An Elementary Proof of the Existence and Uniqueness Theorem for the Navier-Stokes Equations

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1 Introduction

The purpose of this paper is to show that some results concerning solutions of the Navier-Stokes systems can be proven by purely elementary methods. In two-dimensions with periodic boundary conditions, the Navier-Stokes system has the form

$$\begin{aligned}\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} &= \nu\Delta u_1 - \frac{\partial p}{\partial x_1} + f_1(x_1,x_2,t) \ ,\\ \frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} &= \nu\Delta u_2 - \frac{\partial p}{\partial x_2} + f_2(x_1,x_2,t) \ ,\\ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} &= 0 \ .\end{aligned} \tag{1}$$

Here ν is the viscosity, p is the pressure, and f_1, f_2 are the components of an external forcing which may be time-dependent. As our setting is periodic, the functions u_1, u_2, ∇p, f_1, and f_2 are all periodic in x. For simplicity, we take the period to be one.

The first existence and uniqueness theorems for weak solutions of (1) were proven by Leray ([Ler34]) in whole plane $\mathbb{R}^2$. Later these results were extended by E. Hopf (see [Hop51]). In 1962, Ladyzenskaya proved existence and uniqueness results for strong solutions for general two-dimensional domains [Lad69]. V. Yudovich, C. Foias, R. Teman, P. Constantin, and others developed strong methods which provided deep insights into the dynamics described by (1) (see [Yud89, Tem79, Tem95, CF88]).

The purpose of this paper is to present elementary proofs of three theorems. These theorems imply the existence and uniqueness of smooth solutions of (1) and shed some additional light on the dissipative character of the dynamics. We will also discuss what our techniques can give in the three-dimensional setting.

In two-dimensions, it is useful to consider the vorticity $\omega(x_1,x_2,t) = \frac{\partial u_1(x_1,x_2,t)}{\partial x_2} - \frac{\partial u_2(x_1,x_2,t)}{\partial x_1}$. The equation governing ω has the form (see [CM93, DG95])

$$\frac{\partial \omega}{\partial t} + u_1\frac{\partial \omega}{\partial x_1} + u_2\frac{\partial \omega}{\partial x_2} = \nu\Delta\omega + g(x_1,x_2,t) \tag{2}$$

where $g(x_1,x_2,t) = \frac{\partial f_1(x_1,x_2,t)}{\partial x_2} - \frac{\partial f_2(x_1,x_2,t)}{\partial x_1}$. We will need $g(x_1,x_2)$ to posses a modicum of spatial smoothness; this will be made precise shortly.

In our two-dimensional setting, the systems (1) and (2) are equivalent. Expanding ω in Fourier series where $\omega(x_1,x_2,t) = \sum_{k\in\mathbb{Z}^2}\omega_k(t)e^{2\pi i(x,k)}$ with $x=(x_1,x_2)$, we can write a coupled ODE-system for the modes $\omega_k(t)$ (see [DG95]).

$$\frac{d\omega_k}{dt} + 2\pi i\sum_{l_1+l_2=k}\omega_{l_1}\omega_{l_2}\frac{(k,l_2^\perp)}{(l_2,l_2)} = -4\pi^2\nu|k|^2\omega_k + g_k(t) \tag{3}$$

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where $k \in \mathbb{Z}^2$, $|k| = \sqrt{k_1^2 + k_2^2}$, $l^\perp = (l^{(1)}, l^{(2)})^\perp = (-l^{(2)}, l^{(1)})$, and $g_k(t)$ are the spatial Fourier modes of the function $g(x,t)$. Since ω is real, we know $\omega_{-k} = \bar{\omega}_k$. Furthermore, we always assume that $\omega_0 = 0$. The system (3) is the Galerkin system corresponding to (2). A finite dimensional approximation of this Galerkin system can be associated to any finite subset $\mathcal{Z}$ of $\mathbb{Z}^2$ by setting $\omega_k(t) = 0$ for all k outside of $\mathcal{Z}$. In the following, we will implicitly assume that $\mathcal{Z}$ is centrally-symmetric, that is if $k \in \mathcal{Z}$ then $-k \in \mathcal{Z}$.

In fact, we will study a slightly more general version of (2) where the Laplacian is replaced by the operator $|\nabla|^\alpha$ with $\alpha > 1$. This leads to a version of (3) which we index by the choice of α and by the finite index set $\mathcal{Z}$, $\mathcal{Z} \subset \mathbb{Z}^2$, indicating which modes are included in the Galerkin approximation. In short, we consider the finite dimensional ODE system

$$\frac{d\omega_k}{dt} + 2\pi i \sum_{\substack{l_1+l_2=k \\ l_1,l_2\in\mathcal{Z}}} \omega_{l_1}\omega_{l_2}\frac{(k, l_2^\perp)}{(l_2, l_2)} = -4\pi^2\nu|k|^\alpha\omega_k + g_k\ . \qquad (3^\alpha_{\mathcal{Z}})$$

We now state the assumptions on the coefficients $g_k(t)$ to be used at various times during our discussion.

Assumption 1. *The forcing $f(x,t) = (f_1(x,t), f_2(x,t))$ is such that $g^* = \sup_{t\in[0,\infty)}|g(\cdot,t)|_{L^2} < \infty$.*

Assumption 2. *For some r, there exists a constant $\mathcal{G}(r) > 0$ such that*

$$\sup_{t\in[0,\infty)}|g_k(t)| \le \frac{\mathcal{G}(r)}{|k|^{r-\alpha+\epsilon}}$$

for some $\epsilon > 0$ and all $k \in \mathbb{Z}^2\backslash 0$. The constant α is the same as in ($3^\alpha_{\mathcal{Z}}$).

Assumption 3. *For some r and $\gamma > 0$, there exists a constant $\mathcal{G}(r,\gamma) > 0$ such that*

$$\sup_{t\in[0,\infty)}|g_k(t)| \le \frac{\mathcal{G}(r,\gamma)}{|k|^{r-\alpha+\epsilon}}e^{-\gamma|k|^{1+\delta}}$$

for some $\delta > 0$, $\epsilon > 0$, and all $k \in \mathbb{Z}^2\backslash 0$. Again, the constant α is the same as in ($3^\alpha_{\mathcal{Z}}$).

Observe that assumption 3 implies assumption 2. Critical to our discussion is that for ($3^\alpha_{\mathcal{Z}}$) we have the so-called enstrophy estimate. Namely, if $\mathcal{E}(0) = \int \omega^2(x_1,x_2,0)dx_1dx_2 = \sum_k|\omega_k(0)|^2 < \infty$ then one can find $\mathcal{E}^*$ depending only on $\mathcal{E}(0)$, ν, $\sup_{t\in[0,\infty)}|g(\cdot,t)|_{L^2}$, and α such that $\mathcal{E}(t) = \int \omega^2(x_1,x_2,t)dx_1dx_2 \le \mathcal{E}^*$ for all solutions to ($3^\alpha_{\mathcal{Z}}$). It is important to note that $\mathcal{E}^*$ is independent of the set $\mathcal{Z}$ which defines the Galerkin approximation. This enstrophy estimate holds if the forcing satisfies assumption 1 (see e.g. [CF88, DG95, Tem79]).

Now we are ready to formulate our theorems.

Theorem 1. *Assume the forcing satisfies assumption 1 and 2 for some $r > 1$ and $\mathcal{G}(r) > 0$. If for some $\mathcal{D}_1 < \infty$*

$$|\omega_k(0)| \le \frac{\mathcal{D}_1}{|k|^r}$$

then one can find a $\mathcal{D}_1' < \infty$, depending only on $\mathcal{D}_1$, ν, g^*, and $\mathcal{G}$, such that any solution to ($3^\alpha_{\mathcal{Z}}$) with these initial conditions satisfies

$$|\omega_k(t)| \le \frac{\mathcal{D}_1'}{|k|^r}$$

for all $t > 0$. In particular, $\mathcal{D}_1'$ is independent of the set $\mathcal{Z}$ defining the Galerkin approximation.

An existence and uniqueness theorem for (3) follows from theorem 1 by now standard considerations (see [CF88, DG95, Tem79]). We briefly recall the general line of the argument. By the Sobolev embedding theorem, the Galerkin approximations are trapped in a compact subset of L^2 of the 2-torus. This guarantees the existence of a limit point which can be shown to satisfy (3). Using the the regularity inherited from the Galerkin approximations, one then shows that there is a unique solution to (3). Gallavotti [Gal96] contains a similar proof of a similar statement.

Theorem 2. *Assume that assumption 3 holds for some $r > 1$, $\gamma > 0$, and $\mathcal{G}(r,\gamma) > 0$. If the initial conditions satisfy*

$$|\omega_k(0)| \leq \frac{\mathcal{D}_2}{|k|^r} e^{-\gamma_2 |k|}$$

for some $\mathcal{D}_2 < \infty$ and $\gamma_2 > 0$, then one can find a $\mathcal{D}_2' < \infty$ and a $\gamma_2' > 0$, depending only on $\mathcal{D}_2$, γ, r, ν, $g^*, \mathcal{G}$, such that any solution to ($\mathcal{F}_{\mathcal{Z}}^{\alpha}$) starting from these initial conditions satisfies

$$|\omega_k(t)| \leq \frac{\mathcal{D}_2'}{|k|^r} e^{-\gamma_2' |k|}$$

for all $t > 0$. In particular, the constants $\mathcal{D}_2'$ and γ' are independent of the set $\mathcal{Z}$ defining the Galerkin approximation.

Theorem 2 shows that equation (2) preserves the class of real analytic functions on the 2-torus.

Theorem 3. *Assume that assumption 3 holds for some $r > 1$, $\gamma > 0$, and $\mathcal{G}(r,\gamma) > 0$. If the initial conditions satisfy*

$$|\omega_k(0)| \leq \frac{\mathcal{D}_3}{|k|^r}$$

then for any $t_0 > 0$, one can find a $\mathcal{D}_3' > 0$ and a $\gamma_3' > 0$ such that any solution to ($\mathcal{F}_{\mathcal{Z}}^{\alpha}$) with these initial conditions satisfies

$$|\omega_k(t_0)| \leq \frac{\mathcal{D}_3'}{|k|^r} e^{-\gamma_3' |k|} \ .$$

As before, the constant $\mathcal{D}_3'$ is independent of the set $\mathcal{Z}$ defining the Galerkin approximation.

Theorem 3 shows that if the initial conditions $\omega(x,0)$ for (2) are smooth enough then, the solution $\omega(x,t_0)$ is real analytic for arbitrarily small time t_0, . Then according to theorem 2, it remains with in this class for all $t > t_0$. Statements close to these were proven in the works by C. Foias and R. Temam [FT89], C. Doering and E. Titi [DT95] and H. Kreiss [Kre88]. Theorem 1 is proven in §2 and theorems 2 and 3 are proven in §3.

The proofs of all of the theorems in this paper share a common structure. We consider the system of coupled ODEs for the Fourier coefficients. Then we construct a subset Ω of the phase space (the set of possible configurations of the Fourier modes) so that all points in Ω possess the desired decay properties. In addition, Ω is constructed so that it contains the initial data in its interior. Then we endeavor to show that the dynamics never cause the sequence of Fourier modes to leave the subset Ω. How this is done can be understood geometrically. It amounts to showing that the vector field on the boundary of Ω points into the interior of Ω. If this is true, then the solution can never escape Ω.

2 Proof of Theorem 1

Fixing an arbitrary Galerkin approximation corresponding to the modes in some finite subset $\mathcal{Z}$ of $\mathbb{Z}^2$, we write the real version of (3). As we already mentioned, we assume $\omega_0 = 0$ and, because the velocity is real, we also have $\omega_{-k} = \bar{\omega}_k$. Setting $\omega_k = \omega_k^{(1)} + i\omega_k^{(2)}$, we separate the equations for $\omega_k^{(1)}$ and $\omega_k^{(2)}$ obtaining

$$\begin{aligned}
\frac{d\omega_k^{(1)}}{dt} =& 2\pi \sum_{\substack{l_1+l_2=k \\ l_1,l_2\in\mathcal{Z}}} \left[\omega_{l_1}^{(1)}\omega_{l_2}^{(2)} + \omega_{l_1}^{(2)}\omega_{l_2}^{(1)}\right] \frac{(k,l_2^{\perp})}{(l_2,l_2)} - 4\pi\nu|k|^{\alpha}\omega_k^{(1)} + g_k^{(1)} \\
\frac{d\omega_k^{(2)}}{dt} =& -2\pi \sum_{\substack{l_1+l_2=k \\ l_1,l_2\in\mathcal{Z}}} \left[\omega_{l_1}^{(1)}\omega_{l_2}^{(1)} + \omega_{l_1}^{(2)}\omega_{l_2}^{(2)}\right] \frac{(k,l_2^{\perp})}{(l_2,l_2)} - 4\pi\nu|k|^{\alpha}\omega_k^{(2)} + g_k^{(2)}
\end{aligned} \tag{4}$$

where $g_k = g_k^{(1)} + ig_k^{(2)}$.

It follows from the enstrophy estimate that $\sum_k \left[\left(w_k^{(1)}(t)\right)^2 + \left(w_k^{(2)}(t)\right)^2\right] \le \mathcal{E}^*$ and thus $|w_k^{(1)}(t)| \le \sqrt{\mathcal{E}^*}$ and $|w_k^{(2)}(t)| \le \sqrt{\mathcal{E}^*}$ for all $k \in \mathbb{Z}^2$ and $t > 0$. Hence, for any $K_0 > 0$, we can find a $\mathcal{D}_1' = \mathcal{D}_1'(K_0)$ such that for any $t \ge 0$ $|w_k^{(1)}(t)|, |w_k^{(2)}(t)| < \frac{\mathcal{D}_1'}{|k|^r}$ for all $k \in \mathbb{Z}^2$ with $|k| \le K_0$. We also require $\mathcal{D}_1'$ to be greater than $\mathcal{G}$ so later estimates will arrange themselves nicely. Recall that $\mathcal{G}(r)$ was the constant from assumption 2. Since $\mathcal{G}$ is given and only K_0 is ours to vary, we will suppress the dependence of $\mathcal{D}_1'$ on $\mathcal{G}$.

Now consider the subset

$$\Omega_1(K_0) = \left\{ (\omega_k^{(1)}, \omega_k^{(2)})_{k \in \mathbb{Z}^2} : |\omega_k^{(j)}| \le \frac{\mathcal{D}_1'(K_0)}{|k|^r} \text{ for all } j \in \{1,2\},\, k \in \mathbb{Z}^2 \backslash 0 \right\}$$

of $(\mathbb{R}^2)^{\mathbb{Z}^2}$. Its boundary is the subset

$$\partial\Omega_1(K_0) = \left\{ (\omega_k^{(1)}, \omega_k^{(2)})_{k \in \mathbb{Z}^2} : \begin{array}{c} |\omega_k^{(j)}| \le \dfrac{\mathcal{D}_1'}{|k|^r} \text{ for all } j \in \{1,2\}, k \in \mathbb{Z}^2 \backslash 0 \\ \text{and equality holds for some } \bar{k} \text{ and } \bar{\jmath}\ . \end{array} \right\} .$$

We shall also need the subset of this boundary

$$\overline{\partial\Omega_1}(K_0) = \left\{ (\omega_k^{(1)}, \omega_k^{(2)})_{k \in \mathbb{Z}^2} : \begin{array}{c} |\omega_k^{(j)}| \le \dfrac{\mathcal{D}_1'}{|k|^r} \text{ for all } j \in \{1,2\}, k \in \mathbb{Z}^2 \backslash 0 \\ \text{and equality for some } \bar{k} \text{ and } \bar{\jmath} \text{ with } |\bar{k}| > K_0. \end{array} \right\} .$$

Showing that the trajectories of our system remain inside of Ω_1 is equivalent to the statement of the theorem. Recall that using the enstrophy estimate, we picked a $\mathcal{D}_1'(K_0)$ such that if $|k| \le K_0$ then $|w_k^{(1)}(t)|$ and $|w_k^{(2)}(t)|$ were bounded by $\frac{\mathcal{D}_1'}{|k|^r}$ for all $t \in [0, \infty)$. Thus, the only remaining way for a trajectory to leave $\Omega_1(K_0)$, is through the section of the boundary $\overline{\partial\Omega_1}(K^0)$ introduced above. Our basic idea is to show that if K_0 is greater than a specific K_{crit}, then the vector field on $\overline{\partial\Omega_1}(K^0)$ points inward. In other words, the dynamics of (4) can never move the system configuration through $\partial\Omega_1(K_0)$. In still different words, Ω_1 is a trapping region. Since the initial data begins in Ω_1, proving this picture would prove the theorem.

To show that the vector field points inward, fix a point on $\overline{\partial\Omega_1}(K_0)$. For definiteness, consider the case when $\omega_{\bar{k}}^{(1)} = \frac{\mathcal{D}_1'}{|k|^r}$ for some $\bar{k}$ with $|\bar{k}| > K_0$, $|\omega_{k'}^{(1)}| \le \frac{\mathcal{D}_1'}{|k'|^r}$ for all $k' \in \mathbb{Z}\backslash 0$ with $k' \ne \bar{k}$, and $|\omega_k^{(2)}| \le \frac{\mathcal{D}_1'}{|k|^r}$ for all $k \in \mathbb{Z}^2 \backslash 0$. The other cases, namely where $\omega_{\bar{k}}^{(1)} = -\frac{\mathcal{D}_1'}{|k|^2}$ or $\omega_{\bar{k}}^{(2)} = \pm\frac{\mathcal{D}_1'}{|k|^2}$, are handled in the same manner. We have to show that,

$$2\pi \left| \sum_{l_1 + l_2 = \bar{k}} \left[\omega_{l_1}^{(1)}\omega_{l_2}^{(1)} + \omega_{l_1}^{(2)}\omega_{l_2}^{(2)}\right] \frac{(\bar{k}, l_2^{\perp})}{(l_2, l_2)} \right| + \left| g_{\bar{k}}^{(2)} \right| < 4\pi\nu|\bar{k}|^{\alpha} \left|\omega_{\bar{k}}^{(2)}\right| \ . \tag{5}$$

We shall see that the restriction that $|\bar{k}| \ge K_0 > K_{crit}$ does not depend on $\mathcal{D}_1$ only on $\mathcal{E}$.

Consider the following three sums which together bound the first abolute value on the left-hand side of (5):

$$\Sigma_1 = \sum_{\substack{l_1 + l_2 = \bar{k} \\ |l_2| \le |\frac{\bar{k}}{2}|}} \left| \left[\omega_{l_1}^{(1)}\omega_{l_2}^{(1)} + \omega_{l_1}^{(2)}\omega_{l_2}^{(2)}\right] \right| \left| \frac{(\bar{k}, l_2^{\perp})}{(l_2, l_2)} \right|$$

$$\Sigma_2 = \sum_{\substack{l_1 + l_2 = \bar{k} \\ |\frac{\bar{k}}{2}| < |l_2| \le 2|\bar{k}|}} \left| \left[\omega_{l_1}^{(1)}\omega_{l_2}^{(1)} + \omega_{l_1}^{(2)}\omega_{l_2}^{(2)}\right] \right| \left| \frac{(\bar{k}, l_2^{\perp})}{(l_2, l_2)} \right|$$

$$\Sigma_3 = \sum_{\substack{l_1 + l_2 = \bar{k} \\ |l_2| > 2|\bar{k}|}} \left| \left[\omega_{l_1}^{(1)}\omega_{l_2}^{(1)} + \omega_{l_1}^{(2)}\omega_{l_2}^{(2)}\right] \right| \left| \frac{(\bar{k}, l_2^{\perp})}{(l_2, l_2)} \right|$$

We treat each sum separately. For Σ_1, using the Cauchy-Schwartz inequality and the inequalities $\left|\frac{(\bar{k},l_2^\perp)}{(l_2,l_2)}\right| \le \frac{|\bar{k}|}{|l_2|}$, $|l_1| \ge \frac{|\bar{k}|}{2}$, and $|\omega_{l_1}^{(1)}| \le 2^r \mathcal{D}_1' \frac{1}{|\bar{k}|^r}, |\omega_{l_1}^{(2)}| \le 2^r \mathcal{D}_1' \frac{1}{|\bar{k}|^r}$ produces

$$
\begin{aligned}
|\Sigma_1| &\le 2^r \frac{\mathcal{D}_1'}{|\bar{k}|^r} |\bar{k}| \sum_{|l_2| \ge |\frac{\bar{k}}{2}|} \left[|\omega_{l_2}^{(1)}| + |\omega_{l_2}^{(2)}| \right] \frac{1}{|l_2|} \\
&\le 2^r \frac{\mathcal{D}_1' |\bar{k}|}{|\bar{k}|^r} \left(\sqrt{\sum |\omega_{l_2}^{(1)}|^2} + \sqrt{\sum |\omega_{l_2}^{(2)}|^2} \right) \sqrt{\sum_{|l_2| \le |\frac{\bar{k}}{2}|} \frac{1}{|l_2|^2}} \\
&\le 2^{r+1} (\text{const}) \left(\sqrt{\mathcal{E}^*} \right) |\bar{k}| \left(\sqrt{\ln |\bar{k}|} \right) \left(\frac{\mathcal{D}_1'}{|\bar{k}|^r} \right) . \qquad (6)
\end{aligned}
$$

The (const) in the final line is from the inequality

$$
\sum_{|l_2| \le |\frac{\bar{k}}{2}|} \frac{1}{|l_2|^2} \le (\text{const})^2 \ln |\bar{k}| .
$$

To estimate Σ_2, we use the inequalities $\left|\frac{(k,l_2^\perp)}{(l_2,l_2)}\right| \le 2$, $|\omega_{l_2}^{(1)}| \le 2^r \frac{\mathcal{D}_1'}{|\bar{k}|^r}$, and $|\omega_{l_2}^{(2)}| \le 2^r \frac{\mathcal{D}_1'}{|\bar{k}|^r}$ obtaining

$$
\begin{aligned}
|\Sigma_2| &\le 2^{r+1} \frac{\mathcal{D}_1'}{|\bar{k}|^r} \sum_{|l_1| \le 3|\bar{k}|} \left[|\omega_{l_1}^{(1)}| + |\omega_{l_1}^{(2)}| \right] \\
&\le 2^{r+1} \frac{\mathcal{D}_1'}{|\bar{k}|^r} \left[\sqrt{\sum_{|l_1| \le 3|\bar{k}|} |\omega_{l_1}^{(1)}|^2} + \sqrt{\sum_{|l_1| \le 3|\bar{k}|} |\omega_{l_1}^{(2)}|^2} \right] (6|\bar{k}| + 1) \\
&\le 2^{r+2} \mathcal{E} (6|\bar{k}| + 1) \frac{\mathcal{D}_1'}{|\bar{k}|^r} . \qquad (7)
\end{aligned}
$$

The factor $(6|\bar{k}|+1)$ arises as an estimate of the square root of the number of lattice points $l_1 \in \mathbb{Z}^2$ for which $|l_1| \le 3|\bar{k}|$.

In estimating Σ_3, we use $\left|\frac{(\bar{k},l_2^\perp)}{(l_2,l_2)}\right| \le \left|\frac{\bar{k}}{l_2}\right|$ producing

$$
\begin{aligned}
|\Sigma_3| &\le |\bar{k}| \sum_{\substack{l_1+l_2=\bar{k} \\ |l_2| \ge 2|\bar{k}|}} \left[|\omega_{l_1}^{(1)}||\omega_{l_2}^{(2)}| + |\omega_{l_1}^{(2)}||\omega_{l_2}^{(1)}| \right] \frac{1}{|l_2|} \\
&\le |\bar{k}| \left[\left(\sum_{|l_1| \ge \bar{k}} (\omega_{l_1}^{(1)})^2 \right)^{\frac{1}{2}} \left(\sum_{|l_2| \ge 2\bar{k}} \frac{(\omega_{l_2}^{(2)})^2}{|l_2|^2} \right)^{\frac{1}{2}} + \left(\sum_{|l_1| \ge \bar{k}} (\omega_{l_1}^{(2)})^2 \right)^{\frac{1}{2}} \left(\sum_{|l_2| \ge 2\bar{k}} \frac{|\omega_{l_2}^{(1)}|^2}{|l_2|} \right)^{\frac{1}{2}} \right] \\
&\le 2\sqrt{\mathcal{E}^*} |\bar{k}| \mathcal{D}_1' \Big(\sum_{|l_2| \ge 2|\bar{k}|} \frac{1}{|l_2|^{2(r+1)}} \Big)^{\frac{1}{2}} \le 2\sqrt{\mathcal{E}^*} (\text{const}) |\bar{k}| \frac{\mathcal{D}_1'}{|\bar{k}|^r} \qquad (8)
\end{aligned}
$$

where (const) is defined by the inequality

$$
\sum_{|l_2| \ge 2|\bar{k}|} \frac{1}{|l_2|^{2(r+1)}} \le (\text{const})^2 \frac{1}{|\bar{k}|^{2r}} .
$$

Adding (6), (7), and (8) together, we obtain the needed bound on the right hand side of (5):

$$2\pi \sum_{l_1+l+2=\bar{k}} |\omega^{(1)}_{l_1}||\omega^{(2)}_{l_2}| + |\omega^{(2)}_{l_1}||\omega^{(1)}_{l_2}| \le \Big[2^{r+1}(\text{const})\sqrt{\mathcal{E}^*}|\bar{k}|\sqrt{\ln|\bar{k}|} + 2^{r+2}\mathcal{E}^*(6|\bar{k}|+1)$$
$$+ 2\sqrt{\mathcal{E}^*}(\text{const})|k|\Big]\frac{\mathcal{D}'_1}{|\bar{k}|^r}$$
$$\le 2^{r+2}\mathcal{E}^*(\overline{\text{const}})|\bar{k}|\sqrt{\ln|\bar{k}|}\frac{\mathcal{D}'_1}{|\bar{k}|^r} \tag{9}$$

where $(\overline{\text{const}})$ is a new constant.

By assumption 2 and our requirement that the $\mathcal{D}'_1$ be greater than $\mathcal{G}$ (the constant from assumption 2), we know that $|g_k| \le \frac{\mathcal{D}'_1}{|k|^{r-\alpha+\epsilon}}$. Thus, inequality (5) will be satisfied if

$$\left[2^{r+2}\mathcal{E}^*(\overline{\text{const}})\frac{|\bar{k}|\sqrt{\ln|\bar{k}|}}{|\bar{k}|^\alpha} + \frac{1}{|\bar{k}|^\epsilon}\right]\frac{\mathcal{D}'_1}{|\bar{k}|^{r-\alpha}} \le 4\pi\nu\frac{\mathcal{D}'_1}{|\bar{k}|^{r-\alpha}} \ . \tag{10}$$

From this we see that for all $\alpha > 1$, there exists K_{crit} so that if $|\bar{k}| \ge K_{crit}$ then (10) holds. Also notice that K_{crit} is independent of our choice of $\mathcal{D}'_1$ except for the condition that $\mathcal{D}'_1 > \mathcal{G}$. Thus we can find K_{crit} first and then fix K_0 which determines $\mathcal{D}'_1$.

3 Proofs of Theorems 2 and 3

We begin by stating the central estimate on which both theorems rely. It requires estimates similar in spirit to the previous theorem and will be proven at the end of the section. We present a d-dimensional version of the lemma because it will be useful in the discussions of the 3-dimensional setting in the next section.

Lemma 1. *Let $\{a_k\}$ and $\{b_k\}$ be two sequences with $k \in \mathbb{Z}^d$. If for some $r > d-1$ and some $\mathcal{C} > 0$*

$$|a_k| \le \frac{\mathcal{C}}{|k|^r} \qquad |b_k| \le \frac{\mathcal{C}}{|k|^r}$$

then for all $k \in \mathbb{Z}^d$

$$\sum_{\substack{l_1+l_2=k \\ l_1,l_2\in\mathbb{Z}^d}} |a_{l_1}|\,|b_{l_2}|\,\frac{|k|}{|l_2|} \le (const)\left(2^r|k| + 2^{r+1}(6|k|+1)^{\frac{d}{2}} + \frac{1}{2}|k|^{d-1-r}\right)\frac{\mathcal{C}^2}{|k|^r}$$

where the constant depends only on r and not on k.

We now turn to the proof of theorem 2.

Proof of theorem 2. If $|\omega^{(1)}_k(0)| \le \frac{\mathcal{D}_2}{|k|^r}e^{-\gamma_2|k|}$, $|\omega^{(2)}_k(0)| \le \frac{\mathcal{D}_2}{|k|^r}e^{-\gamma_2|k|}$ then surely $|\omega^{(1)}_k|, |\omega^{(2)}_k| \le \frac{\mathcal{D}_2}{|k|^r}$. Therefore by theorem 1, one can find a constant $\bar{\mathcal{D}}_2$ such that $|\omega^{(1)}_k(t)|, |\omega^{(2)}_k(t)| \le \frac{\bar{\mathcal{D}}_2}{|k|^r}$ for all $k \in \mathbb{Z}^2\backslash\{0\}$. Let us set $\mathcal{D}'_2 = \max(2\bar{\mathcal{D}}_2, \mathcal{G})$ where $\mathcal{G}$ is the constant from assumption 3. The numerical factor 2 is somewhat arbitrary. We could chose any factor greater than 1; we take 2 for simplicity.

Choose $K_0 \ge 0$ and consider the set

$$\Omega_2(K_0) = \left\{(\omega^{(1)}_k, \omega^{(2)}_k)_{k\in\mathbb{Z}^2\backslash\{0\}} : |\omega^{(1)}_k| \le \frac{\mathcal{D}'_2}{|k|^r}e^{-\gamma'_2|k|}, |\omega^{(2)}_k| \le \frac{\mathcal{D}'_2}{|k|^r}e^{-\gamma'_2|k|}\right\}$$

The value of $\gamma'_2 = \gamma'_2(K_0)$ is chosen in such a way that the inequalities $|\omega^{(i)}_k(t)| \le \frac{\bar{\mathcal{D}}_2}{|k|^r}$ given by theorem 1 imply that $|\omega^{(i)}_k(t)| \le \frac{\mathcal{D}'_2}{|k|^r}e^{-\gamma'_2|k|}$ for all k, $|k| \le K_0$, and that for $|k| \ge K_0$, $e^{-\gamma'_2|k|} \ge e^{-\gamma|k|^{1+\delta}}$. Here γ and δ are the constants from assumption 3.

As in the proof of theorem 1, we shall show that for sufficiently large K_0 the vector field corresponding to (5) is directed inside $\Omega_2(K_0)$ along the part of the boundary $\partial\Omega_2(K_0)$ where $|\omega_k^{(i)}| \le \frac{\mathcal{D}_2'}{|k|^r} e^{-\gamma_2'|k|}$ for all $k \in \mathbb{Z}^2\backslash\{0\}$ with $|k| \ge K_0$ and for at least one of these, say $\bar{k}$, we have equality. It will be shown that our restriction from below on K_0, needed to ensure the vector field points inward, will not depend on γ_2'. This will yield the stated result.

As in theorem 1, consider for definiteness the case where $\omega_{\bar{k}}^{(1)} = \frac{\mathcal{D}_2'}{|\bar{k}|^2} e^{-\gamma_2'|\bar{k}|}$. The other cases are handled in the same manner. As before, we have to show that the vector field points inward. This would be assured if

$$2\pi \left| \sum_{l_1+l_2=\bar{k}} [\omega_{l_1}^{(1)}\omega_{l_2}^{(2)} + \omega_{l_2}^{(2)}\omega_{l_2}^{(1)}] \frac{(\bar{k}, l_2^{\perp})}{(l_2, l_2)} \right| + |g_{\bar{k}}| < 4\pi^2 |\bar{k}|^{\alpha} \frac{\mathcal{D}_2'}{|\bar{k}|^r} e^{-\gamma_2'|\bar{k}|} \ . \tag{11}$$

This time we do not use the enstrophy estimate as previously. Instead, we use the estimates $|\omega_{l_1}^{(1)}| \le \frac{\mathcal{D}_2'}{|l_1|^r} e^{-\gamma_2'|l_1|}$ and $|\omega_{l_1}^{(2)}| \le \frac{\mathcal{D}_2'}{|l_2|^r} e^{-\gamma_2'[|l_2|}$.

Let us put $v_k^{(j)} = e^{\gamma_2'|k|}\omega_k^{(j)}$, $j = 1, 2$, $k \in \mathbb{Z}^2\backslash 0$. In terms of v, (11) becomes

$$2\pi \left| \sum_{l_1+l_2=\bar{k}} [v_{l_1}^{(1)} v_{l_2}^{(2)} + v_{l_2}^{(2)} v_{l_2}^{(1)}] \frac{e^{-\gamma_2'|l_1|-\gamma_2'|l_2|}}{e^{-\gamma_2'|\bar{k}|}} \frac{(\bar{k}, l_2^{\perp})}{(l_2, l_2)} \right| + |g_{\bar{k}}| e^{\gamma_2'|\bar{k}|} < 4\pi^2 |\bar{k}|^{\alpha} \frac{\mathcal{D}_2'}{|\bar{k}|^r} \ . \tag{12}$$

First notice that $\frac{e^{-\gamma_2'|l_1|} e^{-\gamma_2'|l_2|}}{e^{-\gamma_2'|\bar{k}|}} \le 1$ so it may be neglected. Second notice that for $v_k^{(j)}$, we have the estimate $|v_k^{(j)}| \le \frac{\mathcal{D}_2'}{|k|^r}$ for $k \in \mathbb{Z}^2\backslash 0$. Lastly, we know that $|\frac{(\bar{k}, l_2^{\perp})}{(l_2, l_2)}| \le \frac{|k|}{|l_2|}$. These estimates allow us to apply lemma 1, producing

$$\begin{aligned} 2\pi \Big| \sum \left[v_{l_1}^{(1)} v_{l_2}^{(2)} + v_{l_2}^{(2)} v_{l_2}^{(1)} \right] \frac{e^{-\gamma_2'|l_1|-\gamma_2'|l_2|}}{e^{-\gamma_2'|\bar{k}|}} \frac{(\bar{k}, l_2^{\perp})}{(l_2, l_2)} \Big| \\ \le 2\pi \ \text{const} \ \left(2^{r+1}|\bar{k}| + 2^{r+2}(6|\bar{k}|+1) + |\bar{k}|^{1-r} \right) \mathcal{D}_2' \frac{\mathcal{D}_2'}{|\bar{k}|^r} \ . \end{aligned} \tag{13}$$

From this estimate, we see that if

$$2\pi(\text{const}) \left(2^{r+1}|\bar{k}| + 2^{r+2}(6|\bar{k}|+1) + 2|\bar{k}|^{1-r} \right) \mathcal{D}_2' + \frac{\mathcal{G}}{\mathcal{D}_2'} \frac{e^{-\gamma|\bar{k}|^{1+\delta}}}{e^{-\gamma_2'|\bar{k}|}} |\bar{k}|^{\alpha-\epsilon} < 4\pi^2 \nu |\bar{k}|^{\alpha} \tag{14}$$

then we have established (12), which was our goal. Notice that we chose $\mathcal{D}_2' \ge \mathcal{G}$ and γ_2' such that $\frac{e^{-\gamma|\bar{k}|^{1+\delta}}}{e^{-\gamma_2'|\bar{k}|}} \le 1$ for all k with $|k| \ge K_0$. Since $\alpha > 1$ by picking K_0 large enough, we can force (14) to hold. This is the criteria which sets the level of K_{crit}. The proof of theorem 2 is concluded. □

We now present the proof of theorem 3. Its structure is very similar to the previous proof and also employs lemma 1 but uses a slightly different change of variable.

Proof of theorem 3. Let $\mathcal{D}_1'$ be the constant given by theorem 1, that is such that $|\omega_k(t)| \le \frac{\mathcal{D}_1'}{|k|^2}$ for all $k \in \mathbb{Z}^2\backslash 0$ and all t. Let us put $v_k^{(j)} = \omega_k^{(j)} e^{\gamma_3 t|k|}, j = 1, 2$ where the constant γ_3 will be determined later. The evolution of the $v_k^{(1)}(t)$ are described by the following ODEs

$$\begin{aligned} \frac{dv_k^{(1)}(t)}{dt} =& \gamma_3|k| v_k^{(1)}(t) - 4\pi^2\nu|k|^{\alpha} v_k^{(1)}(t) + g_k^{(1)} e^{\gamma_3 t|k|} \\ & - 2\pi \sum_{l_1+l_2=k} \left[v_{l_1}^{(1)}(t) v_{l_2}^{(2)}(t) + v_{l_1}^{(2)}(t) v_{l_2}^{(1)}(t) \right] \frac{e^{-\gamma_3 t|l_1|} e^{-\gamma_3 t|l_2|}}{e^{-\gamma_3 t|k|}} \frac{(k, l_2^{\perp})}{(l_2, l_2)} \ . \end{aligned} \tag{15}$$

The analogous equations describe the evolution of the $v_k^{(2)}(t)$.

The methods of the previous section can be applied to this coupled system. We fix a time $t_0 > 0$ and an arbitrary positive constant γ_0. For $t = 0$, we have the inequalities

$$|v_k^{(1)}(0)| \leq \frac{\mathcal{D}_3}{|k|^r} \qquad |v_k^{(2)}(0)| \leq \frac{\mathcal{D}_3}{|k|^r}$$

for all k. In light of the definition of $v_k(t)$, theorem 3 would be proven if we show that

$$|v_k^{(1)}(t_0)| \leq \frac{\mathcal{D}'_3}{|k|^r} \qquad |v_k^{(2)}(t_0)| \leq \frac{\mathcal{D}'_3}{|k|^r} \tag{16}$$

for some appropriate $\mathcal{D}'_3$.

As in the proof of theorem 3, we put $\mathcal{D}'_3 = \max(2\mathcal{D}'_1, \mathcal{G})$ where $\mathcal{G}$ is again the constant from assumption 3. For any fixed K_0, we can find a γ_3 so that the following three conditions hold. First, the inequalities $|\omega_k^{(j)}(t)| \leq \frac{\mathcal{D}'_1}{|k|^r}$, imply $|v_k^{(j)}(t)| \leq \frac{\mathcal{D}'_3}{|k|^r}$ for $j = 1, 2$, $t \in [0, t_0]$, and $|k| \leq K_0$. Second, so $e^{\gamma_3 t|k|} \leq e^{\gamma|k|^{1+\delta}}$ for $k \in \mathbb{Z}^2$ with $|k| \geq K_0$ and $t \in [0, t_0]$. In this condition the constants γ and δ are again from assumption 3. Third, we can always assume that $\gamma_3 \leq \gamma_0$. (This last assumption is to simplify the exposition and is not really needed as γ_3 decreases as we increase K_0.)

Now consider the set

$$\Omega_3(K_0) = \left\{ \left(v_k^{(1)}, v_k^{(2)}\right)_{k \in \mathbb{Z}^2 \setminus 0} \text{ with } |v_k^{(j)}| \leq \frac{\mathcal{D}'_3}{|k|^r} \text{ for } j = 1, 2 \text{ and } |k| > K_0 \right\} .$$

Again we will show that if K_0 is greater than some K_{crit}, the vector field along the boundary of $\Omega_3(K_0)$ points inward. The calculation parallels that in theorem 2. For definiteness, we assume that $v_{\bar{k}}^{(1)}(t) = \frac{\mathcal{D}'_3}{|k|^r}$ for some $\bar{k}$ with $|\bar{k}| > K_0$ and that the inequality bounds which define Ω_3 hold for all other k. The other cases proceed analogously.

We wish to show that the vector field points inward. Since $\gamma_3 \leq \gamma_0$, from (15), we see that it is sufficient to show that for $t \in [0, t_0]$

$$\begin{aligned}(4\pi^2\nu|\bar{k}|^\alpha - \gamma_0|\bar{k}|)v_{\bar{k}}^{(1)} > 2\pi &\left| \sum_{l_1 + l_2 = \bar{k}} \left[v_{l_1}^{(1)}(t) v_{l_2}^{(2)}(t) + v_{l_1}^{(2)}(t) v_{l_2}^{(1)}(t) \right] \frac{(k, l_2^\perp)}{(l_2, l_2)} \right| \\ &+ |g_{\bar{k}}^{(1)}| e^{\gamma_3 t|\bar{k}|} \ .\end{aligned} \tag{17}$$

Here, as before, we have neglected the factor $\frac{e^{-\gamma_3 t|l_1|} e^{-\gamma_3 t|l_2|}}{e^{-\gamma_3 t|k|}}$ as it is always less than 1. After applying the inequalities $\mathcal{G} \leq \mathcal{D}'_3$, $e^{\gamma_3 t|k|} \leq e^{\gamma|k|^{1+\delta}}$ and lemma 1, we see that (17) holds if

$$4\pi^2\nu > \gamma_0 \frac{|\bar{k}|}{|\bar{k}|^\alpha} + (\text{const})\mathcal{D}'_3 \left[2^{r+1} \frac{|\bar{k}|}{|\bar{k}|^\alpha} + 2^{r+2} \frac{7|\bar{k}|}{|\bar{k}|^\alpha} + \frac{|\bar{k}|^{1-r}}{|\bar{k}|^\alpha} \right] + \frac{\mathcal{G}}{\mathcal{D}'_3} \frac{1}{|\bar{k}|^\alpha} \ . \tag{18}$$

Because $\alpha > 1$ and $r > 2$, by making $\bar{k}$ large enough we can force (18) to hold. This shows that the solution to any Galerkin approximation stays in Ω_3 until the time t_0 and thus (16) holds and the proof is complete. □

Proof of Lemma 1: As in the proof of theorem 1, we estimate separately three sums.

$$\Sigma_1 = \sum_{\substack{|l_2| \leq |\frac{k}{2}| \\ l_1 + l_2 = k}} |a_{l_1}|\,|b_{l_2}| \frac{|k|}{|l_2|}$$

$$\Sigma_2 = \sum_{\substack{|\frac{k}{2}| < |l_2| \leq 2|k| \\ l_1 + l_2 = k}} |a_{l_1}|\,|b_{l_2}| \frac{|k|}{|l_2|}$$

$$\Sigma_3 = \sum_{\substack{|l_2| > 2|k| \\ l_1 + l_2 = k}} |a_{l_1}|\,|b_{l_2}| \frac{|k|}{|l_2|}$$

Since in Σ_1, the norm of $|l_1| \geq |\frac{\bar{k}}{2}|$, we can write

$$\Sigma_1 \leq \sum_{|l_2| \leq |\frac{k}{2}|} |a_{l_1}||b_{l_2}| \frac{|k|}{|l_2|} \leq \frac{2^r (\mathcal{C})^2}{|k|^r} |k| \sum_{|l_2| \leq |\frac{k}{2}|} \frac{1}{|l_2|^{r+1}} \leq 2^r (\text{const}) |k| \frac{\mathcal{C}^2}{|k|^r} \tag{19}$$

where the constant is defined by the inequality

$$\sum_{l_2 \in \mathbb{Z}^d \backslash 0} \frac{1}{|l_2|^{r+1}} \leq \text{const} \ .$$

For this sum to be finite, we need $r + 1 > d$. For Σ_2 we have $\frac{|k|}{|l_2|} \leq 2$ and hence

$$\begin{aligned} \Sigma_2 &\leq 2 \sum_{|\frac{k}{2}| < |l_2| \leq 2|k|} \frac{\mathcal{C}^2}{|l_1|^r |l_2|^r} \leq \frac{2^{r+2} (\mathcal{C})^2}{|\bar{k}|^r} \sum_{|l_1| \leq 3|k|} \frac{1}{|l_1|^r} \\ &\leq \frac{2^{r+2} (\mathcal{C})^2}{|\bar{k}|^r} \Big(\sum_{|l_1| \leq 3|k|} \frac{1}{|l_1|^{2r}} \Big)^{\frac{1}{2}} \Big(\sum_{|l_1| \leq 3|k|} 1 \Big)^{\frac{1}{2}} \leq 2^{r+1} (\text{const}) \frac{\mathcal{C}^2 (6|k| + 1)}{|k|^r} \ . \end{aligned} \tag{20}$$

Here the constant is the absolute constant defined by

$$\sum_{l_1 \in \mathbb{Z}^d \backslash 0} \frac{1}{|l_1|^{2r}} \leq \text{const} \ .$$

For this sum to be finite, we need $2r > d$. For Σ_3 we have $\frac{|k|}{|l_2|} \leq \frac{1}{2}$. Hence, we can write

$$\begin{aligned} \Sigma_3 &\leq \frac{1}{2} \sum_{|l_2| \geq 2|\bar{k}|} \frac{\mathcal{C}^2}{|l_1|^r |l_2|^{r+1}} \leq \frac{\mathcal{C}^2}{2} \left(\sum_{\substack{|l_1| > |k| \\ l_1 \in \mathbb{Z}^d \backslash 0}} \frac{1}{|l_1|^{2r}} \right)^{\frac{1}{2}} \left(\sum_{\substack{|l_2| \geq 2|k| \\ l_2 \in \mathbb{Z}^d \backslash 0}} \frac{1}{|l_2|^{2r+2}} \right)^{\frac{1}{2}} \\ &\leq (\text{const}) \frac{|k|^{d-1-r}}{2} \frac{\mathcal{C}^2}{|k|^r} \ . \end{aligned} \tag{21}$$

Collecting together (19),(20),(21), we obtain the lemma. □

4 The three-dimensional setting

This paper is mainly concerned with presenting an elementary proof of existence and uniqueness results in the two-dimensional setting. However, these techniques can also be used to gain some insight into the three-dimensional setting. On the three torus, the Navier-Stokes equations take the form

$$\begin{aligned} \frac{\partial u_i}{\partial t} + \sum_{j=1,2,3} u_j \frac{\partial u_i}{\partial x_j} &= \nu \Delta u_j - \frac{\partial p}{\partial x_i} + f_i \qquad i = 1,2,3 \\ \sum_{i=1,2,3} \frac{\partial u_i}{\partial x_i} &= 0 \end{aligned} \tag{22}$$

where $\nu > 0$ is again the viscosity, p is the pressure, and the f_i are the components of the external, time-dependent forcing. As before, we introduce the vorticity $\omega(x,t) = (\omega_1(x,t), \omega_2(x,t), \omega_3(x,t)) = (\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}, \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}, \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1})$. The vorticity obeys the equation

$$\frac{\partial \omega_i}{\partial t} + \sum_j u_j \frac{\partial \omega_i}{\partial x_j} = \sum_j \omega_j \frac{\partial u_i}{\partial x_j} + \nu \Delta \omega_i + g_i \qquad i = 1,2,3 \tag{23}$$

where the g_i are the components of curl$f$. Moving to Fourier space where

$$u(x,t) = \sum_{k\in\mathbb{Z}^3} u_k(t)e^{2\pi i(k,x)} \quad \text{and} \quad \omega(x,t) = \sum_{k\in\mathbb{Z}^3} \omega_k(t)e^{2\pi i(k,x)} \ ,$$

we obtain

$$\frac{d\omega_k(t)}{dt} \quad = \quad -2\pi i \sum_{l_1+l_2=k} \Big[\big(u_{l_1}(t), l_2\big)\omega_{l_2}(t) \ - \ \big(\omega_{l_1}(t), l_2\big)u_{l_2}(t)\Big] \ - \ 4\pi^2\nu|k|^2\omega_k(t) \ + \ g_k(t) \quad . \tag{24}$$

Here the $g_k(t)$ are the Fourier components of the forcing $g(x,t)$. In addition, we can replace the Laplacian with the more general differential operator $|\nabla|^\alpha$ with $\alpha > 1$.

The incompressibility condition implies that

$$u_k(t) \perp k \tag{25}$$

for all $k \in \mathbb{Z}^3$. Similarly, it follows that $\omega_k(t) \perp k$, $\omega_k(t) \perp u_k(t)$, and $|\omega_k(t)| = |k||u_k(t)|$. Hence, (k, u_k, ω_k) is a right-handed orthogonal (but not orthonormal) frame.

Since $(u_{l_1}(t), l_1) = (\omega_{l_1}(t), l_1) = 0$, we can rewrite (24) as

$$\frac{d\omega_k(t)}{dt} \quad = \quad -2\pi i \sum_{l_1+l_2=k} \Big[\big(u_{l_1}(t), k\big)\omega_{l_2}(t) \ - \ \big(\omega_{l_1}(t), k\big)u_{l_2}(t)\Big] \ - \ 4\pi^2\nu|k|^\alpha\omega_k(t) \ + \ g_k(t) \quad . \tag{26}$$

As before, we begin by restricting our attention to a finite subset $\mathcal{Z} \subset \mathbb{Z}^3$. The finite-dimensional Galerkin system corresponding to $\mathcal{Z}$ is

$$\frac{d\omega_k(t)}{dt} \quad = \quad -2\pi i \sum_{\substack{l_1+l_2=k \\ l_1,l_2\in\mathcal{Z}}} \Big[\big(u_{l_1}(t), k\big)\omega_{l_2}(t) \ - \ \big(\omega_{l_1}(t), k\big)u_{l_2}(t)\Big] \ - \ 4\pi^2\nu|k|^\alpha\omega_k(t) \ + \ g_k(t) \quad . \tag{$26_{\mathcal{Z}}^{\alpha}$}$$

Furthermore, to simplify the arguments, we assume that the forcing $g(x,t)$ is a trigonometric polynomial which implies that all but a finite number of the g_k are identically zero. We will always analyze wave numbers above the band which is directly forced; hence, we may neglect the g_k. This is only for convenience. The forcing can be included in the same way as it was in the two-dimensional setting.

Our development is based upon the basic energy estimate (see [CF88, DG95, Tem79]). It states that given any initial data such that $\sum_{k\in\mathbb{Z}^3} |u_k(0)|^2 = E_0 < \infty$ then there exists a constant E^* depending only on E_0, ν, $\sup_t |g(\cdot,t)|_{L^2}$ such that for any finite-dimensional Galerkin approximation, defined by $\mathcal{Z} \subset \mathbb{Z}^3$, we have $\sum_{k\in\mathcal{Z}} |u_k(t)|^2 < E^*$ for all $t > 0$.

When $\alpha = 2$, the system ($26_{\mathcal{Z}}^{\alpha}$) corresponds to the Navier-Stokes equations. Unfortunately, we are unable to prove the theorems in this setting analogous to theorems 1, 2, and 3 when $\alpha = 2$. However, if we increase α, we can.

Theorem 4. *Consider the system ($26_{\mathcal{Z}}^{\alpha}$) with an $\alpha > 2.5$ and satisfying assumption 2. If the initial data $\{\omega_k(0)\}$ are such that*

$$|\omega_k(0)| \leq \frac{\mathcal{D}_4}{|k|^r}$$

for all $k \in \mathbb{Z}^3$ with $r > 1.5$ then there exists a constant $\mathcal{D}_4'$, independent of $\mathcal{Z}$, so that

$$|\omega_k(t)| \leq \frac{\mathcal{D}_4'}{|k|^r}$$

for all $k \in \mathbb{Z}^3$ and $t \geq 0$.

Theorem 5. *Consider the system ($26^\alpha_{\mathcal{Z}}$) with an $\alpha > 2.5$ and satisfying assumption 3. If the initial data $\{\omega_k(0)\}$ are such that*

$$|\omega_k(0)| \leq \frac{\mathcal{D}_5}{|k|^r} e^{-\gamma_5 |k|}$$

for all $k \in \mathbb{Z}^3$ with $r > 2$ then there exists constants $\mathcal{D}'_5 < \infty$ and $\gamma'_5 > 0$, both independent of $\mathcal{Z}$, so that

$$|\omega_k(t)| \leq \frac{\mathcal{D}'_5}{|k|^r} e^{-\gamma'_5 |k|}$$

for all $k \in \mathbb{Z}^3$ and $t \geq 0$.

Theorem 6. *Consider the system ($26^\alpha_{\mathcal{Z}}$) with an $\alpha > 2.5$ and satisfying assumption 3. If the initial data $\{\omega_k(0)\}$ are such that*

$$|\omega_k(0)| \leq \frac{\mathcal{D}_6}{|k|^r}$$

for all $k \in \mathbb{Z}^3$ with $r > 2$ then for any $t_0 > 0$ there exists constants $\mathcal{D}'_6 < \infty$ and $\gamma'_6 > 0$, both independent of $\mathcal{Z}$, so that

$$|\omega_k(t_0)| \leq \frac{\mathcal{D}'_6}{|k|^r} e^{-\gamma'_6 |k|}$$

for all $k \in \mathbb{Z}^3$.

Of these three theorems, we will only give the proof of the first. The second two will be the consequence of two more general theorems given below. They apply to all $\alpha > 1.5$ but require the additional assumption that the enstrophy of all Galerkin approximations, starting from a given set of initial data, remains uniformly bounded in time. This is not known in general. However, when $\alpha > 2.5$, theorem 4 implies this bound. Hence, the two theorems below apply to ($26^\alpha_{\mathcal{Z}}$) when $\alpha > 2.5$ without any assumption on $\mathcal{E}(t)$. In light of theorem 4, theorem 7 and 8 respectively yield theorem 5 and 6 when $\alpha > 2.5$.

Theorem 7. *Let $\{u_k(t)\}$ be a solution to ($26^\alpha_{\mathcal{Z}}$) with $\alpha > 1.5$ such that $\sum_{\mathbb{Z}^3} |\omega_k(t)|^2 < \mathcal{E}^* < \infty$ for all $t > 0$. If $|\omega_k(0)| \leq \frac{\mathcal{D}_7}{|k|^r}$ for some $\mathcal{D}_7 < \infty$ and $r > 2$ then for any $t_1 > 0$ there exists a $\gamma_7 > 0$ and $\mathcal{D}'_7 < \infty$ such that*

$$|\omega_k(t_1)| \leq \frac{\mathcal{D}'_7}{|k|^r} e^{-\gamma_7 |k|} \ .$$

Theorem 8. *Let $\{u_k(t)\}$ be a solution to ($26^\alpha_{\mathcal{Z}}$) with $\alpha > 1.5$ such that $\sum_{\mathbb{Z}^3} |\omega_k(t)|^2 < \mathcal{E}^* < \infty$ for all $t > 0$. If for some $\mathcal{D}_8 < \infty$, $\gamma_8 > 0$, and $r > 2$, $|\omega_k(0)| \leq \frac{\mathcal{D}_8}{|k|^r} e^{-\gamma_8 |k|}$ then there exists a $\gamma'_8 > 0$ and $\mathcal{D}'_8 < \infty$ such that for all $t > 0$*

$$|\omega_k(t)| \leq \frac{\mathcal{D}'_8}{|k|^r} e^{-\gamma'_8 |k|} \ .$$

The above two theorems apply to ($26^\alpha_{\mathcal{Z}}$) for $\alpha > 1.5$. In particular, this means that they cover the standard Navier-Stokes equation which corresponds to $\alpha = 2$. (One can lower the restriction on α to $\alpha > 1$ at the cost of raising the restriction on r to $r > 3$. Similarly, one lowers the restriction on r to $r > 1.5$ at the cost of making $\alpha > 2.5$.)

In proving these two theorems, it was necessary to assume that $\sum_{\mathbb{Z}^3} |\omega_k(t)|^2$ remained uniformly bounded in time. Without such an assumption, we are forced to consider only α which do not correspond to the Navier-Stokes equation. Notice that theorem 4 implies that $\sum_{k \in \mathbb{Z}^3} |\omega_k(t)|^2 < \text{const} < \infty$ for all $t > 0$ and hence theorems 7 and 8 apply showing that the solution is analytic after $t = 0$.

In proving the above results, it is again convenient to split the system ($26^{\alpha}_{\mathcal{Z}}$) into the equations for the real and imaginary parts of $\{u_k\}_k$ and $\{\omega_k\}_k$. Letting $u_k(t) = u_k^{(1)}(t) + iu_k^{(2)}(t)$, $\omega_k(t) = \omega_k^{(1)}(t) + i\omega_k^{(2)}(t)$, and $g_k(t) = g_k^{(1)}(t) + ig_k^{(2)}(t)$; we obtain

$$\begin{aligned}\frac{d\omega_k^{(1)}(t)}{dt} =& 2\pi \sum_{\substack{l_1+l_2=k\\ l_1,l_2\in\mathcal{Z}}} \Big[\big(u_{l_1}^{(1)}(t),k\big)\omega_{l_2}^{(2)}(t) + \big(u_{l_1}^{(2)}(t),k\big)\omega_{l_2}^{(1)}(t) - \big(\omega_{l_1}^{(2)}(t),k\big)u_{l_2}^{(1)}(t) - \big(\omega_{l_1}^{(1)}(t),k\big)u_{l_2}^{(2)}(t)\Big]\\ & - 4\pi^2\nu|k|^{\alpha}\omega_k^{(1)}(t) + g_k^{(1)}(t) \qquad (26^{\alpha}_{\mathcal{Z}}{}^{(1)})\\ \frac{d\omega_k^{(2)}(t)}{dt} =& -2\pi \sum_{\substack{l_1+l_2=k\\ l_1,l_2\in\mathcal{Z}}} \Big[\big(u_{l_1}^{(1)}(t),k\big)\omega_{l_2}^{(1)}(t) - \big(u_{l_1}^{(2)}(t),k\big)\omega_{l_2}^{(2)}(t) - \big(\omega_{l_1}^{(1)}(t),k\big)u_{l_2}^{(1)}(t) + \big(\omega_{l_1}^{(2)}(t),k\big)u_{l_2}^{(2)}(t)\Big]\\ & + g_k^{(2)}(t) - 4\pi^2\nu|k|^{\alpha}\omega_k^{(2)}(t) \qquad (26^{\alpha}_{\mathcal{Z}}{}^{(2)})\end{aligned}$$

Proof of Theorem 4. By the energy estimate, we know that $|u_k^{(j)}(t)| \le \sqrt{E^*}$ for all $t \ge 0$ and $j = 1, 2$. Hence, $|\omega_k^{(j)}(t)| \le |k|\sqrt{E^*}$. Fixing a K_0, set $\mathcal{D}'_4(K_0) = K_0\mathcal{D}_4$. With this choice, $|\omega_k^{(j)}(t)| \le \mathcal{D}'_4(K_0)$ for all $t \ge 0$, $j = 1, 2$, and $k \in \mathbb{Z}^3$ with $|k| \le K_0$. As before, consider the set

$$\Omega_4(K_0) = \left\{(\omega_k^{(1)}, \omega_k^{(2)})_{k\in\mathbb{Z}^3} : |\omega_k^{(1)}| \le \frac{\mathcal{D}'_4(K_0)}{|k|^r}, |\omega_k^{(2)}| \le \frac{\mathcal{D}'_4(K_0)}{|k|^r} \text{ for all } k,\ |k| > K_0\right\} .$$

We have to show that if K_0 is taken to be sufficiently large, the vector field points inward along $\partial\Omega_4$.

We pick a point on the boundary. For definiteness, we will again consider the case when $\omega_{\bar{k}}^{(1)} = \frac{\mathcal{D}'_4}{|k|^r}$ and $\omega_{\bar{k}}^{(2)} \le \frac{\mathcal{D}'_4}{|\bar{k}|^r}$ for some $\bar{k}$ with $|\bar{k}| \ge K_0$ and $\omega_k^{(j)} \le \frac{\mathcal{D}'_4}{|k|^r}$ for all other k with $k \ne \bar{k}$. The theorem will be proven if we can show that there exists a K_{crit}, independent of $\mathcal{D}'_4$, so that if $|\bar{k}| \ge K_0 > K_{crit}$ then

$$\left|2\pi \sum_{\substack{l_1+l_2=\bar{k}\\ l_1,l_2\in\mathcal{Z}}} \Big[\big(u_{l_1}^{(1)}(t),\bar{k}\big)\omega_{l_2}^{(2)}(t) + \big(u_{l_1}^{(2)}(t),\bar{k}\big)\omega_{l_2}^{(1)}(t) - \big(\omega_{l_1}^{(2)}(t),\bar{k}\big)u_{l_2}^{(1)}(t) - \big(\omega_{l_1}^{(1)}(t),\bar{k}\big)u_{l_2}^{(2)}(t)\Big]\right| < 4\pi^2\nu|\bar{k}|^{\alpha}\omega_{\bar{k}}^{(1)}(t) .$$

Other boundary points have the same structure so we will only show the details of the calculation for this case.

We need to estimate the summation. The total sum is made of smaller sums which are dominated by sums of the form $\sum_{l_1+l_2=\bar{k}} |u_{l_1}^{(a)}||\omega_{l_2}^{(b)}||k|$ with $a, b \in \{1, 2\}$. As before, we split this sum into three parts:

$$\begin{aligned}\Sigma_1 &= \sum_{|l_1|\le|\frac{\bar{k}}{2}|} |u_{l_1}^{(a)}||\omega_{l_2}^{(b)}||k|\\ \Sigma_2 &= \sum_{|\frac{\bar{k}}{2}|<|l_1|\le 2|\bar{k}|} |u_{l_1}^{(a)}||\omega_{l_2}^{(b)}||k|\\ \Sigma_3 &= \sum_{2|\bar{k}|<|l_1|} |u_{l_1}^{(a)}||\omega_{l_2}^{(b)}||k|\end{aligned}$$

In Σ_1, $|l_2| \ge |\frac{\bar{k}}{2}|$ and hence

$$\begin{aligned}\Sigma_1 &\le \frac{\mathcal{D}'_4}{|\bar{k}|^r} 2^r|\bar{k}| \left(\sum_{|l_1|\le|\frac{\bar{k}}{2}|} |u_{l_1}^{(a)}|^2\right)^{\frac{1}{2}} \left(\sum_{|l_1|\le|\frac{\bar{k}}{2}|} 1\right)^{\frac{1}{2}}\\ &\le \frac{\mathcal{D}'_4}{|\bar{k}|^r} 2^r(\text{const})\sqrt{E^*}|\bar{k}|^{\frac{5}{2}}\end{aligned}$$

The constant is defined by

$$\Big(\sum_{|l_1|\le|\frac{\bar k}{2}|} 1\Big) \le (\mathrm{const})^2|\bar k|^3 \ .$$

For Σ_2, we know that $|l_2|\le 3|\bar k|$ and $|u^{(a)}_{l_1}|\le \frac{\mathcal{D}_4'}{|l_1|^{r+1}}$ which gives

$$\begin{aligned}\Sigma_2 &\le \frac{\mathcal{D}_4'}{|\bar k|^r}2^r\frac{2}{|k|}|k|\sum_{|l_2|\le 3|\bar k|}|\omega^{(b)}_{l_2}| \le \frac{\mathcal{D}_4'}{|\bar k|^r}2^{r+1}\sum_{|l_2|\le 3|\bar k|}|l_2||u^{(a)}_{l_2}| \\ &\le \frac{\mathcal{D}_4'}{|\bar k|^r}2^{r+1}3|\bar k|\Big(\sum_{|l_2|\le 3|\bar k|}|u^{(a)}_{l_1}|^2\Big)^{\frac12}\Big(\sum_{|l_2|\le 3|\bar k|}1\Big)^{\frac12} \\ &\le \frac{\mathcal{D}_4'}{|\bar k|^r}2^{r+1}3(\mathrm{const})\sqrt{E^*}|\bar k|^{\frac52} \ .\end{aligned}$$

Here the constant is the analogue of the constant in the estimation of Σ_1. For Σ_3, we know that $|l_2|\ge|\bar k|$ and thus

$$\begin{aligned}\Sigma_3 &\le |\bar k|\Big(\sum_{|l_1|\ge 2|\bar k|}|u_{l_1}|^2\Big)^{\frac12}\Big(\sum_{|l_2|\ge|\bar k|}|\omega_{l_2}|^2\Big)^{\frac12} \le |\bar k|\mathcal{D}_4'\sqrt{E^*}\Big(\sum_{|l_2|\ge|\bar k|}\frac{1}{|l_2|^{2r}}\Big)^{\frac12} \\ &\le |\bar k|\mathcal{D}_4'\sqrt{E^*}\frac{(\mathrm{const})}{|\bar k|^{r-\frac32}} \le \frac{\mathcal{D}_4'}{|\bar k|^r}(\mathrm{const})\sqrt{E^*}|\bar k|^{\frac52} \ .\end{aligned}$$

Collecting the three estimates together we see that there is a constant, depending only on r, so that

$$\sum_{l_1+l_2=\bar k}|u^{(a)}_{l_1}||\omega^{(b)}_{l_2}||k| \le (\mathrm{const})\frac{\mathcal{D}_4'}{|\bar k|^r}\sqrt{E^*}|\bar k|^{\frac52} \ . \tag{27}$$

Using this estimate, we see that the condition in (4) will hold if

$$8\pi(\mathrm{const})\sqrt{E^*}|\bar k|^{\frac52}\frac{\mathcal{D}_4'}{|\bar k|^r} < 4\pi^2\nu|\bar k|^\alpha\frac{\mathcal{D}_8'}{|\bar k|^r} \ .$$

Since $\alpha>\frac52$, this will hold for all $\bar k$ sufficiently large; this sets the level of K_{crit}. Notice that it does not depend on $\mathcal{D}_8'$ as was required. □

Proof of Theorem 7. The proof of this theorem is similar to the proof of theorem 3. From the assumptions, we know that $|\omega_k(t)|\le\sqrt{\sum_{\mathbb{Z}^3}|\omega_l(t)|^2}<\sqrt{\mathcal{E}^*}$ for all $t>0$. We set $a^{(j)}_k(t)=u^{(j)}_k e^{\gamma_7 t|k|}$ and $b^{(j)}_k(t)=\omega^{(j)}_k e^{\gamma_7 t|k|}$ for $j=1,2$, where γ_7 is a constant we will set later.

Set $\mathcal{D}_7'=2\max(\sqrt{\mathcal{E}^*},\mathcal{D}_7)$. Fixing a K_0, choose $\gamma_7(K_0)$ so that for all $t\in[0,t_1]$, $j\in\{1,2\}$, and k with $|k|\le K_0$, one has $|b^{(j)}_k(t)|\le\frac{\mathcal{D}_7'}{|k|^r}$. Notice that by the assumption on the initial conditions, we have $|b^{(j)}_k(0)|\le\frac{\mathcal{D}_7'}{|k|^r}$ for all k. Consider the set,

$$\Omega_7(K_0)=\Big\{\Big(b^{(1)}_k,b^{(2)}_k\Big)_{k\in\mathbb{Z}^2\setminus 0} \text{ with } |b^{(j)}_k|\le\frac{\mathcal{D}_7'}{|k|^r} \text{ for } j=1,2 \text{ and } |k|>K_0\Big\} \ .$$

As before we will show that, if K_0 is chosen large enough, a point starting in Ω_7 cannot leave Ω_7 because the vector field along $\partial\Omega_7$ is pointing inward.

We pick a boundary point. For simplicity, we pick the point where $b^{(1)}_{\bar k}=\frac{\mathcal{D}_7'}{|k|^r}$ and all other variables satisfy the inequalities defining Ω_7. In terms of the new variables, the relevant equation of motion reads

$$\begin{aligned}\frac{db^{(1)}_k(t)}{dt} &= (\gamma_7|k|-4\pi^2\nu|k|^\alpha)b^{(1)}_k(t) - 2\pi\sum_{\substack{l_1+l_2=k\\ l_1,l_2\in\mathcal{Z}}}\Big[\big(a^{(1)}_{l_1}(t),k\big)b^{(2)}_{l_2}(t) \\ &\quad + \big(a^{(2)}_{l_1}(t),k\big)b^{(1)}_{l_2}(t) - \big(b^{(2)}_{l_1}(t),k\big)a^{(1)}_{l_2}(t) - \big(b^{(1)}_{l_1}(t),k\big)a^{(2)}_{l_2}(t)\Big]\frac{e^{-\gamma_7 t|l_1|}e^{-\gamma_7 t|l_2|}}{e^{-\gamma_7 t|k|}} \ .\end{aligned}$$

Since $|a_k^{(j)}(t)| = \frac{|b_k^{(j)}(t)|}{|k|}$, to insure that the vector field points inward it is sufficient to show that

$$2\pi \sum |b_{l_1}^{(1)}||b_{l_2}^{(2)}|\frac{|\bar{k}|}{|l_1|} + |b_{l_1}^{(2)}||b_{l_2}^{(1)}|\frac{|\bar{k}|}{|l_1|} + |b_{l_1}^{(2)}||b_{l_2}^{(1)}|\frac{|\bar{k}|}{|l_2|} + |b_{l_1}^{(1)}||b_{l_2}^{(2)}|\frac{|\bar{k}|}{|l_2|} < (4\pi^2\nu|\bar{k}|^\alpha - \gamma_7|\bar{k}|)\frac{\mathcal{D}'_7}{|\bar{k}|^r} .$$

Each of the terms in the above sum can be estimated with the aid of lemma 1. This transforms the previous condition into

$$8\pi(\text{const})\left(2^r|\bar{k}| + 2^{r+1}(6|\bar{k}|+1)^{\frac{3}{2}} + \frac{1}{2}|\bar{k}|^{2-r}\right)\frac{(\mathcal{D}'_7)^2}{|\bar{k}|^r} < (4\pi^2\nu|\bar{k}|^\alpha - \gamma_7|\bar{k}|)\frac{\mathcal{D}'_7}{|\bar{k}|^r} .$$

By picking K_0 large enough, we can force this condition to hold. The fact that γ_7 depends on K_0 is not a problem since it decreases as K_0 increases.

This establishes that the vector field points inward along the boundary of Ω_7 for all times in the interval $[0, t_1]$. Thus at time t_1, the trajectory is still in Ω_7. By returning to the original variables, we have the desired estimate at time t_1. □

Proof of Theorem 8. The proof of this theorem begins as the above theorem and then proceeds as the proof of theorem 2. We change variables to $a_k^{(j)}(t) = u_k^{(j)}e^{\gamma_8|k|}$ and $b_k^{(j)}(t) = \omega_k^{(j)}e^{\gamma_8|k|}$. We use the assumption on $\sqrt{\sum_{\mathbb{Z}^3}|\omega_l(t)|^2}$ to control the lower modes. Then we use the estimates from lemma 1 to control the nonlinearity. We omit the details. □

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References

[CF88] Peter Constantin and Ciprian Foiaş. *Navier-Stokes Equations*. University of Chicago Press, Chicago, 1988.

[CM93] Alexandre J. Chorin and Jerrold E. Marsden. *A mathematical introduction to fluid mechanics*, volume 4 of *Texts in Applied Mathematics*. Springer-Verlag, New York, third edition, 1993.

[DG95] Charles R. Doering and J. D. Gibbon. *Applied analysis of the Navier-Stokes equations*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1995.

[DT95] Charles R. Doering and Edriss S. Titi. Exponential decay rate of the power spectrum for solutions of the Navier-Stokes equations. *Phys. Fluids*, 7(6):1384–1390, 1995.

[FT89] C. Foiaş and R. Temam. Gevrey class regularity for the solutions of the Navier-Stokes equations. *J. Funct. Anal.*, 87(2):359–369, 1989.

[Gal96] G.Gallavotti. *Ipotesi per una introduzione alla Meccanica Dei Fluidi* Gruppo Nazionale Di Fisica Matematica, 1996.

[Hop51] Eberhard Hopf. Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Math. Nachr.*, 4:213–231, 1951.

[Kre88] Heinz-Otto Kreiss. Fourier expansions of the solutions of the Navier-Stokes equations and their exponential decay rate. In *Analyse mathématique et applications*, pages 245–262. Gauthier-Villars, Paris, 1988.

[Lad69] O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow.* Gordon and Breach, New York, 1969.

[Ler34] J. Leray. Essai sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63:193–248, 1934.

[Tem79] Roger Temam. *Navier-Stokes equations:Theory and numerical analysis*, volume 2 of *Studies in Mathematics and its Applications.* North-Holland Publishing Co., Amsterdam-New York, revised edition, 1979.

[Tem95] Roger Temam. *Navier-Stokes equations and nonlinear functional analysis*, volume 66 of *CBMS-NSF Regional Conference Series in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 1995.

[Yud89] V. I. Yudovich. *The linearization method in hydrodynamical stability theory.* American Mathematical Society, Providence, RI, 1989. Translated from the Russian by J. R. Schulenberger.