}$ is being transformed to π in $n-1$ steps via the sequence of prices. We refer to the change in price from π^s to π^{s+1} as the s'th step. Let $G_s^h = \{1, \ldots, s\}$ and $G_s^t = \{s+1, \ldots, n\}$. G_s^h is the subset of goods whose prices remain fixed during the s'th step, and G_s^t is the complement step. Using WGS, it is easy to argue that $Z_j(\pi^{s+1}) \geq Z_j(\pi^s)$ for $j \in G_s^h$, and $Z_j(\pi^{s+1}) \leq Z_j(\pi^s) \leq 0$ for $j \in G_s^t$.

The claim for $j \in G_s^h$ is immediate from the definition of WGS, as in going from π^s to π^{s+1} we keep j's price fixed and only increase some other prices. For $j \in G_s^t$, let us consider the vector

$$\frac{t_s}{t_{s+1}}\pi_{s+1} = (\frac{t_s}{t_{s+1}}\pi_1, \frac{t_s}{t_{s+1}}\pi_2, \ldots, \frac{t_s}{t_{s+1}}\pi_s, t_s\hat{\pi}_{s+1}, t_s\hat{\pi}_{s+2}, \ldots, t_s\hat{\pi}_n).$$

Since this is a scaling of π_{s+1}, the demand for any good at this vector is the same as at π_{s+1}. In going from $\frac{t_s}{t_{s+1}}\pi_{s+1}$ to π_s, the prices in G_s^t are fixed while the prices in G_s^h can only increase, so the demand for any $j \in G_s^t$ only increases. Thus $Z_j(\pi^{s+1}) \le Z_j(\pi^s)$. Furthermore, since $j \in G_s^t$ implies that $j \in G_{s'}^t$ for $s' < s$, we have

$$Z_j(\pi^{s+1}) \le Z_j(\pi^s) \le \cdots \le Z_j(\pi^1).$$

For the sake of clarity, let us divide the rest of the proof into two steps.

Step 1. Here we will show that

$$\hat{\pi} \cdot \left[Z(\pi^{s+1}) - Z(\pi^s)\right] \ge 0 \ \text{ for each } \ 1 \le s \le n-1. \tag{4}$$

From Walras' law, we have

$$\begin{aligned} 0 &= \pi^{s+1} \cdot Z(\pi^{s+1}) - \pi^s \cdot Z(\pi^s) \\ &= \sum_{j \in G_s^h} \pi_j \left[Z_j(\pi^{s+1}) - Z_j(\pi^s)\right] + \sum_{j \in G_s^t} t_{s+1}\hat{\pi}_j Z_j(\pi^{s+1}) - \sum_{j \in G_s^t} t_s \hat{\pi}_j Z_j(\pi^s) \\ &= t_{s+1} \sum_j \hat{\pi}_j \left[Z_j(\pi^{s+1}) - Z_j(\pi^s)\right] - \sum_{j \in G_s^h} (t_{s+1}\hat{\pi}_j - \pi_j) \left[Z_j(\pi^{s+1}) - Z_j(\pi^s)\right] + \sum_{j \in G_s^t} (t_{s+1} - t_s)\hat{\pi}_j Z_j(\pi^s) \end{aligned}$$

Rearranging, we get

$$t_{s+1}\hat{\pi} \cdot \left[Z(\pi^{s+1}) - Z(\pi^s)\right] \ = \ \sum_{j \in G_s^h} (t_{s+1}\hat{\pi}_j - \pi_j) \left[Z_j(\pi^{s+1}) - Z_j(\pi^s)\right] - \sum_{j \in G_s^t} (t_{s+1} - t_s)\hat{\pi}_j Z_j(\pi^s).$$

Since $t_{s+1}\hat{\pi}_j - \pi_j = t_{s+1}\hat{\pi}_j - t_j\hat{\pi}_j \ge t_{s+1}\hat{\pi}_j - t_s\hat{\pi}_j$ for $j \in G_s^h$, $Z_j(\pi^{s+1}) - Z_j(\pi^s) \ge 0$ for $j \in G_s^h$, and $Z_j(\pi^s) \le 0$ for $j \in G_s^t$, we obtain

$$t_{s+1}\hat{\pi} \cdot \left[Z(\pi^{s+1}) - Z(\pi^s)\right] \ \ge \ (t_{s+1} - t_s) \sum_{j \in G_s^h} \hat{\pi}_j \left[Z_j(\pi^{s+1}) - Z_j(\pi^s)\right].$$

Rearranging, we obtain

$$\hat{\pi} \cdot \left[Z(\pi^{s+1}) - Z(\pi^s)\right] \ge \frac{t_{s+1} - t_s}{t_{s+1}} \sum_{j \in G_s^h} \hat{\pi}_j \left[Z_j(\pi^{s+1}) - Z_j(\pi^s)\right]. \tag{5}$$

Since the right hand side of the inequality is non-negative, we have shown (4).

Step 2. Here we show that for at least one s, the right hand side of inequality (5) is significantly larger than 0.

We say that the k'th step is a *big* step if $t_{k+1} - t_k \ge \frac{\varepsilon t_1}{3n}$. We first claim that there must be a big step. For otherwise, we have $t_n - t_1 \le \frac{\varepsilon t_1}{3}$, and this implies that for each j,

$$t_1\hat{\pi}_j \le t_j\hat{\pi}_j = \pi_j \le t_n\hat{\pi}_j \le (1 + \varepsilon/3)t_1\hat{\pi}_j.$$

Applying Lemma 2.3 to the vectors $t_1\hat{\pi}$ and π, we see that the demand at $t_1\hat{\pi}$ is a $(1+\varepsilon)$-approximate demand at π. Since the market clears with the demand at equilibrium $t_1\hat{\pi}$, this implies that π is a weak $(1+\varepsilon)$-approximate equilibrium, a contradiction.

Suppose that the s'th step is a big step. We have the following lower bound on the increase of the income of the $(m+1)$'th trader when prices change from π^s to π^{s+1}.

$$\begin{aligned}\pi^{s+1}\cdot w_{m+1} - \pi^s \cdot w_{m+1} &\geq \pi_n^{s+1} w_{m+1,n} - \pi_n^s w_{m+1,n} \\ &= (t_{s+1}-t_s)\hat{\pi}_n w_{m+1,n}.\end{aligned}$$

Recall that the $(m+1)$'th trader is a Cobb-Douglas trader with a utility function that ensures that she spends $\frac{1}{n}$th of her income on each good. As a result we have

$$\begin{aligned}x_{m+1,1}(\pi^{s+1}) - x_{m+1,1}(\pi^s) &= \frac{\pi^{s+1}\cdot w_{m+1}}{n\pi_1^{s+1}} - \frac{\pi^s\cdot w_{m+1}}{n\pi_1^s} \\ &= \frac{1}{n\pi_1}(\pi^{s+1}\cdot w_{m+1} - \pi^s\cdot w_{m+1}) \\ &\geq \frac{t_{s+1}-t_s}{n\pi_1}\hat{\pi}_n w_{m+1,n}.\end{aligned}$$

Since the original market M satisfies WGS and $1 \in G_s^h$, we have

$$\sum_{i=1}^m x_{i,1}(\pi^{s+1}) - \sum_{i=1}^m x_{i,1}(\pi^s) \geq 0.$$

Adding the two inequalities, we get

$$Z_1(\pi^{s+1}) - Z_1(\pi^s) \geq \frac{t_{s+1}-t_s}{n\pi_1}\hat{\pi}_n w_{m+1,n}.$$

Since the transformed market $\hat{M}$ satisfies WGS (Lemma 3.2), we have $Z_j(\pi^{s+1}) - Z_j(\pi^s) \geq 0$ for each $j \in G_s^h$ and $j \neq 1$. Thus, we have

$$\sum_{j\in G_s^h} \hat{\pi}_j (Z_j(\pi^{s+1}) - Z_j(\pi^s)) \geq \frac{t_{s+1}-t_s}{n\pi_1}\hat{\pi}_1\hat{\pi}_n w_{m+1,n}.$$

This completes Step 2 of the proof. Plugging this inequality into Equation 5, and using the fact that $t_{s+1} - t_s \geq \frac{\varepsilon t_1}{3n}$, we have

$$\begin{aligned}\hat{\pi}\cdot\left[Z(\pi^{s+1}) - Z(\pi^s)\right] &\geq \frac{t_{s+1}-t_s}{t_{s+1}} \sum_{j\in G_s^h} \hat{\pi}_j\left[Z_j(\pi^{s+1}) - Z_j(\pi^s)\right] \\ &\geq \frac{t_{s+1}-t_s}{t_{s+1}}\frac{t_{s+1}-t_s}{n\pi_1}\hat{\pi}_1\hat{\pi}_n w_{m+1,n} \\ &\geq \frac{\varepsilon^2 t_1^2}{9n^3\pi_1 t_{s+1}}\hat{\pi}_1\hat{\pi}_n w_{m+1,n}.\end{aligned}$$

Adding to this inequality the sequence of inequalities

$$\hat{\pi} \cdot \left[Z(\pi^{s'+1}) - Z(\pi^{s'}) \right] \geq 0 \quad \text{for each} \quad 1 \leq s' \leq n-1, \; s' \neq s,$$

we obtain

$$\hat{\pi} \cdot \left[Z(\pi^n) - Z(\pi^1) \right] \geq \frac{\varepsilon^2 t_1^2}{9n^3 \pi_1 t_{s+1}} \hat{\pi}_1 \hat{\pi}_n w_{m+1,n}.$$

Since $\pi^n = \pi$, and $\hat{\pi} \cdot Z(\pi^1) = \hat{\pi} \cdot Z(t_1 \hat{\pi}) = 0$ by Walras' law,

$$\hat{\pi} \cdot Z(\pi) \geq \frac{\varepsilon^2 t_1^2}{9n^3 \pi_1 t_{s+1}} \hat{\pi}_1 \hat{\pi}_n w_{m+1,n}.$$

Since $\pi, \hat{\pi} \in \Delta^+$, each component of these vectors is bounded above by a polynomial in n and $\frac{1}{\eta}$, and below by the inverse of such a polynomial. The same is therefore true for the t_i. Note that $w_{m+1,n} = \eta$. Thus the proof of the lemma is completed. ■

5 Computing Equilibria via Tâtonnement

We now present an efficient algorithm, which is a discrete version of tâtonnement, for computing an approximate equilibrium for a market M. The algorithm does this by computing an approximate equilibrium for the transformed market $\hat{M}$. We start with an arbitrary price vector in the region Δ. In each iteration, if the current vector is an approximate equilibrium, we are done. Otherwise, we compute the excess demand vector at the current price and take a step in the direction of the excess demand vector. A special case occurs when the current vector is not in Δ^+; in this case, we "manually" move to a vector within Δ.

5.1 The Algorithm

Let π^0, the initial price, be any point in Δ. Suppose we have computed a sequence of prices $\pi^0, \ldots, \pi^{i-1}$. We compute π^i as follows. If $\pi^{i-1} \notin \Delta^+$, we let π^i be the point in Δ closest to π^{i-1}. In other words, $\pi^i_j = \pi^{i-1}_j$ if $\eta/2n \leq \pi^{i-1}_j \leq 1$; $\pi^i_j = 1$ if $\pi^{i-1}_j > 1$; $\pi^i_j = \eta/2n$ if $\pi^{i-1}_j < \eta/2n$.

If $\pi^{i-1} \in \Delta^+$, then we use the demand oracle to compute a vector $Y^{i-1} = (Y^{i-1}_1, \ldots, Y^{i-1}_n)$ such that $|Y^{i-1}_j - Z_j(\pi^{i-1})| \leq \delta/4n$ for each j. We let

$$\pi^i = \pi^{i-1} + \frac{\delta}{2} \cdot \frac{1}{(9n^2/\eta)^2} Y^{i-1}.$$

5.2 Analysis

Let us fix an equilibrium π^* of $\hat{M}$ in Δ. We argue that in each iteration, the distance to π^* falls significantly so long as we don't encounter an approximate equilibrium in Δ^+. If the current iteration started off from a vector not in Δ^+, this decrease in distance follows from direct inspection. On the other hand, if the current iteration started off from a vector in Δ^+, the decrease in distance is derived from Lemma 4.2. The details follow.

Let us consider the i'th iteration where we move from π^{i-1} to π^i. If $\pi^{i-1} \notin \Delta^+$, we have $|\pi_j^{i-1} - \pi_j^*| - |\pi_j^i - \pi_j^*| \geq 0$ for each j, while $|\pi_j^{i-1} - \pi_j^*| - |\pi_j^i - \pi_j^*| \geq \eta/4n$ for some j. From this it follows that

$$||\pi^* - \pi^{i-1}||^2 - ||\pi^* - \pi^i||^2 \geq (\eta/4n)^2.$$

Now suppose that $\pi^{i-1} \in \Delta^+$ and that π^{i-1} is not a weak $(1+\varepsilon)$-approx equilibrium for $\hat{M}$. By Lemma 4.2, $\pi^* \cdot Z(\pi^{i-1}) \geq \delta$. Since $\pi^{i-1} \cdot Z(\pi^{i-1}) = 0$ by Walras' Law, we have $(\pi^* - \pi^{i-1}) \cdot Z(\pi^{i-1}) \geq \delta$. Now

$$\begin{aligned}
(\pi^* - \pi^{i-1}) \cdot Y^{i-1} &\geq (\pi^* - \pi^{i-1}) \cdot Z(\pi^{i-1}) - \sum_j |Y_j^{i-1} - Z_j(\pi^{i-1})| * |\pi_j^* - \pi_j^{i-1}| \\
&\geq \delta - \sum_j \frac{\delta}{4n} \cdot 2 \\
&\geq \delta/2.
\end{aligned}$$

Also note that since $||Z(\pi^{i-1})||_2 \leq 8n^2/\eta$ (Lemma 4.1), and $||Z(\pi^{i-1}) - Y^{i-1}||_2 \leq 1$, the triangle inequality implies that $||Y^{i-1}||_2 \leq 9n^2/\eta$.

Let q denote the vector $\frac{\delta}{2}\frac{1}{(9n^2/\eta)^2}Y^{i-1}$, the step taken in the i'th iteration. We have

$$\begin{aligned}
(\pi^* - \pi^{i-1} - q) \cdot q &= (\pi^* - \pi^{i-1}) \cdot q - q \cdot q \\
&= \frac{\delta}{2}\frac{1}{(9n^2/\eta)^2}\left((\pi^* - \pi^{i-1}) \cdot Y^{i-1} - \frac{\delta}{2}\frac{1}{(9n^2/\eta)^2} Y^{i-1} \cdot Y^{i-1}\right) \\
&\geq \frac{\delta}{2}\frac{1}{(9n^2/\eta)^2}\left(\delta/2 - \frac{\delta}{2}\frac{1}{(9n^2/\eta)^2}(9n^2/\eta)^2\right) \\
&\geq 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
||\pi^* - \pi^{i-1}||^2 - ||\pi^* - \pi^i||^2 &= ||\pi^* - \pi^{i-1}||^2 - ||\pi^* - \pi^{i-1} - q||^2 \\
&= (\pi^* - \pi^{i-1}) \cdot q + (\pi^* - \pi^{i-1} - q) \cdot q \\
&\geq (\pi^* - \pi^{i-1}) \cdot q \\
&= \frac{\delta}{2} \cdot \frac{1}{(9n^2/\eta)^2}(\pi^* - \pi^{i-1}) \cdot Y^{i-1} \\
&\geq \frac{\delta^2}{4(9n^2/\eta)^2},
\end{aligned}$$

Suppose that every vector in the sequence $\pi^0, \ldots, \pi^k$ is either (a) not in Δ^+, or (b) in Δ^+ but is not a weak $(1+\varepsilon)$-approx equilibrium. We then have

$$||\pi^* - \pi^{i-1}||^2 - ||\pi^* - \pi^i||^2 \geq \min\{\frac{\delta^2}{4(9n^2/\eta)^2}, (\eta/4n)^2\},$$

for $1 \leq i \leq k$. Let μ denote $\min\{\frac{\delta^2}{4(9n^2/\eta)^2}, (\eta/4n)^2\}$. Adding these inequalities, we get

$$k\mu \leq ||\pi^* - \pi^0||^2 - ||\pi^* - \pi^k||^2 \leq n.$$

Thus $k \leq \frac{n}{\mu}$. We conclude that within $\frac{n}{\mu} + 1$ iterations, the algorithm computes a price in Δ^+ which is a weak $(1+\varepsilon)$-approximate equilibrium for $\hat{M}$. It can be verified that the bound on μ is polynomial in the input size of the original market M, $1/\varepsilon$, and $1/\eta$. Setting $\eta = \varepsilon$ in the transformation corresponding to Lemma 3.2, and putting everything together, we obtain:

Theorem 5.1 *Let M be an exchange market whose excess demand function satisfies WGS, and suppose that M is equipped with a demand oracle. For any $0 < \varepsilon < 1$, the tâtonnement based algorithm computes, in time polynomial in the input size of M and $1/\varepsilon$, a sequence of prices one of which is a weak $(1+\varepsilon)$-approx equilibrium for M.*

In order to actually pick the approximate equilibrium price from the sequence of prices, we need an efficient algorithm that recognizes an approximate equilibrium of M. In fact, it is sufficient for this algorithm to assert that a given price π is a weak $(1+2\varepsilon)$-approx equilibrium provided π is a weak $(1+\varepsilon)$-approx equilibrium. Since the problem of recognizing an approximate equilibrium is an explicitly presented convex programming problem, such an algorithm is generally quite easy to construct.

6 Algorithm Based on Ellipsoid Method

In this section, we describe an algorithm that computes a $(1+\varepsilon)$-approximate equilibrium for market M in time that is polynomial in the input size of M and in $\log \frac{1}{\varepsilon}$. This is in contrast to the tâtonnement based algorithm of the previous section, whose running time depends polynomially on $\frac{1}{\varepsilon}$. As before, we will focus on computing a $(1+\varepsilon)$-approximate equilibrium for the transformed market $\hat{M}$. We will use the Ellipsoid algorithm. In order to apply this method, we will first define a suitable size convex body that contains an equilibrium price vector.

Let π^* be some equilibrium for $\hat{M}$ in Δ^+. Let

$$\lambda = \min\{\frac{\eta}{4n}, \frac{\delta}{4n(9n^2/\eta)}\},$$

where δ is as in Lemma 4.2. Let D denote the cube

$$\{\sigma \in \mathbf{R}^n \mid |\sigma_j - \pi_j^*| \leq \lambda\}.$$

The cube D is small enough to have several useful properties listed below, while having large enough volume for the Ellipsoid method.

Lemma 6.1 *We have (i) $D \subseteq \Delta^+$; (ii) for any $\pi \in \Delta^+$ that is not a weak $(1+\varepsilon)$-approximate equilibrium, $\sigma \in D$, and $Y \in \mathbf{R}^n$ such that $|Y_j - Z_j(\pi)| \leq \frac{\delta}{8n}$ for $1 \leq j \leq n$, we have $\pi \cdot Y \leq \sigma \cdot Y$.*

Proof: That $D \subseteq \Delta^+$ is a consequence of $\pi^* \in \Delta$ and $\lambda \leq \frac{\eta}{4n}$. To show (ii), we first note that

$$\pi^* \cdot Y \geq \pi^* \cdot Z(\pi) - \pi^* \cdot (Y - Z(\pi)) \geq \pi^* \cdot Z(\pi) - \sum_j \frac{\delta \pi_j^*}{8n} \geq \delta - \sum_j \frac{2\delta}{8n} \geq \frac{3\delta}{4},$$

where in the penultimate inequality we use Lemma 4.2 and the fact that that $\pi_j^* \leq 2$. Similarly,

$$\pi \cdot Y \leq \pi \cdot Z(\pi) + \sum_j \frac{\delta \pi_j}{8n} \leq 0 + \frac{\delta}{4} \leq \frac{\delta}{4}.$$

Finally, for $\sigma \in D$, we bound

$$\begin{aligned} \sigma \cdot Y &\geq \pi^* \cdot Y - |(\sigma - \pi^*) \cdot Y| \\ &\geq \frac{3\delta}{4} - n\lambda * ||Y||_2 \\ &\geq \frac{3\delta}{4} - \frac{\delta}{4} \\ &\geq \frac{\delta}{2}. \end{aligned}$$

where we used the bound from Lemma 4.1:

$$||Y||_2 \leq ||Z(\pi)||_2 + ||Y - Z(\pi)||_2 \leq \frac{8n^2}{\eta} + 1 \leq \frac{9n^2}{\eta}.$$

Since $\sigma \cdot Y \geq \frac{\delta}{2} > \frac{\delta}{4} \geq \pi \cdot Y$, we have completed the proof of (ii). ■

The Ellipsoid Application

We are now ready to apply the central-cut ellipsoid method, Theorem 3.21 from [10]. Here is the central-cut ellipsoid theorem, slightly modified to suit our purpose:

Theorem 6.2 *There is an algorithm, called the central-cut ellipsoid method, that solves the following problem:*

Input*: A rational number $\mu > 0$ and a closed convex set $C \subseteq \mathbf{R}^n$ contained in a ball of radius R. There is an oracle that for any $\pi \in \mathbf{Q}^n$ either accepts π or finds a vector $c \in \mathbf{Q}^n$ such that $c \cdot \sigma \leq c \cdot \pi$ for any $\sigma \in C$.*

Output*: Either (i) a vector $a \in \mathbf{Q}^n$ that the oracle accepts, or (ii) an ellipsoid E such that $C \subseteq E$ and $vol(E) \leq \mu$.*

The number of calls that the algorithm makes to the oracle is polynomial in n and the encoding length of its input parameters R and μ. The number of bits used to represent the rational numbers in the vectors given to the oracle is also bounded by such a polynomial.

To apply the theorem, we let the closed convex set C be the cube D. We choose $R = 2n$ since $D \subseteq \Delta^+$ is contained in a ball of radius $2n$ centered at the origin. We let μ be a positive rational number that is smaller than the volume of D, and its decimal representation has a number of bits that is polynomial in n, $\log 1/\eta$, and $\log 1/\varepsilon$. Note that such a μ does exist and is readily computed.

As for the oracle, it works as follows. If $\pi \in \Delta^+$ and π is a $(1+\varepsilon)$-approximate equilibrium for $\hat{M}$, the oracle accepts. Otherwise, there are two cases:

- $\pi \notin \Delta^+$: Then there is a j such that either $\pi_j < \frac{\eta}{4n}$ or $\pi_j > 1 + \frac{\eta}{4n}$. If $\pi_j < \frac{\eta}{4n}$, the oracle returns $c = -e_j$, where e_j is the coordinate vector in the j-th direction. If $\pi_j > 1 + \frac{\eta}{4n}$, the oracle returns $c = e_j$. From Lemma 6.1 (i), it follows that $c \cdot \sigma \leq c \cdot \pi$ for any $\sigma \in D$.

- $\pi \in \Delta^+$: We consult the demand oracle to find a $Y \in \mathbf{R}^n_+$ such that $|Y_j - Z_j(\pi)| \leq \frac{\delta}{8n}$ for $1 \leq j \leq n$. The oracle returns $c = -Y$. From Lemma 6.1 (ii), it follows that $c \cdot \sigma \leq c \cdot \pi$ for any $\sigma \in D$.

What is the output of the central-cut ellipsoid method on the input we have described? One possibility is that it is an ellipsoid E such that $D \subseteq E$ and $vol(E) \leq \mu$. But this is impossible since $\mu < vol(D)$. Thus we can conclude that the ellipsoid algorithm produces a vector that the oracle accepts, that is, a weak $(1+\varepsilon)$-approximate equilibrium in Δ^+.

As for the running time, we observe that the number of calls that the ellipsoid algorithm makes to its oracle is bounded by a polynomial in n, $\log 1/\eta$, and $\log 1/\varepsilon$. This oracle may in turn call the demand oracle, but the overall running time of this call is bounded by a polynomial in the input size of market M, $\log 1/\eta$, and $\log 1/\varepsilon$. We may conclude that the running time of the overall algorithm is also bounded by such a polynomial.

Setting $\eta = \varepsilon$, we obtain a weak $(1 + O(\varepsilon))$-approximate equilibrium for the original market M (Lemmas 3.1 and 3.2).

Theorem 6.3 *Let M be an exchange market whose excess demand function satisfies WGS, and suppose that M is equipped with a demand oracle. For any $0 < \varepsilon < 1$, the above algorithm computes, in time polynomial in the input size of M and in $\log 1/\varepsilon$, a sequence of prices one of which is a weak $(1+\varepsilon)$-approx equilibrium for M.*

7 Conclusions

We have developed a quite general framework which allowed us to introduce two efficient algorithms for the computation of equilibria in exchange markets where the traders have linear, Cobb-Douglas, or some CES utility functions. We expect our framework to work or be readily adaptable to handle other exchange markets, provided that the utility functions satisfy weak gross substitutability.

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