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Impact of users inter-contact times on information dissemination in pervasive social networks

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Abstract

It is commonly perceived that the design principles of the future Internet might be drastically different from today. It is natural to ask what will be the impact of such evolution on the design of future Online Social Networks (OSNs). There is evidence that *human* social networks may be invariant with respect to the underlying online technology supporting them. Furthermore, the increasing pervasiveness of communication technologies is likely to enable any two users to communicate anytime and anywhere. Thus, a possible evolution of OSN design could map directly the structure of human social networks, and build future OSN services on top of a network whose edges represent "communication channels" between users sharing social relationships, and activated when they interact because of their social ties. In this paper we look, in the perspective of future OSN designed according to this concept, at how the patterns of interactions between people in human social networks impact on information dissemination properties. Based on well-established theories from the anthropology field, we study the properties of inter-contact times between users, i.e. the time between successive communication opportunities. This is a crucial feature for information dissemination, as previous results obtained in a conceptually similar environment have shown that the distribution of inter-contact times determines the convergence properties of information diffusion protocols. In the paper we investigate, by analysis, simulation and experimental results, the impact of different users interaction patterns on the properties of inter-contact times and, thus, on the convergence properties of information dissemination protocols.

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Categories and Subject Descriptors C.4 [*Performance of Systems*]; C.2.1 [*Network Architecture and Design*]

General Terms Performance

Keywords online social networks, human social networks, modelling, information diffusion

1. Introduction

Current Online Social Networks (OSNs) are a striking example of the potentiality of a tight synergy between Internet and services/applications naturally supporting human social interactions. It is commonly argued that the Internet technology may drastically change in the (near) future, due to the ever increasing diffusion of pervasive devices with communication capabilities and emerging paradigms such as content-centric [Koponen 2007] and opportunistic [Pelusi 2006] networking. It is thus sensible to consider the impact of these possible evolutions on the design of future OSNs.

There is, on the other hand, significant evidence suggesting that *human* social networks (i.e. the set of social relationships people maintain with each other) are not particularly affected by specific communication technologies [Pollet 2010]. Therefore, it is reasonable to see the properties and structures of human social networks as an invariant with respect to the evolution of the underlying means supporting social interactions. Assuming that the diffusion of pervasive mobile technologies will enable, in principle, communication between any two users anytime and anywhere, it might thus be possible to map the structures of human social networks in the core design of future OSNs technologies. Specifically, it would be possible to form a communication topology supporting OSNs, in which edges correspond to communication channels activated because of a social relationship between the two endpoints (users), and only when those users communicate due to their social relationship. Any OSN service/application would then be built on top of such a topology. The advantage of such an approach would be to make future OSNs less dependent on the specific communication technologies, and closer to the social interactions they are designed to support. Another advantage of such an OSN design paradigm would be that activated communication channels will naturally inherit the trust level between their users. As establishing trust between communication endpoints might be hard in a pervasive networking environment where everyone could communicate with anyone else, this would be another significant advantage.

In this paper we refer to this possible evolution of OSNs as *pervasive social networks*, and start investigating some fundamental properties of information diffusion in pervasive social networks. Specifically, we study conditions under which information diffusion protocols may *diverge*, i.e. yielding infinite expected delay in delivering information, when implemented on top of pervasive social networks. Similar to the concept of "systemic communication" highlighted in [Kossinets 2008], information diffusion in pervasive social networks will occur by exploiting *contacts* between users, i.e. communication events between social peers. The literature on opportunistic networking has analysed the properties of information diffusion in a similar environment, i.e., when diffusion happens via direct contacts between user devices coming within single-hop communication range (e.g. [Chaintreau 2007, Karagiannis 2010, Passarella 2011]). It has been found that the distribution of individual pairs inter-contact times (i.e. the time between two communication events between a pair of users) plays a key role in determining the convergence properties of multi-hop forwarding protocols. In pervasive social networks, the concept of contact is generalised, as physical proximity is not necessary. However, similar convergence problems may be present, as communication events will still be separated by inter-contact times between users. Therefore, in this paper we start analysing the possible effect of inter-contact times on information diffusion protocols in pervasive social networks.

[Chaintreau 2007] has shown that when individual pairs inter-contact times feature a particularly heavy tail (such as a Pareto distribution with shape $\alpha < 2$), a large family of forwarding protocols may not converge. This has been a foundational result in the opportunistic networking literature. In order to characterise the distribution of inter-contact times, real traces have been analysed extensively by the subsequent literature. The vast majority of the literature focused on the distribution of *aggregate* inter-contact times, i.e., the distribution of *all* inter-contact times between any two pairs considered altogether. This distribution, which is clearly much simpler to measure and analyse than the distributions of individual pairs, has been considered as representative of any pair's distribution, such that finding a heavy tail in the aggregate distribution has been perceived as an indication of possible divergence of forwarding protocols. More recently [Passarella 2011] has characterised much more precisely the dependence between individual pairs and aggregate inter-contact times in heterogenous networks (where not all individual pairs inter-contact times are distributed the same), highlighting when this assumption is accurate and when it is not. Characterising this dependence and highlighting when the aggregate inter-contact time can be the right figure to analyse is very important, as the aggregate distribution is a much easier and more compact figure to describe the network, with respect to the distributions of all the individual pairs' inter-contact times.

Differently from [Passarella 2011], in this paper we study the dependence between individual pairs and aggregate inter-contact times in pervasive social networks, i.e., when contact events are not determined by the users movement patterns, but by their social relationships and, thus, by the properties of the underlying human social network. Specifically (as described in Section 2) we focus on well established models of human social networks available in the anthropology literature. Based on these models we derive an analytical model showing the dependence between individual pairs and aggregate inter-contact times in pervasive social networks (Section 3). Then, we exploit the model to highlight under which conditions a heavy tail in the aggregate distribution is representative (or not) of heavy tails in the individual pairs inter-contact times distributions (Section 4). Overall, we find that also in the case of pervasive social networks the distribution of aggregate inter-contact times is not necessarily representative of individual pairs distributions, and that a heavy tailed aggregate distribution may emerge from non-heavy tailed individual pairs distributions. Beyond the specific results presented in Section 4, the key contribution of this paper is to fully characterise the dependence between individual pairs and aggregate inter-contact times, thus providing a design tool for understanding which distribution to analyse on a case-by-case basis, in order to assess the convergence properties of information diffusion protocols.

1.1 Related work

Although with a different focus than this paper, properties of information diffusion in social networks have been analysed, e.g., in [Holme 2005, Kossinets 2008, Onnela 2007]. For example, [Kossinets 2008] considers real social network traces, and studies how information disseminates through multi-hop social paths. Furthermore, [Gruhl 2004, Kempe 2003] analyse the locations in a social networks where to place information to optimise the diffusion process. With respect to this body of work, we focus on a problem not yet analysed, i.e. the impact of inter-contact times distributions on fundamental information diffusion properties.

Information diffusion properties in mobile social networks have also been analysed (see, e.g.,[Boldrini 2010, Ioannidis 2009a;b] and references herein). Specifically, [Ioannidis 2009b] studies optimal strategies for disseminating information through encountered nodes in opportunistic networks. The work in [Boldrini 2010] tackles a similar problem, and investigates how information about social relationships can be exploited from this standpoint. Finally, [Ioannidis 2009a] analyses the specific impact of weak social ties in the information dissemination problem.

To the best of our knowledge, none of the above work studied information diffusion problems considering models of human social networks as we do in this paper. Furthermore, the dependence between the different distributions characterising inter-contact times in pervasive social network, and the resulting impact on information diffusion protocols, has not been analysed before.

2. Human social networks model

For the purpose of this paper, we consider a particular model of human social networks, based on the concept of ego network. An ego network is the network seen from the standpoint of a single individual (ego). It includes only other people (alters) the ego has social relationships with (represented by an edge in the ego network).

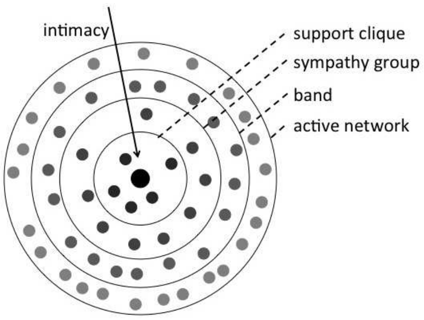

Figure 1. Ego-network's hierarchical structure.

Ego networks have been extensively studied in the anthropology literature [Dunbar 1995; 1998, Hill 2003, Roberts 2010, Zhou 2005], resulting in a detailed model of their structure (Figure 1). [Zhou 2005] has shown that ego networks can be represented as a series of concentric layers centred around the ego. Starting from the inner-most layer, layers are characterised by a decreasing level of *intimacy* with the ego. On the other hand, the *size* of the layers (the number of alters within the layer) increases with a factor approximately equal to 3. Extensive studies have identified four layers, i.e. the support clique, the sympathy group, the band and the active network, with size approximately equal to 5, 15, 45 and 150 [Dunbar 1995; 1998, Hill 2003]. The size of the active network (150) is usually referred to as the *Dunbar's number*, and represents the maximum number of alters an ego can - on average - maintain social relationships with [Hill 2003]. This is a limit related to cognitive capabilities of the human brain [Dunbar 1998]. Note that this hierarchical structure depends very little on the communication means supporting social relationships [Pollet 2010].

[Hill 2003] has also shown that the *emotional closeness* of the ego with a given alter is the key parameter determining the position of the alter in the layers. Furthermore, [Hill 2003, Roberts 2010] show that there is a strong correlation between the emotional closeness and the frequency of communication between the ego and the alter. Therefore, it follows that the structure of the ego network depicted in Figure 1 naturally determines the contact rates between the ego and alters in its social network. Specifically, contacts are more frequent with alters in the inner-most layer (usually referred to as *strong ties*), while the frequency progressively declines for external layers, resulting in *weaker* ties. This property is one of the starting points of the analysis presented in Section 3.

Finally, it is worth pointing out that, for our purposes, focusing on ego networks is sufficient. In general a social network contains more information than the set of ego networks of its members, as the latter does not capture correlations. However, it is straightforward to note that inter-contact times between any pair of users can be fully described by looking at ego networks only, because they depend on the relationship between the users only.

3. Inter-contact times model

In this section we study, through an analytical model, the dependence between the distributions of the individual pairs and aggregate inter-contact times, in a network where contacts can be described with ego network models.

[Karagiannis 2010] has already analysed this dependence in the case when the contact *rates* (the reciprocal of the average inter-contact times) between a given set of pairs is known a priori. To make the model general, in this paper we relax this assumption, and study the dependence when the contact rates are random variables (r.v.) following a known distribution (hereafter Λ_p denotes the contact rate of the generic pair p). Furthermore, we assume that individual pairs inter-contact times are distributed according to a known type of distribution (e.g., Pareto, exponential, ...). For each pair p, the parameters of the inter-contact times distribution are a function of Λ_p, i.e., the parameters are set such that the average inter-contact time is equal to $1/\Lambda_p$. This allows us to model heterogeneous environments in which not all individual inter-contact times are identically distributed, and to control the type of heterogeneity through the r.v. describing the contact rates.

Therefore, three distributions play a key role in our analysis, i.e. i) the distributions of individual pairs inter-contact times (whose CCDF is hereafter denoted as $F_\lambda(x)$), ii) the distribution of individual pairs contact rates (whose density is hereafter denoted as $f(\lambda)$), and iii) the distribution of the aggregate inter-contact times (whose CCDF is hereafter denoted as $\mathcal{F}(x)$).

3.1 Modelling human networks contact patterns

Before deriving the model, we describe how we account for the human social network structures described in Section 2. This is taken into consideration in the definition of the contact rates distribution. Figure 2 provides a schematic representation of a generic distribution. As, in any given ego network, contacts with peers in inner shells occur more frequently than contacts with peers in outer shells, contact rates with peers in inner-most shell should be drawn from the tail of the distribution, while contact rates with peers in the outer-most shell should be drawn from the head. Based on this observation, we divide the possible range of rates in L sectors, where L is the number of layers of a ego network, and layer 1 denotes the inner-most layer. Given the average number of relationships in each layer $n_l, l = 1, \dots, L$ and the total number of relationships N, we can compute the fraction of relationships in each layer as n_l/N (note that $n_L = N$). Let us then denote with $\lambda_0, \dots, \lambda_L$ the values of λ that identify the sectors of the contact rates distribution corresponding to the layers. The values of $\lambda_i, i = 1, \dots, L$ can be computed as the $(1 - \frac{n_l}{N})$-th percentile of the rates distribution (note that λ_L and λ_0 are the minimum and maximum possible values of λ, respectively). Therefore, contact rates with a peer in layer $l = 1, \dots, L$ are drawn from the sector identified by λ_l, λ_{l-1}. It thus follows that the density of contact rates for relationships in layer l is as follows

$$f_l(\lambda) = \begin{cases} 0 & \lambda < \lambda_l \vee \lambda > \lambda_{l-1} \\ C_l f(\lambda) & \lambda_l \le \lambda \le \lambda_{l-1} \end{cases} \quad (1)$$

where C_l is a constant such that $\int_0^\infty f_l(\lambda) d\lambda = 1$, i.e. $C_l = [G(\lambda_{l-1}) - G(\lambda_l)]^{-1}$, $G(\lambda)$ being the CDF of Λ.

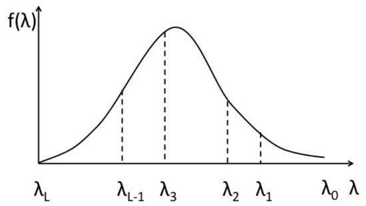

Figure 2. A representative contact rates distribution in human social networks

Note that we consider the distribution of contact rates for alters with a contact rate greater than 0, only. In principle, the distribution of contact rates presents a significant mass probability in 0, corresponding to the fact that an ego "knowns" alters also outside the active network layer, but relationships are so weak that the contact rate is zero.

3.2 General inter-contact times model

The starting point of our model is a result originally presented in [Karagiannis 2010] (and recalled in Lemma 1), which describes the dependence between the distributions of the individual pairs and aggregate inter-contact times, when the contact rates are known a priori. Let assume that P pairs are present in the network, that $n_p(T)$ contact events between pair p occur during an observation time T. Let us denote with $N(T)$ the total number of contact events over T, with θ_p the contact rate of pair p, with θ the total contact rate ($\theta = \sum_p \theta_p$), and with $F_p(x)$ the CCDF of inter-contact times of pair p. Then, the following lemma holds.

LEMMA 1. *In a network where P pairs of nodes exist for which inter-contact times can be observed, the CCDF of the aggregate inter-contact times is:*

$$\mathcal{F}(x) = \lim_{T\to\infty} \sum_{p=1}^{P} \frac{n_p(T)}{N(T)} F_p(x) = \sum_{p=1}^{P} \frac{\theta_p}{\theta} F_p(x) \quad (2)$$

Lemma 1 is rather intuitive. The distribution of aggregate inter-contact times is a mixture of the individual pairs distributions. Each individual pair "weights" in the mixture proportionally to the number of inter-contact times that can be observed in any given interval (or, in other words, proportionally to the rate of inter-contact times).

The result in Lemma 1 can be generalised to the case considered in this paper, where contact rates are r.v. distributed as described in Section 3.1. Specifically, we can derive the following Theorem.

THEOREM 1. *In a pervasive social network where contact rates are determined by the hierarchical structure of ego networks, the CCDF of the aggregate inter-contact times is:*

$$\mathcal{F}(x) = \sum_{l=1}^{L} \frac{p_l C_l}{\sum_{l=1}^{L} p_l E[\Lambda_l]} \int_{\lambda_l}^{\lambda_{l-1}} \lambda f(\lambda) F_\lambda(x) d\lambda \quad (3)$$

where p_l is the probability that a social relationship of any given user is in layer l of its ego network, and Λ_l is a r.v. denoting the contact rates with peers in layer l.

Proof. See Appendix A.

In Appendix A we provide the complete proof of Equation 3. Hereafter, we briefly discuss its physical meaning. First of all, Equation 3 can be seen as the weighted sum of components related to the individual layers of human social networks. Specifically, by defining $\mathcal{F}_l(x)$ as follows:

$$\mathcal{F}_l(x) = \frac{C_l}{E[\Lambda_l]} \int_{\lambda_l}^{\lambda_{l-1}} \lambda f(\lambda) F_\lambda(x) d\lambda \quad (4)$$

we can write $\mathcal{F}(x)$ as

$$\mathcal{F}(x) = \sum_{l=1}^{L} \frac{p_l E[\Lambda_l]}{\sum_{l=1}^{L} p_l E[\Lambda_l]} \mathcal{F}_l(x) \quad (5)$$

Equation 5 highlights an intuitive result. In appendix A we show that $\mathcal{F}_l(x)$ is actually the CCDF of the aggregate inter-contact times over layer l only. Each such component "weights" in the aggregate proportionally to the fraction of pairs falling in the layer (p_l), and to the average contact rates of the layer (i.e., to the average number of inter-contact events that is generated by a pair in that layer).

Besides a more formal derivation shown in Appendix A, the form of the individual layer's component in Equation 4 has a more intuitive derivation, starting from the result in Lemma 1. Specifically, it can be obtained by considering a modified network in which we assume that all rates $\lambda \in [\lambda_l, \lambda_{l-1}]$ are possibly available (for pairs in layer l), each with a probability $f_l(\lambda)d\lambda$. $\mathcal{F}_l(x)$ is thus the aggregate over all the resulting individual pairs inter-contact times distributions. As the number of such distributions becomes infinite and is indexed by Λ_l (a continuous random variable), the summation in Equation 2 becomes an integral over λ. Furthermore, the weight of each distribution (θ_p in Equation 2) becomes $\lambda \cdot p(\lambda) = \lambda f_l(\lambda)d\lambda$, while the total rate ($\theta$ in Equation 2) becomes $\int_0^\infty \lambda f_l(\lambda)d\lambda = E[\Lambda_l]$. The expression in Equation 4 follows immediately.

Theorem 1 shows the dependence between the three distributions that characterise the properties of inter-contact times. The key property we study in the following is under which conditions, and starting from which distributions of individual inter-contact times and contact rates, the distribution of aggregate inter-contact times features a heavy tail. This allows us to check whether focusing on the aggregate inter-contact times is sufficient for assessing the convergence properties of information dissemination, or not. To this end, it is sufficient to study the aggregate inter-contact times distribution over individual layers only, provided by Equation 4. It is, in fact, sufficient that one such aggregate presents a heavy tail for the whole aggregate to be heavy tailed. Thus, Equation 4 is the key starting point for the following analysis.

4. Study of representative pervasive social networks

[Passarella 2011] has analysed the relationship between individual pairs' inter-contact times and aggregate inter-contact times for face-to-face contacts in mobile opportunistic networks. When individual inter-contact times are exponentially distributed, very interesting results about the distribution of the aggregate inter-contact times can be highlighted when the distribution of contact rates is, respectively, gamma, exponential and Pareto. Therefore, in the following we consider the same distributions for contact rates.

First of all, we analyse the dataset presented in [Roberts 2010], which has been one of the basis for the results summarised in Section 2. The dataset collects information about 251 ego networks. Each relationship in each network provides a sample of contact rate. We fit the resulting empirical distribution to the reference distributions of this paper using the Maximum Likelihood (ML) method XXX, and compare the fitted distributions against the data using the Akaike Information Criterion (AIC, [Akaike 1974]). As we find that a gamma distribution provides the best fit, we carry on a detailed analysis of this case (Section 4.1). For completeness the study with the other reference contact rates distributions is presented in Section 4.2.

4.1 Study of a measured case

Figure 3 shows a visual comparison of the samples obtained from [Roberts 2010] and the ML fittings of the considered contact rates distributions (ML estimators of the parameters are provided in Table 1). As for the gamma distribution we consider the following definition (for the density)

$$f(\lambda) = \frac{\lambda^{\alpha-1} b^\alpha e^{-b\lambda}}{\Gamma(\alpha)} \tag{6}$$

where α and b are the shape and rate parameters, respectively. For the exponential distribution we considered the standard definition (resulting in the density in Equation 7)

$$f(\lambda) = be^{-b\lambda} \tag{7}$$

where b is the rate parameter. As for the Pareto distribution, we consider the two possible definitions resulting in the CCDFs below:

$$\begin{aligned} F(\lambda) &= \left(\tfrac{b}{\lambda}\right)^\alpha, \alpha > 0,\ \lambda > b \\ F(\lambda) &= \left(\tfrac{b}{b+\lambda}\right)^\alpha, \alpha > 0,\ \lambda > 0 \end{aligned} \tag{8}$$

where α and b are the shape and scale parameters. The difference between the two forms is that in the first case λ cannot take values arbitrarily close to 0, while in the second it can. We will show that this has a profound impact on the distribution of the aggregate inter-contact times. Hereafter, we denote with "Pareto" the first form, and with "Pareto0" the second form.

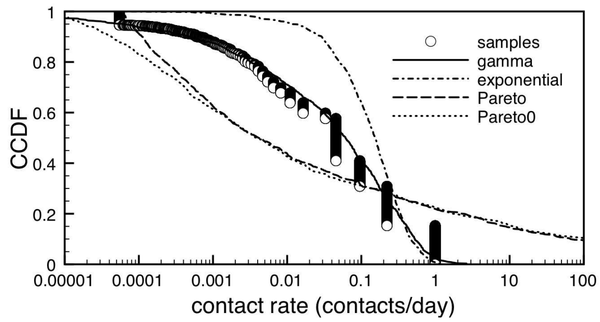

Figure 3. Fitting distributions

The intuition from Figure 3 is that the gamma distribution is the best fit. This is confirmed by the AIC test, whose values are shown in Table 1. Remember that in AIC tests the best alternative is the one with the lowest AIC value [Akaike 1974].

Based on this result, we study in detail the properties of the aggregate inter-contact times distribution assuming that the contact rates distribution is gamma, and the individual inter-contact times distributions are exponential[1]. Lemma 2 and Theorem 2 characterises the distribution of the aggregate inter-contact times in this case.

[1] Note that exponential individual inter-contact times have been found, for example, in face-to-face contacts traces, e.g. [Conan 2007, Gao 2009].

Distribution	Best fit parameters	AIC value
Gamma	$\alpha = 0.34$, $b = 1.63$	-50280.62
Exponential	$b = 4.86$	-23505.08
Pareto	$\alpha = 0.16$, $b = 5.5\text{x}10^{-5}$	-31289.34
Pareto0	$\alpha = 0.16$, $b = 5.5\text{x}10^{-5}$	-28841.84

Table 1. AIC values for the tested distributions.

LEMMA 2. *When contact rates follow a gamma distribution and individual inter-contact times an exponential distribution, the CCDFs of inter-contact times aggregated over individual layers ($\mathcal{F}_l(x)$) all decay, for large x, faster than a power law with exponential cutoff, but the CCDF corresponding to the outer-most layer, which decays as a power law. Specifically, if the contact rates follow a gamma distribution with shape α and rate b, the following relations hold true, for large x:*

$$\begin{cases} \mathcal{F}_l(x) \leq \frac{Re^{-\lambda_l(b+x)}}{x} & l = 1, \dots, L-1 \\ \mathcal{F}_L(x) \simeq \frac{K}{x^{\alpha+1}} \end{cases} \tag{9}$$

where R and K do not depend on x.

Proof. See Appendix B.

THEOREM 2. *In a pervasive social network where individual pairs inter-contact times are exponentially distributed and contact rates follow a gamma distribution, the distribution of the aggregate inter-contact times features a heavy tail. Specifically, the following relation holds true:*

$$f(\lambda) = \frac{\lambda^{\alpha-1} b^\alpha e^{-b\lambda}}{\Gamma(\alpha)}, F_\lambda(x) = e^{-\lambda x}$$
$$\Rightarrow \mathcal{F}(x) \simeq \frac{K}{x^{\alpha+1}} \text{ for large } x$$

where K does not depend on x.

Proof. This follows immediately from Lemma 2, by recalling the relationships between $\mathcal{F}_l(x)$ and $\mathcal{F}(x)$ in Equation 5, and noting that $\mathcal{F}_L(x)$ dominates over all the other components for large x.

Theorem 2 and Lemma 2 provide two interesting insights. First, the presence of aggregate inter-contact times with a heavy tail distribution does not necessarily mean that information dissemination protocols risk divergence, as such a heavy tail can emerge starting from exponentially distributed individual pairs. Therefore, when the contact rates follow a gamma distribution, looking at the distribution of aggregate inter-contact times is not sufficient to check whether information dissemination protocols may diverge or not. Instead, the distributions of individual pairs inter-contact times must be analysed. Second, the power law of $\mathcal{F}(x)$ appears because of the power law of the inter-contact times aggregated over the outer-most layers, $\mathcal{F}_L(x)$. Due to the shape of the gamma distribution, in the outer-most layers contact rates can be arbitrarily close to 0, thus resulting in arbitrarily large inter-contact times. Intuitively, this actually suggests a more general behaviour: Whenever the distribution of the contact rates is such that rates arbitrarily close to 0 can be drawn, the distribution of the aggregate inter-contact times features a heavy tail. This behaviour is confirmed also in the cases with Pareto contact rates.

To validate our analysis, we compare the result of Theorem 2 with simulations. Specifically, we simulate an ego-network of 150 alters. Ego and each alter meet with exponential inter-contact times, with rates drawn from a gamma distribution. For each alter we generate at least 100 inter-contact times. Specifically, each simulation run reproduces an observation of the network for a time interval T, defined according to the following algorithm. For each alter, we first generate 100 inter-contact times, and then compute the total observation time after 100 inter-contact times, T_a, as the sum of the pair inter-contact times. T is defined as the maximum of $T_a, a = 1, ..., 150$. To guarantee that all alters are observed for the same amount of time, we generate additional inter-contact times for each alter until T_a reaches T. Simulations have been replicated 20 times with independent seeds, and confidence intervals (with 99% confidence level) have been computed. Figure 4 shows a very good agreement between the analytical and the simulation models. Recall that the analysis predicts that the tail of the aggregate inter-contact times distribution decays as $\frac{1}{x^{\alpha+1}}$ where α is the shape parameter of the contact rates distribution. Figure 4 shows that - as also found in the analysis - the lower the shape of the contact rates distribution, the heavier the tail of the aggregate inter-contact times. This results from the fact that lower shape parameters result in a higher mass of probability of contact rates around 0, i.e., in an increasing probability of very long inter-contact times.

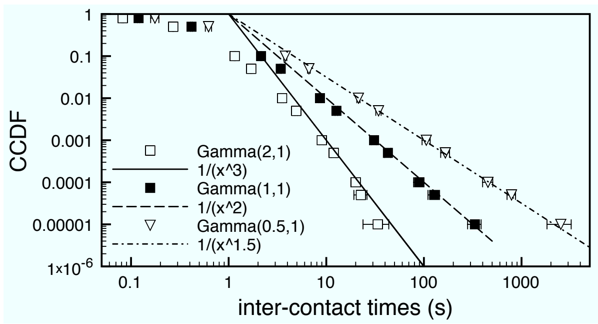

Figure 4. Aggregate inter-contact times with gamma contact rates

4.2 Study of other relevant cases

In this section we focus on the rest of the contact rates distributions considered in [Passarella 2011]. Specifically, as the case of an exponential distribution is a special case of a gamma distribution, we limit our analysis to Pareto distributions, in both variants, "Pareto" and "Pareto0", i.e. when contact rates arbitrarily close to 0 are not and are allowed, respectively.

The case "Pareto" is analysed in Lemma 3 and Theorem 3. With respect to the sectors corresponding to the layers of the ego networks, recall that in this case $\lambda_0 = \infty$ and $\lambda_L = b$ where b is the minimum possible value of the "Pareto" distribution.

LEMMA 3. *When contact rates follow a Pareto distribution whose CCDF is in the form $F(\lambda) = \left(\frac{b}{\lambda}\right)^{\alpha}$, $\lambda > b$ and individual inter-contact times are exponential, the CCDFs of inter-contact times aggregated over individual layers ($\mathcal{F}_l(x)$) all decay, for large x, at least as fast as a power law with exponential cutoff. Specifically, the following relations hold true for large x:*

$$\begin{cases} \mathcal{F}_1(x) \simeq \frac{Re^{-\lambda_1 x}}{x} \\ \mathcal{F}_l(x) \leq \frac{Ke^{-\lambda_l x}}{x} \quad l = 2, \dots, L \end{cases} \tag{10}$$

where R and K do not depend on x.

Proof. See Appendix B.

THEOREM 3. *When contact rates follow a Pareto distribution whose CCDF is in the form $F(\lambda) = \left(\frac{b}{\lambda}\right)^{\alpha}$, $\lambda > b$ and individual inter-contact times are exponential, the CCDF of the aggregate inter-contact times decays, for large x, faster than a power law with exponential cutoff. Specifically, the following relation holds true*

$$F(\lambda) = \left(\frac{b}{\lambda}\right)^{\alpha}, F_\lambda(x) = e^{-\lambda x}$$
$$\Rightarrow \mathcal{F}(x) \leq \frac{Ke^{-bx}}{x} \text{ for large } x$$

where K does not depend on x.

Proof. This comes immediately from Lemma 3, by noticing that the slowest decaying component of $\mathcal{F}(x)$ is the one related to the outer-most layer, and using the corresponding expression from Equation 10.

Lemma 3 and Theorem 3 show that even considering contact rates with an heavy tail (such as a "Pareto") is not sufficient to obtain a heavy tail in the aggregate inter-contact times distribution. This is due to the fact that the "Pareto" distribution does not admit contact rates arbitrarily close to 0.

Finally, Lemma 4 and Theorem 4 analyse the case "Pareto0", i.e. the case where the contact rates can be arbitrarily close to 0.

LEMMA 4. *When contact rates follow a Pareto distribution whose CCDF is in the form $F(\lambda) = \left(\frac{b}{b+\lambda}\right)^{\alpha}$, $\lambda > 0$ and individual inter-contact times are exponential, the CCDFs of inter-contact times aggregated over individual layers ($\mathcal{F}_l(x)$) all decay, for large x, at least as fast as a power law with exponential cutoff, but the CCDF corresponding to the outer-most layer, which decays as a power law. Specifically, the following relations hold true for large x:*

$$\begin{cases} \mathcal{F}_l(x) \leq \frac{Re^{-\lambda_l x}}{x} + \frac{Qe^{-\lambda_{l-1} x}}{x} \quad l = 1, \dots, L-1 \\ \mathcal{F}_L(x) \simeq \frac{K}{x^2} \end{cases} \tag{11}$$

where R, Q and K do not depend on x.

Proof. See Appendix B.

THEOREM 4. *When contact rates follow a Pareto distribution whose CCDF is in the form $F(\lambda) = \left(\frac{b}{b+\lambda}\right)^{\alpha}$, $\lambda > 0$ and individual inter-contact times are exponential, the CCDF of the aggregate inter-contact times decays, for large x, as a power law with shape equal to 2. Specifically, the following relation holds true*

$$F(\lambda) = \left(\frac{b}{b\lambda}\right)^{\alpha}, F_\lambda(x) = e^{-\lambda x}$$
$$\Rightarrow \mathcal{F}(x) \simeq \frac{K}{x^2} \text{ for large } x$$

where K does not depend on x.

Proof. This comes immediately from Lemma 4 by noticing that the slowest decaying component of $\mathcal{F}(x)$ is the one corresponding to the outer-most layer, and using the corresponding expression from Equation 11.

As anticipated, in the case "Pareto0" the aggregate inter-contact times distribution features a heavy tail. Intuitively, this is a side effect of the fact that contact rates can be arbitrarily close to 0. Again, note that this is another example (qualitatively similar to those of Theorem 2) where a heavy tail in the aggregate inter-contact times distribution is not necessarily a symptom of possible divergence of information dissemination protocols, as it can emerge from exponentially distributed individual inter-contact times.

As a final validation check of the analytical results presented in this section, Figure 5 shows the CCDFs of aggregate inter-contact times in the "Pareto" and "Pareto0" cases, comparing analytical and simulation results (simulations where run as explained in Section 4.2). Also in this case the agreement between the analytical and simulation results is very good (remember that the analytical model describes the behaviour of the tail of the aggregate inter-contact times distribution).

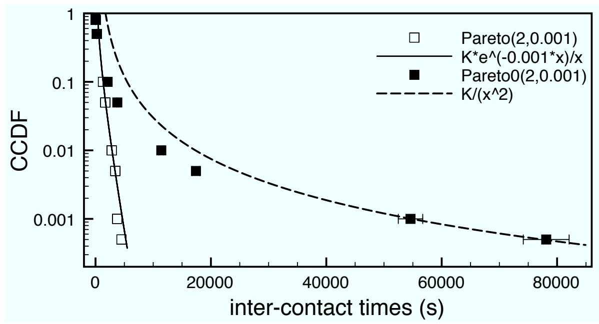

Figure 5. Aggregate inter-contact times with Pareto contact rates

5. Conclusion

In this paper we have studied fundamental properties of information diffusion algorithms in pervasive social networks. Pervasive social networks are a possible evolution of current Online Social Networks, where social network services/applications are designed on top of a communication network that maps directly *human* social networks, i.e., where edges are activated between users that share social relationships, when they communicate because of their social tie. In pervasive social networks, information diffusion will exploit *contact events* between users, i.e. events of communication between them. In such a scenario, it is fundamental to characterise the properties of inter-contact times (i.e., the time between two consecutive contact events), as this has been shown to play a key role in determining convergence properties of information diffusion algorithms.

In this paper we characterise the dependence between the distribution of individual pairs inter-contact times (which determine the convergence of information diffusion) and the distribution of aggregate inter-contact times (which is typically assumed to be the key feature to analyse). We show specific cases where the latter is not representative of the former, and where, therefore, focusing on the latter only is not sufficient. From this standpoint, we highlight that the *heterogeneity* of the network is a fundamental aspect to take into consideration, as, together with the individual pairs distributions, it determines the distribution of the aggregate inter-contact times.

Beyond the specific results described in the paper, the key contribution of this work is providing an analytical description of the relationship between the individual pairs and the aggregate inter-contact times distribution, thus providing a design tool for understanding, on a case-by-case basis, the expected convergence properties of information diffusion algorithms in pervasive social networks.

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A. Proof of Theorem 1

THEOREM 1. *In a pervasive social network where contact rates are determined by the hierarchical structure of human social networks, the CCDF of the aggregate inter-contact times is:*

$$\mathcal{F}(x) = \sum_{l=1}^{L} \frac{p_l C_l}{\sum_{l=1}^{L} p_l E[\Lambda_l]} \int_{\lambda_l}^{\lambda_{l-1}} \lambda f(\lambda) F_\lambda(x) d\lambda$$

where p_l is the probability that a social relationship of any given user is in layer l of its human social network, and Λ_l is a r.v. denoting the contact rates with peers in layer l.

Proof. In the proof we focus on a given ego-network only. As users are supposed to be statistically equivalent as far as their social network is concerned, the distributions of inter-contact times aggregated over a given user or over all the users are the same. With respect to the expression of $\mathcal{F}(x)$ in Lemma 1, contact rates are not known, but are drawn from a set of r.v. with density $f_l(\lambda)$ (Equation 1). The expression of $\mathcal{F}(x)$ can be derived conditioning to a specific set of rates $\lambda_1, \ldots, \lambda_N$, and applying the law of total probability. Without loss of generality, we can assume that the pairs $\{1, \ldots, N\}$ are ordered according to their membership to layers, i.e., the first n_1 pairs belong to the inner-most layer, etc. We thus obtain

$$\begin{aligned} \mathcal{F}(x) &= \int_{\lambda_1} \cdots \int_{\lambda_N} \mathcal{F}(x|\lambda_1,\ldots,\lambda_N) f(\lambda_1,\ldots,\lambda_N) d\lambda_1 \ldots d\lambda_N \\ &= \int_{\lambda_1} \cdots \int_{\lambda_N} \frac{\sum_{p=1}^{N} \lambda_p F_p(x)}{\sum_{p=1}^{N} \lambda_p} f_1(\lambda_1) \ldots f_N(\lambda_N) d\lambda_1 \ldots d\lambda_N \ , \end{aligned}$$

where we have assumed that rates of individual pairs inter-contact times are independent. For a sufficiently large number of pairs in each layer, we can apply the law of large numbers, and approximate $\sum_{i=1}^{n_l} \lambda_i$ as $n_l E[\Lambda_l]$, and $\sum_{p=1}^{N} \lambda_p$ as $\sum_{l=1}^{L} n_l E[\Lambda_l]$. Swapping the integrals and the summations, and substituting $p_l = \frac{n_l}{N}$, we further obtain:

$$\begin{aligned} \mathcal{F}(x) &= \frac{1}{\sum_l n_l E[\Lambda_l]} \\ &\quad \sum_p \int_{\lambda_1} \cdots \int_{\lambda_N} \lambda_p F_p(x) f_1(\lambda_1) \ldots f_N(\lambda_N) d\lambda_1 \ldots d\lambda_N = \\ &= \frac{1}{\sum_l n_l E[\Lambda_l]} \sum_{l=1}^{L} n_l \int_0^\infty \lambda F_\lambda(x) f_l(\lambda) d\lambda = \\ &= \sum_{l=1}^{L} \frac{p_l}{\sum_{l=1}^{L} p_l E[\Lambda_l]} \int_0^\infty \lambda f_l(\lambda) F_\lambda(x) d\lambda \\ &= \sum_{l=1}^{L} \frac{p_l C_l}{\sum_{l=1}^{L} p_l E[\Lambda_l]} \int_{\lambda_l}^{\lambda_{l-1}} \lambda f(\lambda) F_\lambda(x) d\lambda \ , \end{aligned}$$

where we have exploited the assumption that rates of individual pairs inter-contact times of the same layer are identically distributed, and that individual pairs inter-contact times of the same layer follow the same type of distribution, $F_\lambda(x)$.

Note that the above methodology can also be applied to show that $\mathcal{F}_l(x)$ in Equation 4 is the CCDF of the inter-contact times aggregated over layer l only. Specifically, exploiting again Lemma 1, we can condition $\mathcal{F}_l(x)$ to a known set of rates $\lambda_1, \ldots, \lambda_{n_l}$. Thus, we can write:

$$\begin{aligned} \mathcal{F}_l(x) &= \int_{\lambda_1} \cdots \int_{\lambda_{n_l}} \mathcal{F}(x|\lambda_1,\ldots,\lambda_{n_l}) f(\lambda_1,\ldots,\lambda_{n_l}) d\lambda_1 \ldots d\lambda_{n_l} \\ &= \int_{\lambda_1} \cdots \int_{\lambda_{n_l}} \frac{\sum_{i=1}^{n_l} \lambda_i F_i(x)}{\sum_{i=1}^{n_l} \lambda_i} f_1(\lambda_1) \ldots f_{n_l}(\lambda_{n_l}) d\lambda_1 \ldots d\lambda_{n_l} \ . \end{aligned}$$

By approximating $\sum_i \lambda_i$ as $n_l E[\Lambda_l]$, and by recalling that the contact rates in layer l are assumed to be identically distributed, we obtain

$$\begin{aligned} \mathcal{F}_l(x) &= \frac{1}{n_l E[\Lambda_l]} n_l \int_0^\infty \lambda f_l(\lambda) F_\lambda(x) d\lambda \\ &= \frac{C_l}{E[\Lambda_l]} \int_{\lambda_l}^{\lambda_{l-1}} \lambda f(\lambda) F_\lambda(x) d\lambda \end{aligned}$$

B. Proof of Lemmas in Section 4

LEMMA 2. *When contact rates follow a gamma distribution and individual inter-contact times an exponential distribution, the CCDFs of inter-contact times aggregated over individual layers ($\mathcal{F}_l(x)$) all decay, for large x, faster than a power law with exponential cutoff, but the CCDF corresponding to the outer-most layer, which decays as a power law. Specifically, if the contact rates follow a gamma distribution with shape α and rate b, the following relations hold true, for large x:*

$$\begin{cases} \mathcal{F}_l(x) \leq \frac{Re^{-\lambda_l(b+x)}}{x} & l = 1, \ldots, L-1 \\ \mathcal{F}_L(x) \simeq \frac{K}{x^{\alpha+1}} \end{cases}$$

where R and K do not depend on x.

Proof. First of all, it should be noted that when contact rates follow a gamma distribution, the values of λ that limits the sectors corresponding to the layers of the ego network are such that $\lambda_0 = \infty$ and $\lambda_L = 0$. Let us focus on the CCDF of aggregate inter-contact times on intermediate layers (i.e., excluding the inner- and the outer-most layers) first. From Equation 4, by substituting the expressions of $f(\lambda)$ and $F_\lambda(x)$ we obtain:

$$\begin{aligned} \mathcal{F}_l(x) &= H \int_{\lambda_l}^{\lambda_{l-1}} \lambda^\alpha e^{-(b+x)\lambda} d\lambda \\ &= H \frac{\Gamma(\alpha+1, \lambda_l(b+x)) - \Gamma(\alpha+1, \lambda_{l-1}(b+x))}{(b+x)^{\alpha+1}} \end{aligned}$$

where H does not depend on x and $\Gamma(\cdot, \cdot)$ is the upper incomplete Gamma function. For large x, $\Gamma(s, x)$ can be approximated as $x^{s-1}e^{-x}$ [Abramowitz 1964]. We thus obtain:

$$\begin{aligned}\mathcal{F}_l(x) &\simeq \frac{x^\alpha(Re^{-\lambda_l(b+x)} - We^{-\lambda_{l-1}(b+x)})}{(b+x)^{\alpha+1}} \\ &\simeq \frac{Re^{-\lambda_l(b+x)} - We^{-\lambda_{l-1}(b+x)}}{x} \leq \frac{Re^{-\lambda_l(b+x)}}{x}\end{aligned}$$

For the inner-most sector, we can write

$$\mathcal{F}_1(x) = H\int_{\lambda_1}^{\infty} \lambda^\alpha e^{-(b+x)\lambda} d\lambda = H\frac{\Gamma(\alpha+1, \lambda_1(b+x))}{(b+x)^{\alpha+1}}$$

For large x, applying the same approximation for $\Gamma(\cdot,\cdot)$, we obtain

$$\mathcal{F}_1(x) \simeq M\frac{x^\alpha e^{-\lambda_1(b+x)}}{(b+x)^{\alpha+1}} \simeq M\frac{e^{-\lambda_1(b+x)}}{x}$$

Finally, for the outer-most sector, $\mathcal{F}_L(x)$ becomes:

$$\begin{aligned}\mathcal{F}_L(x) &= H\int_0^{\lambda_{L-1}} \lambda^\alpha e^{-(b+x)\lambda} d\lambda \\ &= H\frac{\Gamma(\alpha+1) - \Gamma(\alpha+1, \lambda_{L-1}(b+x)}{(b+x)^{\alpha+1}}\end{aligned}$$

Approximating $\Gamma(\cdot,\cdot)$ we obtain

$$\begin{aligned}\mathcal{F}(x) &\simeq W\frac{\Gamma(\alpha+1)}{(b+x)^{\alpha+1}} - A\frac{x^\alpha e^{-\lambda_{L-1}(b+x)}}{(b+x)^{\alpha+1}} \\ &\simeq W\frac{\Gamma(\alpha+1)}{(b+x)^{\alpha+1}} \simeq \frac{K}{x^{\alpha+1}}\end{aligned}$$

LEMMA 3. *When contact rates follow a Pareto distribution whose CCDF is in the form* $F(\lambda) = \left(\frac{b}{\lambda}\right)^\alpha$, $\lambda > b$ *and individual inter-contact times are exponential, the CCDFs of inter-contact times aggregated over individual layers* ($\mathcal{F}_l(x)$) *all decay, for large* x, *at least as fast as a power law with exponential cutoff. Specifically, the following relations hold true for large* x:

$$\begin{cases}\mathcal{F}_1(x) \simeq \frac{Re^{-\lambda_1 x}}{x} \\ \mathcal{F}_l(x) \leq \frac{Ke^{-\lambda_l x}}{x} \quad l = 2,\ldots,L\end{cases}$$

where R *and* K *do not depend on* x.

Proof. Using the same methodology of Lemma 2, we obtain, for all components of $\mathcal{F}(x)$ but the one corresponding to the inner-most layer, the following expression:

$$\begin{aligned}\mathcal{F}_l(x) &= H\int_{\lambda_l}^{\lambda_{l-1}} \frac{e^{-\lambda x}}{\lambda^\alpha} d\lambda \\ &= H\frac{\Gamma(1-\alpha, \lambda_l x) - \Gamma(1-\alpha, \lambda_{l-1}x)}{x^{1-\alpha}}\end{aligned}$$

Applying the usual approximation of $\Gamma(\cdot,\cdot)$ for large x we obtain

$$\mathcal{F}_l(x) \simeq x^{-\alpha}\frac{Ke^{-\lambda_l x} - Qe^{-\lambda_{l-1}x}}{x^{1-\alpha}} \leq \frac{Ke^{-\lambda_l x}}{x}$$

The component corresponding to the innermost layer can be written as

$$\mathcal{F}_1(x) = H\int_{\lambda_1}^{\infty} \frac{e^{-\lambda x}}{\lambda^\alpha} d\lambda = H\frac{\Gamma(1-\alpha, \lambda_1 x)}{x^{1-\alpha}}$$

The expression in the Lemma follows immediately by applying the usual approximation of $\Gamma(\cdot,\cdot)$ for large x.

LEMMA 4. *When contact rates follow a Pareto distribution whose CCDF is in the form* $F(\lambda) = \left(\frac{b}{b+\lambda}\right)^\alpha$, $\lambda > 0$ *and individual inter-contact times are exponential, the CCDFs of inter-contact times aggregated over individual layers* ($\mathcal{F}_l(x)$) *all decay, for large* x, *at least as fast as a power law with exponential cutoff, but the CCDF corresponding to the outer-most layer, which decays as a power law. Specifically, the following relations hold true for large* x:

$$\begin{cases}\mathcal{F}_l(x) \leq \frac{Re^{-\lambda_l x}}{x} + \frac{Qe^{-\lambda_{l-1}x}}{x} \quad l = 1,\ldots,L-1 \\ \mathcal{F}_L(x) \simeq \frac{K}{x^2}\end{cases}$$

where R, Q *and* K *do not depend on* x.

Proof. Let us consider components of $\mathcal{F}(x)$ other than the one corresponding to the outermost layer. The following equation holds true:

$$\begin{aligned}\mathcal{F}_l(x) &= H\int_{\lambda_l}^{\lambda_{l-1}} \frac{\lambda}{(b+\lambda)^{\alpha+1}} e^{-\lambda x} d\lambda \\ &= e^{bx}\left\{\frac{\Gamma(1-\alpha, (\lambda_l+b)x) - \Gamma(1-\alpha, (\lambda_{l-1}+b)x)}{x^{1-\alpha}} + \right. \\ &+ \left. bx\frac{\Gamma(-\alpha, (\lambda_{l-1}+b)x) - \Gamma(-\alpha, (\lambda_l+b)x)}{x^{1-\alpha}}\right\}\end{aligned}$$

Applying the usual approximation of $\Gamma(\cdot,\cdot)$ for large x it is easy to obtain the following relation

$$\begin{aligned}\mathcal{F}_l(x) &\leq e^{bx}\left\{\frac{[(\lambda_l+b)x]^{-\alpha}e^{-(\lambda_l+b)x}}{x^{1-\alpha}} + \right. \\ &+ \left. \frac{bx[(\lambda_{l-1}+b)x]^{-\alpha-1}e^{-(\lambda_{l-1}+b)x}}{x^{1-\alpha}}\right\}\end{aligned}$$

from which it is straightforward to derive the expression in Equation 11. As for the component of $\mathcal{F}(x)$ corresponding to the outermost layer, we obtain

$$\begin{aligned}\mathcal{F}_L(x) &= H\int_{\lambda_0}^{\lambda_{L-1}} \frac{\lambda}{(b+\lambda)^{\alpha+1}} e^{-\lambda x} d\lambda \\ &= e^{bx}\left\{\frac{\Gamma(1-\alpha, bx) - \Gamma(1-\alpha, (\lambda_{L-1}+b)x)}{x^{1-\alpha}} + \right. \\ &+ \left. bx\frac{\Gamma(-\alpha, (\lambda_{L-1}+b)x) - \Gamma(-\alpha, bx)}{x^{1-\alpha}}\right\}\end{aligned}$$

This time it is necessary to apply an approximation of $\Gamma(s,x)$ that considers higher order terms (for large x), i.e.

$x^{s-1}e^{-x}\left(1+\frac{s-1}{x}\right)$ [Abramowitz 1964]. We thus obtain the final result shown in Equation 11:

$$
\begin{aligned}
\mathcal{F}_L(x) &\simeq e^{bx}\frac{(bx)^{-\alpha-1}e^{-bx}(\alpha+1)-(bx)^{-\alpha-1}e^{-bx}\alpha}{x^{1-\alpha}} \\
&= \frac{K}{x^2}
\end{aligned}
$$