Geometric ergodicity of a bead-spring pair with stochastic Stokes forcing

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Abstract

We consider a simple model for the fluctuating hydrodynamics of a flexible polymer in dilute solution, demonstrating geometric ergodicity for a pair of particles that interact with each other through a nonlinear spring potential while being advected by a stochastic Stokes fluid velocity field. This is a generalization of previous models which have used linear spring forces as well as white-in-time fluid velocity fields.

We follow previous work combining control theoretic arguments, Lyapunov functions, and hypo-elliptic diffusion theory to prove exponential convergence via a Harris chain argument. In addition we allow the possibility of excluding certain "bad" sets in phase space in which the assumptions are violated but from which the system leaves with a controllable probability. This allows for the treatment of singular drifts, such as those derived from the Lennard-Jones potential, which is a novel feature of this work.

1 Introduction

The study of polymer stretching in random fluids has been identified as a first step in the much larger project of modeling and understanding drag reduction in polymer solutions [Che00] and theoretical focus has been brought on the dynamics of simple dumbbell models [LMV02], [CMV05], [AV05]. Of particular interest is the experimentally observed phenomenon called the coiled state / stretched state phase transition [GCS05]. Mathematically this transition has been characterized by seeking models which admit solutions that are ergodic for only certain regions

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of parameter space [CMV05]. In this paper we address the topic of how to prove ergodicity for a wide range of models that generalize preceding work.

Let $X_1(t)$ and $X_2(t)$ denote the respective positions in $\mathbb{R}^2$ of two polymer "beads" connected by a "spring" at time t. Depending on the scale of interest, these beads may be thought of as consecutive segments (consisting of something like 50 monomers) in a polymer chain [DE86, Ö96], or as the ends of a full polymer chain [BHAC77, CMV05, AV05]. Having made this caveat, the canonical Langevin model for two spherical particles in a passive polymer system is given by

$$m\ddot{X}_i = -\nabla_{X_i}\Phi(X_1 - X_2) + \zeta(u(X_i(t), t) - \dot{X}_i(t)) + \kappa\dot{W}(t) \tag{1}$$

for $i = 1, 2$. The mass m is considered to be vanishingly small and so the inertial term, $m\ddot{X}_i$, will be ignored. On the right hand side, the first term is the restorative force exerted on the beads due to the potential energy of the polymer's current configuration. The function Φ denotes the configuration potential for the two beads. The second term is an expression for the drag force exerted by a time-dependent fluid velocity field u with friction coefficient $\zeta := 6\pi a\eta$. This follows from the Stokes drag law for a spherical particle of radius a in a fluid with viscosity η. The final term is the force due to thermal fluctuations in the fluid where $W(t)$ is a standard Brownian motion. The diffusive constant κ is often taken to be $\kappa = \sqrt{2k_B T\zeta}$, where k_B is the Boltzmann constant and T is the temperature of the system in Kelvin, in accordance with the fluctuation-dissipation theorem [CMV05].

The goal of the present work is to achieve rigorous results about the ergodicity of the *connector* process

$$R(t) := \frac{1}{2}(X_1(t) - X_2(t))$$

in both $\kappa = 0$ and $\kappa \neq 0$ regimes with nonlinear spring interaction in the presence of a spatially and temporally correlated incompressible fluid velocity field.

In the simplest possible setting, one ignores the fluid and assumes a Hookean (quadratic) spring potential Φ. In this case, equation (1) is a simplification of the classical Rouse model [DE86]. For the choice of $\Phi(r) = \frac{\gamma}{2}|r|^2$ the particle dynamics satisfy the system of SDE

$$\begin{aligned} dX_1(t) &= \gamma\,[X_2(t) - X_1(t)]\,dt + \kappa\,dW_1(t) \\ dX_2(t) &= \gamma\,[X_1(t) - X_2(t)]\,dt + \kappa\,dW_2(t) \end{aligned}$$

where W_1 and W_2 are independent standard Brownian motions. The dynamics of the connector $R(t)$ are given by

$$dR(t) = -2\gamma R(t) + \frac{\kappa}{\sqrt{2}}dW(t).$$

where $W = \frac{1}{\sqrt{2}}(W_1 - W_2)$ is a standard Brownian motion. We see that each of the connector components is an Ornstein-Uhlenbeck process which therefore has the unique invariant measure $R^i(t) \sim N\left(0, \frac{\kappa^2}{8\gamma}\right)$. This exactly solvable model does not yield physical results, so one must adopt nonlinear models for either or both of the spring potential and fluid forces.

Significant theoretical advances exist for the dynamics of a single tracer particle convected by a wide variety of fluid models [MK99]. One popular fluid model for non-interacting two-point motions [BCH07] [MWD$^+$05] as well as for Hookean bead-spring systems [Che00, LMV02, CMV05] is a time-dependent random field satisfying the statistics of the Kraichnan-Batchelor ensemble [Bat59] [Kra68]. Such a fluid is still statistically white in time, but is colored in space.

In the case where $\kappa = 0$ with non-interacting beads, the spatial correlations in the convecting fluid velocity field allow for concentration and aggregation phenomena [SS02b] [MWD$^+$05] [BCH07]. This happens because when the two beads are very close together, the fluid forces on the respective beads are so strongly correlated there is no force encouraging separation.

The presence of a diffusive term with $\kappa \neq 0$ prevents such aggregation and the long term behavior of the connector depends on so-called Weissenberg number Wi $= \zeta/2\gamma = \kappa^2/4k_BT\gamma$ [CMV05]. It is shown that when Wi < 1 the connector R will have a non-trivial stationary distribution, dubbed the "coiled" state. For Wi > 1, the connector does not have a stationary distribution and is called "stretched." The authors express interest in the case where the fluid is not assumed to be white-in-time.

In this work we use the incompressible stochastic Stokes equations to generate a fluid that is colored in space and time (see Section 1.2). In the Hookean spring case (among other potentials with no repulsive force between the beads) with $\kappa = 0$, this model leads to degenerate dynamics (Proposition A.1). However, in a more general setting with a nonlinear spring potential that includes a repulsive force, we show that dynamics are nondegenerate, although the coiled / stretched state dichotomy discussed in [CMV05] is not present. We find that $R(t)$ is ergodic regardless of the physical parameters (Theorem 2.1).

The method used here to establish ergodicity builds on the Harris Chain theory developed in [Har56, Has80, Num84]. It is particularly indebted to the uniform ergodic results in weighted norms developed in [MT93a, MT93b]. The argument follows the path outlined in [MS02, MSH02] for unique ergodicity of degenerate diffusions, but requires some nontrivial extensions to deal with the multiplicative nature of the noise and to permit the type of singular vector fields that arise as natural choices for the spring potential Φ. We build a framework around a general ergodic result from [HM11] and then develop the needed analysis to apply this framework.

Mathematically, as in [MSH02, MS02], this paper combines control theory with techniques from the theory of hypoelliptic diffusions to invoke results in the spirit of [MT93a, MT93b]. Ergodicity is obtained by proving a minorization condition on a class of "small sets" (see [MT93a, MT93b]) while simultaneously establishing a matching Lyapunov function. However, our problem has a number of difficulties which prevent the application of the results [MSH02] directly. A central issue that needs to be addressed is that the spring potential, and hence the drift term, is permitted to have a singularity (Assumption 1). Therefore the natural candidates for "small sets" are not compact. This difficulty is overcome by splitting the small sets into "good" and "bad" sets. On the compact "good" set, defined in Eq. (27), we demonstrate uniform controllability as in [MSH02, MS02]. On the bad set, one cannot obtain uniform control; however, the deterministic dynamics move the system into the good set in finite time so that geometric ergodicity still holds (Section 2.2). Allowing the spring potential to be singular extends the applicability of the theory to many interesting, physically important potentials such as the Lennard-Jones potential. Related ideas have been also recently been used to prove ergodic and homogenization results in different settings (see [Bub09, HP08]).

1.1 Structure of paper and overview of results

We will conclude Section 1 by proposing the model, leaving the proof of global existence and uniqueness to the Appendix. It is important to point out that without a repulsive force between the beads, this model is degenerate. As an example, we consider in Proposition A.1 a pair of particular choices – including the Hookean spring model – for the spring potential that do not introduce a repulsive force between the beads. We find that the distance between the beads $R(t)$ almost surely tends to 0 as $t \to \infty$ if the spring constant γ is sufficiently strong relative to a quantity that depends on the typical spatial gradients in the random forcing.

In Section 2, we quote an abstract result from the classical ergodic theory literature. The quoted result requires proving a minorization condition and the existence of a Lyapunov function. Section 2.1 contains a general prescription for how to deduce the minorization condition from the existence of a continuous transition density and a weak form of topological irreducibility for the Markov process. In Section 2.2 the needed topological irreducibility is proven via a control theoretic argument. In Section 2.3 we invoke Hörmander's "sum of squares" theorem to prove that the associated hypoelliptic diffusion has a smooth transition density. Section A.4 contains the calculations establishing the existence of a Lyapunov function and Section 2.4 contains a number of generalizations and implications of the preceding results. The appendix contains the derivation of the model used.

Before preceding, we note that among the class of models we propose, the closest to that of Celani, et. al. [CMV05] is the canonical Langevin Equation (1) where the spring potential is quadratic, the mass m is still 0, but the coefficient of the Brownian motion is nonzero: $\kappa = \sqrt{2k_B T\zeta}$. Our generalization is the replacement of the Kraichnan-ensemble with a finite-dimensional version of the stochastic Stokes equations. In this $\kappa > 0$ setting, the dynamics when $|R|$ is small become greatly simplified. Indeed, when the force separating the beads due to the fluid velocity becomes negligible, the remaining terms constitute an Ornstein-Uhlenbeck process. By standard ergodic properties of such processes, R quickly leaves any small neighborhood of the origin with probability 1. For large values of $|R|$, the quadratic spring potential dominates and the Lyapunov function calculation we present in Section A.4 still holds. Since the diffusion is elliptic, existence of a continuous transition density follows trivially, and all arguments in the derivation of the stochastic δ-ball controllability still apply, and thus the ergodic theorem we present in this work holds for $R(t)$.

This stands in contrast to the results in [CMV05] where it was argued that there exists a range of parameters where no stationary distribution exists. Furthermore, in light of the results we present here, it is not clear to us how to construct a model with colored-in-space-and-time fluid velocity field that supports the "stretched" and "coiled" regimes cited in the physics literature. Unfortunately, we cannot comment directly on the model presented in [CMV05], as our approach is highly dependent on the ability to express the dynamics in terms of a system of SDEs.

1.2 Definition of the model

In the overdamped, highly viscous regime, it is reasonable to neglect the nonlinear term in Navier-Stokes equations. Following [OR89], [MS02], [MSH02] and [SS02a], we consider the bead-spring system advected by a random field $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ satisfying the incompressible time-dependent stochastic Stokes equations. Following [Wal86], [DZ92], [Dal99] and [McK06] the stochastic PDE

$$\partial_t u(x,t) - \nu \Delta u(x,t) + \nabla p(x,t) = F(dx,dt), \quad \nabla \cdot u(x,t) = 0 \tag{2}$$

is well defined under the following conditions. For technical simplicity in the ergodicity arguments to come, we take u to be spatially periodic with period L which is presumed to be very large. We take the space-time forcing $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ to be a white-in-time, spatially periodic and colored-in-space Gaussian process satisfying

$$\mathbb{E}[F(x,t)] = 0, \quad \mathbb{E}\big[F^i(x,t)F^j(y,s)\big] = (t \wedge s)\, 2k_B T \nu \delta_{ij} \Gamma(x-y) \tag{3}$$

where Γ is the spatial covariance function, ν is the viscosity of the fluid, $t \wedge s$ denotes the minimum of t and s, the component indices i and j are $i, j \in \{1, 2\}$ and δ_{ij} is a Kronecker delta function. As is shown in Appendix A, we may take the definition of the noise to be

$$F(x,t) = \frac{\sqrt{2k_B T \nu}}{L} \sum_{k \in \mathbb{Z}^2 \setminus 0} \big(\cos\big(\lambda k \cdot x\big) B_k^1(t) + \sin\big(\lambda k \cdot x\big) B_k^2(t)\big)\sigma_k \tag{4}$$

where we have introduced the inverse length scale $\lambda = 2\pi/L$ and the B_k^i are independent standard 2-d Brownian motions. The coefficients σ_k are related to the spatial correlation function Γ through the Fourier relation $\Gamma(x) = \frac{2}{L^2}\sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \cos\big(\lambda k \cdot x\big)\sigma_k^2$.

This relation is possible because Γ is a covariance function, and therefore positive definite. By Bochner's Theorem, Γ is realizable as the Fourier inverse transform of a positive real valued measure called the *spectral* measure. Often one defines the correlation structure on the spectral domain. For clarity of exposition, we take the set of modes with nonzero σ_k, denoted $\mathcal{K} \subset \mathbb{Z}^2 \setminus (0,0)$ to be finite but containing at least three linearly independent vectors. We use $N = |\mathcal{K}|$ to denote the number of active modes.

As is discussed in the Appendix, Section A, we can express the dynamics of the eigenmodes in terms of the family of independent 1-dimensional Ornstein-Uhlenbeck processes $Z(t) := \{Z_k(t)\}_{k \in \mathcal{K}}$ respectively satisfying

$$dZ_k(t) = -\lambda^2 \nu |k|^2 Z_k(t) dt + \sqrt{2\beta\nu}\lambda\sigma_k\, dW_k(t) \tag{5}$$

where $\beta = k_B T/4\pi^2$ and $\{W_k\}_{k \in \mathcal{K}}$ is a family of iid standard 1-dimensional Brownian motions. For each k, we take the initial condition $Z_k(0)$ to be chosen from its respective stationary distribution, namely $Z_k(0) \sim \mathrm{N}\left(0, \beta\sigma_k^2/|k|^2\right)$.

Our goal will be to rigorously analyze the long-term behavior of the connector process R whose dynamics we will study via an approximate system which is derived in the Appendix, Section A. This entails writing X_1 and X_2 in terms of the configuration vector $R(t)$ and the "center of mass" process $M(t)$. As is discussed in that development, we set $M(t) = 0$ to substantially simplify subsequent calculations. We argue that this assumption can be removed and that all of the relevant results hold for the original system.

We now define our model for $R(t)$. Given the family $Z(t)$ defined by (5), let $R : \mathbb{R} \to \mathbb{R}^2$ satisfy the time-inhomogeneous ODE

$$\frac{d}{dt}R(t) = -\nabla\Phi(R(t)) + \sum_{k\in\mathcal{K}} \sin(\lambda\, k \cdot R(t))\, \frac{k^\perp}{|k|} Z_k(t) \tag{6}$$

where for a given vector $k = (k_1, k_2)$ we denote $k^\perp := (-k_2, k_1)$. The configuration potential $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is discussed below in Assumption 1. The last term of (6) summarizes the influence of the fluid on the separation between the beads. We will write this in terms of the multiplication of the $2 \times N$ Stokes matrix $S(r)$ by the vector $z = (z_1, \dots z_N)$,

$$S(r)z := \sum_{k\in\mathcal{K}} \sin(\lambda\, k \cdot r)\, \frac{k^\perp}{|k|} z_k\,. \tag{7}$$

We discuss the existence and uniqueness of the ODE (6) in Appendix A.3 and will think of the solution R with initial condition r_0 in terms of the mapping

$$R := \Psi(r_0, Z) \tag{8}$$

where $\Psi : \mathbb{R}^2 \times C([0,\infty), \mathbb{R}^N) \to C([0,\infty), \mathbb{R}^2)$ is the solution of the ODE given in (6).

As mentioned earlier, the choice of quadratic potential Φ corresponds to a Hookean spring model. There are a number of canonical choices for nonlinear spring potentials (see [BHAC77] Table 10.1-1) but of particular interest to us are potentials which only allow for a finite maximum extension of the polymer. One common choice is known as the *finite extensible nonlinear elastic* (FENE) [BHAC77, AV05, Thi03] potential:

$$\Phi_{\text{FENE}}(r) = \frac{\gamma\rho_{max}^2}{2} \ln\left(\frac{1}{1 - |r|^2/\rho_{max}^2}\right). \tag{9}$$

The parameter $\rho_{max} > 0$ is the maximal extension of the chain. However, because there is no repulsive force in the potential, we find that systems with these potentials have degenerate dynamics (Proposition A.1). In the sequel, we place the following assumptions on the spring potential.

Assumption 1. *Let* $0 < \rho_{max} \le \infty$ *be given and define*

$$\mathcal{D} := \{r \in \mathbb{R}^2 \text{ such that } |r| \le \rho_{max}\}.$$

We assume that the spring potential $\Phi : \mathcal{D} \to \mathbb{R}_+$ *satisfies* $\Phi(0) = 0$ *and each of the following conditions.*

(i) Radial symmetry. *For some continuously differentiable function* $\phi : (0, \rho_{max}) \to \mathbb{R}_+$*, we have*

$$\Phi(r) = \phi(|r|). \tag{10}$$

(ii) Locally Lipschitz gradient. *For any compact region* $K \subset \mathcal{D} \setminus \{0\}$*, there exists a constant* $C > 0$ *such that for all* $r_1, r_2 \in K$*,*

$$|\nabla\Phi(r_1) - \nabla\Phi(r_2)| \leq C|r_1 - r_2|.$$

(iii) Compact level sets. *For every* $\rho \geq 0$*, the set* $\{r \in \mathcal{D}\ s.t.\ \Phi(r) \leq \rho\}$ *is compact.*

(iv) Growth condition. *The potential satisfies* $\lim_{|r|\to\rho_{max}} \Phi(r) = \infty$ *and there exists a* $\gamma > 0$ *and a* $\rho_0 < \rho_{max}$ *such that for all* $r \in \mathcal{D}$ *with* $|r| \in (\rho_0, \rho_{max})$

$$|\nabla\Phi(r)|^2 \geq \gamma\Phi(r). \tag{11}$$

(v) Repulsive force at the origin. *There exists* $\gamma_0 > 0$ *and* $\epsilon_0 > 0$ *such that for all* $r \in \mathcal{D} \setminus \{0\}$ *with* $|r| \leq \epsilon_0$

$$-\nabla\Phi(r) \cdot r \geq \gamma_0. \tag{12}$$

Remark 1.1. It is in this context that we choose the length of the periodicity of the forcing fluid. We take $L \gg 4\rho_0$.

We have in mind potentials that consist of standard choices when the beads are separated by large distances, but that have a singularity at zero. For example, the above assumptions include the families of functions

$$\Phi(r) = \frac{1}{2q}|r|^{2q} + \frac{1}{\alpha|r|^\alpha}, \quad \text{and} \quad \Phi(r) = \Phi_{\text{FENE}}(r) + \frac{1}{\alpha|r|^\alpha}. \tag{13}$$

where α is a positive constant. The choice $\alpha = 12$ corresponds to a Lennard-Jones singularity at zero. One can check that the Growth Condition (iv) is satisfied for such potentials if and only if $q \geq 1$.

2 Ergodicity

In order to state our main result, we must set some notation. Let $X(t) = (R(t), Z(t))$ satisfy the system given by (5) and (6). It follows from Proposition A.3 that the process $X(t)$ is Markov and well-defined on the state space

$$\mathbb{X} := \left\{(r, z) \in \mathcal{D} \times \mathbb{R}^N\right\} .$$

For a bounded, measurable function $\varphi : \mathbb{X} \to \mathbb{R}$, we define the action of the Markov semigroup $\mathcal{P}_t$ by

$$(\mathcal{P}_t\varphi)(x) = \mathbb{E}_x[\varphi(X(t))] \ .$$

To measure convergence to equilibrium, we introduce the following weighted norm on such functions φ relative to a given Lyapunov function $V : \mathbb{X} \to [0, \infty)$,

$$\|\varphi\| := \sup_{x \in \mathbb{X}} \frac{|\varphi(x)|}{1 + V(x)} \ .$$

We note that the Markov semigroup $\mathcal{P}_t$ can be extended to act on all functions φ bounded pointwise above by V. Henceforth, we will use

$$V(x) := \psi(\Phi(r)) + \eta|z|^2 \tag{14}$$

as the Lyapunov function for the Markov process $X(t)$, where $\psi : \mathbb{R} \to \mathbb{R}$ is the function

$$\psi(x) := \begin{cases} 0, & 0 \leq x \leq a \\ c\,(x-a)\, e^{-1/(x-a)^2}, & x > a \end{cases} , \tag{15}$$

where we set $a = \phi(\rho_0)$. The constant ρ_0 is as in Equation (11) of Assumption 1, and the constants c and η are set by an argument in Section A.4. The essential properties of ψ are recorded in Section A.2.

The main result of this article is the following statement about the geometric ergodicity of the Markov process X, which in turn implies the connector process R converges to its unique non-trivial stationary distribution in exponential time.

Theorem 2.1. *Suppose that the set of active modes $\mathcal{K}$ is finite, but contains at least three pairwise linearly independent vectors, and let the spring potential Φ satisfy Assumption 1. Then there exists a unique non-trivial invariant measure π and constants $C > 0$ and $\lambda > 0$ so that for all measurable $\varphi : \mathbb{X} \to \mathbb{R}$ with $\|\varphi\| < \infty$, we have*

$$\|\mathcal{P}_t\varphi - \pi\varphi\| \leq Ce^{-\lambda t}\|\varphi\|$$

where $\pi\varphi = \int \varphi d\pi$.

Let us introduce a family of weighted L^∞-norms that depend on a scale parameter $\beta > 0$. For a measurable $\varphi\colon \mathbb{X} \to \mathbb{R}$ define

$$\|\varphi\|_\beta := \sup_{x \in \mathbb{X}} \frac{|\varphi(x)|}{1 + \beta V(x)} \ .$$

Observe that $\|\cdot\|_1 = \|\cdot\|$ and any two norms in this family are equivalent. Define the corresponding dual metric on probability measures:

$$\rho_\beta(\mu_1,\mu_2) = \sup_{\varphi:\|\varphi\|_\beta\le 1} \int \varphi(x)\mu_1(dx) - \int \varphi(x)\mu_2(dx)$$

for two probability measure μ_1, μ_2 probability measures on $\mathbb{X}$. Note that ρ_β is the usual total variation norm for $\beta = 0$. Theorem 2.1 follows from classical results in [MT93a] and [MT93b] adapted to our setting:

Theorem 2.2. *Suppose that the Lyapunov function $V\colon \mathbb{X} \to [0,\infty)$ has compact level sets with $\lim_{x\to\partial\mathbb{X}} V(x) = \infty$ and that for some $t > 0$, $c_1 > 0$ and $c_0 \in (0,1)$, it satisfies*

$$(\mathcal{P}_t V)(x) \le c_0 V(x) + c_1 \tag{16}$$

for all $x \in \mathbb{X}$. (Here, the boundary set $\partial\mathbb{X}$ includes the point at infinity in unbounded directions.) Furthermore suppose there exists a probability measure ν and constant $\alpha \in (0,1)$ such that

$$\inf_{x\in\mathcal{C}} \mathcal{P}_t(x,\,\cdot\,) \ge \alpha\nu(\,\cdot\,) \tag{17}$$

with $\mathcal{C} := \{x \in \mathbb{X}\colon V(x) \le K\}$ for some $K \ge 2c_1/(1-c_0)$.

Then there exists an $\alpha_0 \in (0,1)$ and $\beta > 0$ so that

$$\rho_\beta(\mathcal{P}_t^*\mu_1, \mathcal{P}_t^*\mu_2) \le \alpha_0\, \rho_\beta(\mu_1,\mu_2)$$

for any two probability measures μ_1 and μ_2 on $\mathbb{X}$.

We begin by fixing the set $\mathcal{C}$ which should be thought of as the "center" of the state space. At the end of the proof of Lemma 2.3 we select a value $\rho_+ \in (\rho_0, \rho_{max})$ which is used to define $\mathcal{C}$:

$$\mathcal{C} := \{x \in \mathbb{X} : V(x) \le \psi(\phi(\rho_+))\}. \tag{18}$$

Recall that the Lyapunov function V is defined by (14) with ϕ and ψ defined by (10) and (15), respectively. As is established by the following lemma, V satisfies the inequality (16). We defer the somewhat standard proof of this lemma to the appendix, Section A.4.

Lemma 2.3 (Lyapunov function)**.** *Fix the values of the constants η, c and ρ_{max} so that they satisfy the constraints imposed by the inequalities* (54)*, and let $V(x)$ be defined as in* (14)*. Then for any $t \ge 1$ there exist constants $c_0 := c_0(t) \in (0,1)$ and $c_1 := c_1(t) \ge 0$ such that* (16) *holds. Moreover we have $\psi(\phi(\rho_+)) \ge \frac{2c_1}{1-c_0}$ as required for the definition* (18) *of $\mathcal{C}$ by Theorem 2.2.*

The remainder of Section 2 is concerned with constructing a minorizing measure, as required by condition (17). The main result is Proposition 2.4. Its proof follows from the topological irreducibility of the transition semigroup established in Proposition 2.10 and the "local smoothing" property proved in Proposition 2.11. The local smoothing property follows from hypoellipticity of the generator of the Markov process X and a version of Hörmander's sum of squares theorem (cf. [Hör85, Str08]).

2.1 Conditions for measure-theoretic irreducibility

In this section we use a very weak form of topological irreducibility to prove the measure-theoretic minorization and irreducibility required in (17).

Proposition 2.4. *Suppose there exists an $x_* \in \mathcal{C}$ such that the following two conditions hold. Then there exists a constant $\alpha \in (0,1)$, a time $t \geq 1$ and a probability measure ν such that* (17) *holds.*

(i) Uniformly Accessible Neighborhood Condition: *For any $\delta > 0$ there exists a constant $r > 0$ and a positive function $\alpha_0 \colon (0,\infty) \to (0,\infty)$ such that*

$$\inf_{x \in \mathcal{C}} \mathcal{P}_{r'}(x, B_\delta(x_*)) \geq \alpha_0(r') \tag{19}$$

for all $r' > r$.

(ii) Continuous Density Condition: *There exists an $s > 0$ and an open set $\mathcal{O} \subset \mathcal{C}$ with $x_* \in \mathcal{O}$, such that for any $x \in \mathcal{O}$ and measurable $A \subset \mathcal{O}$ one has*

$$\mathcal{P}_s(x, A) = \int_A p_s(x,y)dy$$

with $p_s(x,y)$ jointly continuous in (x,y) for $x, y \in \mathcal{O}$ and $p_s(x_*, y_*) > 0$ for some $y_* \in \mathcal{O}$.

Proof. By the continuity assumption on p_s there exists $\delta > 0$ so that $B_\delta(x_*), B_\delta(y_*) \subset \mathcal{O}$ and

$$\inf_{x \in B_\delta(x_*)} \inf_{y \in B_\delta(y_*)} p_s(x,y) \geq \frac{1}{2} p_s(x_*, y_*) > 0 \ .$$

We define the minorizing probability measure ν by $\nu(A) = \lambda(A \cap B_\delta(y_*))/\lambda(B_\delta(y_*))$ where λ is Lebesgue measure and A is any measurable set. With this δ we also fix $r = r(\delta)$ according to the Uniformly Accessible Neighborhood Condition (i).

Now, pick $t \geq 1+r+s$ and define $\alpha(t) = \frac{1}{2}(1 \wedge p_s(x_*, y_*)\alpha_0(t-s)\lambda(B_\delta(y_*)))$, where α_0 is the function given in (19). Then for any measurable set A and $x_0 \in \mathcal{C}$ we have

$$\begin{aligned}\mathcal{P}_t(x_0, A) &= \int_A \int_{\mathbb{R}^{2+N}} \mathcal{P}_{t-s}(x_0, dx)\mathcal{P}_s(x, dy) \\ &\geq \int_{A \cap B_\delta(y_*)} \left(\int_{B_\delta(x_*)} \mathcal{P}_{t-s}(x_0, dx) \right) p_s(x, y)dy \\ &\geq \int_{A \cap B_\delta(y_*)} \alpha_0(t-s)\frac{1}{2}p_s(x_*, y_*)dy \geq \alpha(t)\nu(A) \ ,\end{aligned}$$

which proves the claim.

□

2.2 Topological irreducibility

This section is devoted to proving the Uniformly Accessible Neighborhood Condition (i) stated in Proposition 2.4. This argument consists of first proving that under the spring potential conditions listed in Assumption 1, the system has non-trivial long-term behavior. Unlike the Hookean spring case where the two particles come together as $t \to \infty$ almost surely (Proposition A.1), in the non-linear (with repulsion) spring case we can show that two particles arbitrarily close together have a positive probability of separating in an explicitly defined finite time (Lemma 2.5). Given this separation property, we employ a control argument to show the noise has a positive probability of directing the system to a neighborhood of a specified reference point $x_* \in \mathcal{C}$ (Lemmas 2.6 and 2.10).

2.2.1 A particle separation lemma

Lemma 2.5. *Let $M > m > 0$ be given, suppose $(R(0), Z(0)) = (r_0, z_0) \in \mathcal{D} \times \mathbb{R}^N$, and define*

$$\tau_\epsilon(r_0, z_0) := \inf\{t \geq 0 : |R(t)| \geq \epsilon \textit{ and } |Z(t)| < M\}.$$

Then there exists an $\epsilon \in (0, \epsilon_0]$ where ϵ_0 is defined in Assumption 1 and an $\alpha \in (0, 1)$ such that τ_ϵ satisfies

$$\inf_{\{z_0 : |z_0| < m\}} \inf_{\{r_0 : 0 < |r_0| \leq \epsilon\}} \mathbb{P}\{\tau_\epsilon(r_0, z_0) \leq 1\} \geq \alpha. \qquad (20)$$

Proof. The essence of the argument is that if the noise stays relatively small for sufficiently long, then the repulsive force will dominate the R-dynamics and force the particles away from each other. Without loss of generality, for the remainder of this proof we assume that the initial condition (r_0, z_0) satisfies $r_0 \leq \epsilon_0$ and $|z_0| \leq m$.

We denote the event that the magnitude of the noise stays moderate by $\Omega_z := \big\{ \sup_{t\in[0,1]} |Z(t)| < M \big\}$ and claim there exists an $\epsilon \in (0, \epsilon_0]$ and $\alpha > 0$ such that $\mathbb{P}\{\Omega_z\} \geq \alpha$ and $\mathbb{P}\{\tau_\epsilon \leq 1 \,|\, \Omega_z\} = 1$ and therefore

$$\mathbb{P}\{\tau_\epsilon \leq 1\} \geq \mathbb{P}\{\tau_\epsilon \leq 1 \,|\, \Omega_z\} \cdot \mathbb{P}\{\Omega_z\} \geq 1 \cdot \alpha.$$

We first prove that there exists an $\alpha > 0$ such that

$$\inf_{z_0:|z_0|\leq m} \mathbb{P}\{\Omega_z\} \geq \alpha. \tag{21}$$

Indeed, the noise vector $Z(t) = (Z_1(t), Z_2(t), \ldots, Z_N(t))$ can be written

$$Z(t) = e^{-\Lambda t} z_0 + \int_0^t e^{-\Lambda(t-s)} B dW(s) \tag{22}$$

where Λ is a diagonal matrix whose entries $\{\lambda_k\}_{k\in\mathcal{K}}$ are given by $\lambda_k := \lambda^2\nu|k|^2$ and B is a diagonal matrix whose entries $\{b_k\}_{k\in\mathcal{K}}$ are given by $\sqrt{2\beta\nu}\lambda\sigma_k$.

It follows from (22) that

$$|Z(t)| \leq m + \sum_{k\in\mathcal{K}} \left| e^{-\lambda_k t} \int_0^t e^{\lambda_k s} b_k dW_k(s) \right|.$$

Since $M_k(t) := \int_0^t e^{\lambda_k s} b_k dW_k(s)$ is a continuous martingale with quadratic variation $\langle M_k, M_k\rangle_t = b_k^2(e^{2\lambda_k t} - 1)/2\lambda_k$, then for any $t > 0$, $M_k(t)$ has the same distribution as $\tilde{W}(\langle M_k, M_k\rangle_t)$ where $\tilde{W}$ is a standard Brownian motion. It follows that

$$\alpha_k := \mathbb{P}\Big\{ \sup_{t\in[0,1]} \big|e^{-\lambda_k t} \int_0^t e^{\lambda_k s} dW_k(s)\big| \leq \frac{M-m}{N} \Big\} \geq \mathbb{P}\Big\{ \sup_{t\in[0,t_k]} |\tilde{W}(t)| \leq \frac{M-m}{N} \Big\}$$

where $t_k = b_k^2(e^{2\lambda_k} - 1)/2\lambda_k$. Since a Brownian motion will stay within a prescribed tube over an arbitrarily long finite interval with positive probability, we have that $\alpha_k > 0$. Because there are only finitely many modes and they are mutually independent, we have $\mathbb{P}\{\Omega_z\} \geq \prod_{k\in\mathcal{K}} \alpha_k > 0$. To conclude the proof of the claim (21), it remains only to note that this lower bound for $\mathbb{P}\{\Omega_z\}$ does not depend on the initial condition z_0 as long as $|z_0| \leq m$.

We now show that there exists an $\epsilon > 0$ so that

$$\mathbb{P}\{\tau_\epsilon \in [0,1] | \, \Omega_z\} = 1. \tag{23}$$

Let ϵ_0 and γ_0 be the positive constants from (12) of Assumption 1. We fix

$$\epsilon := \epsilon_0 \wedge \sqrt{\frac{(1 - e^{-(NM)^2/\gamma_0})}{(NM)^2}} \tag{24}$$

and define $\sigma_\epsilon := \inf\{t \geq 0 : |R(t)| \geq \epsilon\}$. Conditioned on the event Ω_z, we have $\tau_\epsilon = \sigma_\epsilon$, and so to prove (23) it suffices to show $\sigma_\epsilon \leq 1$ on Ω_z.

Recall the ODE (6) defining R and the notation S for the Stokes matrix, see (7). For any $t \in [0, \sigma_\epsilon]$ and for any $\vartheta > 0$ we have the differential inequality

$$\frac{d}{dt}\frac{1}{2}|R|^2 = -\nabla\Phi(R) \cdot R + (S(R)Z) \cdot R \geq \gamma_0 - \vartheta|S(R)Z|^2 - \frac{1}{4\vartheta}|R|^2$$

where we have applied the inequality (12) from Assumption 1 to the first term and the polarization inequality $x \cdot y \geq -(\vartheta|x|^2 + \frac{1}{4\vartheta}|y|^2)$ to the second term. Furthermore $|S(R)Z| \leq \|S(R)\|_F|Z|$ where $\|\cdot\|_F$ is the matrix Frobenius norm. The contribution of each column (respectively associated to an eigenmode k) of the Stokes matrix to its Frobenius norm is exactly $\sin^2(\lambda k \cdot R)$. It follows that $\|S(R)\|_F \leq N$. Hence for all $t \in [0, \sigma_\epsilon]$,

$$\frac{d}{dt}\frac{1}{2}|R(t)|^2 \geq -\frac{1}{4\vartheta}|R(t)|^2 + (\gamma_0 - \vartheta N^2|Z(t)|^2).$$

Restricting to the event Ω_z and fixing $\vartheta = \gamma_0/2(NM)^2$, we have

$$\frac{d}{dt}|R(t)|^2 \geq -\frac{(NM)^2}{\gamma_0}|R(t)|^2 + \gamma_0.$$

For any $t \in [0, 1]$, integrating the preceding estimate on Ω_z yields

$$\begin{aligned}|R(t \wedge \sigma_\epsilon)|^2 &\geq e^{-(t\wedge\sigma_\epsilon)(NM)^2/\gamma_0}|r_0|^2 + \gamma_0 \int_0^{t\wedge\sigma_\epsilon} e^{-[(t\wedge\sigma_\epsilon)-s](NM)^2/\gamma_0} ds \\ &\geq \frac{\gamma_0^2}{(NM)^2}\left(1 - e^{-(t\wedge\sigma_\epsilon)(NM)^2/\gamma_0}\right) .\end{aligned}$$

We want to show that on Ω_z, $\sigma_\epsilon \leq 1$ with probability one. Suppose that $\sigma_\epsilon > 1$. Then the last estimate implies that

$$|R(1 \wedge \sigma_\epsilon)|^2 = |R(1)|^2 \geq (NM)^{-2}(1 - e^{-(NM)^2/\gamma_0}) \geq \epsilon^2$$

and hence $\sigma_\epsilon \leq 1$. We conclude the claim (23), which completes the proof. □

2.2.2 Topological irreducibility via control

By Assumption 1, the spring potential Φ has a (possibly non-unique) global minimum $r_{\min}$, which satisfies $|r_{\min}| \le \rho_0$ where ρ_0 was the constant from Assumption 1. We choose a global minimum closest to the origin and denote it by r_*. Since the global minimum of the noise norm $|\cdot|$ is achieved at the origin, $z_* = 0$, we set the global reference point

$$x_* := (r_*, 0) \tag{25}$$

which is a minimum of the Lyapunov function V.

We wish to use the Z process to drive the R process to the reference point r_*. However, due to the possible singularity at the origin (see Assumption 1) the differential equation (6) for R may have unbounded coefficients which presents a genuine difficulty in applying control theoretic arguments. We therefore will designate a region of bad control, $\mathcal{B}$, within the center $\mathcal{C}$ (see (18)), as well as a compact region of good control, $\mathcal{G}$.

In Lemma 2.5 we demonstrated that the R process has a positive probability of escaping from a neighborhood of 0 in unit time. Let ϵ_1 be the constant derived from applying Lemma 2.5 with $m = \psi(\phi(\rho_+))$ and $M = m/\sqrt{\eta}$, where η is given in (54). Since $\eta \le 1/2$ we have $M > m > 0$ as required by the hypothesis of Lemma 2.5. We define the set of "bad" points in $\mathcal{C}$ by

$$\mathcal{B} = \Big\{ (r, z) \in \mathcal{C} : |r| < \epsilon_1 \Big\} \,. \tag{26}$$

Next, we define the set of "good" points $\mathcal{G}$ to be

$$\mathcal{G} = \mathcal{G}_r \times \mathcal{G}_z := \Big\{ (r, z) \in \mathbb{X} : |r| \in \big[\epsilon_1, \rho_+\big], |z|^2 \le \psi(\phi(\rho_+))/\eta \Big\} \,. \tag{27}$$

Note that $\mathcal{C} \subset \mathcal{G} \cup \mathcal{B}$.

We now use a controllability argument to establish the weak form of uniform topological irreducibility on $\mathcal{G}$ given (for the set $\mathcal{C}$) in Eq. (19).

Lemma 2.6 (Topological irreducibility on the "good" set $\mathcal{G}$)**.** *Let $x_* \in \mathcal{C}$ be as given in* (25)*. Then for any $\delta > 0$ there exists $t_1 > 0$ so that for any $t_2 > t_1$ there exists $\alpha_1 > 0$ such that*

$$\inf_{t \in [t_1, t_2]} \inf_{x \in \mathcal{G}} \mathcal{P}_t(x, B_\delta(x_*)) \ge \alpha_1. \tag{28}$$

The proof of the above lemma relies on the following three observations, whose proofs are deferred to the appendix. In what follows, for $f : I \subset \mathbb{R} \mapsto \mathbb{R}^n$, define the sup-norm

$$|f|_\infty := \sup_{t \in I} |f(t)| \,.$$

The first observation is that there is a bounded deterministic control $\tilde{Z}$ that accomplishes the task of moving its associated connector $\tilde{R} = \Psi(r_0, \tilde{Z})$ (recall the definition in Equation (8)) from the initial position r_0 to the reference point r_* at time $t = 1$.

Fact 2.7. (Existence of a deterministic control.) *For any initial position $\tilde{r}_0 \in \mathcal{G}_r$, the set $\mathcal{R} \subset C^\infty([0,1]; \mathcal{G}_r)$ defined by*

$$\mathcal{R} := \left\{ \tilde{R} \;:\; \tilde{R}(0) = \tilde{r}_0,\; \tilde{R}(1) = r_*, \left| \frac{d\tilde{R}}{dt} \right|_\infty \leq 5\rho_+ \right\} \tag{29}$$

is non-empty. Furthermore, there exists an $M_1 > 0$, which does not depend on $\tilde{r}_0$, such that for any $\tilde{R} \in \mathcal{R}$, there exists a continuous $\tilde{Z} \in C([0,1]; \mathbb{R}^N)$ such that

$$\tilde{R} = \Psi(r_0, \tilde{Z}) \quad \textit{and} \quad |\tilde{Z}|_\infty \leq M_1 \,.$$

Next we notice that the map $(r, Z) \mapsto \Psi(r, Z)$ is continuous when r belongs to the good set $\mathcal{G}$. For $\tilde{Z} \in C([0,T]; \mathbb{R}^N)$ and constants $M, \gamma, \delta_z > 0$, define the set

$$\mathcal{Z}(\tilde{Z}, M, \gamma, \delta_z) := \left\{ Z \;:\; |Z(t) - \tilde{Z}(t)| \leq M e^{-\gamma t} + \delta_z \quad \forall\, t \in [0,T] \right\}. \tag{30}$$

Fact 2.8. (Continuity of the map Ψ.) *Fix any $\tilde{r}_0 \in \mathcal{G}_r$, M_2, $T > 0$ and $\delta_r \in (0, \epsilon_1/2)$ where ϵ_1 is from* (26)*. Suppose that $\tilde{Z} \in C([0,T]; \mathbb{R}^N)$ satisfies $|\tilde{Z}|_\infty \leq M_2$. Then there exist constants $\gamma > 0$, $\delta_0 > 0$ and $\delta_z > 0$ such that*

$$|\Psi(r_0, Z) - \Psi(\tilde{r}_0, \tilde{Z})|_\infty \leq \delta_r$$

for all $(r_0, Z) \in \left\{ \mathcal{G}_r \cap \{ r : |r - \tilde{r}_0| \leq \delta_0 \} \right\} \times \mathcal{Z}(\tilde{Z}, M_2, \gamma, \delta_z)$.

Finally, we observe that OU processes stay in a tubular neighborhood with positive probability.

Fact 2.9. (Approximation by OU processes.) *Let a set $\mathcal{Z} = \mathcal{Z}(\tilde{Z}, M, \gamma, \delta_z)$ be given. Then there exists a $p > 0$ such that*

$$\inf_{z_0 \in \mathcal{G}_z} \mathbb{P}_{z_0}\left\{ Z \in \mathcal{Z} \right\} \geq p$$

where $Z = (Z_1, \ldots, Z_N)$ is the solution to (5) *with $Z(0) = z_0$.*

With these observations we now prove Lemma 2.6.

Proof of Lemma 2.6. Fix an initial condition $x_0 = (r_0, z_0) \in \mathcal{G}$ and $\delta > 0$. The argument proceeds in two steps. First we construct a bounded deterministic control $\tilde{Z}$ that accomplishes the task of moving its associated connector $\tilde{R} = \Psi(r_0, \tilde{Z})$ from the initial position r_0 to the reference point r_* at time $t = 1$. Any instance of the noise Z that approximates $\tilde{Z}$ sufficiently well, as in the definition of $\mathcal{Z}$ above, will have an associated connector $R = \Psi(r_0, Z)$ that has a terminal position $R(1)$ near r_*. Demonstrating that such an event has positive probability is not sufficient to prove (28). This is because $Z(1)$ may not be close to $Z_* = 0$. Therefore in the second step of the proof we show that, conditioned on success during the time interval $t \in [0, 1]$, the noise has a positive probability of entering a small neighborhood of the origin rapidly enough so that the connector process does not move far from r_*.

To make these statements precise, we set some notation. Let M_1 be the constant from Fact 2.7 and $m/\sqrt{\eta}$ be the radius of the N-sphere $\mathcal{G}_z$. We define $M_2 = (m/\sqrt{\eta}) + M_1$. For a given tolerance, δ_r, which is set immediately before Equation (35), we define the event

$$\Omega_1 := \{|R(1) - r_*| \le \delta_r,\ |Z(t)| \le M_2 + 1; \quad \forall\, t \in [0, 1]\}\,. \tag{31}$$

It is important to note that M_2 does not depend on the choice of δ_r.

Taking $t_1 := 2$ and assuming $|R(1) - r_*| < \delta_r$ is sufficiently small, we can show that for any $t_2 > 2$, the event

$$\Omega_2 := \big\{\ |R(t) - r_*| < \delta/2, |Z(t)| < \delta/2; \quad \forall\, t \in [2, t_2]\ \big\} \tag{32}$$

has positive probability. The structure of the proof is therefore summarized by:

$$\inf_{t\in[2,t_2]} \mathcal{P}_t(x_0, B_\delta(x_*)) \ge \mathbb{P}_{x_0}\big\{\Omega_2\big\} \ge \mathbb{P}_{x_0}\big\{\Omega_2 \,|\, \Omega_1\big\}\, \mathbb{P}_{x_0}\big\{\Omega_1\big\} \ge p_2 p_1 \tag{33}$$

for some $p_1 > 0$ and $p_2 > 0$ that are independent of the initial condition $x_0 \in \mathcal{G}$.

We begin by showing $\inf_{x_0\in\mathcal{G}} \mathbb{P}_{x_0}\{\Omega_1\} \ge p_1$. Let $\tilde{R}$ be a smooth path in $\mathcal{R}$ which was defined in (29). By Fact 2.7 there exists a bounded deterministic control $\tilde{Z}$ such that $\tilde{R} = \Psi(r_0, \tilde{Z})$ over the interval $t \in [0, 1]$. The initial value of the control, $\tilde{Z}(0)$, satisfies

$$|z_0 - \tilde{Z}(0)| \le |z_0| + |\tilde{Z}(0)| \le (m/\sqrt{\eta}) + M_1$$

where we recall that $m/\sqrt{\eta}$ is the radius of $\mathcal{G}_z$. In order to apply Fact 2.8 we set $M_2 = (m/\sqrt{\eta}) + M_1$ and $T = 1$ while noting that $\tilde{R}(0) = r_0$. Then for a given $\delta_r > 0$, there exist positive constants γ_1 and $\delta_{z,1}$ such that if an instance Z of the noise satisfies

$$|Z(t) - \tilde{Z}(t)| \le M_2 e^{-\gamma_1 t} + \delta_{z,1},\ \forall\, t \in [0, 1] \tag{34}$$

then the corresponding connector process $R = \Psi(r_0, Z)$ satisfies

$$|R(t) - \tilde{R}(t)| \le \delta_r, \, \forall \, t \in [0,1].$$

From Fact 2.9, it follows that

$$p_1 := \mathbb{P}_{z_0}\Big\{Z : |Z(t) - \tilde{Z}(t)| \le M_2 e^{-\gamma_1 t} + \delta_{z,1}, \, \forall \, t \in [0,1]\Big\} > 0$$

and p_1 does not depend on z_0 or r_0. We note that by virtue of the proof of Fact 2.8 $\delta_{z,1}$ can be chosen to be less than or equal to 1. Setting $M = M_2 + 1$ we have shown that $\inf_{x_0 \in \mathcal{G}} \mathbb{P}_{x_0}\{\Omega_1\} \ge p_1$.

Next we prove that $\inf_{x_0 \in \mathcal{G}} \mathbb{P}_{x_0}\{\Omega_2 \mid \Omega_1\} > 0$. As mentioned earlier we must show that ensuing at time $t = 1$, it is possible to rapidly bring the noise near the origin without significantly perturbing R. To this end, we extend the previous deterministic control $\tilde{Z}$ to include the definition $\tilde{Z}(t) = 0$ for all $t \in [1, t_2]$. We also extend the definition of the associated connector so that $\tilde{R} = \Psi(r_0, \tilde{Z})$ is now well-defined over the full interval $t \in [0, t_2]$. By hypothesis, $\tilde{R}(1) = r_*$ is a global minimum of the spring potential and therefore the controlled process experiences zero forcing from both the controlled noise and the spring potential. It follows that $\tilde{R}(t) = r_*$ for all $t \in [1, t_2]$.

We seek to apply Fact 2.8 again to show that R remains close to r_* for all $t \in [1, t_2]$. Even though $Z(1)$ is not necessarily close to the control initial value $\tilde{Z}(1) = 0$, conditioned on Ω_1, $|Z(1)| \le M_2 + 1$. At this point, we fix the value of $\delta_r > 0$ given in the definition of Ω_1. By Fact 2.8, there exist positive constants $\delta_{z,2} \in (0, 1/2)$, $\gamma_2 > 0$ and $\delta_r > 0$ such that if the connector process satisfies $|R(1) - r_*| \le \delta_r$, and if an instance of the noise satisfies

$$|Z(t)| \le (M_2 + 1)e^{-\gamma_2(t-1)} + \delta_{z,2}, \; \forall \; t \in [1, t_2], \tag{35}$$

we have $|R(t) - r_*| \le \delta/2, \; \forall \; t \in [1, t_2]$. Conditioning on Ω_1 and using the Markov property of the system to shift time values appropriately, Fact 2.9 ensures that the noise satisfies (35) with probability $p_2 > 0$.

It remains to require that $|Z(t)| < \delta/2$ for all $t \in [2, t_2]$. From (35), it suffices to find a $\gamma_3 \ge \gamma_2$ sufficiently large that $\exp(-\gamma_3(t-1)) + \delta_{z,2} \le \delta/2$ for all $t \in [2, t_2]$. Indeed, this is the case if we choose $\gamma_3 \ge \ln\left(\frac{\delta}{2} - \delta_{z,2}\right)^{-1}$ and we are done. □

In order to complete the proof of the Uniformly Accessible Neighborhood Condition of Lemma 2.4 we need to extend Lemma 2.6 to apply to all initial conditions in $\mathcal{C}$. To do this, we need the particle separation property from Lemma 2.5.

Lemma 2.10 (Topological irreducibility on $\mathcal{C}$)**.** *Given a $\delta > 0$, there exists a $t_1' > 0$ so that for any $t \geq t_1'$ there is an $\alpha_1' > 0$ with*

$$\inf_{x_0 \in \mathcal{C}} \mathcal{P}_t(x_0, B_\delta(x_*)) \geq \alpha_1'$$

Proof. Set $t_1' = t_1 + 1$ where t_1 is the constant from Lemma 2.6 and let $\tau := \inf\{t > 0 : (R(t), Z(t)) \in \mathcal{G}\}$. Now for any $t \geq t_1'$ and fixed $x_0 \in \mathcal{B}$ we have

$$\begin{aligned}\mathcal{P}_t(x_0, B_\delta(x_*)) &\geq \Big(\mathbb{P}_{x_0}\{X_t \in B_\delta(x_*) | \tau \leq 1\}\Big)\Big(\mathbb{P}_{x_0}\{\tau \leq 1\}\Big) \\ &\geq \Big(\inf_{x \in \mathcal{G}} \inf_{s \in [0,1]} \mathcal{P}_{t-s}(x, B_\delta(x_*))\Big)\Big(\mathbb{P}_{x_0}\{\tau \leq 1\}\Big) \geq \alpha_1 \mathbb{P}_{x_0}\{\tau \leq 1\}\end{aligned}$$

where α_1 is from Lemma 2.6. Finally, we take the $\inf$ over all initial conditions $x_0 \in \mathcal{B}$. Applying Lemma 2.5 with $m = \psi(\phi(\rho_+))$ and $M = m/\sqrt{\eta}$, we conclude there exists an $\alpha > 0$ such that

$$\inf_{x_0 \in \mathcal{B}} \mathcal{P}_t(x_0, B_\delta(x_*)) \geq \alpha_1 \inf_{x_0 \in \mathcal{B}} \mathbb{P}_{x_0}\{\tau \leq 1\} \geq \alpha_1 \alpha > 0.$$

Setting $\alpha_1' = \alpha\alpha_1$ completes the proof. □

2.3 Measure Theoretic Irreducibility via Hörmander's Condition

Lemma 2.11 (Absolute continuity of the transition density)**.** *Let $\{X(t) = (R(t), Z(t))\}_{t \geq 0}$ be a Markov process with transition kernel $\mathcal{P}_t(x, U)$. Then for any $t > 0$, there exists a smooth function $p_t(x, y)$, such that*

$$\mathcal{P}_t(x, U) = \int_U p_t(x, y) dy$$

for every $U \in \mathscr{B}(\mathcal{C})$, where $p_t(x, y)$ is jointly continuous in $(x, y) \in \mathcal{C} \times \mathcal{C}$.

Remark 2.12. In fact, the system has a density for all $(x, y) \in \mathbb{X} \times \mathbb{X}$. However, due to the periodicity of our forcing, proving this would require an additional small argument. Since we do not need this fact, we refrain.

Proof. The claim follows from a now classical theorem of Hörmander which states that if a diffusion on an open manifold satisfies a certain algebraic condition then $L_1 = \partial_t - \mathcal{L}$ and $L_2 = \partial_t - \mathcal{L}^*$ are both hypoelliptic in $\mathcal{C}$ where $\mathcal{L}$ is the generator of the diffusion $X(t)$ and $\mathcal{L}^*$ is its adjoint. A combination of Itô's formula and the fact that we have shown that the singularities of the potential are unattainable demonstrates that $L_1 u = 0$ and $L_2 u = 0$ have distribution-valued solutions. Hypoellipticity of the operators ensures first that these distribution-valued solutions are in fact smooth functions. Furthermore, hypoellipticity implies the existence of fundamental solution, which in turn yields continuity in the second variable throughout the center of the space $\mathcal{C}$.

The fact that the density is jointly continuous follows after a little more work. The argument is laid out in its entirety for $\mathbb{R}^N$ valued diffusions in Section 7.4 of [Str08]. In particular, see Theorem 7.4.3 and Theorem 7.4.20. Essentially, the same proofs follow in our setting since we have shown the system is a well defined diffusion on the manifold $\mathbb{X}$ with distribution-valued solution. Hypoellipticity and the properties which follow are local statements, and therefore still apply. The needed results in the general setting, as opposed to $\mathbb{R}^N$, can be found in Chapter 22 of [Hör85], noting in particular Theorem 22.2.1. However, the presentation in [Str08] is closer to the exact statements we need.

We now turn to the explicit calculations needed to show that Hörmander's condition is satisfied. We recast the system of equations (5) and (6) as

$$dX(t) = A(X(t))\,dt + BdW(t)$$

where $A(x) \in \mathbb{R}^{2+N}$ and $B \in \mathbb{R}^{(2+N)\times(2+N)}$ with

$$A(x) = \begin{pmatrix} -\nabla\Phi(r) + S(r)z \\ -\lambda^2\nu|k|^2 Z \end{pmatrix}\,, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{B} \end{pmatrix}\,.$$

where $\tilde{B}$ is an $N \times N$ diagonal matrix with diagonal entries $\sqrt{2\beta\nu}\lambda\sigma_k$. In this notation, the generator $\mathcal{L}$ of the diffusion is given in terms of a test function φ by

$$(\mathcal{L}\varphi)(x) = (A\cdot\nabla)\varphi(x) + \frac{1}{2}\sum_{k\in\mathcal{K}}(B_k\cdot\nabla)^2\varphi(x)$$

where B_k is the column of B associated with the mode direction $k \in \mathcal{K}$.

For two vector fields A, B let $[A, B] := AB - BA$ denote their the commutator or Lie bracket. In our simplified setting where B_k is a constant vector-field one has

$$[A(X), B_k] = \frac{\partial}{\partial z_k}A(X) = \begin{pmatrix} \sin(\lambda k\cdot R)\frac{k^\perp}{|k|} \\ -\lambda^2\nu|k|^2\,\mathbf{e}_k \end{pmatrix}$$

where $\mathbf{e}_k$ is the is the unit basis vector in $\mathbb{R}^N = \mathbb{R}^{|\mathcal{K}|}$ associated to the mode direction $k \in \mathcal{K}$. Moreover all the iterated Lie brackets of B_k and $A(x)$ are 0. Thus to satisfy the Hörmander's condition at the point x, it is required that

$$\text{span}\Big\{B_k, [A(x), B_k] : k\in\mathcal{K}\Big\} = \mathbb{R}^{2+N}.$$

The set $\{[A(x), B_k]\}_{k\in\mathcal{K}}$ will span $\mathbb{R}^{2+N}$ if and only if the set $\{\sin(k\cdot r)k^\perp\}_{k\in\mathcal{K}}$ spans $\mathbb{R}^2$ since the set $\{\mathbf{e}_k : k \in \mathcal{K}\}$ spans $\mathbb{R}^N$. We recall that by assumption $\mathcal{K}$ contains at least three pairwise independent vectors which we label k_1, k_2, and k_3. One may note that due to the periodicity of the forcing, $\sin(\lambda k\cdot r) = 0$ for all $r \in L\mathbb{Z}^2$. Taking $L \gg \rho_0^2$ will ensure that all of these points lie outside of $\mathcal{C}$. Thus restricting to $x \in \mathcal{C}$ at least two of $r \cdot k_i$ are nonzero and the lemma is proved. □

2.4 Ergodicity of generalizations

In the derivation of the model equations (5) and (6) we imposed the simplifying assumption that the center of mass $M(t) := \frac{1}{2}(X_1(t) + X_2(t))$ is held at zero (see Appendix). This greatly simplified the presentation and did not affect the conclusion that the bead-spring system has an ergodic connector process $R(t)$. Indeed the fluid velocity term with nonzero $M(t)$ is given by Eq. (41):

$$\begin{aligned}\frac{1}{2}&[u(X_1(t),t) - u(X_2(t),t)] \\ &= \sum_{k\in\mathcal{K}} [\cos(\lambda k\cdot M)Z_k - \sin(\lambda k \cdot M)Y_k]\sin(\lambda k\cdot R)\frac{k^\perp}{|k|}\end{aligned}$$

where the $\{Y_k\}$ are a second set of OU-processes defined exactly as the $\{Z_k\}$.

Because the M terms appear inside of cosines and sines, there is no new significant contribution to the Lyapunov function calculation. For the Hörmander condition, the additional terms in the coefficients of the noise introduce more "dead spots" in the forcing, but still one needs only *four* pairwise linearly independent vectors k_i in the mode set $\mathcal{K}$ to ensure that at least two of the vectors

$$\big\{ [\cos(\lambda k_i\cdot M) - \sin(\lambda k_i\cdot M)]\sin(\lambda k_i\cdot R)k_i^\perp \big\}$$

are nonzero. This guarantees the existence of a continuous transition density and it remains to show the δ-ball controllability as in Lemma 2.6. While the calculation is more involved, the principle of identifying the region of good control $\mathcal{G}$, where the coefficients of the R-differential equation are uniform, still applies. Furthermore, since the differential equation for R is linear in the $\{Y_k\}$ and $\{Z_k\}$, we may still solve for stochastic control explicitly in terms of the desired path Γ as long as the new Stokes matrix is non-degenerate. Again, this is guaranteed by the hypothesis that $\mathcal{K}$ contains at least four pairwise linearly independent vectors.

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A Derivation of the model

In the overdamped, highly viscous regime, it is reasonable to neglect the nonlinear term in Navier-Stokes equations [OR89]. Following [Wal86], [DZ92], [Dal99] and [McK06] we have the stochastic PDE given in Section 1, Eq. 2,

$$\partial_t u(x,t) - \nu\Delta u(x,t) + \nabla p(x,t) = F(dx,dt), \quad \nabla \cdot u(x,dt) = 0$$

with periodic boundary conditions on the rectangle $[0,L]\times[0,L]$ where L is presumed to be very large. For this development (see also [SS02a]) we assume that the space-time forcing is a mean zero complex-valued Gaussian process with covariance

$$\mathbb{E}\Big[F^\alpha(x,t)\overline{F^\beta(y,s)}\Big] = (t\wedge s)2k_BT\nu\delta_{\alpha\beta}\Gamma(x-y)$$

where $\alpha,\beta\in\{1,2\}$ and $\delta_{\alpha\beta}$ is a Kronecker delta function. It follows that

$$F(x,t) = \frac{\sqrt{2k_BT\nu}}{L}\sum_{k\in\mathbb{Z}^2\setminus 0} e^{\lambda i k\cdot x}\sigma_k B_k(t)$$

where $\{B_k\}$ is a collection of complex-valued 2-d Brownian motions and the coefficients σ_k are related to the spatial correlation function Γ through the Fourier relation $\Gamma(x) = \frac{2}{L^2}\sum_{k\in\mathbb{Z}^2\setminus\{0\}} e^{\lambda ik\cdot x}\sigma_k^2$. In order to construct a real-valued noise of the form (4), one can set $\sigma_{-k}=\sigma_k$ and $B_{-k}=\overline{B_k}$ and for all k.

To compute the Fourier transform of the SPDE, we note that the transform of the noise is given by

$$\begin{aligned}\int_{[0,L]^2} e^{-\lambda ik\cdot x}F(x,t)dx &= \int_{[0,L]^2} e^{-\lambda ik\cdot x}\frac{\sqrt{2k_BT\nu}}{L}\sum_{j\in\mathbb{Z}^2\setminus 0} e^{\lambda ij\cdot x}\sigma_j B_j(t)dx \\ &= \frac{\sqrt{2k_BT\nu}}{L}\sum_{j\in\mathbb{Z}^2\setminus 0}\sigma_j B_j(t)\int_{[0,L]^2} e^{-\lambda i(k-j)\cdot x}dx \\ &= \sqrt{2k_BT\nu}L\sum_{j\in\mathbb{Z}^2\setminus 0}\sigma_j B_j(t)\delta_{kj} = \sqrt{2k_BT\nu}\,L\,\sigma_k\,B_k(t).\end{aligned}$$

The SPDE transforms into the infinite dimensional system

$$d\hat{u}_k(t) + \lambda^2\nu|k|^2\hat{u}_k(t) + \lambda ik\hat{p}_k(t) = \sqrt{2k_BT\nu}L\sigma_k dB_k(t), \tag{36}$$

$$\lambda ik \cdot \hat{u}_k(t) = 0. \tag{37}$$

For the sake of completing the formal argument, suppose for the moment that the forcing term is smooth with derivative f. By taking the dot product of k with the terms of equation (36), the first two terms vanish – via incompressibility condition (37) – leaving the identity

$$\lambda i|k|^2\hat{p}_k(t) = \sqrt{2k_BT\nu}L\sigma_k k \cdot f(t). \tag{38}$$

Substituting back into (36) and gathering $f(t)$ terms on the right-hand side yields

$$d\hat{u}_k(t) + \lambda^2\nu|k|^2\hat{u}_k(t) = \sqrt{2k_BT\nu}L\sigma_k\Big(f(t) - \frac{k \cdot f(t)}{|k|^2}k\Big). \tag{39}$$

The projection on the right hand side has two standard representations:

$$f - \frac{k \cdot f}{|f|^2}k = \Big(I - \frac{k \otimes k}{|k|^2}\Big)f = \frac{f \cdot k^\perp}{|k|^2}k^\perp,$$

where $k^\perp := \binom{-k_2}{k_1}$. Applying Duhamel's principle and assuming initial condition is taken from the stationary distribution, we have the following representation for solutions to the fluid mode equations

$$\begin{aligned}\hat{u}_k(t) &= e^{-\lambda^2\nu|k|^2t}\hat{u}_k(0) + \sqrt{2k_BT\nu}\sigma_k L\int_0^t e^{-\lambda^2\nu|k|^2(t-s)}\Big(I - \frac{k \otimes k}{|k|^2}\Big)dB_k(t)\\ &= \Big(I - \frac{k \otimes k}{|k|^2}\Big)\zeta_k(t)\end{aligned}$$

where we define ζ_k to be the appropriate complex valued 2-d Ornstein-Uhlenbeck process,

$$d\zeta_k(t) = -\lambda^2\nu|k|^2\zeta_k(t)dt + \sqrt{2k_BT\nu}L\sigma_k dB_k(t)$$

with $\zeta_k(0)$ normally distributed according to the respective stationary distributions for each k. We therefore have the solution for the fluid velocity field,

$$u(x,t) = \frac{1}{L^2}\sum_{k\in\mathbb{Z}^2\backslash 0} e^{\lambda ik\cdot x}\Big(I - \frac{k \otimes k}{|k|^2}\Big)\zeta_k = \frac{1}{L^2}\sum_{k\in\mathbb{Z}^2\backslash 0} e^{\lambda ik\cdot x}\frac{\zeta_k \cdot k^\perp}{|k|^2}k^\perp.$$

After defining $\xi_k := \frac{1}{L^2}\frac{\zeta_k\cdot k^\perp}{|k|}$, we have the complex valued 1-d OU processes that drive the dynamics

$$d\xi_k(t) = -\frac{4\pi^2\nu|k|^2}{L^2}\xi_k(t)dt + \frac{\sqrt{2k_BT\nu}\sigma_k}{L}dW_k(t)$$

Imposing the condition that we require real-valued solutions, after Fourier inversion we have the following trigonometric expansion for 2-d stochastic Stokes

$$u(x,t) = \sum_{k\in\mathbb{Z}^2\setminus 0} (\cos(\lambda\, k\cdot x)Y_k + \sin(\lambda\, k\cdot x)Z_k)\,\frac{k^\perp}{|k|}. \tag{40}$$

where the Y_k and Z_k are the real and imaginary parts of ξ respectively.

In this paper, we study the dynamics of the two beads in normal coordinates: $M(t) = \frac{1}{2}(X_1(t) + X_2(t))$ and $R(t) = \frac{1}{2}(X_1(t) - X_2(t))$,

$$\begin{aligned}\frac{d}{dt}M(t) &= \frac{1}{2}[u(X_1(t),t) + u(X_2(t),t)]\\ \frac{d}{dt}R(t) &= -\nabla\Phi(R(t)) + \frac{1}{2}[u(X_1(t),t) - u(X_2(t),t)].\end{aligned}$$

In light of equation (2), we may write the radial process and the noise together as a Markovian system of SDE with two degenerate directions. In order to write the system in this form, we first record the identity

$$\begin{aligned}&\frac{1}{2}[u(X_1(t),t) - u(X_2(t),t)] \\ &\quad= \sum_{k\in\mathcal{K}} [\cos(\lambda k\cdot M(t))z_k(t) - \sin(\lambda k\cdot M(t))y_k(t)]\sin(\lambda k\cdot R(t))\frac{k^\perp}{|k|}.\end{aligned} \tag{41}$$

For the majority of the paper, we used the simplification $M(t) = 0$ for all t. This does not have any effect on the ergodic results as is discussed in Section 2.4, but it does significantly streamline the presentation. Altogether we have the definition of the dynamics given in Section 1, Eq. (6).

A.1 Degeneracy when there is no repulsive force

Putting aside existence and uniqueness for a moment, we make a quick calculation that reveals a degeneracy for the bead-spring model with a Hookean or FENE spring potential with truncated stochastic Stokes forcing. Namely, under mild conditions, when the two beads come close together, the fluid velocity vectors they respectively see will become so correlated, the beads will never separate.

Proposition A.1 (Degeneracy of the non-repulsive case)**.** *Let R and the family $\{Z_k\}_{k\in\mathcal{K}}$ satisfy the system of differential equations* (5) *and* (6)*. Let the spring potential be given by $\Phi(r) = \frac{\gamma}{2}|r|^2$ or $\Phi(r) = \Phi_{FENE}(r)$ as defined by* (9)*. Then there exists a γ_0 so that if $\gamma > \gamma_0$ then*

$$\lim_{t\to\infty} R(t) = 0$$

almost surely.

Proof. We first note that for all r satisfying $|r| \in (0, \rho_{max})$

$$\nabla\Phi_{\text{FENE}}(r) \cdot r = \frac{\gamma|r|^2}{1 - |r|^2/\rho_{max}^2} \geq \gamma|r|^2.$$

It follows that both the Hookean and FENE potential cases, the process $|R(t)|^2$ satisfies the following pathwise ODE bound,

$$\begin{aligned}\frac{d}{dt}|R(t)|^2 &= -2\nabla\Phi(R(t)) \cdot R(t) + 2\sum_{k\in\mathcal{K}} \sin(\lambda k \cdot R(t))\, \frac{k^\perp \cdot R(t)}{|k|} Z_k(t) \\ &\leq -2\gamma|R(t)|^2 + 2\lambda \sum_{k\in\mathcal{K}} |k \cdot R(t)||k^\perp \cdot R(t)| \frac{|Z_k(t)|}{|k|} \\ &\leq -2\gamma|R(t)|^2 + 2\lambda|R(t)|^2\|Z(t)\|_1\end{aligned}$$

where $\|Z(t)\|_1 := \sum_{k\in\mathcal{K}} |k||Z_k(t)|$.

This differential inequality implies

$$|R(t)|^2 \leq |R(0)| \exp\left[-2\gamma t + 2\lambda \int_0^t \|Z(s)\|_1 ds\right]. \tag{42}$$

Recall that in its stationary distribution, the law of each $Z_k(t)$ is normal with mean zero and variance $\beta\sigma_k^2/|k|^2$ and therefore $\mathbb{E}[|Z_k|] = \sqrt{\frac{2\beta}{\pi}}\frac{\sigma_k}{|k|}$. By the Law of Large Numbers

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t |Z_k(s)| ds = \sqrt{\frac{2\beta}{\pi}}\, \frac{\sigma_k}{|k|} \tag{43}$$

almost surely and so

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \|Z(s)\|_1 ds = \sqrt{\frac{2\beta}{\pi}} \sum_{k\in\mathcal{K}} \sigma_k$$

almost surely. Since we are only considering a finite number of modes, the above sum is finite. Therefore, if $\gamma > \gamma_0 := \lambda\sqrt{\frac{2\beta}{\pi}}\sum_k \sigma_k$, then $|R(t)|^2 \to 0$ almost surely as $t \to \infty$.

□

A.2 A note on the mollifier function ψ

Recall the mollifier function ψ that appeared in the Lyapunov function (14) and in the global estimate in the existence and uniqueness Proposition A.3,

$$\psi(x) := \begin{cases} 0, & 0 \leq x \leq a, \\ c\,(x-a)\exp\left(\frac{-1}{(x-a)^2}\right), & x > a. \end{cases}$$

where $a = \Phi(\rho_0)$. Since $\lim_{x\to 0} x^\alpha e^{-1/x^2} = 0$ for any $\alpha \in \mathbb{R}$, it follows that for any $n \in \mathbb{N}$, the n-th derivative of ψ satisfies $\lim_{x\to a} \psi^{(n)}(x) = 0$. Therefore ψ and all of its derivatives are continuous for all $x \in \mathbb{R}_+$. Furthermore, we have the following proposition.

Proposition A.2. *There exists a constant $C > 0$ such that*

$$\psi(x) \le x\psi'(x) \le \psi(x) + C \tag{44}$$

for all $x \in \mathbb{R}_+$. Furthermore, $\|\psi'\|_\infty < \infty$

Proof. This is trivially true for all $x \in [0, a]$, since $\psi(x) = x\psi'(x) = 0$ for all x in this range. For $x > a$, we compute that $x\psi'(x) = \psi(x)+r(x)$ where the remainder term is given by $r(x) = c\left(a + 2x(x-a)^{-2}\right)\exp\left(-(x-a)^{-2}\right)$. This remainder term is always positive, is continuous for all $x > a$ and satisfies $\lim_{x\to a} r(x) = 0$ and $\lim_{x\to\infty} r(x) = a$. It follows that there exists a $C > 0$ for all $x \ge a$ we have $0 \le r(x) \le C$. The inequalities (44) follow. ☐

A.3 Existence, uniqueness of the bead-spring model

We confirm the global existence and uniqueness of the bead-spring model proposed by Equations (5) and (6). Since we assume that $|\mathcal{K}| = N \in \mathbb{N}$ throughout the main part of this paper, we retain that assumption here.

Proposition A.3. *Suppose that the spring potential Φ satisfies Assumption 1. Let $\{Z_k(t) : t \ge 0\}_{k\in\mathcal{K}}$ be a solution to the family of SDEs* (5) *with initial conditions $Z_k(0) = z_k \in \mathbb{R}$ for all $k \in \mathcal{K}$. Then, almost surely, there exists a unique global solution to the 2-dimensional ODE*

$$\frac{d}{dt}R(t) = -\nabla\Phi(R(t)) + \sum_{k\in\mathcal{K}} \sin(\lambda\, k\cdot R(t))\,\frac{k^\perp}{|k|}Z_k(t) \tag{45}$$

with the initial condition $R(0) = r_0 \in \mathcal{D}\setminus\{0\}$.

Proof. Let $\epsilon > 0$ be given and define the stopping stopping time $\tau_\epsilon := \inf\{t > 0 : |R(t)| < \epsilon$ or $\psi(\Phi(R(t))) > \epsilon^{-1}\}$ where ψ is the function defined in the previous section. We will first prove there exists a unique stopped solution $R(t\wedge\tau_\epsilon)$ to (45). Subsequently we show that $\sup\{\tau_\epsilon\} = \infty$ almost surely.

We rewrite (45) in terms of the Stokes matrix defined by (7),

$$\frac{d}{dt}R(t) = -\nabla\Phi(R(t)) + S(R(t))Z(t). \tag{46}$$

In order to apply the standard Picard-Lindelöf Theorem (see for example, [Hal69]), we think of the vector $Z(t) = (Z_0(t), Z_1(t), \ldots, Z_N(t))$ as a time-inhomogeneous coefficient. To prove that there exists a unique local solution to (45) it is sufficient to show that the functions $\nabla\Phi(r)$ and $S(r)Z(t)$ are continuous in $\mathcal{D} \times \mathbb{R}_+ \setminus \{0 \times \mathbb{R}_+\}$ and locally Lipschitz in the variable r. By Assumption 1, this condition is satisfied by $\nabla\Phi(r)$. For the last term in (46), given an instance of Z, we have

$$\begin{aligned} |S(r_1)Z(t) - S(r_2)Z(t)| &\leq \sum_{k\in\mathcal{K}} |\sin(\lambda k \cdot r_1) - \sin(\lambda k \cdot r_2)||Z_k(t)| \\ &\leq \lambda |r_1 - r_2| \|Z(t)\|_1 \end{aligned}$$

where we recall $\|Z(t)\|_1 := \sum_{k\in\mathcal{K}} |k||Z_k(t)|$. The function $S(r)Z(t)$ is continuous in t almost surely since $|S(r)Z(t_1) - S(r)Z(t_2)| \leq \|S(r)\|_F |Z(t_1) - Z(t_2)|$ and the vector OU process $Z(t)$ is continuous almost surely.

We now show that the process cannot blow up to ρ_{max} in finite time. To this end we consider the process $\psi(\Phi(R(t)))$ which is constant inside a radius of size ρ_0 but then grows to infinity with the potential function as $|R|$ tends to ρ_{max}. By showing $\psi(\Phi(R(t)))$ is bounded above by a 1-d linear ODE, this suffices to show global existence and uniqueness. For a given instance of the noise $Z(t)$, we have

$$\frac{d}{dt}\psi(\Phi(R(t))) = \psi'(\Phi(R(t)))\left(-|\nabla\Phi(R(t))|^2 + \nabla\Phi(R(t)) \cdot [S(R(t))Z(t)]\right)$$

For given values $r \in \mathbb{R}^2$ and $z \in \mathbb{R}^N$ we bound the Stokes forcing term by applying Young's inequality followed by the matrix form of Cauchy-Schwarz:

$$\begin{aligned} \nabla\Phi(r) \cdot (S(r)z) &\leq \frac{1}{2}|\nabla\Phi(r)|^2 + \frac{1}{2}|S(r)z|^2 \leq \frac{1}{2}|\nabla\Phi(r)|^2 + \frac{1}{2}\|S(r)\|_F^2|z|^2 \\ &\leq \frac{1}{2}|\nabla\Phi(r)|^2 + \frac{1}{2}N^2|z|^2. \end{aligned}$$

The inequality $\|S(r)\|_F \leq N$ is given in the proof of Lemma 2.5.

To estimate the first term of the mollified ODE, we consider two cases: (i) $|r| \leq \rho_0$ and (ii) $|r| > \rho_0$. In case (i), $\psi'(\Phi(r)) = 0$ and the entire term disappears. Trivially, $-\psi'(\Phi(r))|\nabla\Phi(r)|^2 = 0 = -\gamma\psi(\Phi(r))$.

For case (ii), we employ the spring potential assumption (11) that for some $\gamma > 0$ if $|r| > \rho_0$ then $|\nabla\Phi(r)|^2 \geq \gamma\Phi(r)$. Furthermore, by Proposition A.2, the mollifier ψ satisfies $\psi'(\Phi(r))\Phi(r) \geq \psi(\Phi(r))$. We obtain

$$-\psi'(\Phi(r))|\nabla\Phi(r)|^2 \leq -\gamma\psi'(\Phi(r))\Phi(r) \leq -\gamma\psi(\Phi(r)). \tag{47}$$

Altogether, we have the differential inequality

$$\frac{d}{dt}\psi(\Phi(R(t))) \leq -\frac{\gamma}{2}\psi(\Phi(R(t))) + \frac{N^2}{2}\|\psi'\|_\infty |Z(t)|^2 \tag{48}$$

Define $Y(t)$ to be the solution to the linear ODE

$$\frac{d}{dt}Y(t) = -\frac{\gamma}{2}Y(t) + \frac{N^2}{2}\|\psi'\|_\infty |Z(t)|^2$$

with $Y(0) = \psi(\Phi(R(0))$. By definition, $\psi(\Phi(R(t)) \leq Y(t)$. By virtue of the fact that the forcing term is positive, $Y(t) > 0$ for all t and, defining $\tau_M = \inf\{t > 0 : Y(t) > M\}$, standard properties of linear ODEs and global existence of the N-dimensional Ornstein-Uhlenbeck imply that $\sup_{M>0} \tau_M = \infty$.

We now show that $\sup_{\epsilon>0} \tau_\epsilon = \infty$ almost surely by demonstrating that the R-dynamics do not hit zero in finite time. The idea here is that for the connecter process to hit zero, the noise must blow up in finite time and this is not possible since our noise is bounded on any finite time interval. Indeed, by Assumption 1, there exists $\epsilon_0 > 0$ such that $-\nabla\Phi(r) \cdot r \geq \gamma_0 > 0$ for all r with $|r| < \epsilon_0$. Suppose $R(T) = 0$ for some $T \in \mathbb{R}_+$. From the above discussion and Equation (45) it follows that $\frac{d}{dt}|R(t)|^2$ is almost surely continuous. Thus $\frac{d}{dt}|R(t)|^2 < 0$ in a subinterval of the set $[T - \delta, T]$ for some $\delta > 0$. Without loss of generality, we may assume that $|R(t)| < \epsilon_0$ for $[T - \delta, T]$. Let $M := \sup_{t\in[T-\delta,T]} \|Z(t)\|_1$. In this regime, we have

$$\begin{aligned}\frac{d}{dt}|R(t)|^2 &= -2\nabla\Phi(R(t)) \cdot R(t) + 2\sum_{k\in\mathcal{K}} \sin(\lambda k \cdot R(t))\, \frac{k^\perp \cdot R(t)}{|k|} Z_k(t) \\ &\geq 2\gamma_0 - 2\lambda \sum_{k\in\mathcal{K}} |k \cdot R(t)||k^\perp \cdot R(t)| \frac{|Z_k(t)|}{|k|} \geq 2\gamma_0 - 2\lambda MN|R(t)|^2.\end{aligned}$$

However, the right-hand side is positive when $|R(t)|^2 \leq \gamma_0/(\lambda MN)$, contradicting the hypothesis that $\frac{d}{dt}|R(t)|^2 < 0$ in a subinterval of $[T-\delta, T]$ when $|R(t)|$ is small enough. Therefore the origin is unattainable in finite time. □

A.4 The Lyapunov function

The proof for the Lyapunov estimate, Lemma 2.3, proceeds similarly to the proof of the upper bound in the Existence and Uniqueness Proposition A.3. The only differences arise from the need to treat the $R(t)$ and $Z(t)$ dynamics simultaneously. For the sake of easy reference, we recall the definition of the Lyapunov function $V(r, z) = \psi(\Phi(r)) + \eta|z|^2$ where ψ is defined in Section A.2 and η is to be defined in the following proof.

Proof of Lemma 2.3. The generator $\mathcal{L}$ for the Markov process $X(t) := (R(t), Z(t))$ is given by

$$\mathcal{L} := (-\nabla\Phi(r) + S(r)z) \cdot \nabla_r + \nu\lambda^2\Big(\sum_{k\in\mathcal{K}} -|k|^2 z_k \frac{\partial}{\partial z_k} + \beta\sigma_k^2 \frac{\partial^2}{\partial z_k^2}\Big)$$

It suffices to find an $a > 0$ and $b > 0$ such that

$$\mathcal{L}V(x) \leq -aV(x) + b. \tag{49}$$

From (49), using Ito's formula and Gronwall's inequality one can show that $(\mathcal{P}_t V)(x) \leq e^{-at}V(x) + b/a$. Thus we have $c_0 = e^{-at}$ with $c_1 = b/a$. The restriction on the constants (c_0, c_1, ρ_+) from Theorem 2.2 (in light of of the definition of $\mathcal{C}$ in Equation 18) translates to the following constraint on (a, b, ρ_+):

$$b < \frac{1}{2}a\psi(\phi(\rho_+))(1 - e^{-at}). \tag{50}$$

Applying $\mathcal{L}$ to the Lyapunov function V yields:

$$\begin{aligned}\mathcal{L}V(r,z) = \psi'(\Phi(r))\left(-|\nabla\Phi(r)|^2 + (S(r)z)\cdot\nabla\Phi(r)\right)\\ + 2\eta\nu\lambda^2\sum_{k\in\mathcal{K}}\left(-|k|^2 z_k^2 + \beta\sigma_k^2\right).\end{aligned}$$

In bounding the Stokes forcing term we must make a slightly sharper estimate than the one used in the proof of Proposition A.3. We apply Young's inequality (with $\delta \in (0,1)$ to be chosen below) followed by the matrix form of Cauchy-Schwarz and the inequality $\|S(r)\|_F \leq N$ which is given in the proof of Lemma 2.5:

$$(S(r)z)\cdot\nabla\Phi(r) \leq \frac{1}{4\delta}|S(r)z|^2 + \delta|\nabla\Phi(r)|^2 \leq \frac{1}{4\delta}N^2|z|^2 + \delta|\nabla\Phi(r)|^2.$$

Denoting $\hat{k} := \min_{k\in\mathcal{K}}\{|k|\}$ and $\|\sigma\|_0^2 = \sum_{k\in\mathcal{K}}\sigma_k^2$, after collecting terms we have

$$\begin{aligned}\mathcal{L}V(x) \leq -(1-\delta)\psi'(\Phi(r))|\nabla\Phi(r)|^2 + 2\eta\nu\lambda^2\beta\|\sigma\|_0^2\\ + (N^2\psi'(\Phi(r))/4\delta - 2\eta\nu\lambda^2\hat{k}^2)|z|^2.\end{aligned} \tag{51}$$

We estimate the first term as in the proof of Proposition A.3 equation 47, $-(1-\delta)\psi'(\Phi(r))|\nabla\Phi(r)|^2 \leq -(1-\delta)\gamma\psi(\Phi(r))$ for all $r \in \mathbb{R}^2$.

Regardless of the value of r, we require that the coefficient of $|z|^2$ in (51) satisfy the constraint $N^2\psi'(\Phi(r))/4\delta - 2\eta\nu\lambda^2\hat{k}^2 \leq -\eta\gamma(1-\delta)$, which is true for all η satisfying

$$\eta \geq \frac{N^2}{2\nu\lambda^2\hat{k}^2 - \gamma(1-\delta)}\frac{\|\psi'(\cdot)\|_\infty.}{4\delta} \tag{52}$$

By choosing the δ close to 1, we can ensure that the denominator is positive. Applying these estimates, Equation (51) becomes

$$\mathcal{L}V \leq -(1-\delta)\gamma V + 2\eta\nu\lambda^2\beta\|\sigma\|_0^2.$$

Our final restriction involves the constant terms given in Equation (50), with $a = (1-\delta)\gamma$ and $b = 2\eta\nu\lambda^2\beta\|\sigma\|_0^2$. We obtain the constraint

$$\eta \le \frac{(1-\delta)\gamma\psi(\phi(\rho_+))(1-e^{-(1-\delta)\gamma t})}{4\nu\lambda^2\beta\|\sigma\|_0^2}. \tag{53}$$

Since $t \ge 1$ it is enough to have $\eta \le \frac{(1-\delta)\gamma\psi(\phi(\rho_+))(1-e^{-(1-\delta)\gamma})}{4\nu\lambda^2\beta\|\sigma\|_0^2}$. Combining (52) and (53), we need to find η such that

$$\frac{N^2\|\psi'(\cdot)\|_\infty}{4\delta(2\nu\lambda^2\hat{k}^2 - \gamma(1-\delta))} \le \eta \le \frac{(1-\delta)\gamma\psi(\phi(\rho_+))(1-e^{-(1-\delta)\gamma})}{4\nu\lambda^2\beta\|\sigma\|_0^2}. \tag{54}$$

At this point, all parameters have been fixed except for the choice of the constant c in the definition of ψ, and the choice of ρ_+. By choosing c to be sufficiently small, we can diminish $\|\psi'\|_\infty$ enough that the left hand side is less than $1/4$. Subsequently we observe that regardless of the value of c, $\lim_{\rho\to\rho_{max}} \psi(\rho) = \infty$ and so we can choose ρ_+ in such a way that the right-hand side is arbitrarily large. For simplicity, we pick it so that the right-hand side is 1/2. ☐

B Topological Irreducibility

Proof of Fact 2.7 . Any two points r_0 and r_* in $\mathcal{G}_r$ can be connected by a path consisting of two parts, $r_0 \to |r_*|r_0/|r_0| \to r_*$, a line segment (connecting r_0 to $|r_*|\,r_0/|r_0|$) and then a circular arc (connecting $|r_*|\,r_0/|r_0|$ to r_*). The length of the linear segment is less than ρ_0 and the length of the circular arc will be less than $\pi\rho_0$. Qualitatively speaking, by smoothing out the corner, there exists a smooth curve from r_0 to r_* with arclength less than $(1+\pi)\rho_0$. It follows that there exists a parametrization $\tilde{R}$ of such a curve, and furthermore, the $\mathcal{R}$ defined by Equation (29) in the statement of Fact 2.7 in non-empty.

Given this $\tilde{R}$, we consider the linear (in $\tilde{Z}$) system

$$\frac{d}{dt}\tilde{R}(t) = -\nabla\Phi(\tilde{R}(t)) + S(\tilde{R}(t))\tilde{Z}(t)$$

for every $t \in [0,1]$. There exists a unique minimal norm solution

$$\tilde{Z}(t) = S^\dagger(\tilde{R}(t))\Big(\nabla\Phi(\tilde{R}(t)) + \frac{d}{dt}\tilde{R}(t)\Big)$$

where $S^\dagger := S^*(SS^*)^{-1}$ is the Moore-Penrose pseudoinverse [BIG80] and S^* is the transpose of S. We claim that $\tilde{Z}$ is continuous and therefore bounded over the interval $t \in [0,1]$. Indeed, by hypothesis, both $\nabla\Phi(\tilde{R})$ and $\frac{d}{dt}\tilde{R}$ are continuous, so we only must show that $S^\dagger(\tilde{R}(\cdot))$ is continuous.

As a finite sum of sines, S is a continuous function on $\mathbb{R}^2$. It follows that both S^* and SS^* are continuous as well, and $(SS^*)^{-1}$ is continuous in any domain in which its determinant satisfies $|\det(S(r)S^*(r))| > 0$ for all r in the domain. Because SS^* is a 2×2 matrix

$$SS^* = \begin{pmatrix} |S_1|^2 & S_1 \cdot S_2 \\ S_1 \cdot S_2 & |S_2|^2 \end{pmatrix}$$

where S_1 and S_2 are the first and second rows of S respectively, the determinant simplifies to $\det(S(r)) = |S_1(r)|^2|S_2(r)|^2\big(1 - \cos^2(\theta(r))\big)$ where θ is the angle between the vectors S_1 and S_2. Noting that θ is a continuous function of r while recalling that each $S_i(r)$ is continuous and that $\mathcal{G}_r$ is compact, it suffices to show that that $S_1(r)$ and $S_2(r)$ are linearly independent for all $r \in \mathcal{G}_R$. Because the row space and column space of a matrix have the same dimension, this reduces to showing the column rank of $S(r)$ is two. This follows immediately from the hypothesis that the active mode vector set $\mathcal{K}$ contains at least three pairwise linearly independent vectors, which we label k_1, k_2 and k_3. Among the three columns $\{\sin(\lambda k_j \cdot r)k_j^\perp\}_{j=1}^3$ at most one of the sine coefficients is zero, leaving at least two linearly independent columns.

We conclude that the control $\tilde{Z}(\cdot)$ is well-defined, continuous and has a magnitude which is bounded above by

$$|\tilde{Z}(t)| \leq M_1 := \sup_{r \in \mathcal{G}_r} \|S^\dagger(r)\|_F(|\nabla\Phi(r)| + 5\rho_+)$$

for all $t \in [0,1]$. ☐

Proof of Fact 2.8. Let the constants $\delta_r \in (0, \epsilon_1/2)$, $T > 0$ and $M_2 > 0$ be given. Suppose $\tilde{Z} \in C([0,T], \mathbb{R}^N)$ is a deterministic control with $|Z|_\infty \leq M_2$ such that $\tilde{R} = \Psi(\tilde{r}_0, \tilde{Z})$ satisfies $\tilde{R}(t) \in \mathcal{G}_r$ for all $t \in [0,T]$.

We will show that there exist positive constants γ, δ_0, and δ_z such that if $|r_0 - \tilde{r}_0| \leq \delta_0$ and $Z(\cdot) \in \mathcal{Z}(\tilde{Z}, M_2, \gamma, \delta_z)$, then

$$\sup_{t \in [0,T]} |R(t) - \tilde{R}(t)| \leq \delta_r. \tag{55}$$

To this end, define $H(t) := R(t) - \tilde{R}(t)$. Then H satisfies the integral equation

$$H(t) = H(0) + \int_0^t \nabla\Phi(R(s)) - \nabla\Phi(\tilde{R}(s))ds + \int_0^t S(R(s))Z(s) - S(\tilde{R}(s))\tilde{Z}(s)ds$$

As functions of R, both $\nabla\Phi$ and S are locally Lipschitz. Let $\mathcal{G}_r^+ \subset \mathbb{R}^2$ be the annulus centered at the origin with inner radius $\epsilon_1/2$ and outer radius $\rho_0 + \epsilon_1/2$. Although the deterministic control is defined so that $\tilde{R}$ stays in $\mathcal{G}_r$, instances of a the actual connector process R may wander slightly out of the good region. It is with respect to this enlarged set that we take the local Lipschitz constants, $\lambda_\Phi > 0$ and $\lambda_S > 0$ such that for all $r, \tilde{r} \in \mathcal{G}^+$,

$$|\nabla\Phi(r) - \nabla\Phi(\tilde{r})| \leq \lambda_\Phi |r - \tilde{r}|, \quad \|S(r) - S(\tilde{r})\|_F \leq \lambda_S |r - \tilde{r}|.$$

Observing that $|S(r)z - S(\tilde{r})\tilde{z}| \leq \lambda_S |r-\tilde{r}||z| + \|S(\tilde{r})\|_F |z - \tilde{z}|$ for all $r, \tilde{r} \in \mathcal{G}_r^+$ yields

$$|H(t)| \leq |H(0)| + \int_0^t (\lambda_\Phi + \lambda_S |Z(s)|)|H(s)|ds + \int_0^t \|S(\tilde{R}(s))\|_F |Z(s) - \tilde{Z}(s)|ds.$$

By virtue of the assumption that $Z \in \mathcal{Z}(\tilde{Z}, M_2, \gamma, \delta_z)$, defined in (30) the second integral satisfies the bound

$$\int_0^t \|S(\tilde{R}(s))\|_F |Z(s) - \tilde{Z}(s)|ds \leq \sup_{r\in\mathcal{G}_r} \|S(r)\|_F \int_0^t M_2 e^{-\gamma s} + \delta_z ds,$$

and so after simplifying we have $|H(t)| \leq \int_0^t \beta |H(s)|ds + g(t)$ where $\beta = \lambda_\Phi + (2M_2 + \delta_z)\lambda_S$ and $g(t) = \delta_0 + \sup_{r\in\mathcal{G}} \|S(r)\|_F \big(\frac{M_2}{\gamma} + \delta_z t\big)$. Using the integral form of Gronwall's Inequality yields $|H(t)| \leq g(t) + \int_0^t g(s)\beta e^{\beta(t-s)}ds$. After substituting in the values of g and β and integrating, we see that for all $t \in [0, T]$,

$$|H(t)| \leq \Big[\delta_0 + \sup_{r\in\mathcal{G}_r} \|S(r)\|_F \Big(\frac{M_2}{\gamma} + \frac{\delta_z}{\lambda_\Phi + M_2\lambda_S}\Big)\Big] e^{(\lambda_\Phi + M_2\lambda_S)T}$$

Taking δ_0 and δ_z sufficiently small while taking γ sufficiently large yields (55). ☐

Proof of Fact 2.9. Let the constants $\gamma > 0$, $\delta_z > 0$ and $M > 0$ be given, along with $\tilde{Z} \in C([0,T]; \mathbb{R}^N)$ satisfying $|\tilde{Z}|_\infty < M$. As in the proof of Lemma 2.5 the noise vector $Z(t) = (Z_1(t), Z_2(t), \ldots, Z_N(t))$ can be written $Z(t) = e^{-\Lambda t} z_0 + \int_0^t e^{-\Lambda(t-s)} B dW(s)$ where Λ is a diagonal matrix whose entries $\{\lambda_k\}_{k\in\mathcal{K}}$ are given by $\lambda_k := \lambda^2 \nu |k|^2$ and B is a diagonal matrix whose entries $\{b_k\}_{k\in\mathcal{K}}$ are given by $\sqrt{2\beta\nu}\lambda\sigma_k$.

Again view the stochastic integral as a time change of a Brownian motion. As before $M_k(t) := \int_0^t e^{\lambda_k s} b_k dW_k(s)$ is a continuous martingale with quadratic variation $\langle M_k, M_k \rangle_t = b_k^2 (e^{2\lambda_k t} - 1)/2\lambda_k$, we observe that for any $t > 0$, $M_k(t)$ has the same distribution as $\tilde{W}(\langle M_k, M_k\rangle_t)$ where $\tilde{W}$ is a standard Brownian motion. For any continuous curve Γ with $\Gamma(0) = 0$, $\tilde{T} > 0$ and $\delta > 0$

$$\mathbb{P}\Big\{ \sup_{t\in[0,\tilde{T}]} |\tilde{W}(t) - \Gamma(t)| \le \delta \Big\} > \tilde{p}$$

for some $\tilde{p} > 0$ (see [Dur96] for example). Since we have assumed there are only a finite number of active modes, and the modes are independent, Fact 2.9 follows immediately from the union bound. ☐

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