Uniformly Area Expanding Flows in Spacetimes

by

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Lenhard L. Ng

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2014

ABSTRACT

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Abstract

The central object of study of this thesis is inverse mean curvature vector flow of two-dimensional surfaces in four-dimensional spacetimes. Being a system of forward-backward parabolic PDEs, inverse mean curvature vector flow equation lacks a general existence theory. Our main contribution is proving that there exist infinitely many spacetimes, not necessarily spherically symmetric or static, that admit smooth global solutions to inverse mean curvature vector flow. Prior to our work, such solutions were only known in spherically symmetric and static spacetimes. The technique used in this thesis might be important to prove the Spacetime Penrose Conjecture, which remains open today.

Given a spacetime $(N^4, \overline{g})$ and a spacelike hypersurface M. For any closed surface Σ embedded in M satisfying some natural conditions, one can "steer" the spacetime metric $\overline{g}$ such that the mean curvature vector field of Σ becomes tangential to M while keeping the induced metric on M. This can be used to construct more examples of smooth solutions to inverse mean curvature vector flow from smooth solutions to inverse mean curvature flow in a spacelike hypersurface.

Dedicated to my parents: Sihong and Juhua.

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1

Introduction

1.1 Motivation: Mass in General Relativity

General relativity is the study of large scale structures of the universe. One fundamental object in general relativity is the notion of mass. Pointwise energy density and total mass of a spacetime are both well-defined in general relativity. However, the local mass of a given region in a spacetime (called quasi-local mass), as well as the relationship between local mass and pointwise energy density and total mass of the spacetime are still not very well understood.

Despite of many attempts in defining the quasi-local mass (e.g. [2, 3, 5, 15, 16, 24, 47]), none of the proposed functionals satisfy all the desired properties. One such natural property is that the total mass of the spacetime should be bounded from below by the mass of a region in it, assuming some positivity condition on the pointwise energy density (e.g. dominant energy condition).

Given a spacetime $(N^4, \overline{g})$ and a complete asymptotically flat spacelike hypersurface M^3 (also called a *slice*) with the induced Riemannian metric g. Let k be the second fundamental form of M. The triple (M^3, g, k) is called a *Cauchy data* of this hypersurface (see Figure 1.1 below).

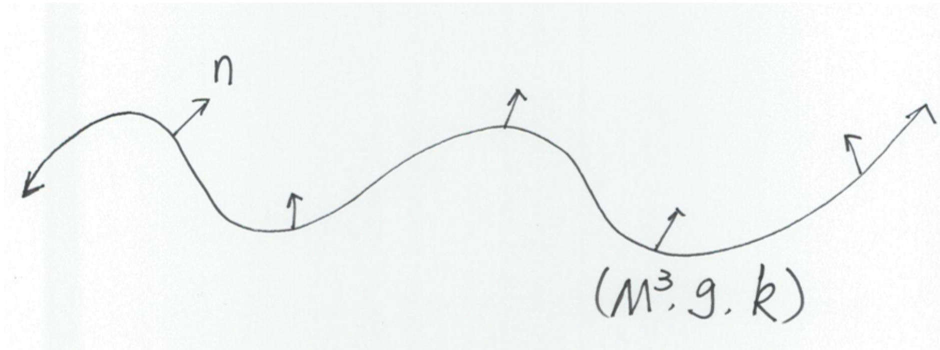

FIGURE 1.1: Slice in a spacetime with Cauchy data.

There is a well-defined quantity called the *ADM mass* (defined by R. Arnowitt, S. Deser and C. Misner in [1]) that measures the total mass of this hypersurface. Suppose M has a compact outermost minimizing surface Σ. Physically, Σ can be viewed as the apparent horizon of blackholes.

In the case that M is *totally geodesic*, i.e. $k = 0$, then the pointwise energy density equals the scalar curvature of M. In this case, the Riemannian Penrose Inequality states that:

Theorem 1.1 (Riemannian Penrose Inequality)**.** *Let M and Σ be given as above. If the scalar curvature of (M, g) is non-negative, then its ADM mass is greater than or equal to $\sqrt{|\Sigma|/16\pi}$, where $|\Sigma|$ is the total area of Σ (see Figure 1.2).*

Penrose [37] first conjectured this inequality in 1973, and he gave a heuristic proof based on physical considerations, explained as follows. It turns out that the lower bound $\sqrt{|\Sigma|/16\pi}$ in the Riemannian Penrose Inequality equals the *Hawking mass*, which is a quasi-local mass functional proposed by Hawking [24], of the minimal surface Σ. This can be viewed as the mass of the blackholes inside Σ. Thus, the Riemannian Penrose Inequality states that the total mass of M should be at least the mass contributed by the blackholes, assuming that the energy density (which equals

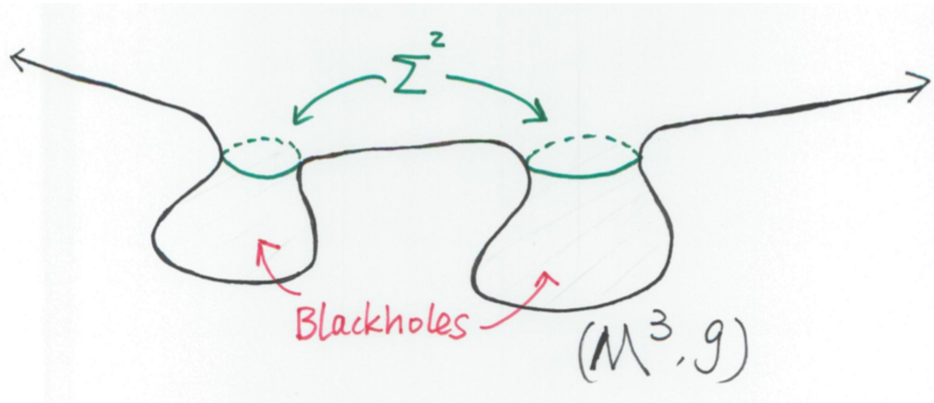

FIGURE 1.2: Totally geodesic spacelike hypersurface with compact outermost minimal surface Σ.

the scalar curvature in the case of a totally geodesic hypersurface) is non-negative everywhere.

Geroch [22], Jang and Wald [26] first discovered a monotone property of Hawking mass of surfaces under smooth *inverse mean curvature flow*. Based on this, Huisken and Ilmanen [25] gave a proof of this inequality in the case of a single blackhole (i.e. Σ is connected). In the same year, Bray [5] proved the full Riemannian Penrose Inequality using a different technique.

In the case of no blackholes, the Riemannian Penrose Inequality is also known as the Riemannian Positive Mass Theorem:

Theorem 1.2 (Riemannian Positive Mass Theorem)**.** *Given a complete asymptotically flat Riemannian manifold* (M^3, g) *with non-negative scalar curvature. The ADM mass of* M *is non-negative.*

In 1979, Schoen and Yau [42] proved this result using a variational method. In the same year, they [41] generalized this result to Riemannian manifolds of dimension less than eight. In 1981, they [43] removed the assumption that M is totally geodesic and proved the Riemannian Positive Mass Theorem for an arbitrary spacelike hypersurface in a spacetime that satisfies the dominant energy condition.

1.2 Inverse Mean Curvature Vector Flow

So far all the discussions assume that our spacetime has a hypersurface with zero second fundamental form, in which case inverse mean curvature flow naturally bridges the Hawking mass of an apparent horizon of blackholes and the ADM mass of the hypersurface. However, a spacetime in general does not admit such a totally geodesic hypersurface. This is because that the second fundamental form of a hypersurface in a spacetime has six components, but the hypersurface only has one degree of freedom. Thus it is not generic to have all six components vanish. Therefore it is desirable to obtain a similar bound on the total mass of the spacetime by the mass of the blackholes without this assumption. This leads to the general Spacetime Penrose Conjecture, which is still open today.

A viable candidate for proving this conjecture is the codimension-two analogue of inverse mean curvature flow, called the *inverse mean curvature vector flow*. However, there are two major problems with this flow. First, unlike the inverse mean curvature flow, which is a forward parabolic PDE, inverse mean curvature vector flow is a system of forward-backward parabolic PDEs: forward parabolic in spacelike directions, and backward parabolic in timelike directions (see [25]). Backward parabolic equations lack a general existence theory. For instance, the reverse heat flow is backward-parabolic. Given many initial conditions, the reverse heat flow would develop singularities instantaneously. However, the reverse heat flow would exist for time $t > 0$ if we first perform the heat flow for time t and then start flowing backwards.

Second, for some initial surfaces, even the inverse mean curvature vector flow exist, their Hawking mass still won't give us a lower bound on the total mass of the spacetime as in the inverse mean curvature flow case simply because the former is too large. To illustrate this, take a t = constant slice in the Minkowski spacetime.

The round sphere in that slice has zero Hawking mass. Spacial perturbations will decrease the Hawking mass making it negative, whereas timelike perturbations will increase the Hawking mass making it positive. During inverse mean curvature vector flow, the spacial "wiggles" will smooth out due to the parabolic nature of the flow. However, timelike "wiggles" will get amplified since the flow is reverse parabolic in the timelike directions. With these surfaces with positive Hawking mass, inverse mean curvature vector flow will not provide a lower bound for the ADM mass of Minkowski space, which is zero.

However, these two problems seem to solve each other because they are both suggesting that solutions to inverse mean curvature vector flow exist only when the "right" initial surface is given. The important question is then: Given a spacetime. Do such "right" initial surfaces always exist? The answer is affirmative if the spacetime is *spherically symmetric* or *static*.

Inverse mean curvature vector flow of surfaces in spherically symmetric spacetimes was first studied by E. Malec, and N. ÓMurchadha [31]. They showed that inverse mean curvature vector flow of spherically symmetric spheres exist for all time. Intuitively, the spherical symmetries prevent timelike "wiggles" to occur. Moreover, the Hawking mass is monotonically non-decreasing under inverse mean curvature vector flow of spacelike surfaces with spacelike mean curvature vectors, assuming the spacetime satisfies the dominant energy condition.

Later Frauendiener [21] showed that, in an arbitrary spacetime that satisfies the dominant energy condition, if smooth inverse mean curvature vector flow exists, then the Hawking mass is monotonically non-decreasing. In 2004, H. Bray, S. Hayward, M. Mars and W. Simom [8] showed that we can in fact flow along a one-parameter family of directions and the Hawking mass is still monotone.

In spherically symmetric spacetimes, the "right" initial surfaces for inverse mean curvature vector flow are spherically symmetric spheres. What about spacetimes that are not necessarily spherically symmetric?

Bray and Ye Li were trying to develop a general existence theory for inverse mean curvature vector flow back in 2009, and one of their intuitions was that if one can somehow control the flow of the surfaces so that they stays tangential to a spacelike slice, then the flow might not develop singularities. In fact it has been shown that:

Proposition 1.3 ([8])**.** *The family of closed embedded spacelike surfaces* $\{\Sigma_s\}$ *is a solution to the smooth inverse mean curvature vector flow with spacelike inverse mean curvature vector* $\vec{I}_{\Sigma_s}$ *everywhere on the surfaces if and only if there exists a spacelike hypersurface* $M^3 \subset N$*, such that the mean curvature vector* $\vec{H}_{\Sigma_s}$ *is tangential to* M *at all* (x, s)*, and* $\{\Sigma_s\}$ *is a solution to the smooth inverse mean curvature flow in* M*.*

Following the intuition, we prove the main theorem in Chapter 4:

Theorem 1.4 (**Main Theorem**)**.** *There exist infinitely many non-spherically symmetric, non-static spacetimes that admit inverse mean curvature vector flow coordinate charts. Given such a spacetime* U *with an inverse mean curvature vector flow coordinate chart* (t, r, θ, ϕ) *and the constructed spacetime metric* $\overline{g}$*. The coordinate spheres* $S_{t,r}$ *contained in each* $t =$ *constant slice, when reparameterized by* $r^2 = e^s$*, are smooth global solutions to the inverse mean curvature vector flow equation.*

This theorem is restated and proved in Theorem 4.4. The proof is based on explicit constructions of spacetime metrics that admit inverse mean curvature vector flow coordinate charts, defined in Chapter 4. Theorem 1.4 seems to suggest that spacetimes that admit smooth solutions to inverse mean curvature vector flow exist generically. However, this general problem of find solutions to inverse mean curvature vector flow (i.e. the "right" initial surface) in arbitrary spacetimes is still open.

There also exists a coordinate-free analogue of Theorem 1.4:

Theorem 1.5. *Given a spacetime* $(N^4, \overline{g})$*, a spacelike hypersurface* M^3 *and a closed embedded surface* $(\Sigma^2, g_\Sigma) \subset M$*. Suppose* Σ *is area expanding (defined in (4.3.5)), then there exists a unique smooth steering parameter* $Q = Q_\Sigma \in C^\infty(N)$*, such that in the steered spacetime metric* $\overline{g}_Q$ *(defined in (4.3.1)),* $\vec{H}_\Sigma$ *is tangential to* M *everywhere on* Σ*.*

This can be used to generate more examples of solutions to inverse mean curvature vector flow. Consider a smooth solution to inverse mean curvature flow in a spacelike hypersurface M. One can then smoothly adjust the spacetime metric along the flow such that the mean curvature vector of each flow surface becomes tangential to M. These steered surfaces are then solutions to inverse mean curvature vector flow, since the area expanding condition is already satisfied, and now the mean curvature vectors are tangential to a spacelike hypersurface (see Proposition 1.3).

1.3 Uniformly Area Expanding Straight Out Flows and Time Flat Surfaces

Inverse mean curvature vector flow is a type of flow that has bad existence theory, but very good properties: the Hawking mass is monotone under smooth inverse mean curvature vector flow. There is another flow studied in Chapter 5, called *uniformly area expanding straight out flow* (or simply *straight out flow*), that has solutions with a wide class of initial surfaces. In that chapter, we try to construct spacetimes that admit a coordinate chart in which straight out flow of coordinate spheres exists for all time. Partial results have been obtained while complete understanding of this problem is still work in progress:

Proposition 1.6. *Suppose a spacetime* $(N^4, \overline{g})$ *admits a coordinate chart* $\{t, r, \theta, \phi\}$

such that the coordinate representation of $\overline{g}$ *is*

$$\overline{g} = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} -v^2 & d & e & f \\ d & u^2 & 0 & 0 \\ e & 0 & a & c \\ f & 0 & c & b \end{pmatrix} \end{array} \tag{1.3.1}$$

with $ab - c^2 = r^4 \sin^2\theta$*. Then* $e_r := \frac{1}{u}\frac{\partial}{\partial r}$ *is straight out if and only if* d *satisfies a second order elliptic PDE in* d*:* $\Delta_{g_S} d + G(d, d') = 0$*, where* G *is given by (5.2.30) and (5.2.31).*

We conjecture that solutions to the above elliptic PDE always exist.

In addition to general existence, another reason for studying the straight out flow is that the Hawking mass is also monotonically non-decreasing under such flow (e.g. see [9], [10]) if the spacetime also satisfies the dominant energy condition.

The general existence of straight out flows can serve as a disadvantage since we can even flow surfaces with positive Hawking mass, too large to be used as a lower bound of the total mass of some spacetimes, in straight out directions. To see this, again consider the Minkowski spacetime. All surfaces that are contained in a spacelike plane have non-positive Hawking mass. Thus, non-planer surfaces have positive Hawking mass. For such surfaces, inverse mean curvature vector flow would not work since there are time "wiggles". However, those surfaces can still flow in straight out directions. Since the total mass of Minkowski space is zero, having a surface with positive Hawking mass is not going to give us a lower bound for the total mass since the Hawking mass is monotone.

Inverse mean curvature vector flow and uniformly area expanding straight out flow are two special cases of *uniformly area expanding flows*: orthogonal flows such that the rate of change of the area form of each flow surface equals the area form itself. The Hawking mass is not necessarily monotone under general uniformly area expanding flows. H. Bray, J. Jauregui and M. Mars very recently ([9], [10]) obtained a variational formula of the Hawking mass under general uniformly area expanding flows, which consists of four major terms (see [10]). The first three terms are non-negative if the spacetime satisfies the dominant energy condition. The fourth term is an integral term with integrant a function of the spacetime multiplied by the divergence of the connection one-form associated with their mean curvature vector of the flow surfaces. Thus, if the connection one-form is divergence free, then the fourth term vanishes and the Hawking mass is monotone. Surfaces with divergence free connection one-form associated with the mean curvature vectors are called *time-flat* (defined in [9], [10]). While the conditions on inverse mean curvature vector flow coordinate chart can be viewed as a global "flatness" condition on the surface, the time-flat condition is a local "flatness" condition.

The organization of this thesis is given as follows. In Chapter 2, we study the monotonicity of Hawking mass under smooth inverse mean curvature flow. In Chapter 3, we study inverse mean curvature vector flow in spherically symmetric spacetimes. The notations used in this thesis are also introduced in that chapter. In Chapter 4, we prove the main theorems 1.4 and 1.5. In Chapter 5, we study uniformly area expanding straight out flows, and prove Proposition 1.6. Finally in Chapter 6, some open problems and future works are discussed.

2

Huisken-Ilmanen Inverse Mean Curvature Flow and Monotonicity of Hawking Mass

In this chapter we study *inverse mean curvature flow* of a closed embedded surface Σ^2 in an *asymptotically flat* Riemannian manifold (M^3, g). The motivation is the Riemannian Penrose Inequality:

Theorem 2.1 (Riemannian Penrose Inequality)**.** *Let (M^3, g) be a complete, asymptotically flat Riemannian manifold with non-negative scalar curvature and a compact outermost minimal surface Σ of total area $|\Sigma|$, then*

$$m_{ADM} \geqslant \sqrt{\frac{|\Sigma|}{16\pi}} \tag{2.0.1}$$

with equality if and only if (M^3, g) is isometric to the Schwarzschild metric with mass $m > 0$:

$$\left(\mathbb{R}^3 \backslash \{0\}, \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}\right) \tag{2.0.2}$$

outside their respective outermost minimal surfaces.

Σ here can be viewed at the apparent horizon of blackholes. The lower bound for the ADM mass, $\sqrt{\frac{|\Sigma|}{16\pi}}$ has the physical interpretation as the mass of the blackholes. Penrose [37] first conjectured the Riemannian Penrose Inequality in 1973, and he gave a heuristic proof based on the physical considerations. In 2001, Huisken and Ilmanen [25] proved this inequality using inverse mean curvature flow in the case of a single blackhole. In the same year, H. Bray [5] proved this inequality using conformal flow of metrics that works for any number of blackholes. In 2009, H. Bray and D. Lee [13] generalized the inequality to all dimensions less than eight. In 2010, Lam [28] proved the Riemannian Penrose Inequality for graphs in all dimensions. In 2011, Schwartz [45] proved a volumetric version of the Penrose inequality for conformally flat manifolds. The general Spacetime Penrose Conjecture is still open today (see [33, 11, 12, 33] for more discussions of this conjecture).

In Section 2.1, we define asymptotically flat manifolds, ADM mass and Hawking mass of closed surfaces. In Section 2.2, we study the Geroch, Jang-Wald monotonicity formula of Hawking mass under smooth inverse mean curvature flow. In Section 2.3, we briefly discuss Huisken and Ilmanen's proof of the Riemannian Penrose Inequality using such flows.

2.1 Asymptotically Flat Manifolds, ADM Mass and Hawking Mass

Definition 2.1.1. *An n-dimensional Riemannian manifold (M^n, g) is called* asymptotically flat *if it satisfies the following two conditions:*

(1) There exists a compact set $K \subset M$ and a diffeomorphism

$$\Phi : E := M \backslash K \longrightarrow \mathbb{R}^n \backslash \bar{B}_1,$$

where B_1 is the unit open ball in $\mathbb{R}^n$; and

(2) In the coordinate chart $(x^1, x^2, \cdots, x^n)$ on E induced by the above diffeomorphism Φ, called an asymptotically flat coordinate chart, the metric components g_{ij} and the scalar curvature R satisfy the following decay conditions at any point $x \in E$, $i, j, k, l = 1, 2, \cdots, n$:

(1) $g_{ij}(x) = \delta_{ij}(x) + O(|x|^{-p})$;

(2) $|x||g_{ij,k}(x)| + |x|^2|g_{ij,kl}(x)| = O(|x|^{-p})$;

(3) $|R(x)| = O(|x|^{-q})$,

with some constants $p > \frac{n-2}{2}$ and $q > n$. Here $g_{ij,k}$ and $g_{ij,kl}$ are coordinate derivatives.

E is called an asymptotically flat end *of M. An asymptotically flat manifold can have multiple asymptotically flat ends.*

Definition 2.1.2. *Given an asymptotically flat Riemannian manifold (M^n, g) and asymptotically flat coordinate chart. The ADM mass of of M is:*

$$m_{ADM}(M,g) := \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\nu^j dS_r \tag{2.1.1}$$

where ω_{n-1} is the volume of the $(n-1)$-dimensional round sphere; S_r is the coordinate sphere of radius r; ν is the outward unit normal along S_r; and dS_r is the volume form of S_r.

The ADM mass of an asymptotically flat manifold was defined by Richard Arnowitt, Stanley Deser and Charles W. Misner [1]. They proved that the above definition is independent of the choice of asymptotically flat coordinate charts. Thus, the notion of the ADM mass is well-defined. We sometimes simply write $m_{ADM}(g)$ instead of $m_{ADM}(M, g)$ if the underlying manifold is clear.

In dimension 3, we have

$$m_{ADM}(M^3, g) = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \sum_{i,j=1}^{3} (g_{ij,i} - g_{ii,j})\nu^j dS_r. \tag{2.1.2}$$

Definition 2.1.3. *Given a Riemannian manifold (M^3, g) and a closed embedded surface (Σ^2, g_Σ) with the induced metric. The Hawking mass of Σ is defined to be:*

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma |H_\Sigma|^2 \, dA_\Sigma\right), \tag{2.1.3}$$

where H_Σ is the scalar mean curvature of Σ in M.

Example 2.1.1 (Euclidean Space)**.** *$\mathbb{R}^n$ with the standard Euclidean metric is an asymptotically flat manifold with zero ADM mass.*

Example 2.1.2 (Conformal Transformation of Metric)**.** *Given an asymptotically flat manifold (M^n, g). Consider a conformal transformation $\overline{g} = u^{\frac{4}{n-2}} g$ of the metric g, with $u \in C^\infty(M)$, $u > 0$. By Equation (A.5.14) the scalar curvatures $\overline{R}$ and R of $\overline{g}$ and g respectively, are related by:*

$$\overline{R} = u^{-\frac{n+2}{n-2}} \left(Ru - \frac{4(n-1)}{n-2} \Delta_g u\right). \tag{2.1.4}$$

If u and its coordinate derivatives satisfy the following decay conditions, $i, j, k = 1, 2, \cdots, n$:

(1) u *tends to* 1 *at* ∞*;*

(2) $u_{,i} = O(|x|^{-p-1})$*;*

(3) $u_{,jk} = O(|x|^{-p-2})$*;*

(4) $\Delta_g u = O(|x|^{-q})$

in an asymptotically flat coordinate chart of (M^n, g) *for some constants* $p > \frac{n-2}{2}$ *and* $q > n$*, then* $(M^n, \overline{g})$ *is also asymptotically flat in that coordinate chart. Moreover,*

$$m_{ADM}(\overline{g}) = m_{ADM}(g) - \lim_{r\to\infty} \frac{2}{(n-2)\omega_{n-1}} \int_{S_r} \frac{\partial u}{\partial \nu} dS_r, \tag{2.1.5}$$

where $\frac{\partial u}{\partial \nu}$ *is the outward normal derivative of* u *along* S_r*.*

Example 2.1.3 (Schwarzschild Manifold)**.** *Combining Example 2.1.1 and 2.1.2, consider the following one-parameter family of conformal transformations of* $(\mathbb{R}^n \backslash \{0\}, \delta_{ij})$*, parameterized by a constant* $m > 0$*:*

$$\left(\mathbb{R}^n \backslash \{0\}, \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij}\right). \tag{2.1.6}$$

This is called the Schwarzschild manifold *of dimension* n *and* mass m*. It is easy to verify that* $u = 1 + \frac{m}{2|x|^{n-2}}$ *satisfies the desired decay condition to make the resulting metric asymptotically flat.* m *is called the mass because the ADM mass of this metric is exactly* m*. This can be seen quite easily for the three-dimensional Schwarzschild manifold:* $\left(\mathbb{R}^3 \backslash \{0\}, \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}\right)$*. By Equation (2.1.5):*

$$\begin{aligned} m_{ADM}\left(\left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}\right) &= m_{ADM}(\delta) - \lim_{r\to\infty} \frac{1}{2\pi} \int_{S_r} \frac{\partial}{\partial r}\left(1 + \frac{m}{2r}\right) dS_r \\ &= -\lim_{r\to\infty} \frac{1}{2\pi} \int_{S_r} -\frac{m}{2} r^{-2}\, dS_r \\ &= \lim_{r\to\infty} \frac{1}{2\pi} \frac{m}{2} r^{-2} 4\pi r^2 \\ &= m \end{aligned} \tag{2.1.7}$$

We now study further the geometry of the three-dimensional Schwarzschild manifold. Let $r := |x|$*. First, note that the Schwarzschild metric is symmetric under the* *mapping* $r \mapsto \frac{m^2}{4r}$*. Thus the Schwarzschild manifold has two ends, with the center of symmetry being* $\frac{m}{2r} = \frac{2r}{m}$*, that is* $r = \frac{m}{2}$*, which is a two-sphere. Recall that if* $g = u^4\delta$ *for some positive function* u*, then the mean curvature of a sphere of radius* r *with respect to* g *is given by:*

$$H = \frac{1}{u^2}\left(\frac{2}{r} + \frac{4}{u}\frac{du}{dr}\right). \tag{2.1.8}$$

Therefore at $r = \frac{m}{2}$*, the mean curvature is zero. Hence, the sphere* $r = \frac{m}{2}$ *is a minimal surface, which can be viewed as the apparent horizon of a blackhole. The region outside of the blackhole is called the* exterior region *of the Schwarzschild manifold:*

$$\left(\mathbb{R}^3 \backslash B_{\frac{m}{2}}, \left(1 + \frac{m}{r}\right)^4 \delta\right).$$

The exterior region is an asymptotically flat end (see Figure 2.1)

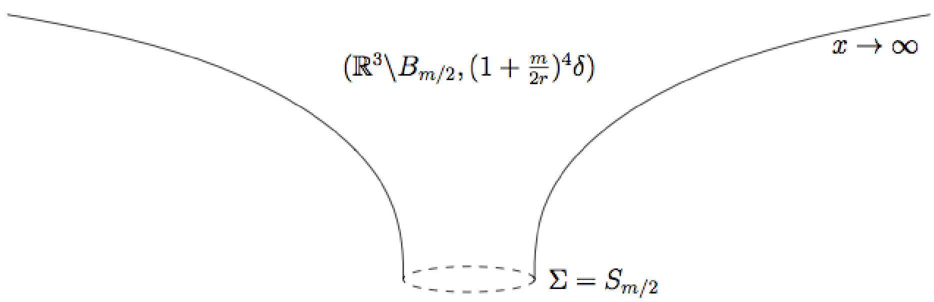

FIGURE 2.1: Exterior region of three-dimension Schwarzschild manifold with boundary the minimal sphere $S_{\frac{m}{2}}$. Figure courtesy of Mau-Kwong G. Lam.

There exists an isometric embedding of the three-dimensional Schwarzschild manifold into $\mathbb{R}^4$ *such that*

$$r = \frac{w^2}{8m} + 2m.$$

The image of this embedding is a parabola (see Figure 2.2), and the minimal sphere $S_{\frac{m}{2}}$ *gets mapped to the sphere* $S_{2m} \subset \mathbb{R}^4$*:*

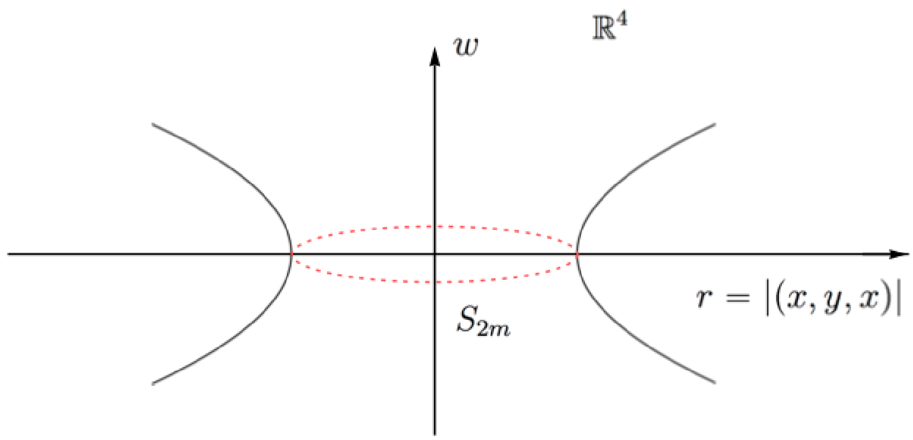

FIGURE 2.2: Isometric embedding of three-dimensional Schwarzschild manifold into $\mathbb{R}^4$.

The area of $S_{\frac{m}{2}}$ *is then given by the Euclidean area of* S_{2m}*:* $|S_{\frac{m}{2}}| = 4\pi(2m)^2 = 16\pi m^2$*. Therefore*

$$m_H(S_{\frac{m}{2}}) = \sqrt{\frac{|S_{\frac{m}{2}}|}{16\pi}} \left(1 - \int_{S_{\frac{m}{2}}} \cancelto{0}{H^2_{S_{\frac{m}{2}}}} \, dA_{S_{\frac{m}{2}}}\right) = m. \tag{2.1.9}$$

Combing this with the ADM mass (2.1.7), we see that

Proposition 2.2. *The ADM mass of the three-dimensional Schwarzschild manifold* $\left(\mathbb{R}^3 \backslash \{0\}, \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}\right)$ *equals the Hawking mass of the minimal sphere* $S_{\frac{m}{2}}$*, which is exactly* m*.*

More generally, Huisken and Ilmanen [25] proved (see also [40]):

Theorem 2.3. *Given an asymptotically flat Riemannian manifold* (M^n, g) *and an asymptotically flat coordinate chart. Then*

$$\lim_{r\to\infty} m_H(S_r) = m_{ADM}(M), \tag{2.1.10}$$

where S_r *is the coordinate sphere of radius* r*.*

2.2 Geroch, Jang-Wald's Approach and Their Monotonicity Formula

Huisken and Ilmanen's proof of the Riemannian Penrose Inequality is based on the monotonicity property of the Hawking mass under smooth inverse mean curvature flow, first discovered by Geroch and Jang-Wald.

Definition 2.2.1 (Inverse Mean Curvature Flow)**.** *Given a Riemannian manifold (M^3, g) and a closed embedded surface Σ^2 in M. A smooth inverse mean curvature flow of Σ in M is a smooth family of surfaces $F : \Sigma \times [0, T] \longrightarrow M$ of Σ such that the following parabolic evolution equation is satisfied:*

$$\frac{\partial F}{\partial t} = \frac{\nu_t}{H_t}, \quad t \in [0, T], \tag{2.2.1}$$

where ν_t and H_t is the unit outward normal vector field and scalar mean curvature of $\Sigma_t := F(\Sigma, t)$, respectively.

A family of closed embedded surfaces $\{\Sigma_t\}$ in M is called a smooth solution to inverse mean curvature flow if they satisfy (2.2.1). Given such a family of surfaces $\{\Sigma_t\}$. The first variation of area formula (A.3.7) implies that

$$\frac{d}{dt}|\Sigma_t| = \int_{\Sigma_t} H_t \frac{1}{H_t} dA_\Sigma = |\Sigma_t|. \tag{2.2.2}$$

Therefore the area of Σ_t grows exponentially under inverse mean curvature flow.

Example 2.2.1 (Inverse Mean Curvature Flow of Spheres)**.** *Consider a round sphere S_{r_0} in $\mathbb{R}^3$ with radius $r_0 > 0$, and flow this sphere out by inverse mean curvature flow. By (2.2.2), the flow surfaces are still round spheres, and the area grows exponentially. Thus $\{S_{e^{t/2} r_0}\}$ is a solution to this flow for all time.*

Geroch [22], Jang and Wald [26] discovered the following nice connection between solutions to inverse mean curvature flow and the Hawking mass:

Theorem 2.4 (Geroch, Jang-Wald)**.** *Given (M^3, g) with non-negative scalar curvature. If a family of closed embedded surfaces $\{\Sigma_t\}$ is a smooth solution to inverse mean curvature flow in M, then for all $t > 0$,*

$$\frac{d}{dt} m_H(\Sigma_t) \geqslant 0, \tag{2.2.3}$$

i.e. the Hawking mass is monotonically non-decreasing.

Proof. Let H_t and dA_t be the scalar mean curvature and the volume form of Σ_t in M, respectively.

$$\frac{d}{dt} m_H(\Sigma_t) = \frac{d}{dt}\left[\sqrt{\frac{|\Sigma_t|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma_t} H_t^2 \, dA_t\right)\right]$$

$$= \frac{d}{dt}\left(\sqrt{\frac{|\Sigma_t|}{16\pi}}\right)\left(1 - \frac{1}{16\pi}\int_{\Sigma_t} H_t^2 \, dA_t\right) + \sqrt{\frac{|\Sigma_t|}{16\pi}}\left(-\frac{1}{16\pi}\frac{d}{dt}\int_{\Sigma_t} H_t^2 \, dA_t\right)$$

$$= \frac{1}{2}\left(\frac{|\Sigma_t|}{16\pi}\right)^{-1/2}\frac{1}{16\pi}|\Sigma_t|\left(1 - \frac{1}{16\pi}\int_{\Sigma_t} H_t^2 \, dA_t\right) \qquad \text{(By Equation (2.2.2))}$$

$$+ \sqrt{\frac{|\Sigma_t|}{16\pi}}\left(-\frac{1}{16\pi}\int_{\Sigma_t}\left[2H_t\frac{d}{dt}(H_t)dA_t + H_t^2\frac{d}{dt}(dA_t)\right]\right)$$

$$= \frac{1}{2}\sqrt{\frac{|\Sigma_t|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma_t} H_t^2 \, dA_t\right) + \sqrt{\frac{|\Sigma_t|}{16\pi}}\left(-\frac{1}{16\pi}\int_{\Sigma_t}\left[2H_t\frac{d}{dt}(H_t)dA_t + H_t^2 dA_t\right]\right)$$

$$= \sqrt{\frac{|\Sigma_t|}{16\pi}}\left\{\frac{1}{2}\left(1 - \frac{1}{16\pi}\int_{\Sigma_t} H_t^2 \, dA_t\right) - \frac{1}{16\pi}\int_{\Sigma_t}\left[2H_t\frac{d}{dt}(H_t) + H_t^2\right]dA_t\right\} \tag{2.2.4}$$

By the first variation of mean curvature formula (A.4.3):

$$\frac{d}{dt}H_t = -\Delta_{\Sigma_t}\left(\frac{1}{H_t}\right) - \frac{1}{H_t}\mathrm{Ric}^M(\nu,\nu) - \frac{1}{H_t}||\mathrm{II}_t||^2, \tag{2.2.5}$$

where Ric^M is the Ricci curvature of M. Plug (2.2.5) into (2.2.4) we get:

$$\frac{d}{dt}m_H(\Sigma_t) = \sqrt{\frac{|\Sigma_t|}{16\pi}}\Bigg\{\frac{1}{2} - \frac{1}{2}\frac{1}{16\pi}\int_{\Sigma_t} H_t^2 \, dA_t$$

$$- \frac{1}{16\pi}\int_{\Sigma_t} 2H_t\left[-\Delta_{\Sigma_t}\left(\frac{1}{H_t}\right) - \frac{1}{H_t}\mathrm{Ric}^M(\nu,\nu) - \frac{1}{H_t}||\mathrm{II}_t||^2\right] + H_t^2 dA_t\Bigg\}$$

$$= \sqrt{\frac{|\Sigma_t|}{16\pi}}\Bigg\{\frac{1}{2} + \frac{1}{16\pi}\int_{\Sigma_t}\left[2H_t\Delta_{\Sigma_t}\left(\frac{1}{H_t}\right) + 2\mathrm{Ric}^M(\nu,\nu) + 2||\mathrm{II}_t||^2 - \frac{3}{2}H_t^2\right] dA_t\Bigg\} \tag{2.2.6}$$

We now compute the first three integral terms in the above. By integration by parts, we get:

$$\int_{\Sigma_t} 2H_t\Delta_{\Sigma_t}\left(\frac{1}{H_t}\right) dA_t = \int_{\Sigma_t} -2\langle\nabla_{\Sigma_t}H, \nabla_{\Sigma_t}\frac{1}{H_t}\rangle dA_t = \int_{\Sigma_t} \frac{2||\nabla_{\Sigma_t}H_t||^2}{H_t^2} dA_t. \tag{2.2.7}$$

Now by the Gauss equation (see e.g. [27]), we have

$$\mathrm{Ric}^M(\nu,\nu) = \frac{1}{2}R^M - K^{\Sigma_t} + \frac{1}{2}H_t^2 - \frac{1}{2}||\mathrm{II}_t||^2, \tag{2.2.8}$$

where K^{Σ_t} is the Gauss curvature of Σ_t.

Next let $\lambda_1(t)$ and $\lambda_2(t)$ be the principal curvatures of Σ_t, then

$$H_t = \lambda_1(t) + \lambda_2(t), \quad ||\mathrm{II}_t||^2 = \lambda_1(t)^2 + \lambda_2(t)^2.$$

Therefore

$$||\mathrm{II}_t||^2 - \frac{1}{2}H_t^2 = \lambda_1(t)^2 + \lambda_2(t)^2 - \frac{[\lambda_1(t) + \lambda_2(t)]^2}{2} = \frac{[\lambda_1(t) - \lambda_2(t)]^2}{2}. \tag{2.2.9}$$

Plug them back into (2.2.6), we get:

$$\begin{aligned}
&\frac{d}{dt}m_H(\Sigma_t) = \sqrt{\frac{|\Sigma_t|}{16\pi}}\left\{\frac{1}{2} + \frac{1}{16\pi}\int_{\Sigma_t}\left[2H_t\Delta_{\Sigma_t}\left(\frac{1}{H_t}\right) + 2\mathrm{Ric}^M(\nu,\nu) + 2||\mathrm{II}_t||^2 - \frac{3}{2}H_t^2\right]\right\} \\
&= \sqrt{\frac{|\Sigma_t|}{16\pi}}\left\{\frac{1}{2} + \frac{1}{16\pi}\int_{\Sigma_t}\frac{2||\nabla_{\Sigma_t}H_t||^2}{H_t^2} + R^M - 2K^{\Sigma_t} + H_t^2 + ||\mathrm{II}_t||^2 - \frac{3}{2}H_t^2 dA_t\right\} \\
&\geqslant \sqrt{\frac{|\Sigma_t|}{16\pi}}\left\{\frac{1}{2} + \frac{1}{16\pi}\int_{\Sigma_t} -2K^{\Sigma_t} + \frac{(\lambda_1(t)-\lambda_2(t))^2}{2}dA_t\right\} \qquad (R^M \geqslant 0) \\
&\geqslant \sqrt{\frac{|\Sigma_t|}{16\pi}}\left\{\frac{1}{2} - \frac{1}{8\pi}\int_{\Sigma_t} K^{\Sigma_t}dA_t\right\} \\
&\geqslant 0 \qquad (2.2.10)
\end{aligned}$$

where the last inequality follows from the Gauss-Bonnet formula:

$$\int_{\Sigma_t} K^{\Sigma_t}dA_t \leqslant 2\pi\chi(\Sigma_t) = 2\pi(2 - 2\cdot \mathrm{genus}(\Sigma_t)) \leqslant 4\pi.$$

□

Using this, Geroch, Jang-Wald discovered a possible approach to prove the Riemannian Penrose Inequality via the following steps:

• Let Σ be the outermost minimal surface in M. Its Hawking mass is $\sqrt{\frac{|\Sigma|}{16\pi}}$ since its mean curvature is zero. Notice that this is the lower bound in the Riemannian Penrose Inequality.

• Flow Σ out by inverse mean curvature flow, and *assume* that the flow is smooth and exists for all time. Let $\{\Sigma_t\}$ be the flow surfaces. Theorem 2.4 implies that the Hawking mass is non-decreasing.

• Let S_r be the coordinate sphere of radius r in an asymptotically flat coordinate chart of M. Theorem 2.3 implies that $\lim_{r\to\infty} m_H(S_r) = m_{ADM}(M)$.

Here is the upshot: If smooth inverse mean curvature flow of Σ in M exists for all time, and the flow surfaces approach large coordinate spheres near infinity sufficiently fast, then:

$$m_{ADM}(M) = \lim_{t\to\infty} m_H(\Sigma_t) \geqslant m_H(\Sigma_0) = m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}, \tag{2.2.11}$$

and this would prove the Riemannian Penrose Inequality. However, smooth solutions to inverse mean curvature flow do not always exist. In fact, in the case that the flow surface becomes minimal (i.e. mean curvature is zero), the flow is no longer defined since the flow speed is the reciprocal of the mean curvature (see (2.2.1)). There are other cases where singularities can occur:

Example 2.2.2 (Inverse Mean Curvature Flow of Disjoint Spheres)**.** *Suppose* Σ *is a disjoint union of two spheres. Inverse mean curvature flow of* Σ *will develop self-intersection in finite time.*

Example 2.2.3 (Inverse Mean Curvature Flow of Thin Torus)**.** *Consider a thin torus in* $\mathbb{R}^3$*, obtained as the boundary of an* ϵ*-neighborhood of a large round circle. Thus its mean curvature is positive everywhere. Now starting flowing the torus by inverse mean curvature flow (see Figure 2.3). By the first variation of the mean curvature (A.4.3) and the parabolic maximum principle, the flow speed has a lower bound. As a consequence the torus will fatten up and eventually the mean curvature will become negative in the hole of the torus. Thus, the mean curvature must be zero at some point. However, the flow is not defined when the mean curvature is zero.*

2.3 Huisken-Ilmanen's Approach and Level Set Formulation of Inverse Mean Curvature Flow with Jumps

Because inverse mean curvature flow does not always have solutions, Huisken and Ilmanen defined a generalized inverse mean curvature flow which *always* has solutions.

FIGURE 2.3: Inverse mean curvature flow of a thin torus which develops a singularity in finite time. Picture courtesy of Andrew Goetz.

The basic idea is that, in this generalized flow, when a surface is enclosed by another surface of less area, it *jumps* outward to its outermost minimal area enclosure (see [25, 5, 6]), and then resume inverse mean curvature flow. Huisken and Ilmanen used a level set formulation to characterize this jumping phenomenon. Consider a scalar-valued function f on M, and let Σ_t be the level set of f:

$$\Sigma_t = \{x \in M | f(x) = t\}.$$

In this setting, the inverse mean curvature flow equation (2.2.1) becomes:

$$\operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right) = |\nabla f|. \tag{2.3.1}$$

In the above, notice that the left hand side is the mean curvature of Σ_t, and the right hand side is the reciprocal of the flow speed. Thus when $|\nabla f| \neq 0$, equation (2.3.1) describes inverse mean curvature flow of the level sets. Note that this formulation allows jumps in a natural way, since if f is constant on some region of M, then the level sets of f would just jump over that region during the flow. Huisken and Ilmanen defined a weak solution to (2.3.1) using an energy minimization principle, and proved existence of such a weak solution by regularizing the degenerate elliptic equation (2.3.1). They showed that the Hawking mass is still monotone as in the smooth inverse mean curvature flow case. In particular, the Hawking mass is non-decreasing during jumps. In this way, they proved the Riemannian Penrose Inequaltiy (2.1) in the case of a single blackhole (i.e. the outermost minimal surface Σ is connected).

3

Inverse Mean Curvature Vector Flow in Spherically Symmetric Spacetimes

In this chapter, we study inverse mean curvature vector flows in spherically symmetric spacetimes. The motivation comes from the fact that despite lack of general existence theory, inverse mean curvature vector flow always works in spherical symmetry. In Section 3.1 and 3.2, terminologies and notations used in later discussions are provided. In Section 3.3, we study closed embedded codimension-two surfaces in a spacetime and the geometry of their normal bundles. We define mean curvature vector fields and Hawking mass. We then define inverse mean curvature vector flow of a closed embedded surface. In Section 3.4, we show that spherically symmetric spheres are smooth global solutions to inverse mean curvature vector flow, and the Hawking mass is monotonically non-decreasing if the spacetime also satisfies the dominant energy condition.

3.1 Spacetime, Einstein Equation, Dominant Energy Condition

A *spacetime* $(N^4, \overline{g}, \overline{\nabla})$ considered in this thesis is a connected, smooth, time-oriented, four-dimensional manifold with Lorentian metric $\overline{g}$ of signature $(-,+,+,+)$. $\overline{\nabla}$ is the associated Levi-Civita connection. A tangent vector of N is called (see Figure 3.1)

1. *timelike* if $\overline{g}(v,v) < 0$;

2. *null* if $\overline{g}(v,,v) = 0$;

3. *spacelike* if $\overline{g}(v,,v) > 0$.

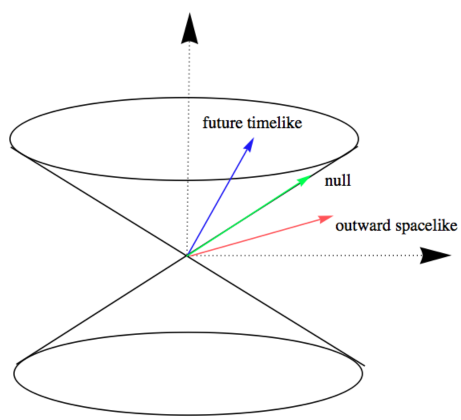

FIGURE 3.1: Timelike, null and spacelike vectors in spacetime.

N is called *time-orientable* if it admits a smooth timelike vector field $\vec{T}$. N is *time-oriented* if such a vector field $\vec{T}$ is chosen. A vector field X is called *future-pointing* if $\overline{g}(X, \vec{T}) > 0$, or *past-pointing* if $\overline{g}(X, \vec{T}) < 0$.

$\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\overline{g}}$ is used to denote the inner product with respect to $\overline{g}$, unless otherwise specified.

A vector field $X \in \Gamma(TN)$ is said to be *timelike* (resp. *null* or *spacelike*) if at every point $p \in N$, $X(p)$ is timelike (resp. null or spacelike). A submanifold M of N

is said to be *timelike* (resp. *null* or *spacelike*) if every tangent vector of M is timelike (resp. null or spacelike).

Let Ric^N and R^N be the Ricci curvature and scalar curvature of the spacetime respectively. The *Einstein curvature tensor* G is defined as:

$$G := \mathrm{Ric}^N - \frac{1}{2}\mathrm{R}^N \cdot \overline{g}. \tag{3.1.1}$$

We assume that the spacetime satisfies the *Einstein Equation*:

$$G = 8\pi T, \tag{3.1.2}$$

where T is the *stress energy tensor*. For any tangent vectors u, v of N, $T(u, v)$ has the physical meaning as the energy density going in the direction of u as observed by someone going in the direction of v. The *dominant energy condition* is:

$$T(u, v) \geqslant 0, \quad \forall\ u, v \text{ future-pointing, timelike.} \tag{3.1.3}$$

3.2 Spacelike Hypersurfaces

Given a spacetime. Consider a spacelike hypersurface M with globally defined, future-timelike unit normal vector field $\mathbf{n}$ on M, and induced Riemannian metric $g = g^M$ by restricting the spacetime metric onto M. M is also called a *slice* in a spacetime. Let k be the second fundamental form of M, then the triple (M^3, g, k) is called a *Cauchy data*. Given a Cauchy data and the unit normal vector field $\mathbf{n}$, we define the *energy density* of M as $\mu := T(\mathbf{n}, \mathbf{n})$. We can compute that

$$\begin{aligned}
\mu = \frac{1}{8\pi}G(\mathbf{n},\mathbf{n}) &= \frac{1}{8\pi}\left(\mathrm{Ric}^N(\mathbf{n},\mathbf{n}) - \frac{1}{2}\mathrm{R}^N \cdot \overline{g}(\mathbf{n},\mathbf{n})\right) \quad &\text{(By the Einstein equation)}\\
&= \frac{1}{8\pi}\left(\mathrm{Ric}^N(\mathbf{n},\mathbf{n}) + \frac{1}{2}\mathrm{R}^N\right) \quad &(\mathbf{n} \text{ is unit time like})\\
&= \frac{1}{16\pi}\left(R^M + (\mathrm{trace}_g k)^2 - ||k||_g^2\right). &(3.2.1)
\end{aligned}$$

where R^M is the scalar curvature of (M, g). We define the *momentum density* of M to be a one-form $J(\cdot)$ on M, such that $J(X) := T(\mathbf{n}, X)$, for any tangent vector field X on M. Then:

$$\begin{aligned} J(\cdot) = T(\mathbf{n}, \cdot) &= \frac{1}{8\pi}\left(\mathrm{Ric}^N(\mathbf{n}, \cdot) - \frac{1}{2}R^N \overline{g}(\mathbf{n}, \cdot)\right) \\ &= \frac{1}{8\pi}\mathrm{Ric}^N(\mathbf{n}, \cdot) \qquad (\mathbf{n} \text{ is normal to } M) \\ &= \frac{1}{8\pi}\mathrm{div}_g((k - \mathrm{trace}_g k) \cdot g). \end{aligned} \tag{3.2.2}$$

Equation (3.2.1) and (3.2.2) follow from the Gauss and Codazzi equations respectively, and they are called the *constraint equations.* The dominant energy condition (3.1.3) implies that

$$\mu \geqslant ||J||_g. \tag{3.2.3}$$

In the time-symmetric case where the second fundamental form $k = 0$, we see that $\mu = \frac{R^M}{16\pi}$, and $J = 0$. The dominant energy condition thus reduces to $R \geqslant 0$.

3.3 Geometry of Codimension Two Surfaces in Spacetime

Let Σ be an *closed, embedded, spacelike* surface in N with codimension two. We assume that Σ is an oriented surface such that at each point the notion of "outward" and "inward" is well-defined. Let g_Σ be the induced metric on Σ from the spacetime metric $\overline{g}$.

3.3.1 Rank-two Normal Bundle Geometry: Normal Connection and Connection One-form

Let $N\Sigma$ be the rank-two normal bundle of Σ. Define a connection on $N\Sigma$, denoted as $\nabla^\perp$, to be the projection of $\overline{\nabla}$ onto $N\Sigma$. Notice that since Σ is spacelike, $N\Sigma$ has an induced metric with signature $(-,+)$. Therefore, at each point $p \in \Sigma$, $N_p\Sigma$ has four quadrants: future-timelike, past-timelike, outward-spacelike and inward-spacelike.

Given a local orthonormal frame $\{e_1, e_2\}$ of $N\Sigma$. Suppose e_1 is outward-spacelike and e_2 is future-time like, then the geometry of $N_p\Sigma$ is depicted by Figure 3.2:

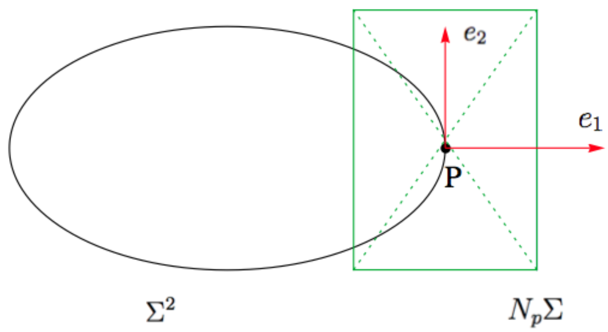

FIGURE 3.2: Rank two normal bundle at a point p on a surface with orthonormal basis $e_1(p)$ and $e_2(p)$.

Define a linear isomorphism, denoted as "$\perp$", on each fiber $N_p\Sigma$ of the normal bundle as follows: for any orthonormal basis $\{u, v\}$ of $N_p\Sigma$ such that u is outward-spacelike and v is future-timelike, set $u^\perp := v$, and then extend linearly to the entire fiber. This definition is independent of the choice of basis, and is an involutive isomorphism. Notice that

$$\langle u^\perp, u^\perp \rangle = \langle v, v \rangle = -1 = -\langle u, u \rangle.$$

This isomorphism can be viewed as analogy of the 90° rotation in Euclidean space (see [9]).

Given any tangent vector field X of Σ. If v is an outward-spacelike unit normal vector field along Σ, then:

$$0 = \nabla^\perp_X \langle v, v \rangle = 2\langle \nabla^\perp_X v, v \rangle.$$

Therefore $\nabla^\perp_X v$ is perpendicular to v, and hence is parallel to $v^\perp$. From this one can define a one-form α_v on Σ, uniquely depends on v, such that:

$$\alpha_v(X) := \langle \nabla^\perp_X v, v^\perp \rangle, \quad \forall\, X \in \Gamma(T\Sigma). \tag{3.3.1}$$

This definition yields the following straightforward corollary (see [9]):

Corollary 3.1. *For a smooth section* $v \in \Gamma(N\Sigma)$*, the associated connection one-form* α_v *vanishes if and only if* v *is parallel with respect to the normal connection* $\nabla^\perp$*, i.e,*

$$\nabla^\perp_X v = 0, \quad \forall\, X \in \Gamma(T\Sigma).$$

3.3.2 Mean Curvature Vector Field, Hawking Mass and Inverse Mean Curvature Vector Flow

We define

$$\vec{\mathrm{II}} : T\Sigma \times T\Sigma \longrightarrow N\Sigma, \quad (X, Y) \mapsto (\overline{\nabla}_X Y)|_{N\Sigma} \tag{3.3.2}$$

to be the *second fundamental form* of Σ, where X, Y are tangent vector fields along Σ, and $(\overline{\nabla}_X Y)|_{N\Sigma}$ is the projection of $\nabla_X Y$ onto the normal bundle of Σ. Define the *mean curvature vector field* of Σ to be the trace of the second fundamental form with respect to g_Σ:

$$\vec{H}_\Sigma := \mathrm{trace}_{g_\Sigma} \vec{\mathrm{II}}. \tag{3.3.3}$$

Therefore, $\vec{H}_\Sigma$ is an *inward-pointing* normal vector field along Σ. Given any normal vector field $\vec{n}$, define the *mean curvature scalar in the direction of* $\vec{n}$ to be:

$$H_{\vec{n}} := -\langle \vec{H}_\Sigma, \vec{n} \rangle. \tag{3.3.4}$$

Given (Σ, g_Σ) and the mean curvature vector $\vec{H}_\Sigma$, the *Hawking mass* of Σ is defined to be:

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma \overline{g}(\vec{H}_\Sigma, \vec{H}_\Sigma)\, dA_\Sigma \right), \tag{3.3.5}$$

where $|\Sigma|$ is the area of the surface Σ.

Definition 3.3.1. *Given a spacetime* $(N, \overline{g})$ *and a surface* (Σ^2, g_Σ). *Define the* inverse mean curvature vector field *of* Σ *to be:*

$$\vec{I}_\Sigma := -\frac{\vec{H}_\Sigma}{\langle \vec{H}_\Sigma, \vec{H}_\Sigma \rangle}. \tag{3.3.6}$$

According to our sign convention for $\vec{H}_\Sigma$, $\vec{I}_\Sigma$ thus defined is *outward-pointing.*

Definition 3.3.2 (Smooth Inverse Mean Curvature Vector Flow)**.** *Given a closed embedded surface* Σ^2 *in a spacetime* $(N^4, \overline{g})$. *A* smooth inverse mean curvature vector flow *of* Σ *is a smooth family of surfaces* $F : \Sigma \times [0, T] \longrightarrow N$ *of* Σ *satisfying the following evolution equation:*

$$\frac{\partial}{\partial s} F(x, s) = \vec{I}_{\Sigma_s}(x, s), \quad s \in [0, T] \textit{ and } (x, s) \in \Sigma_s := F(\Sigma, s). \tag{3.3.7}$$

$T > 0$ could also be ∞. By the first variation of area formula (see Equation A.3.7 in appendix A.3), the rate of change of area form of the flow surfaces under smooth inverse mean curvature vector flow is given by:

$$\frac{d}{ds} dA_{\Sigma_s} = -\langle \vec{H}_{\Sigma_s}, \vec{I}_{\Sigma_s} \rangle dA_{\Sigma_s} = dA_{\Sigma_s}. \tag{3.3.8}$$

That is, the rate of the area form of each surface is the area form itself, everywhere on each surface. This is a special case of a *uniformly area expanding flow* first defined by H. Bray, J. Jauregui and M. Mars in [10].

3.4 Model Spacetime: Spherically Symmetric Spacetime

In this section, we study spherically symmetric spacetimes. The main motivation comes from the fact that, even though inverse mean curvature vector flow lacks a general existence theory, smooth solutions still exist in many spherically symmetric spacetimes. Moreover, the Hawking mass is monotonically non-decreasing if the spacetime satisfies the dominant energy condition. Thus, it is critical to understand the geometry of spherical symmetry.

Definition 3.4.1. *A spacetime* $(N^4, \overline{g})$ *is said to be* spherically symmetric *if its isometry group* $Isom(N^4)$ *contains a subgroup* G *that is isomorphic to the rotation group* $SO(3)$*; moreover for any point* $p \in N$*, the orbit of* p *under the action of* G *is a two-sphere with metric a multiple of the standard round metric.*

From the above definition, N^4 and standard sphere S^2 share $SO(3)$ as a subgroup in their isometry groups, thus N^4 share some of the symmetries with S^2, and hence the term *spherically symmetric.*

Proposition 3.2. *If* $(N^4, \overline{g})$ *is a spherically symmetric spacetime that admits a coordinate chart* $\{t, r, \theta, \phi\}$*, such that* $\overline{g}$ *takes the form:*

$$g = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} \quad t \quad & \quad r \quad & \theta & \quad \phi \end{array} \\ \begin{pmatrix} -v^2(t,r) & 0 & 0 & 0 \\ 0 & u^2(t,r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \end{array} \tag{3.4.1}$$

where u *and* v *are smooth functions of* t *and* r *only. Then within each* t = *constant slice, smooth inverse mean curvature vector flow of coordinate sphere* $S_{t,r}$ *exists for all time with monotonically non-decreasing Hawking mass.*

Remark. *Roughly speaking, all spherically symmetric spacetimes outside blackholes admit such metrics as in* (3.4.1)*.*

Proof. Let $g_{t,r}$ be the metric on $S_{t,r}$, then

$$g_{t,r} = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

and its inverse is given by:

$$g_{t,r}^{-1} = \frac{1}{r^2} d\theta^2 + \frac{1}{r^2 \sin^2\theta} d\phi^2.$$

Let $\vec{H}_{t,r}$ be the mean curvature vector of $S_{t,r}$. Notice that $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial r}\}$ forms a frame for the normal bundle of $S_{t,r}$, and thus $H_{t,r}$ can be computed as follows:

$$\begin{aligned}
\vec{H}_{t,r} &= g_{t,r}^{ij} (\overline{\nabla}_{\partial_i} \partial_j)^{\text{nor}} \\
&= g_{t,r}^{ij} \left(\frac{\langle \overline{\nabla}_{\partial_i} \partial_j, \partial_t \rangle}{\langle \partial_t, \partial_t \rangle} \partial_t + \frac{\langle \overline{\nabla}_{\partial_i} \partial_j, \partial_r \rangle}{\langle \partial_r, \partial_r \rangle} \partial_r \right) \\
&= g_{t,r}^{\theta\theta} \left(\frac{\langle \overline{\nabla}_{\partial_\theta} \partial_\theta, \partial_t \rangle}{\langle \partial_t, \partial_t \rangle} \partial_t + \frac{\langle \overline{\nabla}_{\partial_\theta} \partial_\theta, \partial_r \rangle}{\langle \partial_r, \partial_r \rangle} \partial_r \right) + 2 g_{t,r}^{\theta\phi} \left(\frac{\langle \overline{\nabla}_{\partial_\theta} \partial_\phi, \partial_t \rangle}{\langle \partial_t, \partial_t \rangle} \partial_t + \frac{\langle \overline{\nabla} \partial_\theta \partial_\phi, \partial_r \rangle}{\langle \partial_r, \partial_r \rangle} \partial_r \right) \\
&+ g_{t,r}^{\phi\phi} \left(\frac{\langle \overline{\nabla}_{\partial_\phi} \partial_\phi, \partial_t \rangle}{\langle \partial_t, \partial_t \rangle} \partial_t + \frac{\langle \overline{\nabla}_{\partial_\phi} \partial_\phi, \partial_r \rangle}{\langle \partial_r, \partial_r \rangle} \partial_r \right) \\
&= g_{t,r}^{\theta\theta} (\overline{\Gamma}_{\theta\theta}^t \partial_t + \overline{\Gamma}_{\theta\theta}^r \partial_r) + 0 + g_{t,r}^{\phi\phi} (\overline{\Gamma}_{\phi\phi}^t \partial_t + \overline{\Gamma}_{\phi\phi}^r \partial_r) && (g_{t,r}^{-1} \text{ is diagonal}) \\
&= \frac{1}{r^2} \left(-\frac{r}{u^2} \right) \partial_r + \frac{1}{r^2 \sin^2\theta} \left(-\frac{r \sin^2\theta}{u^2} \right) \partial_r && (\text{See Section A.1}) \\
&= -\frac{2}{r} \frac{1}{u^2} \partial_r. && (3.4.2)
\end{aligned}$$

That is

$$\vec{H}_{t,r} = -\frac{2}{r} \frac{1}{u^2} \partial_r. \qquad (3.4.3)$$

The inverse mean curvature vector is:

$$\vec{I}_{t,r} = -\frac{\vec{H}_{t,r}}{\langle \vec{H}_{t,r}, \vec{H}_{t,r} \rangle} = \frac{\frac{2}{r}\frac{1}{u^2}\partial_r}{\langle -\frac{2}{r}\frac{1}{u^2}\partial_r, -\frac{2}{r}\frac{1}{u^2}\partial_r \rangle} = \frac{\frac{2}{r}\frac{1}{u^2}\partial_r}{\frac{4}{r^2}\frac{1}{u^4}u^2} = \frac{r}{2}\partial_r. \qquad (3.4.4)$$

Therefore, inverse mean curvature vector flow of $S_{t,r}$ is a reparametrization of radial flow, and hence is smooth and exits for all time.

To prove monotonicity of Hawking mass, recall that:

$$m_H(S_{t,r}) = \sqrt{\frac{|S_{t,r}|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{S_{t,r}} \overline{g}(\vec{H}_{t,r}, \vec{H}_{t,r})\, dA_{t,r}\right), \tag{3.4.5}$$

where $dA_{t,r}$ is the area form of $S_{t,r}$, and $|S_{t,r}|$ is the area of $S_{t,r}$. Note that

$$\begin{aligned}
|S_{t,r}| &= \int_{S_{t,r}} dA_{t,r} = \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta\, d\theta d\phi \\
&= r^2 \int_0^{2\pi} 2\, d\phi \\
&= 4\pi r^2.
\end{aligned}$$

Plug the mean curvature vector of $S_{t,r}$ (3.4.3) into the Hawking mass equation (3.4.5), we get:

$$\begin{aligned}
m_H(S_{t,r}) &= \sqrt{\frac{4\pi r^2}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{S_{t,r}} \overline{g}\left(-\frac{2}{r}\frac{1}{u^2}\partial_r, -\frac{2}{r}\frac{1}{u^2}\partial_r\right) dA_{t,r}\right) \\
&= \frac{r}{2}\left(1 - \frac{1}{16\pi}\int_{S_{t,r}} \frac{4}{r^2}\frac{1}{u^4}\overline{g}(\partial_r, \partial_r)\, dA_{t,r}\right) \\
&= \frac{r}{2}\left(1 - \frac{1}{16\pi}\int_0^{2\pi}\int_0^{\pi} \frac{4}{r^2}\frac{1}{u^2}\, r^2 sin\theta\, d\theta\, d\phi\right) \\
&= \frac{r}{2}\left(1 - \frac{1}{4\pi}\frac{1}{u^2}4\pi\right) \\
&= \frac{r}{2}\left(1 - \frac{1}{u^2}\right).
\end{aligned} \tag{3.4.6}$$

By Equation (3.4.4), the variation of the Hawking mass of $S_{t,r}$ along inverse mean curvature vector flow is given by:

$$\vec{I}_{t,r}(m_H(S_{t,r})) = \frac{r}{2}\frac{\partial m_H(S_{t,r})}{\partial_r} = \frac{r}{2}\left[\frac{1}{2}\left(1 - \frac{1}{u^2}\right) + r\frac{u_{,r}}{u^3}\right].$$

Let G be the Einstein curvature tensor of $(N^4, \overline{g})$. $G(\partial_t, \partial_t)$ can be computed as (see Equation $(A.1.8)$ in Appendix A.1):

$$G(\partial_t, \partial_t) = \frac{2}{r}\frac{u_{,r}}{u^3}v^2 + \frac{1}{r^2}v^2\left(1 - \frac{1}{u^2}\right) = \frac{2v^2}{r^2}\left[\frac{1}{2}\left(1 - \frac{1}{u^2}\right) + r\frac{u_{,r}}{u^3}\right]. \tag{3.4.7}$$

By the dominant energy condition (3.1.3), $G(\partial_t, \partial_t) \geqslant 0$. Therefore

$$\frac{1}{2}\left(1 - \frac{1}{u^2}\right) + r\frac{u_{,r}}{u^3} \geqslant 0, \tag{3.4.8}$$

which implies that $\vec{I}_{t,r}(m_H(S_{t,r})) \geqslant 0$, as desired. □

For any spherically symmetric spacetime in Proposition 3.2, it can then be foliated by $t =$ constant spacelike hyperplanes, and each hyperplane can be foliated by smooth inverse mean curvature vector flow spheres (see Figure 3.3).

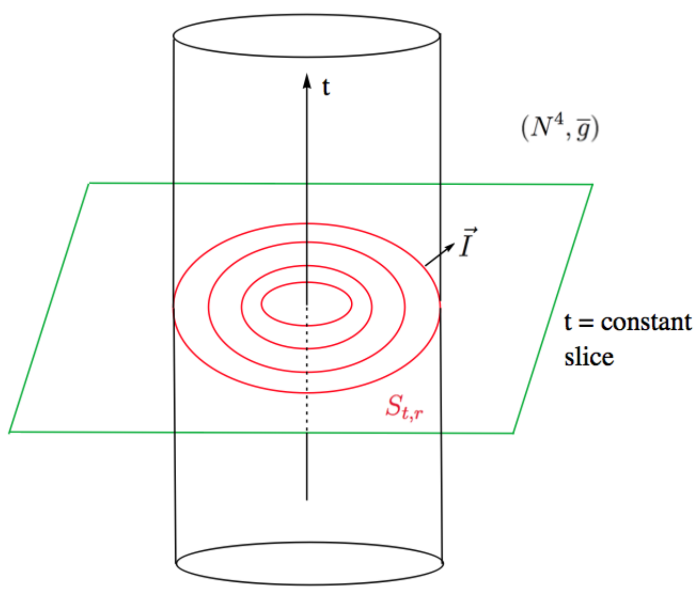

FIGURE 3.3: Inverse mean curvature vector flow of coordinate spheres $S_{t,r}$ in spherically symmetric spacetime (3.4.1).

4

Spacetimes that Admit Inverse Mean Curvature Vector Flow Solutions

In this chapter we construct non-spherically symmetric, non-static spacetimes that admit smooth global solutions to inverse mean curvature vector flows. Surfaces we study here satisfy the topological and geometric setups defined in Section 3.3. In Section 4.1, we recall the definition of inverse mean curvature vector flow and the bad existence theory of such flows. Motivated by the spherically symmetry case, in Section 4.2, we construct spacetimes that admit a special coordinate chart (called an inverse mean curvature vector flow coordinate chart) in which smooth inverse mean curvature vector flow of coordinate spheres exists for all time. We show that there in fact exist infinitely many spacetimes that admit such coordinate charts, and hence admit smooth solutions to inverse mean curvature vector flow.

In Section 4.3, we give a coordinate-free analogue of our construction, and show that we can actually “steer” a spacetime metric in a certain direction to make the mean curvature vector of a surface embedded in a spacelike hypersurface M tangential to M. Finally in Section 4.4, we discuss some generalizations of the technique we use in constructing inverse mean curvature vector flow coordinates.

4.1 Motivations from Spherically Symmetric Spacetimes and Main Results

Given a spacetime $(N^4, \overline{g})$ and a closed codimension two surface Σ with induced metric g_Σ and mean curvature vector $\vec{H}_\Sigma$, recall from Section 3.3.2 that a *smooth inverse mean curvature vector flow* of Σ is a normal variation

$$F : \Sigma \times [0, T] \longrightarrow N, \quad (\Sigma, s) \mapsto F(\Sigma, s) =: \Sigma_s,$$

such that

$$\frac{\partial}{\partial s} F(x, s) = \vec{I}_{\Sigma_s}(x, s), \quad \forall\, (x, s) \in \Sigma_s, \tag{4.1.1}$$

where $\vec{I}_{\Sigma_s}$ is the inverse mean curvature vector of Σ_s defined as

$$\vec{I}_{\Sigma_s} := -\frac{\vec{H}_{\Sigma_s}}{\langle \vec{H}_{\Sigma_s}, \vec{H}_{\Sigma_s} \rangle_{\overline{g}}}.$$

The inverse mean curvature vector flow equation 4.1.1 (same as Equation 3.3.7) is a forward-backward parabolic PDE, forward-parabolic in spacelike directions and backward-parabolic in timelike directions. Such a PDE lacks a general existence theory. However, such PDEs can still have solutions if we start with the "right" initial conditions. In spherically symmetric spacetimes, the "right" initial surfaces are spherically symmetric spheres. Had we chosen some other sphere to start with, inverse mean curvature vector flow is very likely to not exist. This is due to the mean curvature vector computation in Equation 3.4.3 in spherical symmetry: it is radial and has no components in the timelike direction, therefore the inverse mean curvature vector flow of spheres will be contained inside $t =$ constant spacelike slices. Geometrically, spherical symmetry eliminates all the timelike "wiggles" of the flow surfaces, hence restricting the flow direction to be spacelike. Since the inverse mean curvature vector flow equation is backward-parabolic only in timelike directions, inverse mean curvature vector flow exists without running into singularities in spherical symmetry (see Figure 3.3).

How do we generalize the spherically symmetric case to non-symmetric spacetimes? Note that the mean curvature vector $\vec{H}_\Sigma$ of Σ is a section of the normal bundle $N\Sigma$, and thus has a timelike component and a spacelike component. Motivated by the spherically symmetric case, intuitively if the flow surfaces all have "purely *spacelike*" mean curvature vectors, we might hope to get a better existence theory.

Proposition 4.1 ([8])**.** *The family of closed embedded spacelike surfaces $\{\Sigma_s\}$ is a solution to the smooth inverse mean curvature vector flow with spacelike inverse mean curvature vector $\vec{I}_{\Sigma_s}$ everywhere on the surfaces if and only if there exists a spacelike hypersurface $M^3 \subset N$, such that the mean curvature vector $\vec{H}_{\Sigma_s}$ is tangential to M at all (x, s), and $\{\Sigma_s\}$ is a solution to the smooth inverse mean curvature flow in M.*

Proof. Given $\{\Sigma_s\}$ a solution to the smooth inverse mean curvature vector flow with *spacelike* inverse mean curvature vector $\vec{I}_{\Sigma_s}$. Consider the hypersurface M of N defined by the union of all the surfaces Σ_s, i.e. the "sweep out" region of the the flow surfaces. Since the flow is smooth and spacelike, M is a smooth manifold which is spacelike as well. $\vec{H}_{\Sigma_s}$ is tangential to M by the construction of M. Since Σ_s is of codimension one in M, $\vec{H}_{\Sigma_s}$ is parallel to the unit outward normal vector field ν_{Σ_s} along Σ_s, i.e.,

$$\vec{H}_{\Sigma_s} = -\lambda \nu_{\Sigma_s}, \tag{4.1.2}$$

at each $(x, s) \in \Sigma_s$ for some smooth positive function λ. λ is chosen to positive since $\vec{H}_{\Sigma_s}$ points inward. Therefore

$$\vec{I}_{\Sigma_s} = -\frac{\vec{H}_{\Sigma_s}}{\langle \vec{H}_{\Sigma_s}, \vec{H}_{\Sigma_s} \rangle} = \frac{\lambda \nu_{\Sigma_s}}{\langle -\lambda \nu_{\Sigma_s}, -\lambda \nu_{\Sigma_s} \rangle} = \frac{\lambda \nu_{\Sigma_s}}{\lambda^2 \langle \nu_{\Sigma_s}, \nu_{\Sigma_s} \rangle} = \frac{\nu_{\Sigma_s}}{\lambda} = \frac{\nu_{\Sigma_s}}{H_{\Sigma_s}}, \tag{4.1.3}$$

where H_{Σ_s} is the mean curvature scalar of Σ_s in the direction of ν_{Σ_s}, defined by Equation 3.3.4:

$$H_{\Sigma_s} := -\langle \vec{H}_{\Sigma_s}, \nu_{\Sigma_s} \rangle = -\langle -\lambda \nu_{\Sigma_s}, \nu_{\Sigma_s} \rangle = \lambda.$$

From equation (4.1.3), we see that $\{\Sigma_s\}$ indeed is a solution to the smooth inverse mean curvature flow in M.

Conversely, suppose M is a spacelike hypersurface and $\{\Sigma_s\}$ is a solution to the smooth inverse mean curvature flow in M. Assuming that $\vec{H}_{\Sigma_s}$ is tangential to M at each (x, s), we know that $\vec{H}_{\Sigma_s}$ is spacelike as well. Moreover, by reversing the computations in equation (4.1.3), we have

$$\vec{I}_{\Sigma_s} = -\frac{\vec{H}_{\Sigma_s}}{\langle \vec{H}_{\Sigma_s}, \vec{H}_{\Sigma_s} \rangle}.$$

Thus $\{\Sigma_s\}$ is a solution to the smooth inverse mean curvature vector flow equation in N, with spacelike inverse mean curvature vectors. □

Therefore spacelike smooth inverse mean curvature vector flow solutions in N correspond to smooth inverse mean curvature flow solutions in a spacelike hypersurface of N with tangential mean curvature vector fields. Huisken and Ilmanen defined a weak notion of inverse mean curvature flow in which jumps are allowed. This suggests the following definition of a weak solution of inverse mean curvature vector flow:

Definition 4.1.1 (Weak Solution to Inverse Mean Curvature Vector Flow, [8])**.** *A family of spacelike surfaces $\{\Sigma_s\}$ is said to be a weak solution to the inverse mean curvature vector flow equation if there exists a spacelike hypersurface M in N containing Σ_s such that $\vec{H}_{\Sigma_s}$ is tangential to M everywhere for all $s \in [0, T]$, and $\{\Sigma_s\}$ is a solution to Huisken-Ilmanen inverse mean curvature flow in M (i.e. with jumps).*

Now we focus on the problem of finding spacetimes with inverse mean curvature vector flow solutions.

Suppose a spacetime $(N^4, \overline{g})$ admits the following special foliation: N is foliated by spacelike hyperplanes, and then within each hyperplane, smooth inverse mean curvature vector flow of spheres exists and foliates the entire hyperplane. Consequently, the mean curvature vector of the flow spheres are tangential to the hyperplane. If N admits such a special "double" foliation (e.g. spherically symmetric spacetimes), then N has to be topologically equivalent to $(\mathbb{R}^3 \backslash B_1) \times \mathbb{R}$, where B_1 is the closed unit ball in $\mathbb{R}^3$.

Suppose $(N^4, \overline{g})$ admits such a special foliation. We can use this to define coordinates that generalize the spherically symmetric coordinates. We define the t-coordinate by setting each hyperplane as $t =$ constant, thus the t-coordinate tells us which hyperplane we are on. For each inverse mean curvature vector flow sphere, define $A = 4\pi r^2$, where A is the area of that sphere. This defines a very natural r-coordinate. Since inverse mean curvature vector flow is area expanding, the r-coordinate is well-defined. For simplicity we assume that $r \geqslant 1$, i.e. the initial spheres on each hyperplane have area 4π. Then define (θ, ϕ)-coordinates on an initial sphere, $0 < \theta < \pi$ and $0 < \phi < 2\pi$, such that the area form satisfies

$$dA_0 = \frac{A(0)}{4\pi} \sin\theta d\theta d\phi = \sin\theta d\theta d\phi, \tag{4.1.4}$$

where $A(0)$ is the area of the initial sphere. Extend (θ, ϕ) by setting them to be constant along perpendicular directions of the initial sphere, such that (θ, ϕ) coordinates are defined for each sphere. By the extension, $\frac{\partial}{\partial r}$ will be perpendicular to each sphere. The equation for the area form (4.1.4) will be preserved: $dA_r = \frac{A(r)}{4\pi} \sin\theta d\theta d\phi^*$. See

* In smooth inverse mean curvature vector flow, $\frac{d}{ds}(dA_s) = -\langle \vec{I}_s, \vec{H}_s \rangle dA_s = dA_s$, where dA_s is the area form of Σ_s. The solution to this equation is $dA_s = e^s dA_0$. The area of Σ_s is thus given by $A(s) = A(0)e^s = 4\pi r^2$, by the definition of the r-coordinate. Thus $e^s \frac{A(0)}{4\pi} = r^2$. Therefore the area form in the r parameter is $dA_r = e^s dA_0 = e^s \frac{A(0)}{4\pi} \sin\theta d\theta d\phi = r^2 \sin\theta d\theta d\phi = \frac{A(r)}{4\pi} \sin\theta d\theta d\phi$.

Figure 4.1 for an illustration.

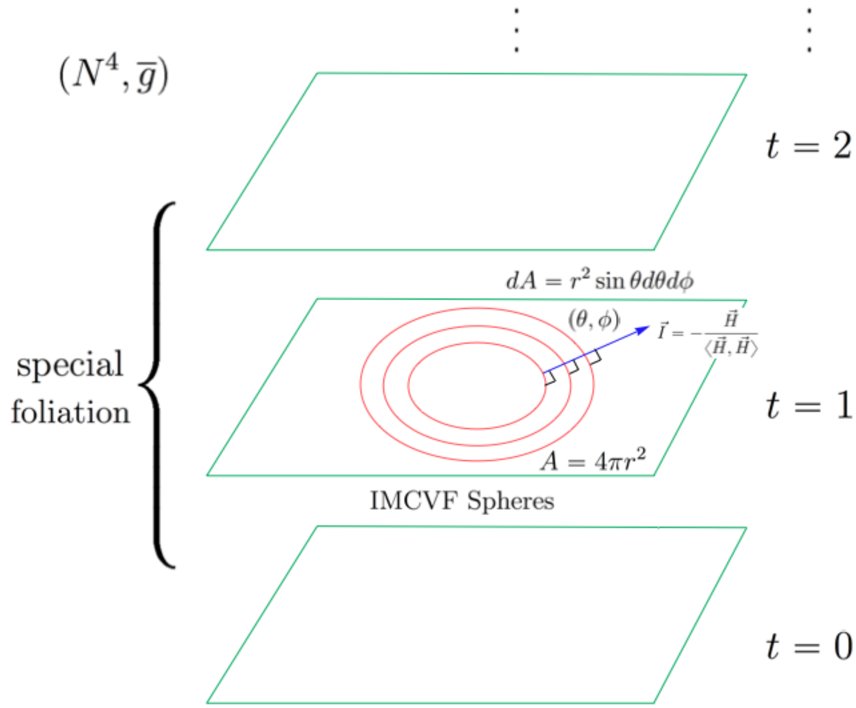

FIGURE 4.1: Special foliation of a spacetime $(N^4, \overline{g})$: first foliated by hyperplanes, then each hyperplane is foliated by inverse mean curvature vector flow of spheres. This generalizes the spherically symmetric case in Figure 3.3.

Therefore we have proved the "only-if" direction of the following theorem:

Theorem 4.2. *A spacetime $(N^4, \overline{g})$ is foliated by spacelike hyperplanes, and each hyperplane is foliated by smooth inverse mean curvature vector flow of spheres if and only if there exists a coordinate chart $\{t, r, \theta, \phi\}$ of N, such that in this coordinate chart the metric has the form:*

$$\overline{g} = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} -v^2 & d & e & f \\ d & u^2 & 0 & 0 \\ e & 0 & a & c \\ f & 0 & c & b \end{pmatrix} \end{array} \tag{4.1.5}$$

where u, v, a, b, c, d, e, f are smooth functions on N, and the following four conditions are satisfied:

$$(1) \quad \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \rangle = 0; \tag{4.1.6}$$

$$(2) \quad \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \phi} \rangle = 0; \tag{4.1.7}$$

$$(3) \quad dA_{t,r} = r^2 \sin\theta d\theta d\phi \ (i.e.\ ab - c^2 = r^4 \sin^2\theta); \tag{4.1.8}$$

$$(4) \quad \vec{H}_{t,r} \text{ is tangential to the } t = \text{constant hyperplane}; \tag{4.1.9}$$

where $dA_{t,r}$ and $\vec{H}_{t,r}$ are the area form and the mean curvature vector of the coordinate sphere $S_{t,r}$, respectively.

Proof of the "if" direction. Given a coordinate chart $\{t, r, \theta, \phi\}$ of $(N, \overline{g})$ such that the $\overline{g}$ satisfies the four conditions, N is then foliated by t = constant slices which are spacelike since the metric has the form (4.1.5). For any t = constant slice, the coordinate spheres $\{S_{t,r}\}$ are solutions of a normal flow since $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \rangle = \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \phi} \rangle = 0$. We reparametrize the flow by setting $s := C + 2\ln r$, where C is a positive constant. Then

$$\frac{d}{dr}(dA) = \frac{d}{ds}(dA)\frac{ds}{dr} = \frac{2}{r}\frac{d}{ds}(dA). \tag{4.1.10}$$

On the other hand by condition (3)

$$\frac{d}{dr}(dA) = \frac{d}{dr}(r^2 \sin\theta d\theta d\phi) = 2r \sin\theta d\theta d\phi = \frac{2}{r} dA. \tag{4.1.11}$$

Combing the two equations above, we have $\frac{d}{ds}(dA) = dA$. Thus, by the first variation of area formula, $\{S_{t,r}\}$ when reparameterized by $r^2 = Ce^s$, are smooth solutions to inverse mean curvature flow. By condition (4), the mean curvature vector of $S_{t,r}$ stays tangential to the slice, therefore $\{S_{t,r}\}$ with the above reparameterization are smooth solutions to inverse mean curvature vector flow. □

Definition 4.1.2 (Inverse Mean Curvature Vector Flow Coordinate Chart)**.** *If a spacetime* $(N^4, \overline{g})$ *admits a coordinate chart* $\{t, r, \theta, \phi\}$ *such that the four conditions (4.1.6), (4.1.7), (4.1.8) and (4.1.9) are satisfied, then* $\{t, r, \theta, \phi\}$ *is called an inverse mean curvature vector flow coordinate chart, and* N *is called a spacetime that admits an inverse mean curvature vector flow coordinate chart.*

We sometimes refer the fourth condition (4.1.9) as the *steering condition*, as it forces the coordinate spheres to stay inside the spacelike hyperplane during inverse mean curvature vector flow.

Many spherically symmetric spacetimes admit an inverse mean curvature vector flow coordinate chart (e.g. coordinate chart (3.4.1) with $d = e = f = c = 0$, and $a = r^2$, $b = r^2 \sin^2 \theta$ and radial mean curvature vector by Equation (3.4.3)). However, given an arbitrary spacetime $(N^4, \overline{g})$, it is generally impossible to reparametrize it with an inverse mean curvature vector flow coordinate chart (e.g. a spacetime that is not topologically equivalent to $(\mathbb{R}^3 \backslash B_1) \times \mathbb{R}$). However, is it possible to *construct* a spacetime that admits such a coordinate chart? In the next section we show that we can actually construct many such spacetimes:

Proposition 4.3 (Existence of Spacetimes That Admit an Inverse Mean Curvature Vector Flow Coordinate Chart)**.** *Let* $U := (\mathbb{R}^3 \backslash B_1) \times \mathbb{R} \subset \mathbb{R}^4$*. There exist infinitely many spacetime metrics of the form of (4.1.5) that admits an inverse mean curvature vector flow coordinate chart.*

Combining Proposition 4.3 and Theorem 4.2, we have the following main theorem of this thesis:

Theorem 4.4 (**Main Theorem**)**.** *There exist infinitely many non-spherically symmetric, non-static spacetimes that admit inverse mean curvature vector flow coordinate charts. Given such a spacetime* U *with an inverse mean curvature vector flow* *coordinate chart (t, r, θ, ϕ) and the constructed spacetime metric $\overline{g}$. The coordinate spheres $S_{t,r}$ contained in each $t =$ constant slice, when reparameterized by $r^2 = e^s$, are smooth global solutions to the inverse mean curvature vector flow equation.*

4.2 Construction of Spacetimes That Admit Inverse Mean Curvature Vector Flow Coordinate Charts

In this section we prove Proposition 4.3. Let $U = (\mathbb{R}^3 \backslash B_1) \times \mathbb{R}$. It is easy to construct a spacetime metric $\overline{g}$ that admits a coordinate chart $\{t, r, \theta, \phi\}$ that satisfies condition (4.1.6) and (4.1.7). Simply define

$$\overline{g} := \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} -v^2 & d & e & f \\ d & u^2 & 0 & 0 \\ e & 0 & a & c \\ f & 0 & c & b \end{pmatrix} \end{array} \tag{4.2.1}$$

where a, b, c, d, e, f, u, v are arbitrary smooth functions on U. Choosing two of the three variables a, b and c such that $ab - c^2 = r^4 \sin^2 \theta$ satisfies condition (4.1.8).

The fourth condition requires $\vec{H}_{t,r}$, the mean curvature vector field of $S_{t,r}$, to be tangential to the $t =$ constant slice. This is equivalent to requiring $\vec{H}_{t,r}$ to be parallel to $\frac{\partial}{\partial r}$. We compute the conditions on the metric components such that this is true.

Lemma 4.5. *The determinant of the spacetime metric $\overline{g}$ in (4.2.1) is given by:*

$$|\overline{g}| := det(\overline{g}) = (-u^2v^2 - d^2)(ab - c^2) + eu^2(cf - be) + fu^2(ce - af) \tag{4.2.2}$$

$$= (-u^2v^2 - d^2)(ab - c^2) + u^2(2cef - be^2 - af^2) \tag{4.2.3}$$

Moreover, the coordinate representation of the inverse $(\overline{g}^{-1})$ *is given by:*

$$
(\overline{g})^{-1} = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array}
\begin{array}{c}
\begin{array}{cccc} t & r & \theta & \phi \end{array} \\
\begin{pmatrix}
\frac{u^2(ab-c^2)}{|\overline{g}|} & \frac{-d(ab-c^2)}{|\overline{g}|} & \frac{u^2(cf-be)}{|\overline{g}|} & \frac{u^2(ce-af)}{|\overline{g}|} \\
\frac{-d(ab-c^2)}{|\overline{g}|} & \frac{-v^2(ab-c^2)+f(ce-af)+e(cf-be)}{|\overline{g}|} & \frac{-d(cf-be)}{|\overline{g}|} & \frac{-d(ce-af)}{|\overline{g}|} \\
\frac{u^2(cf-be)}{|\overline{g}|} & \frac{-d(cf-be)}{|\overline{g}|} & \frac{-u^2v^2b-u^2f^2-bd^2}{|\overline{g}|} & \frac{u^2v^2c+u^2ef+cd^2}{|\overline{g}|} \\
\frac{u^2(ce-af)}{|\overline{g}|} & \frac{-d(ce-af)}{|\overline{g}|} & \frac{u^2v^2c+u^2ef+cd^2}{|\overline{g}|} & \frac{-u^2v^2a-u^2e^2-ad^2}{|\overline{g}|}
\end{pmatrix}
\end{array}
\tag{4.2.4}
$$

Proof. See Section A.2.1 in Appendix A. □

4.2.1 Geometry of $S_{t,r}$ and the Normal Bundle $NS_{t,r}$

Fix a t = constant slice M. Let $S_{t,r}$ be a coordinate sphere in M. We endow $S_{t,r}$ with the induced metric from $\overline{g}$, denoted as g_S. Then in the $\{\theta, \phi\}$ coordinate system, g_S has the following representation:

$$
g_S := g|_{S_{t,r}} = \begin{array}{c} \theta \\ \phi \end{array} \overset{\begin{array}{cc} \theta & \phi \end{array}}{\begin{pmatrix} a & c \\ c & b \end{pmatrix}} \tag{4.2.5}
$$

Thus its inverse metric is:

$$
g_S^{-1} = \frac{1}{ab - c^2} \begin{array}{c} \theta \\ \phi \end{array} \overset{\begin{array}{cc} \theta & \phi \end{array}}{\begin{pmatrix} b & -c \\ -c & a \end{pmatrix}} \tag{4.2.6}
$$

The normal bundle $NS_{t,r}$ of $S_{t,r}$ is of rank-two. $\frac{\partial}{\partial r}$ is a nonzero section of $NS_{t,r}$, and thus can be used as a basis for the normal bundle. Let $\mathbf{n}$ be a complementary basis vector field of the normal bundle that is orthogonal to $\frac{\partial}{\partial r}$. Since $\frac{\partial}{\partial r}$ is outward spacelike, we can assume that $\mathbf{n}$ is future timelike. Therefore using the basis

$\{\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\}$ we can write $\mathbf{n}$ as

$$\mathbf{n} = \frac{\partial}{\partial t} + x\frac{\partial}{\partial r} + y\frac{\partial}{\partial \theta} + z\frac{\partial}{\partial \phi},$$

with x, y, z yet to be determined, such that

- $\langle \mathbf{n}, \frac{\partial}{\partial r} \rangle = 0;$
- $\langle \mathbf{n}, \frac{\partial}{\partial \theta} \rangle = 0;$
- $\langle \mathbf{n}, \frac{\partial}{\partial \phi} \rangle = 0.$

We have three equations and three unknowns which give us

$$\mathbf{n} = \frac{\partial}{\partial t} + \frac{-d}{u^2}\frac{\partial}{\partial r} + \frac{cf - be}{ab - c^2}\frac{\partial}{\partial \theta} + \frac{ce - af}{ab - c^2}\frac{\partial}{\partial \phi}. \tag{4.2.7}$$

Lemma 4.6.

$$\langle \boldsymbol{n}, \boldsymbol{n} \rangle = \frac{det(\overline{g})}{u^2(ab - c^2)} = \frac{det(\overline{g})}{u^2 det(g_S)} =: \frac{|\overline{g}|}{u^2|g_S|}, \tag{4.2.8}$$

where we set $|g_S| := det(g_S)$.

Proof. See Section A.2.2 in Appendix A. □

Remark. *Caution that since* $\boldsymbol{n}$ *is timelike,* $\langle \boldsymbol{n}, \boldsymbol{n} \rangle < 0$*. Thus*

$$||\boldsymbol{n}||_{\overline{g}} = (-\langle \boldsymbol{n}, \boldsymbol{n} \rangle)^{1/2} = \left(\frac{-|\overline{g}|}{u^2|g_S|}\right)^{1/2}. \tag{4.2.9}$$

Let $\{e_r, e_n\}$ be the normalized orthonormal frame obtained from $\{\frac{\partial}{\partial r}, \mathbf{n}\}$:

$$e_r := \frac{\partial}{\partial r}/||\frac{\partial}{\partial r}||_{\overline{g}} = \frac{1}{u}\frac{\partial}{\partial r}, \quad e_n := \frac{\mathbf{n}}{||\mathbf{n}||_{\overline{g}}}.$$

Let

$$\vec{I}_{t,r} = -\frac{\vec{H}_{t,r}}{\langle \vec{H}_{t,r}, \vec{H}_{t,r} \rangle}$$

be the outward-pointing inverse mean curvature vector. The geometry of the normal bundle of $S_{t,r}$ is given by Figure 4.2 below. Recall that $\vec{H}_{t,r}$ points inward by our convention.

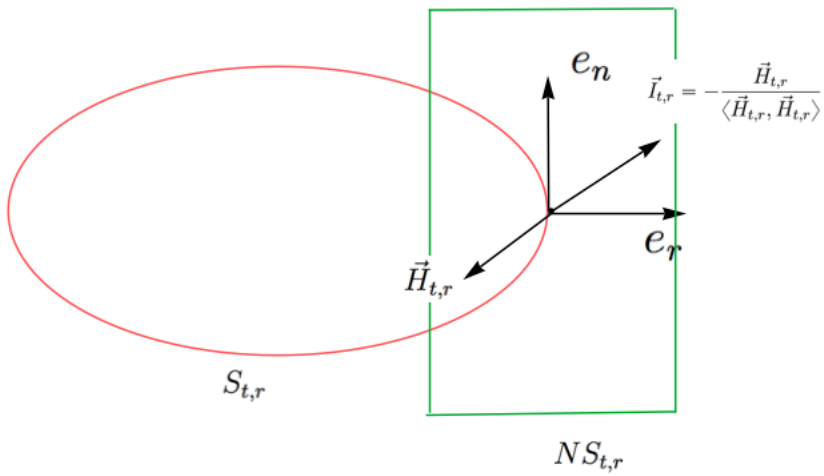

FIGURE 4.2: Normal bundle of coordinate sphere $S_{t,r}$.

4.2.2 Mean Curvature Vector of $S_{t,r}$

Now we have a coordinate sphere $(S_{t,r}, g_S)$ with induced metric g_S in $(U, \overline{g}, \overline{\nabla})$, where $\overline{\nabla}$ is the Levi-Civita connection with respect to $\overline{g}$. We have an orthonormal frame $\{e_r, e_n\}$ of the normal bundle $NS_{t,r}$ defined in the previous subsection. In this subsection we compute the mean curvature vector $\vec{H}_{t,r}$ of $S_{t,r}$. Recall the definition of

$\vec{H}_{t,r}$ from Equation (3.3.3):

$$\vec{H}_{t,r} = \text{trace}_{g_S}\vec{\text{II}} \qquad \text{(By definition of } \vec{H}_{t,r}\text{)}$$

$$= g_S^{ij}\vec{\text{II}}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \qquad \left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j} \in \{\tfrac{\partial}{\partial \theta}, \tfrac{\partial}{\partial \phi}\}\right)$$

$$= g_S^{ij}\left(\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right)\Bigg|_{NS_{t,r}} \qquad \text{(By definition of } \vec{\text{II}}(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j})\text{)}$$

$$= g_S^{ij}\left[\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, e_r\rangle e_r - \langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, e_n\rangle e_n\right]$$

$$(\{e_r, e_n\} \text{ is an orthonormal frame of } NS_{t,r})$$

$$= g_S^{ij}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, e_r\rangle e_r - g_S^{ij}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, e_n\rangle e_n \tag{4.2.10}$$

We compute the two parts in $\vec{H}_{t,r}$ separately. First of all

$$g_S^{ij}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, e_r\rangle = \frac{1}{u}g_S^{ij}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial r}\rangle = \frac{1}{u}g_S^{ij}\langle\overline{\Gamma}^k_{ij}\frac{\partial}{\partial x^k}, \frac{\partial}{\partial r}\rangle$$

$$= \frac{1}{u}g_S^{ij}(\overline{\Gamma}^t_{ij}\overline{g}_{tr} + \overline{\Gamma}^r_{ij}\overline{g}_{rr}) \qquad \left(\tfrac{\partial}{\partial r} \text{ is perpendicular to } \tfrac{\partial}{\partial \theta}, \tfrac{\partial}{\partial \phi}\right)$$

$$= \frac{1}{u}\overline{g}_{tr}(g_S^{\theta\theta}\overline{\Gamma}^t_{\theta\theta} + 2g_S^{\theta\phi}\overline{\Gamma}^t_{\theta\phi} + g_S^{\phi\phi}\overline{\Gamma}^t_{\phi\phi})$$

$$+ \frac{1}{u}\overline{g}_{rr}(g_S^{\theta\theta}\overline{\Gamma}^r_{\theta\theta} + 2g_S^{\theta\phi}\overline{\Gamma}^r_{\theta\phi} + g_S^{\phi\phi}\overline{\Gamma}^r_{\phi\phi})$$

$$= \frac{1}{u}\frac{1}{|g_S|}\left[\overline{g}_{tr}(b\overline{\Gamma}^t_{\theta\theta} - 2c\overline{\Gamma}^t_{\theta\phi} + a\overline{\Gamma}^t_{\phi\phi}) + \overline{g}_{rr}(b\overline{\Gamma}^r_{\theta\theta} - 2c\overline{\Gamma}^r_{\theta\phi} + a\overline{\Gamma}^r_{\phi\phi})\right]$$

$$= \frac{1}{u}\frac{1}{|g_S|}\left[\overline{g}_{tr}(\star) + \overline{g}_{rr}(\star\star)\right] \tag{4.2.11}$$

where

$$(\star) := b\overline{\Gamma}^t_{\theta\theta} - 2c\overline{\Gamma}^t_{\theta\phi} + a\overline{\Gamma}^t_{\phi\phi}, \tag{4.2.12}$$

and

$$(\star\star) := b\overline{\Gamma}^r_{\theta\theta} - 2c\overline{\Gamma}^r_{\theta\phi} + a\overline{\Gamma}^r_{\phi\phi}. \tag{4.2.13}$$

We now compute all the related Christoffel symbols.

$$
\begin{aligned}
\overline{\Gamma}^t_{\theta\theta} &= \frac{1}{2}\Big(\overline{g}^{tt}(2\overline{g}_{\theta t,\theta} - \overline{g}_{\theta\theta,t}) + \overline{g}^{tr}(2\overline{g}_{\theta r,\theta} - \overline{g}_{\theta\theta,r}) + \overline{g}^{t\theta}(2\overline{g}_{\theta\theta,\theta} - \overline{g}_{\theta\theta,\theta}) + \overline{g}^{t\phi}(2\overline{g}_{\theta\phi,\theta} \\
&- \overline{g}_{\theta\theta,\phi})\Big) \\
&= \frac{1}{2}\Big(\overline{g}^{tt}(2\overline{g}_{\theta t,\theta} - \overline{g}_{\theta\theta,t}) - \overline{g}^{tr}\overline{g}_{\theta\theta,r} + \overline{g}^{t\theta}\overline{g}_{\theta\theta,\theta} + \overline{g}^{t\phi}(2\overline{g}_{\theta\phi,\theta} - \overline{g}_{\theta\theta,\phi})\Big) \\
&= \frac{1}{2|\overline{g}|}\Big(u^2(ab-c^2)(2e_{,\theta} - a_{,t}) + d(ab-c^2)a_{,r} + u^2(cf-be)a_{,\theta} + u^2(ce-af)(2c_{,\theta} \\
&- a_{,\phi})\Big) \qquad (4.2.14)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\Gamma}^t_{\theta\phi} &= \frac{1}{2}\Big(\overline{g}^{tt}(\overline{g}_{\phi t,\theta} + \overline{g}_{\theta t,\phi} - \overline{g}_{\theta\phi,t}) + \overline{g}^{tr}(g_{\phi r,\theta} + \overline{g}_{\theta r,\phi} - \overline{g}_{\theta\phi,r}) + \overline{g}^{t\theta}(\overline{g}_{\phi\theta,\theta} + \overline{g}_{\theta\theta,\phi} - \overline{g}_{\theta\phi,\theta}) \\
&+ \overline{g}^{t\phi}(\overline{g}_{\phi\phi,\theta} + \overline{g}_{\theta\phi,\phi} - \overline{g}_{\theta\phi,\phi})\Big) \\
&= \frac{1}{2}\Big(\overline{g}^{tt}(\overline{g}_{\phi t,\theta} + \overline{g}_{\theta t,\phi} - \overline{g}_{\theta\phi,t}) - \overline{g}^{tr}\overline{g}_{\theta\phi,r} + \overline{g}^{t\theta}\overline{g}_{\theta\theta,\phi} + \overline{g}^{t\phi}\overline{g}_{\phi\phi,\theta}\Big) \\
&= \frac{1}{2|\overline{g}|}\Big(u^2(ab-c^2)(f_{,\theta} + e_{,\phi} - c_{,t}) + d(ab-c^2)c_{,r} + u^2(cf-be)a_{,\phi} + u^2(ce-af)b_{,\theta}\Big) \\
&\qquad (4.2.15)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\Gamma}^t_{\phi\phi} &= \frac{1}{2}\Big(\overline{g}^{tt}(2\overline{g}_{\phi t,\phi} - \overline{g}_{\phi\phi,t}) + \overline{g}^{tr}(2\overline{g}_{\phi r,\phi} - \overline{g}_{\phi\phi,r}) + \overline{g}^{t\theta}(2\overline{g}_{\phi\theta,\phi} - \overline{g}_{\phi\phi,\theta}) + \overline{g}^{t\phi}(2\overline{g}_{\phi\phi,\phi} \\
&- \overline{g}_{\phi\phi,\phi})\Big) \\
&= \frac{1}{2}\Big(g^{tt}(2\overline{g}_{\phi t,\phi} - \overline{g}_{\phi\phi,t}) - \overline{g}^{tr}\overline{g}_{\phi\phi,r} + \overline{g}^{t\theta}(2\overline{g}_{\phi\theta,\phi} - \overline{g}_{\phi\phi,\theta}) + \overline{g}^{t\phi}\overline{g}_{\phi\phi,\phi}\Big) \\
&= \frac{1}{2|\overline{g}|}\Big(u^2(ab-c^2)(2f_{,\phi} - b_{,t}) + d(ab-c^2)b_{,r} + u^2(cf-be)(2c_{,\phi} - b_{,\theta}) \\
&+ u^2(ce-af)b_{,\phi}\Big) \qquad (4.2.16)
\end{aligned}
$$

$$\overline{\Gamma}^r_{\theta\theta} = \frac{1}{2}\Big(\overline{g}^{rt}(2\overline{g}_{\theta t,\theta} - \overline{g}_{\theta\theta,t}) + \overline{g}^{rr}(2\overline{g}_{\theta r,\theta} - g_{\theta\theta,r}) + \overline{g}^{r\theta}(2\overline{g}_{\theta\theta,\theta} - \overline{g}_{\theta\theta,\theta}) + \overline{g}^{r\phi}(2\overline{g}_{\theta\phi,\theta}$$

$$- \overline{g}_{\theta\theta,\phi})\Big)$$

$$= \frac{1}{2}\Big(\overline{g}^{rt}(2\overline{g}_{\theta t,\theta} - \overline{g}_{\theta\theta,t}) - \overline{g}^{rr}\overline{g}_{\theta\theta,r} + \overline{g}^{r\theta}\overline{g}_{\theta\theta,\theta} + \overline{g}^{r\phi}(2\overline{g}_{\theta\phi,\theta} - \overline{g}_{\theta\theta,\phi})\Big)$$

$$= \frac{1}{2|\overline{g}|}\Big(- d(ab - c^2)(2e_{,\theta} - a_{,t}) - [-v^2(ab - c^2) + f(ce - af) + e(cf - be)]a_{,r}$$

$$+ (-d)(cf - be)a_{,\theta} + (-d)(ce - af)(2c_{,\theta} - a_{,\phi})\Big) \tag{4.2.17}$$

$$\overline{\Gamma}^r_{\theta\phi} = \frac{1}{2}\Big(\overline{g}^{rt}(\overline{g}_{\phi t,\theta} + \overline{g}_{\theta t,\phi} - \overline{g}_{\theta\phi,t}) + \overline{g}^{rr}(\overline{g}_{\phi r,\theta} + \overline{g}_{\theta r,\phi} - \overline{g}_{\theta\phi,r}) + \overline{g}^{r\theta}(\overline{g}_{\phi\theta,\theta} + \overline{g}_{\theta\theta,\phi} - \overline{g}_{\theta\phi,\theta})$$

$$+ \overline{g}^{r\phi}(\overline{g}_{\phi\phi,\theta} + \overline{g}_{\theta\phi,\phi} - \overline{g}_{\theta\phi,\phi})\Big)$$

$$= \frac{1}{2}\Big(\overline{g}^{rt}(\overline{g}_{\phi t,\theta} + \overline{g}_{\theta t,\phi} - \overline{g}_{\theta\phi,t}) - \overline{g}^{rr}\overline{g}_{\theta\phi,r} + \overline{g}^{r\theta}\overline{g}_{\theta\theta,\phi} + \overline{g}^{r\phi}\overline{g}_{\phi\phi,\theta}\Big)$$

$$= \frac{1}{2|\overline{g}|}\Big(- d(ab - c^2)(f_{,\theta} + e_{,\phi} - c_{,t}) - [-v^2(ab - c^2) + f(ce - af) + e(cf - be)]c_{,r}$$

$$+ (-d)(cf - be)a_{,\phi} + (-d)(ce - af)b_{,\theta}\Big) \tag{4.2.18}$$

$$\overline{\Gamma}^r_{\phi\phi} = \frac{1}{2}\Big(\overline{g}^{rt}(2\overline{g}_{\phi t,\phi} - \overline{g}_{\phi\phi,t}) + \overline{g}^{rr}(2\overline{g}_{\phi r,\phi} - \overline{g}_{\phi\phi,r}) + \overline{g}^{r\theta}(2\overline{g}_{\phi\theta,\phi} - \overline{g}_{\phi\phi,\theta}) + \overline{g}^{r\phi}(2\overline{g}_{\phi\phi,\phi}$$

$$- \overline{g}_{\phi\phi,\phi})\Big)$$

$$= \frac{1}{2}\Big(\overline{g}^{rt}(2\overline{g}_{\phi t,\phi} - \overline{g}_{\phi\phi,t}) - \overline{g}^{rr}\overline{g}_{\phi\phi,r} + \overline{g}^{r\theta}(2\overline{g}_{\phi\theta,\phi} - \overline{g}_{\phi\phi,\theta}) + \overline{g}^{r\phi}\overline{g}_{\phi\phi,\phi}\Big)$$

$$= \frac{1}{2|\overline{g}|}\Big(- d(ab - c^2)(2f_{,\phi} - b_{,t}) - [-v^2(ab - c^2) + f(ce - af) + e(cf - be)]b_{,r}$$

$$+ (-d)(cf - be)(2c_{,\phi} - b_{,\theta}) + (-d)(ce - af)b_{,\phi}\Big) \tag{4.2.19}$$

With all the Christoffel symbols computed, we have:

$$
\begin{aligned}
(\star) &= b\overline{\Gamma}^{t}_{\theta\theta} - 2c\overline{\Gamma}^{t}_{\theta\phi} + a\overline{\Gamma}^{t}_{\phi\phi} \\
&= \frac{1}{2|\overline{g}|}\Big\{u^2(ab-c^2)[2be_{,\theta} - \cancel{a_{,t}b} - 2ce_{,\phi} - 2cf_{,\theta} + \cancel{2ce_{,t}} + 2af_{,\phi} - \cancel{ab_{,t}}] \\
&+ d(ab-c^2)[a_{,r}b - 2cc_{,r} + ab_{,r}] + u^2(cf-be)[a_{,\theta}b - 2a_{,\phi}c + 2ac_{,\phi} - ab_{,\theta}] \\
&+ u^2(ce-af)[2bc_{,\theta} - a_{,\phi}b - 2b_{,\theta}c + ab_{,\phi}]\Big\}
\end{aligned}
\tag{4.2.20}
$$

The terms cancel in the second line above since $ab - c^2 = r^4\sin^2\theta$, and thus is not a function of t.

Next we compute

$$
\begin{aligned}
(\star\star) &= b\overline{\Gamma}^{r}_{\theta\theta} - 2c\overline{\Gamma}^{r}_{\theta\phi} + a\overline{\Gamma}^{r}_{\phi\phi} \\
&= \frac{1}{2|\overline{g}|}\Big\{-d(ab-c^2)[2be_{,\theta} - \cancel{a_{,t}b} - 2ce_{,\phi} - 2cf_{,\theta} + \cancel{2ce_{,t}} + 2af_{,\phi} - \cancel{ab_{,t}}] \\
&- [-v^2(ab-c^2) + f(ce-af) + e(cf-be)][a_{,r}b - 2cc_{,r} + ab_{,r}] \\
&- d(cf-be)[a_{,\theta}b - 2a_{,\phi}c + 2ac_{,\phi} - ab_{,\theta}] - d(ce-af)[2bc_{,\theta} - a_{,\phi}b - 2b_{,\theta}c + ab_{,\phi}]\Big\}
\end{aligned}
\tag{4.2.21}
$$

Now plug (4.2.20) and (4.2.21) back into (4.2.11):

$$g_S^{ij}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, e_r\rangle = \frac{1}{u}\frac{1}{|g_S|}\big[d(\star) + u^2(\star\star)\big]$$

$$= \frac{1}{u}\frac{1}{|g_S|}\frac{1}{2|\overline{g}|}\Big\{[du^2(ab-c^2) - u^2d(ab-c^2)][2be_{,\theta} - 2ce_{,\phi} - 2cf_{,\theta} + 2af_{,\phi}]$$

$$+ [d^2(ab-c^2) + u^2v^2(ab-c^2) - fu^2(ce-af) - eu^2(cf-be)][a_{,r}b - 2cc_{,r} + ab_{,r}]$$

$$+ [du^2(cf-be) - u^2d(cf-be)][a_{,\theta}b - 2a_{,\phi}c + 2ac_{,\phi} - ab_{,\theta}]$$

$$+ [du^2(ce-af) - u^2d(ce-af)][2bc_{,\theta} - a_{,\phi}b - 2b_{,\theta}c + ab_{,\phi}]\Big\}$$

$$= \frac{1}{u}\frac{1}{|g_S|}\frac{1}{2|\overline{g}|}[d^2(ab-c^2) + u^2v^2(ab-c^2) - fu^2(ce-af) - eu^2(cf-be)][a_{,r}b - 2cc_{,r}$$

$$+ ab_{,r}]$$

$$= \frac{1}{u}\frac{1}{|g_S|}\frac{1}{2|\overline{g}|}(-|\overline{g}|)(ab-c^2)_{,r} \qquad \text{(By Equation 4.2.2)}$$

$$= -\frac{1}{u}\frac{4r^3\cancel{\sin^2\theta}}{r^4\cancel{\sin^2\theta}}\frac{1}{2} = -\frac{2}{r}\frac{1}{u}. \tag{4.2.22}$$

Next we compute the second term in (4.2.10):

$$g_S^{ij}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, e_n\rangle = \frac{1}{||\mathbf{n}||_{\overline{g}}}g_S^{ij}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, \mathbf{n}\rangle = \frac{1}{||\mathbf{n}||_{\overline{g}}}g_S^{ij}\langle\overline{\Gamma}^t_{ij}\frac{\partial}{\partial t}, \mathbf{n}\rangle \qquad (\mathbf{n} \perp \tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial\theta}, \tfrac{\partial}{\partial\phi})$$

$$= \frac{1}{||\mathbf{n}||_{\overline{g}}}g_S^{ij}\overline{\Gamma}^t_{ij}\langle\mathbf{n}, \mathbf{n}\rangle \qquad (\mathbf{n} - \tfrac{\partial}{\partial t} \text{ is spacelike})$$

$$= -g_S^{ij}\overline{\Gamma}^t_{ij}||\mathbf{n}||_{\overline{g}} = -||\mathbf{n}||_{\overline{g}}(g_S^{\theta\theta}\overline{\Gamma}^t_{\theta\theta} + 2g_S^{\theta\phi}\overline{\Gamma}^t_{\theta\phi} + g_S^{\phi\phi}\overline{\Gamma}^t_{\phi\phi})$$

$$= -\frac{||\mathbf{n}||_{\overline{g}}}{|g_S|}(\star) \tag{4.2.23}$$

where $(\star)$ is computed in (4.2.20). Now plug (4.2.8) into the above, we get:

$$g_S^{ij}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, e_n\rangle = -\left(\frac{-|\overline{g}|}{u^2|g_S|}\right)^{1/2}\frac{1}{|g_S|}(\star) = -\frac{1}{u}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}(\star). \tag{4.2.24}$$

Combing (4.2.22) and (4.2.24), we have:

Proposition 4.7. *The mean curvature vector $\vec{H}_{t,r}$ of coordinate sphere $S_{t,r}$ in the spacetime metric $\overline{g}$ (4.2.1) is given by:*

$$\vec{H}_{t,r} = -\frac{2}{r}\frac{1}{u}e_r + \frac{1}{u}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}(\star)e_n. \tag{4.2.25}$$

where $(\star)$ is given by (4.2.20).

Corollary 4.8. *The mean curvature vector $\vec{H}_{t,r}$ of coordinate sphere $S_{t,r}$ is parallel to $\frac{\partial}{\partial r}$ everywhere if and only if $(\star) = 0$, where $(\star)$ is given by (4.2.20). Moreover, in this case, the mean curvature vector equals:*

$$\vec{H}_{t,r} = -\frac{2}{r}\frac{1}{u}e_r. \tag{4.2.26}$$

Notice the similarity between the mean curvature vector expression above and the mean curvature vector in the spherically symmetric case in (3.4.3).

Proof. This is quite straightforward since $\vec{H}_{t,r}$ is parallel to $\frac{\partial}{\partial r}$ everywhere if and only if $\langle \vec{H}_{t,r}, e_n \rangle = 0$. Expand this out we get:

$$0 = \langle \vec{H}_{t,r}, e_n \rangle = \langle \frac{1}{u}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}(\star)e_n, e_n \rangle = -\frac{1}{u}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}(\star).$$

Since $u \neq 0$, the above is equivalent to

$$(\star) = 0,$$

as desired. $\square$

Therefore the fourth condition in the definition of inverse mean curvature vector flow coordinate chart is equivalent to $(\star) = 0$, that is:

$$0 = \frac{1}{2|\overline{g}|}\Big\{u^2(ab-c^2)[2be_{,\theta} - 2ce_{,\phi} - 2cf_{,\theta} + 2af_{,\phi}]$$
$$+ d(ab-c^2)(ab-c^2)_{,r} + u^2(cf-be)[a_{,\theta}b - 2a_{,\phi}c + 2ac_{,\phi} - ab_{,\theta}]$$
$$+ u^2(ce-af)[2bc_{,\theta} - a_{,\phi}b - 2b_{,\theta}c + ab_{,\phi}]\Big\}$$

Or equivalently, using $ab - c^2 = r^4 \sin^2\theta$,

$$0 = [2be_{,\theta} - 2ce_{,\phi} - 2cf_{,\theta} + 2af_{,\phi}] + d\frac{4r^3\sin^2\theta}{u^2}$$
$$+ \frac{cf-be}{r^4\sin^2\theta}[a_{,\theta}b - 2a_{,\phi}c + 2ac_{,\phi} - ab_{,\theta}] + \frac{ce-af}{r^4\sin^2\theta}[2bc_{,\theta} - a_{,\phi}b - 2b_{,\theta}c + ab_{,\phi}]. \tag{4.2.27}$$

Here is the upshot: Equation (4.2.27) is *zeroth order* in the metric component d, thus we can choose two of the three variables a, b, c, and e, f, u, v all together 6 variables, and solve for d explicitly:

$$d = -\frac{u^2}{4r^3\sin^2\theta}\Big\{[2be_{,\theta} - 2ce_{,\phi} - 2cf_{,\theta} + 2af_{,\phi}] + \frac{cf-be}{r^4\sin^2\theta}[a_{,\theta}b - 2a_{,\phi}c + 2ac_{,\phi} - ab_{,\theta}]$$
$$+ \frac{ce-af}{r^4\sin^2\theta}[2bc_{,\theta} - a_{,\phi}b - 2b_{,\theta}c + ab_{,\phi}]\Big\}. \tag{4.2.28}$$

Thus all four conditions (4.1.6), (4.1.7) and (4.1.8) and (4.1.9) are solvable with infinitely many solutions. Combing the above, we have proved Proposition 4.3 and hence the main theorem 4.4.

There are six degrees of freedom in constructing our spacetime metric $\overline{g}$ that admits an inverse mean curvature vector flow coordinate chart. Moreover, the six free variables do not need to be spherically symmetric. The spherically symmetric (3.4.1) metric is a special case of this large set of spacetime metrics. It is unknown if perturbations of spherically symmetric spacetime with inverse mean curvature vector flow coordinate chart still have inverse mean curvature vector flow solutions. More specifically, we conjecture that:

Conjecture 4.1. *Given Minkowski space with inverse mean curvature vector flow coordinate chart that can be smoothly extended to the boundary, consider a perturbation of the spacetime metric. The resulting spacetime still admits inverse mean curvature vector flow solutions (in a single spacelike hypersurface) that exist for all time.*

Notice that in Equation (4.2.28) if all the variables are smooth then d will be smooth except possibly when $\sin\theta = 0$, since d is not defined by our formula there (see (4.2.28)). This happens at the north ($\theta = 0$) and south pole ($\theta = \pi$), which are two *coordinate chart singularities*, not metric singularities of the spacetime.

If c, e, f are chosen to be 0 at a neighborhood of the north and the south pole, then the right hand side of (4.2.28) will be zero there. In this way d can be extended smoothly across the two coordinate chart singularities, and will be smooth on the entire spacetime.

Another way to extend d smoothly over the coordinate chart singularities is to choose the metric to be spherically symmetric in a neighborhood of the north and south pole:

$$g = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} -v^2(t,r) & 0 & 0 & 0 \\ 0 & u^2(t,r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \end{array} \tag{4.2.29}$$

This has the advantage that $d = 0$ around the coordinate chart singularities. Then extend a, b, c, e, f smoothly to the entire spacetime while maintaining the condition that $ab - c^2 = r^4\sin^2\theta$. The resulting metric still satisfies the four conditions since smooth inverse mean curvature vector flow of spheres exists for all time in spherically symmetry. d will be smooth since $d = 0$ identically.

One could study more general asymptotic conditions for (4.2.28) to be smooth and bounded as θ approaches 0 or π, but we choose not to discuss it further here.

Remark. *We can actually prove that $\vec{H}_{t,r}$ takes the form (4.2.26) in inverse mean curvature vector flow coordinates without computing it out explicitly. We now show that as a sanity check of our computation. In inverse mean curvature vector flow coordinates $\vec{H}_{t,r}$ is parallel to $\frac{\partial}{\partial r}$, thus let $\lambda = \lambda(t, r, \theta, \phi)$ such that*

$$\vec{H}_{t,r} = \lambda \frac{\partial}{\partial r}.$$

The inverse mean curvature vector is now

$$\vec{I}_{t,r} = -\frac{\vec{H}_{t,r}}{\langle \vec{H}_{t,r}, \vec{H}_{t,r} \rangle} = -\frac{\lambda \frac{\partial}{\partial r}}{\langle \lambda \frac{\partial}{\partial r}, \lambda \frac{\partial}{\partial r} \rangle} = -\frac{1}{\lambda u^2} \frac{\partial}{\partial r}.$$

By the first variation of area formula (A.3.7), the rate of change of the area form of $S_{t,r}$ under outward radial flow *is given by*

$$\begin{aligned} \frac{d}{dr} dA_{S_{t,r}} &= -\langle \vec{H}_{t,r}, \frac{\partial}{\partial r} \rangle dA_{S_{t,r}} \\ &= -\langle \vec{H}_{t,r}, \frac{\partial}{\partial r} \rangle r^2 \sin\theta d\theta d\phi \\ &= -\lambda u^2 r^2 \sin\theta d\theta d\phi \end{aligned} \tag{4.2.30}$$

Notice the left hand side of the above equals to

$$\frac{d}{dr} dA_{S_{t,r}} = \frac{d}{dr}(r^2 \sin\theta d\theta d\phi) = 2r \sin\theta d\theta d\phi.$$

Thus matching the two sides we get:

$$2r \sin\theta d\theta d\phi = -\lambda u^2 r^2 \sin\theta d\theta d\phi,$$

that is $\lambda = -\frac{2}{r}\frac{1}{u^2}$. *Therefore*

$$\vec{H}_{t,r} = \lambda \frac{\partial}{\partial r} = -\frac{2}{r}\frac{1}{u^2}\frac{\partial}{\partial r} = -\frac{2}{r}\frac{1}{u} e_r,$$

which is the same as (4.2.25).

4.3 Coordinate Free Analogue and Steering Parameters

In the previous section we have shown that there exist many spacetimes that admit inverse mean curvature vector flow coordinate chart, in which the coordinates spheres are solutions to the inverse mean curvature vector flow equation. The fourth condition (4.1.9) in the definition of inverse mean curvature vector flow coordinates can be viewed as a *steering condition* that keeps the flow direction of coordinate spheres tangential to a spacelike hypersurface.

Given a spacetime $(N^4, \overline{g}, \overline{\nabla})$, a spacelike hypersurface (M^3, g) with induced metric g, and a closed embedded surface (Σ, g_Σ) in M with induced metric g_S. Assuming the normal bundle of Σ is trivial, there is a unique unit outward normal vector field of Σ in M, denoted as e_r. Let e_t be the unit outward normal vector field of Σ in N that is perpendicular to e_r. Since M is spacelike, e_r is spacelike and e_t is timelike. Define a local coordinate chart $\{\theta, \phi\}$ on Σ, and let $\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\}$ be the local coordinate frame. Figure 4.3 depicts the above set up, in which the mean curvature vector $\vec{H}_\Sigma$ is not necessarily tangential to M. We can extend the frame $\{e_t, e_r, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\}$ to a frame on a neighborhood of Σ in N, and we will identify the frame with its extension.

Since the local frame $\{e_t, e_r, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\}$ is not fully a coordinate frame, the commutator coefficients C_{ij}^k defined as:

$$[\alpha_i, \alpha_j] = C_{ij}^k \alpha_k, \quad \alpha_i \in \{e_t, e_r, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\} \tag{4.3.1}$$

are not necessarily zero. We need these coefficients to compute the connection coefficients with respect this frame later.

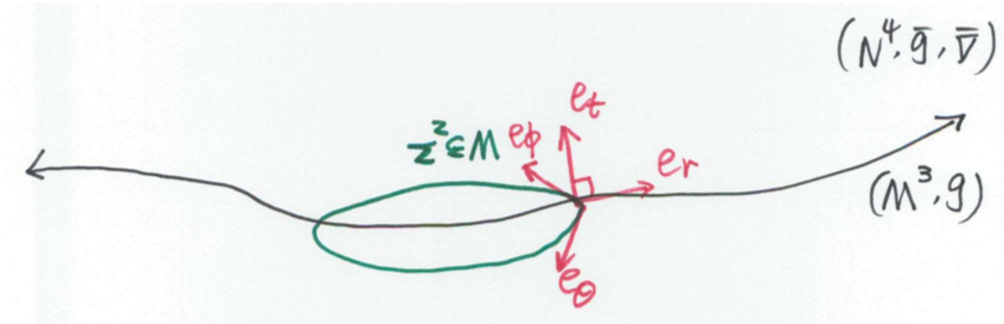

FIGURE 4.3: Setup of inverse mean curvature vector flow steering.

With the local frame $\{e_t, e_r, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\}$ of the tangent bundle of N, let the associated dual frame of the cotangent bundle of N be $\{\beta_t, \beta_r, d\theta, d\phi\}$, where

$$\beta_i(e_j) = \delta_{ij}, \quad i, j \in \{t, r\}.$$

With respect to this dual frame, we can write the spacetime metric $\bar{g}$ as

$$[\bar{g}] = \begin{array}{c} \\ e_t \\ e_r \\ \frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\phi} \end{array} \begin{array}{c} \begin{array}{cccc} e_t & e_r & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & c & b \end{pmatrix} \end{array} \tag{4.3.2}$$

where a, b, c are smooth functions on N.

We want to change the metric so that $\vec{H}_\Sigma$ is tangential to M. Recall that the fourth condition (4.1.9) in the construction of the inverse mean curvature vector flow coordinates is a zeroth order equation for d, the (t, r)-metric component. This motivates the following definition:

Definition 4.3.1 (Steering of Spacetime Metric)**.** *Given a spacetime metric* $\overline{g}$*. A metric* $\overline{g}_Q$ *on* N *is called a* steering *of* $\overline{g}$ *if*

$$\overline{g}_Q := \overline{g} + Q(\beta_t \otimes \beta_r + \beta_r \otimes \beta_t) \tag{4.3.3}$$

for some smooth function $Q \in C^\infty(N)$*.* Q *is called a* steering parameter*.*

Note that the coefficient matrix for the steered metric $\overline{g}_Q$ is given by:

$$[\overline{g}_Q] = \begin{array}{c} \\ e_t \\ e_r \\ \frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\phi} \end{array} \begin{array}{c} \begin{array}{cccc} e_t & e_r & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \end{array} \\ \begin{pmatrix} -1 & Q & 0 & 0 \\ Q & 1 & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & c & b \end{pmatrix} \end{array} \tag{4.3.4}$$

Thus geometrically, e_t and e_r are not necessarily orthogonal to each other in the steered metric (see Figure 4.4).

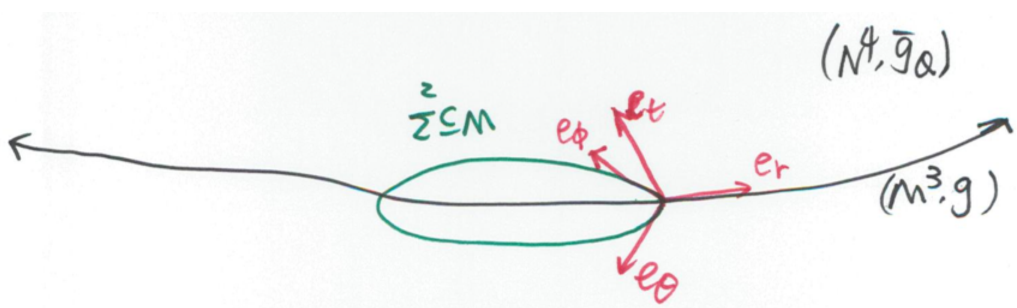

FIGURE 4.4: After steering the spacetime metric $\overline{g}$, e_t and e_r are not necessarily orthogonal to each other. The metric g on M remains the same.

Definition 4.3.2 (Area Expanding Condition)**.** $\Sigma \subset M$ *given as above is said to be area expanding if*

$$e_r(ab - c^2) > 0. \tag{4.3.5}$$

Theorem 4.9. *Let $(N^4, \overline{g})$, a spacelike hypersurface M^3 and a closed embedded surface (Σ^2, g_Σ) in M be given as above. If Σ is area expanding, then there exists a unique smooth steering parameter $Q = Q_\Sigma \in C^\infty(N)$, such that in the steered spacetime metric $\overline{g}_Q$, $\vec{H}_\Sigma$ is tangential to M everywhere on Σ.*

Proof. To show that there exists Q such that $\vec{H}_\Sigma$ is tangential to M, it suffices to find Q such that $\vec{H}_\Sigma$ (with respect to $\overline{g}_Q$) is parallel to e_r.

Notation: In the following the subscript Q will be dropped for simplicity and all the spacetime metric will be referring to the steered metric $\overline{g}_Q$.

The computations are similar to the inverse mean curvature vector flow coordinates case. We divide the computations into five steps:

Step 1: Pick a local normal variation of Σ and let $\{\Sigma_s\}$ be the variational surfaces, $s \in (0, \epsilon)$. Since $\Sigma_0 = \Sigma$ is area expanding, we can assume that this normal variation is also area expanding. Extend $\{e_t, e_r\}$ to a neighborhood of Σ in N such that e_r remains outward unit normal to each Σ_s in M. Define an r-coordinate by requiring the area of Σ_s to be $A(s) =: 4\pi r^2$. Since the variation is area expanding, r is well-defined. Extend $\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\}$ to a neighborhood of Σ in N such that they are constant along normal directions of Σ in M. By this extension $\frac{\partial}{\partial r}$ is perpendicular to each Σ_s. Thus there exists a function λ such that $e_r = \lambda\frac{\partial}{\partial r}$.

$$[e_r, \frac{\partial}{\partial\theta}] = [\lambda\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}] = -\lambda_{,\theta}\frac{\partial}{\partial t}. \tag{4.3.6}$$

This implies that $C^\theta_{r\theta} = 0$. Similarly $C^\phi_{r\phi} = 0$.

Step 2: After steering the metric $\overline{g}$, e_r and e_t are not necessarily orthogonal to each other. Let $\mathbf{n}$ be the normal vector of Σ such that $\langle \mathbf{n}, e_r\rangle = 0$. The computation of $\mathbf{n}$ is the same as the inverse mean curvature vector flow coordinate case (4.2.7), and we obtain:

$$\mathbf{n} = e_t - Qe_r. \tag{4.3.7}$$

Note that $\langle \mathbf{n}, \mathbf{n} \rangle = \langle e_t - Qe_r, e_t - Qe_r \rangle = -1 - 2Q^2 + Q^2 = -(1 + Q^2)$, which agrees with (4.2.8) with our metric $\overline{g}_Q$.

Let $e_n := \frac{\mathbf{n}}{||\mathbf{n}||} = \frac{e_t - Qe_r}{(1+Q^2)^{1/2}}$ be the unit timelike normal vector.

Step 3: With respect to the orthonormal frame $\{e_n, e_r\}$ of the normal bundle $N\Sigma$, we can write the mean curvature vector as

$$\vec{H}_\Sigma = \langle \vec{H}_\Sigma, e_r \rangle e_r - \langle \vec{H}_\Sigma, e_n \rangle e_n. \tag{4.3.8}$$

Therefore $\vec{H}_\Sigma$ is parallel to e_r if and only if $\langle \vec{H}_\Sigma, e_n \rangle = 0$. Now we compute the condition on Q such that this inner product vanishes.

$$\begin{aligned}
\langle \vec{H}_\Sigma, e_n \rangle &= \langle \mathrm{trace}_{g_\Sigma} \mathrm{II}, e_n \rangle = g_\Sigma^{ij} \langle \left(\overline{\nabla}_{\partial_i} \partial_j \right) |_{N\Sigma}, e_n \rangle && (\partial_i, \partial_j \in \{\tfrac{\partial}{\partial \theta}, \tfrac{\partial}{\partial \phi}\}) \\
&= g_\Sigma^{ij} \left\langle \langle \overline{\nabla}_{\partial_i} \partial_j, e_r \rangle e_r - \langle \overline{\nabla}_{\partial_i} \partial_j, e_n \rangle e_n, e_n \right\rangle \\
&= -g_\Sigma^{ij} \langle \overline{\nabla}_{\partial_i} \partial_j, e_n \rangle \langle e_n, e_n \rangle = g_\Sigma^{ij} \langle \overline{\nabla}_{\partial_i} \partial_j, e_n \rangle && (\langle e_n, e_n \rangle = -1) \\
&= \frac{1}{||\mathbf{n}||} g_\Sigma^{ij} \langle \overline{\nabla}_{\partial_i} \partial_j, \mathbf{n} \rangle = \frac{1}{||\mathbf{n}||} g_\Sigma^{ij} \omega_{ij}^t \langle e_t, \mathbf{n} \rangle \\
&= \frac{1}{||\mathbf{n}||} g_\Sigma^{ij} \omega_{ij}^t (-(1 + Q^2)) = -g_\Sigma^{ij} \omega_{ij}^t \frac{1 + Q^2}{(1 + Q^2)^{1/2}} \\
&= -(1 + Q^2)^{1/2} g_\Sigma^{ij} \omega_{ij}^t && (4.3.9)
\end{aligned}$$

where ω is the connection coefficients of the connection $\overline{\nabla}_Q$ with respect to the metric $\overline{g}_Q$ and the local frame $\{e_t, e_r, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\}$, which is not necessarily a coordinate frame. This is an important difference from the inverse mean curvature vector flow coordinates case.

Recall that

$$\omega_{ij}^k = \frac{1}{2} \overline{g}^{kl} (\overline{g}_{il,j} + \overline{g}_{jl,i} - \overline{g}_{ij,l} + C_{lij} + C_{lji} - C_{ijl}), \tag{4.3.10}$$

where $C_{ijl} := \overline{g}_{lm} C_{ij}^m$, and C_{ij}^m is the commutator coefficients defined in (4.3.1).

Notation: we use $\overline{g}_{ij,k}$ to denote $e_k(\overline{g}_{ij})$, not necessarily a coordinate derivative.

Step 4: Now we compute the four connection coefficients as follows:

$$\begin{aligned}
\omega^t_{\theta\theta} &= \frac{1}{2}\overline{g}^{ti}(2\overline{g}_{\theta i,\theta} - \overline{g}_{\theta\theta,i} + 2C_{i\theta\theta} - C_{\theta\theta i}) \\
&= \frac{1}{2}\overline{g}^{ti}(2\overline{g}_{\theta i,\theta} - \overline{g}_{\theta\theta,i} + 2\overline{g}_{\theta j}C^j_{i\theta} - \cancelto{0}{\overline{g}_{ij}C^j_{\theta\theta}}) \qquad (\{\tfrac{\partial}{\partial\theta}, \tfrac{\partial}{\partial\phi}\} \text{ is a coordinate frame}) \\
&= \frac{1}{2}\Big\{\overline{g}^{tt}(\cancelto{0}{2\overline{g}_{\theta t,\theta}} - \overline{g}_{\theta\theta,t} + 2\overline{g}_{\theta j}C^j_{t\theta}) + \overline{g}^{tr}(\cancelto{0}{2\overline{g}_{\theta r,\theta}} - \overline{g}_{\theta\theta,r} + 2\overline{g}_{\theta j}C^j_{r\theta}) + 0 + 0\Big\} \\
&= \frac{1}{2}\Big\{\overline{g}^{tt}(-\overline{g}_{\theta\theta,t} + 2\overline{g}_{\theta\theta}C^\theta_{t\theta} + 2\overline{g}_{\theta\phi}C^\phi_{t\theta}) + \overline{g}^{tr}(-\overline{g}_{\theta\theta,r} + 2\overline{g}_{\theta\theta}C^\theta_{r\theta} + 2\overline{g}_{\theta\phi}C^\phi_{r\theta})\Big\} \\
&= \frac{ab-c^2}{2|\overline{g}|}\Big\{(-a_{,t} + 2aC^\theta_{t\theta} + 2cC^\phi_{t\theta}) - Q(-a_{,r} + 2aC^\theta_{r\theta} + 2cC^\phi_{r\theta})\Big\} \qquad (4.3.11)
\end{aligned}$$

$$\begin{aligned}
\omega^t_{\theta\phi} &= \frac{1}{2}\overline{g}^{ti}(\overline{g}_{\phi i,\theta} + \overline{g}_{\theta i,\phi} - \overline{g}_{\theta\phi,i} + C_{i\theta\phi} + C_{i\phi\theta} - C_{\theta\phi i}) \\
&= \frac{1}{2}\overline{g}^{ti}(\overline{g}_{\phi i,\theta} + \overline{g}_{\theta i,\phi} - \overline{g}_{\theta\phi,i} + \overline{g}_{\phi j}C^j_{i\theta} + \overline{g}_{\theta j}C^j_{i\phi} - \cancelto{0}{\overline{g}_{ij}C^j_{\theta\phi}}) \\
&= \frac{1}{2}\Big\{\overline{g}^{tt}(\cancelto{0}{\overline{g}_{\phi t,\theta}} + \cancelto{0}{\overline{g}_{\theta t,\phi}} - \overline{g}_{\theta\phi,t} + \overline{g}_{\phi\theta}C^\theta_{t\theta} + \overline{g}_{\phi\phi}C^\phi_{t\theta} + \overline{g}_{\theta\theta}C^\theta_{t\phi} + \overline{g}_{\theta\phi}C^\phi_{t\phi}) \\
&\quad + \overline{g}^{tr}\cancelto{0}{\overline{g}_{\phi r,\theta}} + \cancelto{0}{\overline{g}_{\theta r,\phi}} - \overline{g}_{\theta\phi,r} + \overline{g}_{\phi\theta}C^\theta_{r\theta} + \overline{g}_{\phi\phi}C^\phi_{r\theta} + \overline{g}_{\theta\theta}C^\theta_{r\phi} + \overline{g}_{\theta\phi}C^\phi_{r\phi})\Big\} \\
&= \frac{ab-c^2}{2|\overline{g}|}\Big\{(-c_{,t} + cC^\theta_{t\theta} + bC^\phi_{t\theta} + aC^\theta_{t\phi} + cC^\phi_{t\phi}) - Q(-c_{,r} + cC^\theta_{r\theta} + bC^\phi_{r\theta} + aC^\theta_{r\phi} \\
&\quad + cC^\phi_{r\phi})\Big\} \qquad (4.3.12)
\end{aligned}$$

$$\begin{aligned}
\omega^t_{\phi\theta} &= \frac{1}{2}\overline{g}^{ti}(\overline{g}_{\theta i,\phi} + \overline{g}_{\phi i,\theta} - \overline{g}_{\phi\theta,i} + C_{i\phi\theta} + C_{i\theta\phi} - C_{\phi\theta i}) \\
&= \frac{1}{2}\overline{g}^{ti}(\overline{g}_{\theta i,\phi} + \overline{g}_{\phi i,\theta} - \overline{g}_{\phi\theta,i} + \overline{g}_{\theta j}C^j_{i\phi} + \overline{g}_{\phi j}C^j_{i\theta} - \cancelto{0}{\overline{g}_{ij}C^j_{\phi\theta}}) \\
&= \omega^t_{\theta\phi} \qquad (4.3.13)
\end{aligned}$$

$$
\begin{aligned}
\omega^t_{\phi\phi} &= \frac{1}{2}\overline{g}^{ti}(2\overline{g}_{\phi i,\phi} - \overline{g}_{\phi\phi,i} + 2C_{i\phi\phi} - C_{\phi\phi i}) \\
&= \frac{1}{2}\overline{g}^{ti}(2\overline{g}_{\phi i,\phi} - \overline{g}_{\phi\phi,i} + 2\overline{g}_{\phi j}C^j_{i\phi} - \cancelto{0}{\overline{g}_{ij}C^j_{\phi\phi}}) \\
&= \frac{1}{2}\Big\{\overline{g}^{tt}(\cancelto{0}{2\overline{g}_{\phi t,\phi}} - \overline{g}_{\phi\phi,t} + 2\overline{g}_{\phi\theta}C^\theta_{t\phi} + 2\overline{g}_{\phi\phi}C^\phi_{t\phi}) + \overline{g}^{tr}(\cancelto{0}{2\overline{g}_{\phi r,\phi}} - \overline{g}_{\phi\phi,r} + 2\overline{g}_{\phi\theta}C^\theta_{r\phi} \\
&\quad + 2\overline{g}_{\phi\phi}C^\phi_{r\phi})\Big\} \\
&= \frac{ab-c^2}{2|\overline{g}|}\Big\{(-b_{,t} + 2cC^\theta_{t\phi} + 2bC^\phi_{t\phi}) - Q(-b_{,r} + 2cC^\theta_{r\phi} + 2bC^\phi_{r\phi})\Big\} \qquad (4.3.14)
\end{aligned}
$$

Step 5: Now plug (4.3.11), (4.3.12), (4.3.13) and (4.3.14) back into (4.3.9), we get:

$$
\begin{aligned}
\langle \vec{H}_\Sigma, e_n\rangle &= -(1+Q^2)^{1/2}\frac{1}{ab-c^2}(b\omega^t_{\theta\theta} - c\omega^t_{\theta\phi} - c\omega^t_{\phi\theta} + a\omega^t_{\phi\phi}) \\
&= -(1+Q^2)^{1/2}\frac{1}{ab-c^2}(b\omega^t_{\theta\theta} - 2c\omega^t_{\theta\phi} + a\omega^t_{\phi\phi}) \qquad \text{(By (4.3.13))} \\
&= -(1+Q^2)^{1/2}\frac{1}{ab-c^2}\frac{ab-c^2}{2|\overline{g}|}\Big\{(-a_{,t}b + 2abC^\theta_{t\theta} + \cancel{2bcC^\phi_{t\theta}} + 2cc_{,t} - 2c^2C^\theta_{t\theta} - \cancel{2bcC^\phi_{t\theta}} \\
&\quad - \cancel{2acC^\theta_{t\phi}} - 2c^2C^\phi_{t\phi} - ab_{,t} + \cancel{2acC^\theta_{t\phi}} + 2abC^\phi_{t\phi}) - Q(-a_{,r}b + 2abC^\theta_{r\theta} + \cancel{2bcC^\phi_{r\theta}} \\
&\quad + 2cc_{,r} - 2c^2C^\theta_{r\theta} - \cancel{2bcC^\phi_{r\theta}} - \cancel{2acC^\theta_{r\phi}} - 2c^2C^\phi_{r\phi} - ab_{,r} + \cancel{2acC^\theta_{r\phi}} + 2abC^\phi_{r\phi})\Big\} \\
&= -\frac{(1+Q^2)^{1/2}}{2|\overline{g}|}\Big\{ - e_t(ab-c^2) + 2(ab-c^2)(C^\theta_{t\theta} + C^\phi_{t\phi}) \\
&\quad - Q\left(-e_r(ab-c^2) + 2(ab-c^2)\cancelto{0}{(C^\theta_{r\theta} + C^\phi_{r\phi})}\right)\Big\} \\
&= -\frac{(1+Q^2)^{1/2}}{2|\overline{g}|}\Big\{e_r(ab-c^2)Q - e_t(ab-c^2) + 2(ab-c^2)(C^\theta_{t\theta} + C^\phi_{t\phi})\Big\} \qquad (4.3.15)
\end{aligned}
$$

since $C^\theta_{r\theta} = C^\phi_{r\phi} = 0$ by Equation (4.3.6). Therefore $\vec{H}_\Sigma$ is parallel to e_r if and only if

$$
e_r(ab-c^2)Q - e_t(ab-c^2) + 2(ab-c^2)(C^\theta_{t\theta} + C^\phi_{t\phi}) = 0. \qquad (4.3.16)
$$

Notice that Equation (4.3.16) is zeroth order in Q. Since Σ is area expanding, $e_r(ab - c^2) > 0$, and hence we get a unique solution

$$Q = \frac{e_t(ab - c^2) - 2(ab - c^2)(C_{t\theta}^{\theta} + C_{t\phi}^{\phi})}{e_r(ab - c^2)}. \tag{4.3.17}$$

□

Lemma 4.10. *$e_r(ab - c^2) = 0$ if and only if $H_{e_r} = -\langle \vec{H}_\Sigma, e_r \rangle = 0$, i.e. Σ is a minimal surface in M, where H_{e_r} is the mean curvature scalar of Σ in the direction of e_r.*

Proof. For the first claim, note that

$$e_r(dA_\Sigma) = e_r(\sqrt{ab - c^2}\beta_\theta\beta_\phi) = \frac{1}{2\sqrt{ab - c^2}} e_r(ab - c^2)\beta_\theta\beta_\phi = \frac{1}{2(ab - c^2)} e_r(ab - c^2) dA_\Sigma. \tag{4.3.18}$$

On the other hand by the first variation of area formula (A.3.7),

$$e_r(dA_\Sigma) = -\langle \vec{H}_\Sigma, e_r \rangle dA_\Sigma = H_{e_r} dA_\Sigma. \tag{4.3.19}$$

Combing the two equations we get:

$$2H_{e_r} \cdot (ab - c^2) = e_r(ab - c^2). \tag{4.3.20}$$

Therefore $e_r(ab - c^2) = 0$ if and only if $H_{e_r} = 0$. □

An application of Theorem 4.9 is to generate more examples of inverse mean curvature vector flow solutions. Let $(N^4, \overline{g})$ be a spacetime and (M, g) a spacelike hypersurface with induced metric g. Suppose $\{\Sigma_s\}$ is a solution to the smooth inverse mean curvature flow in M. Let $\vec{H}_s$ and dA_s be the mean curvature vector field and area form of Σ_s, respectively. By the first variation formula,

$$\frac{d}{ds}(dA_s) = dA_s. \tag{4.3.21}$$

Theorem 4.9 allows us to steer the metric smoothly along inverse mean curvature flow to keep $\vec{H}_s$ tangential to M. Note that (4.3.21) still holds after the steering since we are not changing the metric on M. Therefore by Proposition 4.1, in the steered metric $\{\Sigma_s\}$ is a solution to smooth inverse mean curvature vector flow equation.

One could generalize this technique to study weak solutions to inverse mean curvature vector flow equation (defined in (4.1.1)) using solutions to Huisken-Ilmanen inverse mean curvature flow (with jumps) in a hypersurface, but we will not give a rigorous treatment here.

4.4 Generalizations

Given a surface (Σ, g_Σ) inside a spacetime $(N^4, \overline{g})$. Let $\vec{I}$ be the inverse mean curvature vector. A more general flow than inverse mean curvature vector flow is to flow out Σ in the following direction:

$$\vec{\xi} := \vec{I} + \beta \vec{I}^{\perp}, \tag{4.4.1}$$

where the $\perp$ operation is a linear isomorphism on the normal bundle defined in Section 3.3.1, and β is a constant on each flow surface (hence is only a function of the flow parameter) such that $-1 \leqslant \beta \leqslant 1$.

Therefore inverse mean curvature vector flow corresponds to the case where $\beta = 0$. The procedure for constructing spacetimes with inverse mean curvature vector flow solutions can be generalized to constructing spacetimes in which this more general flow exists. The idea is to construct a spacetime metric $\overline{g}$ that admit a coordinate chart $\{t, r, \theta, \phi\}$, such that conditions (4.1.6), (4.1.7), (4.1.8) and the fourth condition:

$$\vec{\xi} \text{ is parallel to } \frac{\partial}{\partial r}. \tag{4.4.2}$$

are satisfied.

5

Uniformly Area Expanding Straight Out Flows

In this chapter we study *uniformly area expanding straight out flows*, or simply *straight out flows* of spacelike surfaces in a spacetime. Consider a spacetime $(N^4, \overline{g})$ and a closed embedded surfaces Σ with the induced metric g_Σ. A straight out direction of Σ, first studied by M. Mars, E. Malec and Simon [30], is a normal vector field that has "minimal variations" along Σ. Such a normal vector field is a minimizer of a natural functional defined on the normal bundle. The Hawking mass is also monotonically non-decreasing under smooth straight out flows assuming the spacetime satisfies the dominant energy condition. A condition for a spacetime to admit straight out flow coordinate charts is derived in this chapter. Complete understanding of such spacetimes is still work in progress.

5.1 Background and Straight Out Flow Coordinate Chart

Let $(N^4, \overline{g})$ be a time-oriented spacetime and (Σ^2, g_Σ) be a closed embedded surface. The normal bundle of Σ, $N\Sigma$, has an induced metric of signature $(-,+)$. On each fiber $N_p\Sigma$, the nonzero vectors get partitioned into four quadrants: outward-spacelike, inward-spacelike, future-timelike and past-timelike. Let $U^+N(\Sigma)$ denote the subbundle of $N\Sigma$ that consists of outward-spacelike normal vector fields of unit length. Given a smooth section ν of $U^+N(\Sigma)$, its associated connection one-form α_ν on Σ is defined by:

$$\alpha_\nu(X) := \langle \nabla^\perp_X v, v^\perp \rangle_{\overline{g}}, \quad \forall\, X \in \Gamma(T\Sigma). \tag{5.1.1}$$

where $\perp$ is a fiberwise linear isomorphism defined in Section 3.3. Notice that $\{\nu, \nu^\perp\}$ forms an orthonormal frame of $N\Sigma$.

Given another smooth section $\overline{\nu}$ of $U^+N(\Sigma)$, there exists a constant $\theta > 0$ such that

$$\begin{aligned} \overline{\nu} &= \cosh\theta\nu + \sinh\theta\nu^\perp \\ \overline{\nu}^\perp &= \sinh\theta\nu + \cosh\theta\nu^\perp \end{aligned} \tag{5.1.2}$$

Geometrically $\{\overline{\nu}, \overline{\nu}^\perp\}$ can be viewed as hyperbolic rotation of $\{\nu, \nu^\perp\}$ by angle θ. The associated connection one-forms are related by:

Lemma 5.1 ([9])**.** *Let $\alpha_{\overline{\nu}}$ be the associated connection one-form of $\overline{\nu}$, then $\alpha_{\overline{\nu}}$ is related to α_ν by*

$$\alpha_{\overline{\nu}} = \alpha_v - d\theta. \tag{5.1.3}$$

Let E be an energy functional on $U^+N(\Sigma)$ such that:

$$E(\nu) := \int_\Sigma ||\nabla^\perp \nu||^2 \, dA_\Sigma, \quad \forall\, \nu \in \Gamma\big(U^+N(\Sigma)\big). \tag{5.1.4}$$

H. Bray and J. Jauregui proved the following proposition:

Proposition 5.2 ([9])**.** *$\nu \in \Gamma(N\Sigma)$, outward spacelike and is of unit length, is a minimizer of E if and only if $div_\Sigma(\alpha_\nu) = 0$.*

Intuitively, a minimizer of E is a normal vector field with "minimal variations" along Σ. Bray and Jauregui also proved the existence of such minimizers:

Proposition 5.3 ([9])**.** *Minimizers of E exist, and any two such minimizers differ by a hyperbolic rotation as in (5.1.2) of a constant angle θ.*

Proof. We present the proof in [9]. Fix a section ν of $U^+N(\Sigma)$. Given any other section $\overline{\nu}$, $\overline{\nu}$ and ν are related by a hyperbolic rotation as in (5.1.2). Let α_ν and $\alpha_{\overline{\nu}}$ be the associated connection one-forms of ν and $\overline{\nu}$, respectively. By Lemma 5.1, $\alpha_{\overline{\nu}}$ is a divergence free if and only if

$$0 = \mathrm{div}_\Sigma(\alpha_{\overline{\nu}}) = -d^*\alpha_{\overline{\nu}} = -d^*\alpha_\nu + d^*d\theta = \mathrm{div}_\Sigma(\alpha_\nu) + \Delta_\Sigma\theta, \tag{5.1.5}$$

where d^* is the L^2-adjoint of d on Σ. The above equation is equivalent to:

$$\Delta_\Sigma\theta = \mathrm{div}_\Sigma(\alpha_\nu). \tag{5.1.6}$$

This is a Poisson equation. By the divergence theorem, $\int_\Sigma \mathrm{div}_\Sigma(\alpha_\nu)\, dA_\Sigma = 0$. Therefore Equation (5.1.6) has smooth solutions. Moreover, any two solutions differ by an element in the kernel of Δ_Σ, which consists of constant functions on Σ. □

A minimizer of E can be viewed as a *straight out* direction of the surface Σ, as it tries to level the surface as much as possible. This motivates the following definition:

Definition 5.1.1 (Uniformly Area Expanding Straight Out Flow)**.** *Given a surface Σ in a time-oriented spacetime $(N^4, \overline{g})$. Let $\vec{T}$ be a global timelike vector field. A smooth straight out flow of Σ in N is a smooth normal variation $F : \Sigma \times [0, T] \longrightarrow N$ such that*

$$\frac{\partial}{\partial s}F(x, s) = \beta\nu_s(x), \quad div_{\Sigma_s}(\alpha_{\nu_s}) = 0, \ \int_{\Sigma_s} \langle \vec{T}, \nu_s \rangle\, dA_s = 0, \tag{5.1.7}$$

where β is chosen such that $\frac{d}{ds}(dA_s) = dA_s$; and $s \in [0, T]$, $\Sigma_s := F(\Sigma, s)$, and ν_s is a outward spacelike unit normal vector field along Σ_s with zero divergence.

On each surface of Σ_s in Definition 5.1.1, the outward-spacelike vector ν_s with zero divergence is only unique up to a hyperbolic rotation with angle θ (see Proposition 5.3). The additional condition that $\displaystyle\int_{\Sigma_s} \langle \vec{T}, \nu_s \rangle \, dA_s = 0$ defines each ν_s uniquely. This condition is due to J. Jauregui.

E. Malec, M. Mars and W. Simon have studied such flows in [30], and they have shown that given a smooth solution $\{\Sigma_t\}$ to the straight out flow equation (5.1.7),

$$\frac{d}{dt} m_H(\Sigma_s) \geqslant 0,$$

i.e. that Hawking mass is also monotonically non-decreasing (also see [10]).

The goal in this chapter is to construct spacetimes that admit smooth solutions to straight out flow equation. The idea is similar to the construction in Chapter 4. Given $U := (\mathbb{R}^3 \backslash B_1) \times \mathbb{R}$, where B_1 is the unit ball in $\mathbb{R}^3$, we seek a spacetime metric $\overline{g}$ on U that admit a coordinate chart $\{t, r, \theta, \phi\}$, such that $\overline{g}$ satisfies the following four conditions:

$$(1) \quad \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \rangle = 0; \tag{5.1.8}$$

$$(2) \quad \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \phi} \rangle = 0; \tag{5.1.9}$$

$$(3) \quad \text{Area form of coordinate sphere } S_{t,r} \text{ satisfies} \quad dA_{t,r} = r^2 \sin\theta d\theta d\phi; \tag{5.1.10}$$

$$(4) \quad e_r := \frac{1}{u}\frac{\partial}{\partial r} \text{ is straight out, i.e. } \mathrm{div}_{g_S} \alpha_{e_r} = 0; \tag{5.1.11}$$

where α_{e_r} is the connection one-form associated with e_r.

The following definition is analogous to Definition 4.1.2:

Definition 5.1.2 (Straight Out Flow Coordinate Chart)**.** *If a spacetime $(N^4, \overline{g})$ admits a coordinate chart $\{t, r, \theta, \phi\}$ such that the four conditions (5.1.8), (5.1.9),* *(5.1.10) and (5.1.11) are satisfied, then* $\{t, r, \theta, \phi\}$ *is called a straight out coordinate chart, and* N *is called a spacetime that admits a straight out coordinate chart.*

5.2 Construction of Spacetimes That Admit Uniformly Area Expanding Straight Out Flow Coordinate Charts (work in progress)

Let $U = (\mathbb{R}^3 \backslash B_1) \times \mathbb{R}$. Given a coordinate chart $\{t, r, \theta, \phi\}$, define a spacetime metric $\overline{g}$ to be

$$\overline{g} := \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} -v^2 & d & e & f \\ d & u^2 & 0 & 0 \\ e & 0 & a & c \\ f & 0 & c & b \end{pmatrix} \end{array} \tag{5.2.1}$$

for smooth functions a, b, c, d, e, f, u, v on U. $\overline{g}$ satisfies conditions (5.1.8) and (5.1.9). Choosing two of the three variables a, b and c such that $ab - c^2 = r^4 \sin^2 \theta$ satisfies condition (5.1.10). We show that the fourth condition (5.1.11) is a second order elliptic PDE in the metric component d.

The normal bundle of the coordinate sphere $S_{t,r}$ has an orthonormal frame $\{e_r, e_n\}$, where $e_r = \frac{1}{u}\frac{\partial}{\partial r}$, and $e_n = \frac{\mathbf{n}}{||\mathbf{n}||_{\overline{g}}}$. The normal vector $\mathbf{n}$ is the same as in Equation (4.2.7). We compute the condition on $\overline{g}$ such that e_r is straight out on each $S_{t,r}$.

Notation: For the rest of this chapter, the induced metric on $S_{t,r}$ is denoted as g_S; $\overline{\nabla}$ is the Levi-Civita connection with respect to $\overline{g}$; $|\overline{g}|$ and $|g_S|$ are determinants of $\overline{g}$ and g_S respectively; and the subscript e_r is omitted from α_{e_r} whenever there is no confusion.

5.2.1 Computation of The Connection One Form and Its Divergence

In terms of the frame $\{d\theta, d\phi\}$ for $T^*S_{t,r}$, write $\alpha = \alpha_\theta d\theta + \alpha_\phi d\phi$, where

$$
\begin{aligned}
\alpha_\theta := \alpha\left(\frac{\partial}{\partial\theta}\right) &= \langle \overline{\nabla}^{\perp}_{\frac{\partial}{\partial\theta}} e_r, e_r^{\perp}\rangle && \text{(By definition 5.1.1)}\\
&= \langle \overline{\nabla}^{\perp}_{\frac{\partial}{\partial\theta}} e_r, e_n\rangle = \left\langle \overline{\nabla}^{\perp}_{\frac{\partial}{\partial\theta}}\left(\frac{1}{u}\frac{\partial}{\partial r}\right), \frac{\mathbf{n}}{||\mathbf{n}||}\right\rangle\\
&= \left\langle (1/u)_{,\theta}\frac{\partial}{\partial r} + \frac{1}{u}\overline{\nabla}^{\perp}_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial r}, \frac{\mathbf{n}}{|\mathbf{n}|}\right\rangle\\
&= \frac{1}{u||\mathbf{n}||}\left\langle \overline{\nabla}^{\perp}_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial r}, \mathbf{n}\right\rangle = \frac{1}{u||\mathbf{n}||}\left\langle \overline{\nabla}_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial r}, \mathbf{n}\right\rangle && (\mathbf{n}\text{ is normal})\\
&= \frac{1}{u||\mathbf{n}||}\overline{\Gamma}^t_{\theta r}\left\langle \frac{\partial}{\partial t}, \mathbf{n}\right\rangle && (\mathbf{n}\text{ is perpendicular to } \tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial\theta}, \tfrac{\partial}{\partial\phi})\\
&= \frac{1}{u||\mathbf{n}||}\overline{\Gamma}^t_{\theta r}\cdot\frac{|\overline{g}|}{u^2|g_S|} && \text{(By (4.2.8))}\\
&= \frac{1}{u}\left(\frac{u^2|g_S|}{-|\overline{g}|}\right)^{1/2}\overline{\Gamma}^t_{\theta r}\cdot\frac{|\overline{g}|}{u^2|g_S|} && \text{(By (4.2.9))}\\
&= -\frac{1}{u^2}\overline{\Gamma}^t_{\theta r}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} && (|\overline{g}| = -(|\overline{g}|^2)^{1/2}, \text{ since } |\overline{g}| < 0)
\end{aligned}
$$

Therefore

$$
\alpha_\theta = -\frac{1}{u^2}\overline{\Gamma}^t_{\theta r}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}. \tag{5.2.2}
$$

By a similar computation, we have:

$$
\alpha_\phi := \alpha\left(\frac{\partial}{\partial\phi}\right) = -\frac{1}{u^2}\overline{\Gamma}^t_{\phi r}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}. \tag{5.2.3}
$$

Therefore:

$$
\alpha = -\frac{1}{u^2}\overline{\Gamma}^t_{\theta r}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\cdot d\theta - \frac{1}{u^2}\overline{\Gamma}^t_{\phi r}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\cdot d\phi. \tag{5.2.4}
$$

The divergence of α is defined to be the divergence of its dual vector field, which is $\beta = \beta^\theta \frac{\partial}{\partial\theta} + \beta^\phi \frac{\partial}{\partial\phi}$, where

$$\beta^\theta = g_S^{\theta\theta}\alpha_\theta + g_S^{\theta\phi}\alpha_\phi, \quad \beta^\phi = g_S^{\phi\theta}\alpha_\theta + g_S^{\phi\phi}\alpha_\phi \tag{5.2.5}$$

Recall that

Definition 5.2.1. *For a vector field $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$ in local coordinates, its divergence with respect to a metric g is given by:*

$$div_g X := \frac{1}{\sqrt{|g|}} \sum_{i=1}^n \frac{\partial}{\partial x^i}\left(X^i \sqrt{|g|}\right).$$

Therefore the divergence of α along $S_{t,r}$ is

$$\begin{aligned}
\mathrm{div}_{g_S}\alpha : &= \mathrm{div}_{g_S}\beta = \frac{1}{\sqrt{|g_S|}} \sum_{i=1}^2 \frac{\partial}{\partial x^i}\left(\beta^i \sqrt{|g_S|}\right) \qquad \left(\tfrac{\partial}{\partial x^i} \in \{\tfrac{\partial}{\partial\theta}, \tfrac{\partial}{\partial\phi}\}\right) \\
&= \frac{1}{\sqrt{|g_S|}}\left[\frac{\partial}{\partial\theta}\left(\beta^\theta\sqrt{|g_S|}\right) + \frac{\partial}{\partial\phi}\left(\beta^\phi\sqrt{|g_S|}\right)\right] \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\beta^\theta r^2\sin\theta\right) + \frac{\partial}{\partial\phi}\left(\beta^\phi r^2\sin\theta\right)\right] \\
&= \frac{1}{r^2\sin\theta}\left(\beta^\theta_{,\theta}\cdot r^2\sin\theta + \beta^\theta\cdot r^2\cos\theta + \beta^\phi_{,\phi}\cdot r^2\sin\theta + 0\right)
\end{aligned}$$

that is:

$$\mathrm{div}_{g_S}\alpha = \beta^\theta_{,\theta} + \beta^\phi_{,\phi} + \beta^\theta\cot\theta. \tag{5.2.6}$$

5.2.2 Straight Out Flow Coordinate Chart: Big Picture

We expand out Equation (5.2.6), separating higher order derivatives from lower order derivatives.

$$
\begin{aligned}
\operatorname{div}_{g_S}\alpha &= \beta^{\theta}_{,\theta} + \beta^{\phi}_{,\phi} + \beta^{\theta}\cot\theta \\
&= (g_S^{\theta\theta}\alpha_\theta + g_S^{\theta\phi}\alpha_\phi)_{,\theta} + (g_S^{\phi\theta}\alpha_\theta + g_S^{\phi\phi}\alpha_\phi)_{,\phi} + (g_S^{\theta\theta}\alpha_\theta + g_S^{\theta\phi}\alpha_\phi)\cot\theta \\
&= (g_S^{\theta\theta}\alpha_{\theta,\theta} + g_S^{\theta\phi}\alpha_{\phi,\theta} + g_S^{\phi\theta}\alpha_{\theta,\phi} + g_S^{\phi\phi}\alpha_{\phi,\phi}) \qquad \text{(2nd derivative in } \overline{g}) \\
&+ (g_{S,\theta}^{\theta\theta}\alpha_\theta + g_{S,\theta}^{\theta\phi}\alpha_\phi + g_{S,\phi}^{\phi\theta}\alpha_\theta + g_{S,\phi}^{\phi\phi}\alpha_\phi) + (g_S^{\theta\theta}\alpha_\theta + g_S^{\theta\phi}\alpha_\phi)\cot\theta \\
&\qquad \text{(1st derivative in } \overline{g}) \\
&=: (B) + (l1) \qquad (5.2.7)
\end{aligned}
$$

From (5.2.2) and (5.2.3), α_θ and α_ϕ are first order in the derivative of the space-time metric $\overline{g}$. Therefore the divergence equation (5.2.6) is second order in the derivative of $\overline{g}$. In (5.2.7) above, (B) denotes the second derivative terms, and $(l1)$ denotes the first derivative terms.

To compute (B), notice that:

$$
\begin{aligned}
g_S^{\theta\theta}\alpha_{\theta,\theta} &= g_S^{\theta\theta}\left(-\frac{1}{u^2}\overline{\Gamma}^t_{\theta r}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\right)_{,\theta} \qquad \text{(plug in (5.2.2) for } \alpha_\theta) \\
&= g_S^{\theta\theta}\left(\left(-\frac{1}{u^2}\right)_{,\theta}\overline{\Gamma}^t_{\theta r}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} + \left(-\frac{1}{u^2}\right)\overline{\Gamma}^t_{\theta r,\theta}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\right. \\
&+ \left.\left(-\frac{1}{u^2}\right)\overline{\Gamma}^t_{\theta r}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}_{,\theta}\right) \\
&=: -\frac{1}{u^2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} g_S^{\theta\theta}\cdot\overline{\Gamma}^t_{\theta r,\theta} + \text{lower order derivatives in } \overline{g}.
\end{aligned}
$$

Similarly for the other three terms in (B), the highest order derivative terms are the terms containing derivatives of the Christoffel symbols. Therefore

$$
(B) = (C) + (l2), \qquad (5.2.8)
$$

where

$$(C) := -\frac{1}{u^2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\left(g_S^{\theta\theta}\overline{\Gamma}^t_{\theta r,\theta} + g_S^{\theta\phi}\overline{\Gamma}^t_{\phi r,\theta} + g_S^{\phi\theta}\overline{\Gamma}^t_{\theta r,\phi} + g_S^{\phi\phi}\overline{\Gamma}^t_{\phi r,\phi}\right)$$

$$= -\frac{1}{u^2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\cdot\frac{1}{|g_S|}\left(b\overline{\Gamma}^t_{\theta r,\theta} - c\overline{\Gamma}^t_{\phi r,\theta} - c\overline{\Gamma}^t_{\theta r,\phi} + a\overline{\Gamma}^t_{\phi r,\phi}\right)$$

$$= -\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}\cdot\left(b\overline{\Gamma}^t_{\theta r,\theta} - c\overline{\Gamma}^t_{\phi r,\theta} - c\overline{\Gamma}^t_{\theta r,\phi} + a\overline{\Gamma}^t_{\phi r,\phi}\right)$$

$$=: -\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}\cdot(D) \tag{5.2.9}$$

and

$$(l2) := g_S^{\theta\theta}\left(\left(-\frac{1}{u^2}\right)_{,\theta}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} + \left(-\frac{1}{u^2}\right)\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}_{,\theta}\right)\overline{\Gamma}^t_{\theta r}$$

$$+ g_S^{\theta\phi}\left(\left(-\frac{1}{u^2}\right)_{,\theta}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} + \left(-\frac{1}{u^2}\right)\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}_{,\theta}\right)\overline{\Gamma}^t_{\phi r}$$

$$+ g_S^{\phi\theta}\left(\left(-\frac{1}{u^2}\right)_{,\phi}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} + \left(-\frac{1}{u^2}\right)\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}_{,\phi}\right)\overline{\Gamma}^t_{\theta r}$$

$$+ g_S^{\phi\phi}\left(\left(-\frac{1}{u^2}\right)_{,\phi}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} + \left(-\frac{1}{u^2}\right)\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}_{,\phi}\right)\overline{\Gamma}^t_{\phi r}$$

$$= \left(\left(-\frac{1}{u^2}\right)_{,\theta}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} + \left(-\frac{1}{u^2}\right)\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}_{,\theta}\right)\cdot(g_S^{\theta\theta}\overline{\Gamma}^t_{\theta r} + g_S^{\theta\phi}\overline{\Gamma}^t_{\phi r})$$

$$+ \left(\left(-\frac{1}{u^2}\right)_{,\phi}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} + \left(-\frac{1}{u^2}\right)\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}_{,\phi}\right)\cdot(g_S^{\phi\theta}\overline{\Gamma}^t_{\theta r} + g_S^{\phi\phi}\overline{\Gamma}^t_{\phi r}) \tag{5.2.10}$$

Notice that all the second derivatives of $\overline{g}$ in the divergence equation (5.2.7) lie in (D). Note that:

$$\overline{\Gamma}^t_{\theta r} = \frac{1}{2}\Big(\overline{g}^{tt}(\overline{g}_{\theta t,r} + \overline{g}_{rt,\theta} - \overline{g}_{\theta r,t}) + \overline{g}^{tr}(\overline{g}_{\theta r,r} + \overline{g}_{rr,\theta} - g_{\theta r,r}) + \overline{g}^{t\theta}(\overline{g}_{\theta\theta,r} + \overline{g}_{r\theta,\theta} - \overline{g}_{\theta r,\theta})$$

$$+ \overline{g}^{t\phi}(\overline{g}_{\theta\phi,r} + \overline{g}_{r\phi,\theta} - g_{\theta r,\phi})\Big)$$

$$= \frac{1}{2}\Big(\overline{g}^{tt}(\overline{g}_{\theta t,r} + \overline{g}_{rt,\theta}) + \overline{g}^{tr}\overline{g}_{rr,\theta} + \overline{g}^{t\theta}\overline{g}_{\theta\theta,r} + \overline{g}^{t\phi}\overline{g}_{\theta\phi,r}\Big)$$

$$= \frac{1}{2|\overline{g}|}\Big(u^2(ab-c^2)(e_{,r} + d_{,\theta}) - d(ab-c^2)2uu_{,\theta} + u^2(cf-be)a_{,r} + u^2(ce-af)c_{,r}\Big)$$

$$\overline{\Gamma}^t_{\phi r} = \frac{1}{2}\Big(\overline{g}^{tt}(\overline{g}_{\phi t,r} + \overline{g}_{rt,\phi} - g_{\phi r,t}) + \overline{g}^{tr}(\overline{g}_{\phi r,r} + \overline{g}_{rr,\phi} - \overline{g}_{\phi r,r}) + \overline{g}^{t\theta}(\overline{g}_{\phi\theta,r} + \overline{g}_{r\theta,\phi} - \overline{g}_{\phi r,\theta})$$

$$+ \overline{g}^{t\phi}(\overline{g}_{\phi\phi,r} + \overline{g}_{r\phi,\phi} - \overline{g}_{\phi r,\phi})\Big)$$

$$= \frac{1}{2}\Big(\overline{g}^{tt}(\overline{g}_{\phi t,r} + \overline{g}_{rt,\phi}) + \overline{g}^{tr}\overline{g}_{rr,\phi} + \overline{g}^{t\theta}\overline{g}_{\phi\theta,r} + \overline{g}^{t\phi}\overline{g}_{\phi\phi,r}\Big)$$

$$= \frac{1}{2|\overline{g}|}\Big(u^2(ab-c^2)(f_{,r} + d_{,\phi}) - d(ab-c^2)2uu_{,\phi} + u^2(cf-be)c_{,r} + u^2(ce-af)b_{,r}\Big)$$

We compute (D) as follows:

$$(D) = \frac{b}{2}[\overline{g}^{tt}(\overline{g}_{\theta t,r} + \overline{g}_{rt,\theta}) + \overline{g}^{tr}\overline{g}_{rr,\theta} + \overline{g}^{t\theta}\overline{g}_{\theta\theta,r} + \overline{g}^{t\phi}\overline{g}_{\theta\phi,r}]_{,\theta}$$

$$- \frac{c}{2}[\overline{g}^{tt}(\overline{g}_{\phi t,r} + \overline{g}_{rt,\phi}) + \overline{g}^{tr}\overline{g}_{rr,\phi} + \overline{g}^{t\theta}\overline{g}_{\phi\theta,r} + \overline{g}^{t\phi}\overline{g}_{\phi\phi,r}]_{,\theta}$$

$$- \frac{c}{2}[\overline{g}^{tt}(\overline{g}_{\theta t,r} + \overline{g}_{rt,\theta}) + \overline{g}^{tr}\overline{g}_{rr,\theta} + \overline{g}^{t\theta}\overline{g}_{\theta\theta,r} + \overline{g}^{t\phi}\overline{g}_{\theta\phi,r}]_{,\phi}$$

$$+ \frac{a}{2}[\overline{g}^{tt}(\overline{g}_{\phi t,r} + \overline{g}_{rt,\phi}) + \overline{g}^{tr}\overline{g}_{rr,\phi} + \overline{g}^{t\theta}\overline{g}_{\phi\theta,r} + \overline{g}^{t\phi}\overline{g}_{\phi\phi,r}]_{,\phi}$$

$$= \frac{1}{2}\overline{g}^{tt}(b\overline{g}_{\theta t,r\theta} + b\overline{g}_{rt,\theta\theta} - c\overline{g}_{\phi t,r\theta} - c\overline{g}_{rt,\phi\theta} - c\overline{g}_{\theta t,r\phi} - c\overline{g}_{rt,\theta\phi} + a\overline{g}_{\phi t,r\phi} + a\overline{g}_{rt,\phi\phi})$$

$$+ \frac{1}{2}\overline{g}^{tr}(b\overline{g}_{rr,\theta\theta} - c\overline{g}_{rr,\phi\theta} - c\overline{g}_{rr,\theta\phi} + a\overline{g}_{rr,\phi\phi}) + \frac{1}{2}\overline{g}^{t\theta}(b\overline{g}_{\theta\theta,r\theta} - c\overline{g}_{\phi\theta,r\theta} - c\overline{g}_{\theta\theta,r\phi}$$

$$+ a\overline{g}_{\phi\theta,r\phi}) + \frac{1}{2}\overline{g}^{t\phi}(b\overline{g}_{\theta\phi,r\theta} - c\overline{g}_{\phi\phi,r\theta} - c\overline{g}_{\theta\phi,r\phi} + a\overline{g}_{\phi\phi,r\phi}) + (E)$$

$$=: (I) + (II) + (III) + (IV) + (E) \qquad (5.2.11)$$

where (E) consists of lower (first) order derivatives in $\overline{g}$:

$$
\begin{aligned}
(E) = &\frac{b}{2}[\overline{g}^{tt}_{,\theta} \cdot (\overline{g}_{\theta t,r} + \overline{g}_{rt,\theta}) + \overline{g}^{tr}_{,\theta} \cdot \overline{g}_{rr,\theta} + \overline{g}^{t\theta}_{,\theta} \cdot \overline{g}_{\theta\theta,r} + \overline{g}^{t\phi}_{,\theta} \cdot \overline{g}_{\theta\phi,r}] \\
&- \frac{c}{2}[\overline{g}^{tt}_{,\theta} \cdot (\overline{g}_{\phi t,r} + \overline{g}_{rt,\phi}) + \overline{g}^{tr}_{,\theta} \cdot \overline{g}_{rr,\phi} + \overline{g}^{t\theta}_{,\theta} \cdot \overline{g}_{\phi\theta,r} + \overline{g}^{t\phi}_{,\theta} \cdot \overline{g}_{\phi\phi,r}] \\
&- \frac{c}{2}[\overline{g}^{tt}_{,\phi} \cdot (\overline{g}_{\theta t,r} + \overline{g}_{rt,\theta}) + \overline{g}^{tr}_{,\phi} \cdot \overline{g}_{rr,\theta} + \overline{g}^{t\theta}_{,\phi} \cdot \overline{g}_{\theta\theta,r} + \overline{g}^{t\phi}_{,\phi} \cdot \overline{g}_{\theta\phi,r}] \\
&+ \frac{a}{2}[\overline{g}^{tt}_{,\phi} \cdot (\overline{g}_{\phi t,r} + \overline{g}_{rt,\phi}) + \overline{g}^{tr}_{,\phi} \cdot \overline{g}_{rr,\phi} + \overline{g}^{t\theta}_{,\phi} \cdot \overline{g}_{\phi\theta,r} + \overline{g}^{t\phi}_{,\phi} \cdot \overline{g}_{\phi\phi,r}] \\
&= \frac{1}{2}\overline{g}^{tt}_{,\theta}(be_{,r} + bd_{,\theta} - cf_{,r} - cd_{,\phi}) + \frac{1}{2}\overline{g}^{tt}_{,\phi}(-ce_{,r} - cd_{,\theta} + af_{,r} + ad_{,\phi}) \\
&+ \frac{1}{2}\overline{g}^{tr}_{,\theta}(2buu_{,\theta} - 2cuu_{,\phi}) + \frac{1}{2}\overline{g}^{tr}_{,\phi}(-2cuu_{,\theta} + 2auu_{,\phi}) + \frac{1}{2}\overline{g}^{t\theta}_{,\theta}(ba_{,r} - cc_{,r}) \\
&+ \frac{1}{2}\overline{g}^{t\theta}_{,\phi}(-ca_{,r} + ac_{,r}) + \frac{1}{2}\overline{g}^{t\phi}_{,\theta}(bc_{,r} - cb_{,r}) + \frac{1}{2}\overline{g}^{t\phi}_{,\phi}(-cc_{,r} + ab_{,r}) \qquad \text{(see (5.2.11))}
\end{aligned}
$$

Therefore the fourth condition (5.1.11) becomes:

$$
\begin{aligned}
0 = \operatorname{div}_{g_S}\alpha^{\nu} &= \beta^{\theta}_{,\theta} + \beta^{\phi}_{,\phi} + \beta^{\theta}\cot\theta = (B) + (l1) && \text{(see (5.2.7))} \\
&= (C) + (l2) + (l1) && \text{(see (5.2.8))} \\
&= -\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}} \cdot (D) + (l2) + (l1) && \text{(see (5.2.9))} \\
&= -\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}} \cdot \Big((I) + (II) + (III) + (IV) + (E)\Big) + (l2) + (l1) && (5.2.12)
\end{aligned}
$$

The highest derivatives of $\overline{g}$ (second derivatives) lie in $(I) + (II) + (III) + (IV)$.

5.2.3 Straight Out Flow Coordinate Chart: Complete Form

In this subsection, (5.2.12) is computed explicitly from left to right.

(I):

$$(I) = \frac{u^2(ab-c^2)}{2|\overline{g}|}(be_{,r\theta} + bd_{,\theta\theta} - cf_{,r\theta} - cd_{,\phi\theta} - ce_{,r\phi} - cd_{,\theta\phi} + af_{,r\phi} + ad_{,\phi\phi})$$

$$= \frac{u^2}{2}\frac{|g_S|}{|\overline{g}|}(bd_{,\theta\theta} - 2cd_{,\theta\phi} + ad_{,\phi\phi} + be_{,r\theta} - cf_{,r\theta} - ce_{,r\phi} + af_{,r\phi}). \quad (5.2.13)$$

Therefore

$$-\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}} \cdot (I) = -\frac{1}{\cancel{u^2}}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}} \cdot \frac{\cancel{u^2}}{2}\frac{|g_S|}{|\overline{g}|} \cdot$$

$$(bd_{,\theta\theta} - 2cd_{,\theta\phi} + ad_{,\phi\phi} + be_{,r\theta} - cf_{,r\theta} - ce_{,r\phi} + af_{,r\phi})$$

$$= \frac{1}{2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}\Big((bd_{,\theta\theta} - 2cd_{,\theta\phi} + ad_{,\phi\phi}) + (be_{,r\theta} - cf_{,r\theta} - ce_{,r\phi} + af_{,r\phi})\Big). \quad (5.2.14)$$

(II) :

$$(II) = \frac{-d(ab-c^2)}{2|\overline{g}|}(b(u^2)_{,\theta\theta} - 2c(u^2)_{,\theta\phi} + a(u^2)_{,\phi\phi})$$

$$= \frac{-d|g_S|}{2|\overline{g}|}(b(u^2)_{,\theta\theta} - 2c(u^2)_{,\theta\phi} + a(u^2)_{,\phi\phi}). \quad (5.2.15)$$

Therefore

$$-\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}} \cdot (II) = -\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}} \cdot \frac{-d|g_S|}{2|\overline{g}|}(b(u^2)_{,\theta\theta} - 2c(u^2)_{,\theta\phi} + a(u^2)_{,\phi\phi})$$

$$= -\frac{1}{2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}\frac{d}{u^2}(b(u^2)_{,\theta\theta} - 2c(u^2)_{,\theta\phi} + a(u^2)_{,\phi\phi}). \quad (5.2.16)$$

(III):

$$(III) = \frac{u^2(cf-be)}{2|\overline{g}|}(ba_{,r\theta} - cc_{,r\theta} - ca_{,r\phi} + ac_{,r\phi}). \quad (5.2.17)$$

Therefore

$$-\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}\cdot(III) = -\frac{1}{\cancel{u^2}}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}\cdot\frac{\cancel{u^2}(cf-be)}{2|\overline{g}|}(ba_{,r\theta}-cc_{,r\theta}-ca_{,r\phi}+ac_{,r\phi})$$

$$= \frac{cf-be}{2}\frac{1}{|g_S|^{3/2}}\frac{1}{(-|\overline{g}|)^{1/2}}(ba_{,r\theta}-cc_{,r\theta}-ca_{,r\phi}+ac_{,r\phi})$$

$$= \frac{1}{2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}\frac{cf-be}{ab-c^2}(ba_{,r\theta}-cc_{,r\theta}-ca_{,r\phi}+ac_{,r\phi}). \tag{5.2.18}$$

(IV):

$$(IV) = \frac{u^2(ce-af)}{2|\overline{g}|}(bc_{,r\theta}-cb_{,r\theta}-cc_{,r\phi}+ab_{,r\phi}). \tag{5.2.19}$$

Therefore

$$-\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}\cdot(IV) = -\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}\cdot\frac{u^2(ce-af)}{2|\overline{g}|}(bc_{,r\theta}-cb_{,r\theta}-cc_{,r\phi}+ab_{,r\phi})$$

$$= \frac{ce-af}{2}\frac{1}{|g_S|^{3/2}}\frac{1}{(-|\overline{g}|)^{1/2}}(bc_{,r\theta}-cb_{,r\theta}-cc_{,r\phi}+ab_{,r\phi})$$

$$= \frac{1}{2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}\frac{ce-af}{ab-c^2}(bc_{,r\theta}-cb_{,r\theta}-cc_{,r\phi}+ab_{,r\phi}). \tag{5.2.20}$$

We now compute (E). First we compute the derivative of the inverse metric:

(1)

$$\overline{g}^{tt}_{,\theta} = \left(\frac{u^2|g_S|}{|\overline{g}|}\right)_{,\theta} = \frac{(2uu_{,\theta}|g_S|+u^2|g_S|_{,\theta})|\overline{g}|-u^2|g_S||\overline{g}|_{,\theta}}{|\overline{g}|^2}$$

$$= \frac{2uu_{,\theta}|g_S|+2u^2|g_S|\cot\theta}{|\overline{g}|}-\frac{u^2|g_S||\overline{g}|_{,\theta}}{|\overline{g}|^2} \qquad \text{(By (A.2.17))}$$

$$= \frac{u^2|g_S|}{|\overline{g}|}\left(2\frac{u_{,\theta}}{u}+2\cot\theta-\frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right).$$

Similarly

$$\overline{g}^{tt}_{,\phi} = \left(\frac{u^2|g_S|}{|\overline{g}|}\right)_{,\phi} = \frac{(2uu_{,\phi}|g_S| + u^2 \cancelto{0}{|g_S|_{,\phi}})|\overline{g}| - u^2|g_S||\overline{g}|_{,\phi}}{|\overline{g}|^2} \quad (|g_S| = r^4\sin^2\theta)$$

$$= \frac{2uu_{,\phi}|g_S|}{|\overline{g}|} - \frac{u^2|g_S||\overline{g}|_{,\phi}}{|\overline{g}|^2}$$

$$= \frac{u^2|g_S|}{|\overline{g}|}\left(2\frac{u_{,\phi}}{u} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right).$$

(2)

$$\overline{g}^{tr}_{,\theta} = \left(\frac{-d|g_S|}{|\overline{g}|}\right)_{,\theta} = \frac{(-d_{,\theta}|g_S| - d|g_S|_{,\theta})|\overline{g}| - (-d|g_S|)|\overline{g}|_{,\theta}}{|\overline{g}|^2}$$

$$= \frac{-d_{,\theta}|g_S| - 2d|g_S|\cot\theta}{|\overline{g}|} + \frac{d|g_S||\overline{g}|_{,\theta}}{|\overline{g}|^2} \qquad \text{(By (A.2.17))}$$

$$= \frac{-d|g_S|}{|\overline{g}|}\left(\frac{d_{,\theta}}{d} + 2\cot\theta - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right).$$

Similarly

$$\overline{g}^{tr}_{,\phi} = \left(\frac{-d|g_S|}{|\overline{g}|}\right)_{,\phi} = \frac{(-d_{,\phi}|g_S| - d\cancelto{0}{|g_S|_{,\phi}})|\overline{g}| - (-d|g_S|)|\overline{g}|_{,\phi}}{|\overline{g}|^2}$$

$$= \frac{-d_{,\phi}|g_S|}{|\overline{g}|} + \frac{d|g_S||\overline{g}|_{,\phi}}{|\overline{g}|^2}$$

$$= \frac{-d|g_S|}{|\overline{g}|}\left(\frac{d_{,\phi}}{d} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right).$$

(3)

$$\overline{g}^{t\theta}_{,\theta} = \left(\frac{u^2(cf-be)}{|\overline{g}|}\right)_{,\theta} = \frac{\big(2uu_{,\theta}(cf-be) + u^2(cf-be)_{,\theta}\big)|\overline{g}| - u^2(cf-be)|\overline{g}|_{,\theta}}{|\overline{g}|^2}$$

$$= \frac{2uu_{,\theta}(cf-be) + u^2(cf-be)_{,\theta}}{|\overline{g}|} - \frac{u^2(cf-be)|\overline{g}|_{,\theta}}{|\overline{g}|^2}$$

$$= \frac{u^2(cf-be)}{|\overline{g}|}\left(2\frac{u_{,\theta}}{u} + \frac{(cf-be)_{,\theta}}{cf-be} - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right).$$

Similarly

$$\overline{g}^{t\theta}_{,\phi} = \left(\frac{u^2(cf-be)}{|\overline{g}|}\right)_{,\phi} = \frac{\big(2uu_{,\phi}(cf-be) + u^2(cf-be)_{,\phi}\big)|\overline{g}| - u^2(cf-be)|\overline{g}|_{,\phi}}{|\overline{g}|^2}$$

$$= \frac{2uu_{,\phi}(cf-be) + u^2(cf-be)_{,\phi}}{|\overline{g}|} - \frac{u^2(cf-be)|\overline{g}|_{,\phi}}{|\overline{g}|^2}$$

$$= \frac{u^2(cf-be)}{|\overline{g}|}\left(2\frac{u_{,\phi}}{u} + \frac{(cf-be)_{,\phi}}{cf-be} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right).$$

(4)

$$\overline{g}^{t\phi}_{,\theta} = \left(\frac{u^2(ce-af)}{|\overline{g}|}\right)_{,\theta} = \frac{\big(2uu_{,\theta}(ce-af) + u^2(ce-af)_{,\theta}\big)|\overline{g}| - u^2(ce-af)|\overline{g}|_{,\theta}}{|\overline{g}|^2}$$

$$= \frac{2uu_{,\theta}(ce-af) + u^2(ce-af)_{,\theta}}{|\overline{g}|} - \frac{u^2(ce-af)|\overline{g}|_{,\theta}}{|\overline{g}|^2}$$

$$= \frac{u^2(ce-af)}{|\overline{g}|}\left(2\frac{u_{,\theta}}{u} + \frac{(ce-af)_{,\theta}}{ce-af} - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right).$$

Similarly

$$\overline{g}^{t\phi}_{,\phi} = \left(\frac{u^2(ce-af)}{|\overline{g}|}\right)_{,\phi} = \frac{\big(2uu_{,\phi}(ce-af) + u^2(ce-af)_{,\phi}\big)|\overline{g}| - u^2(ce-af)|\overline{g}|_{,\phi}}{|\overline{g}|^2}$$

$$= \frac{2uu_{,\phi}(ce-af) + u^2(ce-af)_{,\phi}}{|\overline{g}|} - \frac{u^2(ce-af)|\overline{g}|_{,\phi}}{|\overline{g}|^2}$$

$$= \frac{u^2(ce-af)}{|\overline{g}|}\left(2\frac{u_{,\phi}}{u} + \frac{(ce-af)_{,\phi}}{ce-af} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right).$$

Therefore

$$(E) = \frac{1}{2}\frac{u^2|g_S|}{|\overline{g}|}\left(2\frac{u_{,\theta}}{u} + 2\cot\theta - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(be_{,r} + bd_{,\theta} - cf_{,r} - cd_{,\phi})$$

$$+ \frac{1}{2}\frac{u^2|g_S|}{|\overline{g}|}\left(2\frac{u_{,\phi}}{u} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-ce_{,r} - cd_{,\theta} + af_{,r} + ad_{,\phi})$$

$$+ \frac{-du|g_S|}{|\overline{g}|}\left(\frac{d_{,\theta}}{d} + 2\cot\theta - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(bu_{,\theta} - cu_{,\phi}) + \frac{-du|g_S|}{|\overline{g}|}\left(\frac{d_{,\phi}}{d} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-cu_{,\theta}$$

$$+ au_{,\phi}) + \frac{1}{2}\frac{u^2(cf-be)}{|\overline{g}|}\left(2\frac{u_{,\theta}}{u} + \frac{(cf-be)_{,\theta}}{cf-be} - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(ba_{,r} - cc_{,r})$$

$$+ \frac{1}{2}\frac{u^2(cf-be)}{|\overline{g}|}\left(2\frac{u_{,\phi}}{u} + \frac{(cf-be)_{,\phi}}{cf-be} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-ca_{,r} + ac_{,r})$$

$$+ \frac{1}{2}\frac{u^2(ce-af)}{|\overline{g}|}\left(2\frac{u_{,\theta}}{u} + \frac{(ce-af)_{,\theta}}{ce-af} - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(bc_{,r} - cb_{,r})$$

$$+ \frac{1}{2}\frac{u^2(ce-af)}{|\overline{g}|}\left(2\frac{u_{,\phi}}{u} + \frac{(ce-af)_{,\phi}}{ce-af} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-cc_{,r} + ab_{,r}).$$

Therefore

$$-\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}(E) = \frac{1}{2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}\left[\left(2\frac{u_{,\theta}}{u} + 2\cot\theta - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(be_{,r} + bd_{,\theta} - cf_{,r} - cd_{,\phi})\right.$$

$$\left. + \left(2\frac{u_{,\phi}}{u} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-ce_{,r} - cd_{,\theta} + af_{,r} + ad_{,\phi})\right]$$

$$-\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}\frac{d}{u}\left[\left(\frac{d_{,\theta}}{d} + 2\cot\theta - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(bu_{,\theta} - cu_{,\phi}) + \left(\frac{d_{,\phi}}{d} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-cu_{,\theta} + au_{,\phi})\right]$$

$$+ \frac{1}{2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}\frac{cf-be}{ab-c^2}\left[\left(2\frac{u_{,\theta}}{u} + \frac{(cf-be)_{,\theta}}{cf-be} - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(ba_{,r} - cc_{,r})\right.$$

$$\left. + \left(2\frac{u_{,\phi}}{u} + \frac{(cf-be)_{,\phi}}{cf-be} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-ca_{,r} + ac_{,r})\right]$$

$$+ \frac{1}{2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}\frac{ce-af}{ab-c^2}\left[\left(2\frac{u_{,\theta}}{u} + \frac{(ce-af)_{,\theta}}{ce-af} - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(bc_{,r} - cb_{,r})\right.$$

$$\left. + \left(2\frac{u_{,\phi}}{u} + \frac{(ce-af)_{,\phi}}{ce-af} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-cc_{,r} + ab_{,r})\right].$$

That is:

$$-\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}(E) = \frac{1}{2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}\left\{\left(2\frac{u_{,\theta}}{u} + 2\cot\theta - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(be_{,r} + bd_{,\theta} - cf_{,r} - cd_{,\phi})\right.$$

$$+\left(2\frac{u_{,\phi}}{u} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-ce_{,r} - cd_{,\theta} + af_{,r} + ad_{,\phi}) - \frac{2d}{u}\left[\left(\frac{d_{,\theta}}{d} + 2\cot\theta - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(bu_{,\theta} - cu_{,\phi})\right.$$

$$\left.+\left(\frac{d_{,\phi}}{d} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-cu_{,\theta} + au_{,\phi})\right] + \frac{cf - be}{ab - c^2}\left[\left(2\frac{u_{,\theta}}{u} + \frac{(cf-be)_{,\theta}}{cf - be} - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(ba_{,r} - cc_{,r})\right.$$

$$\left.+\left(2\frac{u_{,\phi}}{u} + \frac{(cf-be)_{,\phi}}{cf - be} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-ca_{,r} + ac_{,r})\right] + \frac{ce - af}{ab - c^2}\left[\left(2\frac{u_{,\theta}}{u} + \frac{(ce-af)_{,\theta}}{ce - af} - \frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)\cdot\right.$$

$$\left.\left.(bc_{,r} - cb_{,r}) + \left(2\frac{u_{,\phi}}{u} + \frac{(ce-af)_{,\phi}}{ce - af} - \frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-cc_{,r} + ab_{,r})\right]\right\}. \tag{5.2.21}$$

Next we compute $(l2)$. To do that, we first have

$$\left(-\frac{1}{u^2}\right)_{,\theta}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} + \left(-\frac{1}{u^2}\right)\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}_{,\theta}$$

$$= \frac{2}{u^3}u_{,\theta}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} - \frac{1}{u^2}\frac{1}{2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{-1/2}\left(\frac{-|\overline{g}|_{,\theta}|g_S| + |\overline{g}||g_S|_{,\theta}}{|g_S|^2}\right)$$

$$= \frac{2}{u^3}u_{,\theta}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} - \frac{1}{2u^2}\left(\frac{|g_S|}{-|\overline{g}|}\right)^{1/2}\left(\frac{-|\overline{g}|_{,\theta}}{|g_S|} + \frac{2|\overline{g}||g_S|\cot\theta}{|g_S|^2}\right) \quad \text{(By (A.2.17))}$$

$$= \frac{2}{u^3}u_{,\theta}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} - \frac{1}{2u^2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}(-|\overline{g}|_{,\theta} + 2|\overline{g}|\cot\theta)$$

$$= \frac{1}{u^2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\left(\frac{2u_{,\theta}}{u} - \frac{1}{2}\left(\frac{-1}{|\overline{g}|}\right)(-|\overline{g}|_{,\theta} + 2|\overline{g}|\cot\theta)\right)$$

$$= \frac{1}{u^2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\left(\frac{2u_{,\theta}}{u} + \cot\theta - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|}\right).$$

Similarly we have:

$$\left(-\frac{1}{u^2}\right)_{,\phi}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}+\left(-\frac{1}{u^2}\right)\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}_{,\phi}$$

$$=\frac{2}{u^3}u_{,\phi}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}-\frac{1}{u^2}\frac{1}{2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{-1/2}\left(\frac{-|\overline{g}|_{,\phi}|g_S|+|\overline{g}|\cancelto{0}{|g_S|_{,\phi}}}{|g_S|^2}\right)$$

$$=\frac{2}{u^3}u_{,\phi}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}-\frac{1}{2u^2}\left(\frac{-|g_S|}{|\overline{g}|}\right)^{1/2}\left(\frac{-|\overline{g}|_{,\phi}}{|g_S|}\right)$$

$$=\frac{2}{u^3}u_{,\phi}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}-\frac{1}{2u^2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}(-|\overline{g}|_{,\phi})$$

$$=\frac{1}{u^2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\left(\frac{2u_{,\phi}}{u}-\frac{1}{2}\left(\frac{-1}{|\overline{g}|}\right)(-|\overline{g}|_{,\phi})\right)$$

$$=\frac{1}{u^2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\left(\frac{2u_{,\phi}}{u}-\frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right).$$

Then we need to compute:

$$g_S^{\theta\theta}\overline{\Gamma}^t_{\theta r}+g_S^{\theta\phi}\overline{\Gamma}^t_{\phi r}=\frac{1}{|g_S|}(b\overline{\Gamma}^t_{\theta r}-c\overline{\Gamma}^t_{\phi r})$$

$$=\frac{1}{2}\frac{1}{|g_S||\overline{g}|}\Big(b\big(u^2(ab-c^2)(e_{,r}+d_{,\theta})-d(ab-c^2)2uu_{,\theta}+u^2(cf-be)a_{,r}+u^2(ce-af)c_{,r}\big)$$

$$-c\big(u^2(ab-c^2)(f_{,r}+d_{,\phi})-d(ab-c^2)2uu_{,\phi}+u^2(cf-be)c_{,r}+u^2(ce-af)b_{,r}\big)\Big)$$

$$=\frac{1}{2}\frac{1}{|g_S||\overline{g}|}\Big(u^2|g_S|(be_{,r}+bd_{,\theta}-cf_{,r}-cd_{,\phi})-d|g_S|(2buu_{,\theta}-2cuu_{,\phi})$$

$$+u^2(cf-be)(ba_{,r}-cc_{,r})+u^2(ce-af)(bc_{,r}-cb_{,r})\Big)$$

$$=\frac{u^2}{2}\frac{1}{|\overline{g}|}\Big((be_{,r}+bd_{,\theta}-cf_{,r}-cd_{,\phi})-\frac{2d}{u}(bu_{,\theta}-cu_{,\phi})+\frac{cf-be}{ab-c^2}(ba_{,r}-cc_{,r})$$

$$+\frac{ce-af}{ab-c^2}(bc_{,r}-cb_{,r})\Big).$$

Similarly:

$$g_S^{\phi\theta}\overline{\Gamma}^t_{\theta r} + g_S^{\phi\phi}\overline{\Gamma}^t_{\phi r} = \frac{1}{|g_S|}(-c\overline{\Gamma}^t_{\theta r} + a\overline{\Gamma}^t_{\phi r})$$

$$= \frac{1}{2}\frac{1}{|g_S||\overline{g}|}\Big(- c\big(u^2(ab-c^2)(e_{,r} + d_{,\theta}) - d(ab-c^2)2uu_{,\theta} + u^2(cf-be)a_{,r} + u^2(ce-af)$$

$$\cdot c_{,r}\big) + a\big(u^2(ab-c^2)(f_{,r} + d_{,\phi}) - d(ab-c^2)2uu_{,\phi} + u^2(cf-be)c_{,r} + u^2(ce-af)b_{,r}\big)\Big)$$

$$= \frac{1}{2}\frac{1}{|g_S||\overline{g}|}\Big(u^2|g_S|(-ce_{,r} - cd_{,\theta} + af_{,r} + ad_{,\phi}) - d|g_S|(-2cuu_{,\theta} + 2auu_{,\phi})$$

$$+ u^2(cf-be)(-ca_{,r} + ac_{,r}) + u^2(ce-af)(-cc_{,r} + ab_{,r})\Big)$$

$$= \frac{u^2}{2}\frac{1}{|\overline{g}|}\Big((-ce_{,r} - cd_{,\theta} + af_{,r} + ad_{,\phi}) - \frac{2d}{u}(-cu_{,\theta} + au_{,\phi}) + \frac{cf-be}{ab-c^2}(-ca_{,r} + ac_{,r})$$

$$+ \frac{ce-af}{ab-c^2}(-cc_{,r} + ab_{,r})\Big).$$

Putting them all together, we get:

$$(l2) = \frac{1}{\cancel{u^2}}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\left(\frac{2u_{,\theta}}{u} + \cot\theta - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|}\right)\cdot\frac{\cancel{u^2}}{2}\frac{1}{|\overline{g}|}\Big((be_{,r} + bd_{,\theta} - cf_{,r} - cd_{,\phi})$$

$$- \frac{2d}{u}(bu_{,\theta} - cu_{,\phi}) + \frac{cf-be}{ab-c^2}(ba_{,r} - cc_{,r}) + \frac{ce-af}{ab-c^2}(bc_{,r} - cb_{,r})\Big)$$

$$+ \frac{1}{\cancel{u^2}}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\left(\frac{2u_{,\phi}}{u} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)\cdot\frac{\cancel{u^2}}{2}\frac{1}{|\overline{g}|}\Big((-ce_{,r} - cd_{,\theta} + af_{,r} + ad_{,\phi})$$

$$- \frac{2d}{u}(-cu_{,\theta} + au_{,\phi}) + \frac{cf-be}{ab-c^2}(-ca_{,r} + ac_{,r}) + \frac{ce-af}{ab-c^2}(-cc_{,r} + ab_{,r})\Big).$$

Simplifying $(l2)$, we get:

$$(l2) = -\frac{1}{2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}\Bigg\{\left(\frac{2u_{,\theta}}{u} + \cot\theta - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|}\right)\cdot\Big((be_{,r} + bd_{,\theta} - cf_{,r} - cd_{,\phi})$$

$$- \frac{2d}{u}(bu_{,\theta} - cu_{,\phi}) + \frac{cf - be}{ab - c^2}(ba_{,r} - cc_{,r}) + \frac{ce - af}{ab - c^2}(bc_{,r} - cb_{,r})\Big)$$

$$+ \left(\frac{2u_{,\phi}}{u} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)\cdot\Big((-ce_{,r} - cd_{,\theta} + af_{,r} + ad_{,\phi}) - \frac{2d}{u}(-cu_{,\theta} + au_{,\phi})$$

$$+ \frac{cf - be}{ab - c^2}(-ca_{,r} + ac_{,r}) + \frac{ce - af}{ab - c^2}(-cc_{,r} + ab_{,r})\Big)\Bigg\}. \tag{5.2.22}$$

Next we compute $(l1)$:

$$(l1) = (g^{\theta\theta}_{S,\theta}\alpha_\theta + g^{\theta\phi}_{S,\theta}\alpha_\phi + g^{\phi\theta}_{S,\phi}\alpha_\theta + g^{\phi\phi}_{S,\phi}\alpha_\phi) + (g^{\theta\theta}_S\alpha_\theta + g^{\theta\phi}_S\alpha_\phi)\cot\theta$$

$$= (g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S\cot\theta)\alpha_\theta + (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S\cot\theta)\alpha_\phi$$

$$= -\frac{1}{u^2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}(g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S\cot\theta)\overline{\Gamma}^t_{\theta r}$$

$$+ \left(-\frac{1}{u^2}\right)\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}(g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S\cot\theta)\overline{\Gamma}^t_{\phi r}$$

$$= -\frac{1}{u^2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\Big((g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S\cot\theta)\overline{\Gamma}^t_{\theta r} + (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S\cot\theta)\overline{\Gamma}^t_{\phi r}\Big)$$

$$=: -\frac{1}{u^2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2}\Big((l1a) + (l1b)\Big),$$

where $(l1a) := (g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S\cot\theta)\overline{\Gamma}^t_{\theta r}$, and $(l1b) := (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S\cot\theta)\overline{\Gamma}^t_{\phi r}$.

Note that

$$(l1a) = (g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)\frac{1}{2|\overline{g}|}\Big(u^2(ab-c^2)(e_{,r}+d_{,\theta}) - d(ab-c^2)2uu_{,\theta}$$

$$+ u^2(cf-be)a_{,r} + u^2(ce-af)c_{,r}\Big)$$

$$= \frac{u^2}{2}\frac{|g_S|}{|\overline{g}|}(g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)d_{,\theta} + \frac{1}{2|\overline{g}|}\Big(u^2|g_S|(g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)e_{,r}$$

$$- 2d|g_S|uu_{,\theta}(g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta) + u^2(cf-be)(g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)a_{,r}$$

$$+ u^2(ce-af)(g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)c_{,r}\Big).$$

Similarly:

$$(l1b) = (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)\frac{1}{2|\overline{g}|}\Big(u^2(ab-c^2)(f_{,r}+d_{,\phi}) - d(ab-c^2)2uu_{,\phi}$$

$$+ u^2(cf-be)c_{,r} + u^2(ce-af)b_{,r}\Big)$$

$$= \frac{u^2}{2}\frac{|g_S|}{|\overline{g}|}(g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)d_{,\phi} + \frac{1}{2|\overline{g}|}\Big(u^2|g_S|(g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)f_{,r}$$

$$- 2d|g_S|uu_{,\phi}(g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta) + u^2(cf-be)(g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)c_{,r}$$

$$+ u^2(ce-af)(g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)b_{,r}\Big).$$

Plug them back into $(l1)$, we get:

$$(l1) = -\frac{1}{u^2}\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} \frac{u^2}{2}\frac{|g_S|}{|\overline{g}|}\Big((g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)d_{,\theta}$$

$$+ (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)d_{,\phi}\Big)$$

$$+ \left(-\frac{1}{u^2}\right)\left(\frac{-|\overline{g}|}{|g_S|}\right)^{1/2} \Bigg\{\frac{u^2}{2}\frac{|g_S|}{|\overline{g}|}\Big((g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)e_{,r}$$

$$+ (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)f_{,r}\Big)$$

$$+ \frac{(-2du)}{2}\frac{|g_S|}{|\overline{g}|}\Big((g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)u_{,\theta} + (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)u_{,\phi}\Big)$$

$$+ \frac{u^2(cf-be)}{2|\overline{g}|}\Big((g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)a_{,r} + (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)c_{,r}\Big)$$

$$+ \frac{u^2(ce-af)}{2|\overline{g}|}\Big((g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)c_{,r} + (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)b_{,r}\Big)\Bigg\}. \quad (5.2.23)$$

And

$$(5.2.23) = \frac{1}{2}\left(\frac{|g_S|}{-|\overline{g}|}\right)^{1/2}\Big((g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)d_{,\theta} + (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)d_{,\phi}\Big)$$

$$+ \frac{1}{2}\left(\frac{|g_S|}{-|\overline{g}|}\right)^{1/2}\Bigg\{\Big((g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)e_{,r} + (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)f_{,r}\Big)$$

$$- \frac{2d}{u}\Big((g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)u_{,\theta} + (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)u_{,\phi}\Big)$$

$$+ \frac{cf-be}{ab-c^2}\Big((g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)a_{,r} + (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)c_{,r}\Big)$$

$$+ \frac{ce-af}{ab-c^2}\Big((g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta)c_{,r} + (g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_S \cot\theta)b_{,r}\Big)\Bigg\}.$$

It can be computed that

$$g^{\theta\theta}_{S,\theta} + g^{\phi\theta}_{S,\phi} + g^{\theta\theta}_S \cot\theta = \frac{1}{|g_S|}(b_{,\theta} - b\cot\theta - c_{,\phi});$$

and

$$g^{\theta\phi}_{S,\theta} + g^{\phi\phi}_{S,\phi} + g^{\theta\phi}_{S} \cot\theta = \frac{1}{|g_S|}(-c_{,\theta} + c\cot\theta + a_{,\phi}).$$

Using the above, we can simplify $(l1)$ as follows:

$$\begin{aligned}
(l1) = {} & \frac{1}{2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}\Big((b_{,\theta} - b\cot\theta - c_{,\phi})d_{,\theta} + (-c_{,\theta} + c\cot\theta + a_{,\phi})d_{,\phi}\Big) \\
& + \frac{1}{2}\left(\frac{-1}{|g_S||\overline{g}|}\right)^{1/2}\left[(b_{,\theta} - b\cot\theta - c_{,\phi})\left(e_{,r} - 2\frac{u_{,\theta}}{u}d + \frac{cf - be}{ab - c^2}a_{,r} + \frac{ce - af}{ab - c^2}c_{,r}\right)\right. \\
& \left. +(-c_{,\theta} + c\cot\theta + a_{,\phi})\left(f_{,r} - 2\frac{u_{,\phi}}{u}d + \frac{cf - be}{ab - c^2}c_{,r} + \frac{ce - af}{ab - c^2}b_{,r}\right)\right]. \qquad (5.2.24)
\end{aligned}$$

Plug (5.2.14), (5.2.16), (5.2.18), (5.2.20), (5.2.21), (5.2.22) and (5.2.24) back into the divergence free equation (5.2.12), we get:

$$
\begin{aligned}
2(-|g_S||\overline{g}|)^{1/2}\cdot \mathrm{div}_{g_S}\alpha^\nu &= -2(-|g_S||\overline{g}|)^{1/2}\frac{1}{u^2}\frac{(-|\overline{g}|)^{1/2}}{|g_S|^{3/2}}\cdot\Big((I)+(II)+(III)+(IV)\\
&+(E)\Big)+(l2)+(l1) \qquad \text{(By (5.2.12))}\\
&=\underline{(bd_{,\theta\theta}-2cd_{,\theta\phi}+ad_{,\phi\phi})}+(be_{,r\theta}-cf_{,r\theta}-ce_{,r\phi}+af_{,r\phi})-\frac{d}{u^2}(b(u^2)_{,\theta\theta}-2c(u^2)_{,\theta\phi}\\
&+a(u^2)_{,\phi\phi})+\frac{cf-be}{ab-c^2}(ba_{,r\theta}-cc_{,r\theta}-ca_{,r\phi}+ac_{,r\phi})+\frac{ce-af}{ab-c^2}(bc_{,r\theta}-cb_{,r\theta}-cc_{,r\phi}+ab_{,r\phi})\\
&+\left(2\frac{u_{,\theta}}{u}+2\cot\theta-\frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(be_{,r}+bd_{,\theta}-cf_{,r}-cd_{,\phi})\\
&+\left(2\frac{u_{,\phi}}{u}-\frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-ce_{,r}-cd_{,\theta}+af_{,r}+ad_{,\phi})-\frac{2d}{u}\left[\left(\frac{d_{,\theta}}{d}+2\cot\theta-\frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(bu_{,\theta}-cu_{,\phi})\right.\\
&\left.+\left(\frac{d_{,\phi}}{d}-\frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-cu_{,\theta}+au_{,\phi})\right]+\frac{cf-be}{ab-c^2}\left[\left(2\frac{u_{,\theta}}{u}+\frac{(cf-be)_{,\theta}}{cf-be}-\frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)(ba_{,r}-cc_{,r})\right.\\
&\left.+\left(2\frac{u_{,\phi}}{u}+\frac{(cf-be)_{,\phi}}{cf-be}-\frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-ca_{,r}+ac_{,r})\right]+\frac{ce-af}{ab-c^2}\left[\left(2\frac{u_{,\theta}}{u}+\frac{(ce-af)_{,\theta}}{ce-af}-\frac{|\overline{g}|_{,\theta}}{|\overline{g}|}\right)\cdot\right.\\
&\left.(bc_{,r}-cb_{,r})+\left(2\frac{u_{,\phi}}{u}+\frac{(ce-af)_{,\phi}}{ce-af}-\frac{|\overline{g}|_{,\phi}}{|\overline{g}|}\right)(-cc_{,r}+ab_{,r})\right]\\
&-\left(\frac{2u_{,\theta}}{u}+\cot\theta-\frac{|\overline{g}|_{,\theta}}{2|\overline{g}|}\right)\cdot\Big((be_{,r}+bd_{,\theta}-cf_{,r}-cd_{,\phi})-\frac{2d}{u}(bu_{,\theta}-cu_{,\phi})\\
&+\frac{cf-be}{ab-c^2}(ba_{,r}-cc_{,r})+\frac{ce-af}{ab-c^2}(bc_{,r}-cb_{,r})\Big)-\left(\frac{2u_{,\phi}}{u}-\frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)\cdot\Big((-ce_{,r}-cd_{,\theta}\\
&+af_{,r}+ad_{,\phi})-\frac{2d}{u}(-cu_{,\theta}+au_{,\phi})+\frac{cf-be}{ab-c^2}(-ca_{,r}+ac_{,r})+\frac{ce-af}{ab-c^2}(-cc_{,r}+ab_{,r})\Big)\\
&+\underline{(b_{,\theta}-b\cot\theta-c_{,\phi})d_{,\theta}+(-c_{,\theta}+c\cot\theta+a_{,\phi})d_{,\phi})}\\
&+(b_{,\theta}-b\cot\theta-c_{,\phi})\left(e_{,r}-2\frac{u_{,\theta}}{u}d+\frac{cf-be}{ab-c^2}a_{,r}+\frac{ce-af}{ab-c^2}c_{,r}\right)\\
&+(-c_{,\theta}+c\cot\theta+a_{,\phi})\left(f_{,r}-2\frac{u_{,\phi}}{u}d+\frac{cf-be}{ab-c^2}c_{,r}+\frac{ce-af}{ab-c^2}b_{,r}\right). \qquad (5.2.25)
\end{aligned}
$$

Notice that the terms underlined, up to a $|g_S|$ factor, equal the Laplacian of d along $S_{t,r}$ (see equation (A.2.16) in Appendix A.2.3). We can simplify (5.2.25) as:

$$
\begin{aligned}
(5.2.25) &= |g_S|\Delta_{g_S} d + (be_{,r\theta} - cf_{,r\theta} - ce_{,r\phi} + af_{,r\phi}) \\
&- \frac{d}{u^2}(b(u^2)_{,\theta\theta} - 2c(u^2)_{,\theta\phi} + a(u^2)_{,\phi\phi}) + \frac{cf-be}{ab-c^2}(ba_{,r\theta} - cc_{,r\theta} - ca_{,r\phi} + ac_{,r\phi}) \\
&+ \frac{ce-af}{ab-c^2}(bc_{,r\theta} - cb_{,r\theta} - cc_{,r\phi} + ab_{,r\phi}) + \left(\cot\theta - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|}\right)(be_{,r} + bd_{,\theta} - cf_{,r} - cd_{,\phi}) \\
&- \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}(-ce_{,r} - cd_{,\theta} + af_{,r} + ad_{,\phi}) \\
&- \frac{2d}{u}\left[\left(\frac{d_{,\theta}}{d} - \frac{2u_{,\theta}}{u} + \cot\theta - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|}\right)(bu_{,\theta} - cu_{,\phi}) + \left(\frac{d_{,\phi}}{d} - \frac{2u_{,\phi}}{u} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)(-cu_{,\theta} + au_{,\phi})\right] \\
&+ \frac{cf-be}{ab-c^2}\Bigg[\left(\frac{(cf-be)_{,\theta}}{cf-be} - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|} - \cot\theta\right)(ba_{,r} - cc_{,r}) \\
&+ \left(\frac{(cf-be)_{,\phi}}{cf-be} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)(-ca_{,r} + ac_{,r})\Bigg] \\
&+ \frac{ce-af}{ab-c^2}\Bigg[\left(\frac{(ce-af)_{,\theta}}{ce-af} - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|} - \cot\theta\right)(bc_{,r} - cb_{,r}) \\
&+ \left(\frac{(ce-af)_{,\phi}}{ce-af} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)(-cc_{,r} + ab_{,r})\Bigg] \\
&+ (b_{,\theta} - b\cot\theta - c_{,\phi})\left(e_{,r} - \frac{2d}{u}u_{,\theta} + \frac{cf-be}{ab-c^2}a_{,r} + \frac{ce-af}{ab-c^2}c_{,r}\right) \\
&+ (-c_{,\theta} + c\cot\theta + a_{,\phi})\left(f_{,r} - \frac{2d}{u}u_{,\phi} + \frac{cf-be}{ab-c^2}c_{,r} + \frac{ce-af}{ab-c^2}b_{,r}\right). \qquad (5.2.26)
\end{aligned}
$$

Definition 5.2.2. *We set $L_2(\overline{g}) = L_2(d,e,f,u,a,b,c)$ to be the second derivative terms in the straight out flow equation, that is:*

$$
\begin{aligned}
L_2(\overline{g}) &:= |g_S|\Delta_{g_S} d + (be_{,r\theta} - cf_{,r\theta} - ce_{,r\phi} + af_{,r\phi}) - \frac{d}{u^2}(b(u^2)_{,\theta\theta} - 2c(u^2)_{,\theta\phi} + a(u^2)_{,\phi\phi}) \\
&+ \frac{cf-be}{ab-c^2}(ba_{,r\theta} - cc_{,r\theta} - ca_{,r\phi} + ac_{,r\phi}) + \frac{ce-af}{ab-c^2}(bc_{,r\theta} - cb_{,r\theta} - cc_{,r\phi} + ab_{,r\phi}). \qquad (5.2.27)
\end{aligned}
$$

With this defintion, we further simplify (5.2.26) as follows:

$$
\begin{aligned}
(5.2.26) &= L_2(\overline{g}) + \cot\theta(\cancel{be_{,r}} + bd_{,\theta} - \cancel{cf_{,r}} - cd_{,\phi}) - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|}(be_{,r} + bd_{,\theta} - cf_{,r} - cd_{,\phi}) \\
&- \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}(-ce_{,r} - cd_{,\theta} + af_{,r} + ad_{,\phi}) - \frac{2d}{u}\Big[\left(\frac{d_{,\theta}}{d} - \frac{2u_{,\theta}}{u} + \cot\theta - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|}\right)(bu_{,\theta} - cu_{,\phi}) \\
&+ \left(\frac{d_{,\phi}}{d} - \frac{2u_{,\phi}}{u} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)(-cu_{,\theta} + au_{,\phi})\Big] \\
&+ \frac{cf-be}{ab-c^2}\Big[\left(\frac{(cf-be)_{,\theta}}{cf-be} - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|} - \cot\theta\right)(ba_{,r} - cc_{,r}) + \left(\frac{(cf-be)_{,\phi}}{cf-be} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)\cdot \\
&(-ca_{,r} + ac_{,r})\Big] \\
&+ \frac{ce-af}{ab-c^2}\Big[\left(\frac{(ce-af)_{,\theta}}{ce-af} - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|} - \cot\theta\right)(bc_{,r} - cb_{,r}) + \left(\frac{(ce-af)_{,\phi}}{ce-af} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)\cdot \\
&(-cc_{,r} + ab_{,r})\Big] \\
&+ b_{,\theta}e_{,r} - \frac{2d}{u}b_{,\theta}u_{,\theta} + \frac{cf-be}{ab-c^2}b_{,\theta}a_{,r} + \frac{ce-af}{ab-c^2}b_{,\theta}c_{,r} - \cancel{be_{,r}\cot\theta} + \underline{\frac{2d}{u}bu_{,\theta}\cot\theta} \\
&- \frac{cf-be}{ab-c^2}ba_{,r}\cot\theta - \frac{ce-af}{ab-c^2}bc_{,r}\cot\theta - c_{,\phi}e_{,r} + \frac{2d}{u}c_{,\phi}u_{,\theta} - \frac{cf-be}{ab-c^2}c_{,\phi}a_{,r} \\
&- \frac{ce-af}{ab-c^2}c_{,\phi}c_{,r} - c_{,\theta}f_{,r} + \frac{2d}{u}c_{,\theta}u_{,\phi} - \frac{cf-be}{ab-c^2}c_{,\theta}c_{,r} - \frac{ce-af}{ab-c^2}c_{,\theta}b_{,r} + \cancel{cf_{,r}\cot\theta} \\
&\underline{-\frac{2d}{u}cu_{,\phi}\cot\theta} + \frac{cf-be}{ab-c^2}cc_{,r}\cot\theta + \frac{ce-af}{ab-c^2}cb_{,r}\cot\theta + a_{,\phi}f_{,r} - \frac{2d}{u}a_{,\phi}u_{,\phi} \\
&+ \frac{cf-be}{ab-c^2}a_{,\phi}c_{,r} + \frac{ce-af}{ab-c^2}a_{,\phi}b_{,r}. \qquad (5.2.28)
\end{aligned}
$$

$$
\begin{aligned}
(5.2.28) &= L_2(\overline{g}) + \cot\theta(bd_{,\theta} - cd_{,\phi}) - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|}(be_{,r} + bd_{,\theta} - cf_{,r} - cd_{,\phi}) \\
&- \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}(-ce_{,r} - cd_{,\theta} + af_{,r} + ad_{,\phi}) \\
&- \frac{2d}{u}\Big[\left(\frac{d_{,\theta}}{d} - \frac{2u_{,\theta}}{u} + \cot\theta - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|}\right)(bu_{,\theta} - cu_{,\phi}) + \left(\frac{d_{,\phi}}{d} - \frac{2u_{,\phi}}{u} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)\cdot \\
&(-cu_{,\theta} + au_{,\phi})\Big] \\
&+ \frac{cf-be}{ab-c^2}\Big[\left(\frac{(cf-be)_{,\theta}}{cf-be} - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|} - \cot\theta\right)(ba_{,r} - cc_{,r}) + \left(\frac{(cf-be)_{,\phi}}{cf-be} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)\cdot \\
&(-ca_{,r} + ac_{,r})\Big] \\
&+ \frac{ce-af}{ab-c^2}\Big[\left(\frac{(ce-af)_{,\theta}}{ce-af} - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|} - \cot\theta\right)(bc_{,r} - cb_{,r}) + \left(\frac{(ce-af)_{,\phi}}{ce-af} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)\cdot \\
&(-cc_{,r} + ab_{,r})\Big] \\
&+ (b_{,\theta}e_{,r} - c_{,\phi}e_{,r} - c_{,\theta}f_{,r} + a_{,\phi}f_{,r}) - \frac{2d}{u}(b_{,\theta}u_{,\theta} - c_{,\phi}u_{,\theta} - c_{,\theta}c_{\phi} + a_{,\phi}u_{,\phi}) \\
&+ \frac{cf-be}{ab-c^2}(b_{,\theta}a_{,r} - a_{,r}c_{,\phi} - c_{,r}c_{,\theta} + a_{,\phi}c_{,r}) + \frac{ce-af}{ab-c^2}(b_{,\theta}c_{,r} - c_{,r}c_{,\phi} - b_{,r}c_{,\theta} + a_{,\phi}b_{,r}) \\
&+ \underline{\frac{2d}{u}\cot\theta(bu_{,\theta} - cu_{,\phi})} - \frac{cf-be}{ab-c^2}\cot\theta(ba_{,r} - cc_{,r}) - \frac{ce-af}{ab-c^2}\cot\theta(bc_{,r} - cb_{,r})
\end{aligned}
\tag{5.2.29}
$$

Finally the above can be simplified to:

$$
\begin{aligned}
&L_2(\overline{g}) + \cot\theta(bd_{,\theta} - cd_{,\phi}) - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|}(be_{,r} + bd_{,\theta} - cf_{,r} - cd_{,\phi}) - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}(-ce_{,r} - cd_{,\theta} + af_{,r} \\
&+ ad_{,\phi}) \\
&- \frac{2d}{u}\left[\left(\frac{d_{,\theta}}{d} - \frac{2u_{,\theta}}{u} - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|}\right)(bu_{,\theta} - cu_{,\phi}) + \left(\frac{d_{,\phi}}{d} - \frac{2u_{,\phi}}{u} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)(-cu_{,\theta} + au_{,\phi})\right] \\
&+ \frac{cf - be}{ab - c^2}\left[\left(\frac{(cf-be)_{,\theta}}{cf - be} - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|} - 2\cot\theta\right)(ba_{,r} - cc_{,r}) + \left(\frac{(cf-be)_{,\phi}}{cf - be} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)\cdot \right. \\
&\left.(-ca_{,r} + ac_{,r})\right] \\
&+ \frac{ce - af}{ab - c^2}\left[\left(\frac{(ce-af)_{,\theta}}{ce - af} - \frac{|\overline{g}|_{,\theta}}{2|\overline{g}|} - 2\cot\theta\right)(bc_{,r} - cb_{,r}) + \left(\frac{(ce-af)_{,\phi}}{ce - af} - \frac{|\overline{g}|_{,\phi}}{2|\overline{g}|}\right)\right. \\
&\left.(-cc_{,r} + ab_{,r})\right] \\
&+ (b_{,\theta}e_{,r} - c_{,\phi}e_{,r} - c_{,\theta}f_{,r} + a_{,\phi}f_{,r}) - \frac{2d}{u}(b_{,\theta}u_{,\theta} - c_{,\phi}u_{,\theta} - c_{,\theta}c_{\phi} + a_{,\phi}u_{,\phi}) \\
&+ \frac{cf - be}{ab - c^2}(b_{,\theta}a_{,r} - a_{,r}c_{,\phi} - c_{,r}c_{,\theta} + a_{,\phi}c_{,r}) + \frac{ce - af}{ab - c^2}(b_{,\theta}c_{,r} - c_{,r}c_{,\phi} - b_{,r}c_{,\theta} + a_{,\phi}b_{,r}).
\end{aligned} \tag{5.2.30}
$$

$$=: |g_S|\Delta_{g_S} d + F(d, d').$$

Therefore we have the following characterization of the fourth condition (5.1.11):

Proposition 5.4. *The fourth condition (5.1.11) is a second order elliptic PDE in d:*

$$\Delta_{g_S} d + G(d, d') = 0, \tag{5.2.31}$$

where $G := \frac{F}{|g_S|} = \frac{F}{r^4 \sin^2\theta}$.

If Equation (5.2.31) is solvable, then many spacetimes that admit straight out flow coordinate chart exist, even beyond spherically symmetric examples. A necessary condition for (5.2.31) to be solvable if

$$\int_{S_{t,r}} G(d, d')\ dA_{t,r} = 0. \tag{5.2.32}$$

We conjecture that this is always the case. The verification of (5.2.32) is still work in progress.

6

Conclusions and Open Problems

We have constructed many examples of non-spherically symmetric, non-static spacetimes that admit smooth global solutions to inverse mean curvature vector flow. Prior to our work, such solutions were only known in spherically symmetric and static spacetimes. Our work seems to suggest that spacetimes that admit inverse mean curvature vector flow solutions might exist generically. However, this more general problem is still open:

Problem 6.1. *Given an arbitrary spacetime. Can we always find a "right" initial surface such that inverse mean curvature vector flow starting with this surface exists for all time?*

Going to the big picture of relating local and global notions of mass, it is still unknown that:

Problem 6.2. *Given a spacetime that is sufficiently asymptotically flat (e.g. Schwarzschild outside a compact set). Does the Hawking of inverse mean curvature vector flow surfaces approach the total mass of the spacetime?*

A natural next step is to consider the following problem:

Problem 6.3. *Given a spherically symmetric spacetime that admits an inverse mean curvature vector flow coordinate chart. Consider a perturbation of the spacetime metric. Does the perturbed metric admit an inverse mean curvature vector flow coordinate chart as well?*

We conjecture that this is always the case for some Minkowski spacetimes:

Conjecture 6.1. *Given Minkowski space with inverse mean curvature vector flow coordinate chart that can be smoothly extended to the boundary, consider a perturbation of the spacetime metric. The resulting spacetime still admits inverse mean curvature vector flow solutions (in a single spacelike hypersurface) that exist for all time.*

Appendix A

Geometric Calculations

A.1 Ricci, Scalar and Einstein Curvature of Spherically Symmetric Spacetime

In this subsection, we are going to compute the Ricci, Scalar and Einstein curvature of the spherically symmetric space time $(N^4, \overline{g})$, with coordinates (t, r, θ, ϕ), such that g has the local coordinate representation as in (3.4.1):

$$\overline{g} = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} -v^2(t,r) & 0 & 0 & 0 \\ 0 & u^2(t,r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \end{array} \tag{A.1.1}$$

Note that $\{\partial_t, \partial_r, \partial_\theta, \partial_\phi\}$ form a local frame of the tangent bundle. We assume that the connection on $(N^4, \overline{g})$ is the Levi-Civita connection. The Einstein summation convention will be used, and colons will always denote the coordinate chart derivative whereas semicolons will denote covariant derivatives.In the following, we will use Latin letters i, j, k, l and so on to be indices taking values in $\{t, r, \theta, \phi\}$. Recall the following formulas from Riemannian geometry:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(g_{jl,i} + g_{il,j} - g_{ij,l}) \qquad \text{(Christoffel symbols)}$$

$$\mathrm{Ric}_{ij} = \mathrm{Ric}(\partial_i, \partial_j) = \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{km}\Gamma^m_{ij} - \Gamma^k_{jm}\Gamma^m_{ik} \qquad \text{(Ricci curvature)}$$

$$\mathrm{R} = \mathrm{tr}_g\mathrm{Ric} \qquad \text{(scalar curvature)}$$

We will write the Christoffel symbols in four matrices $\Gamma^t, \Gamma^r, \Gamma^\theta$, and Γ^ϕ as in Alan Parry's survey paper [[35]]. Now we compute them in sequence, using the fact that the metric g is diagonal in our coordinates.

1. The computation of Γ^t:

   - $\Gamma^t_{tt} = \frac{1}{2}g^{tt}(g_{tt,t}) = \frac{1}{2}\frac{1}{v^2}2vv_{,t} = \frac{v_{,t}}{v}$.
   - $\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2}g^{tt}(g_{rt,t} + g_{tt,r} - g_{tr,t}) = \frac{1}{2}\frac{1}{v^2}(2vv_{,r}) = \frac{v_{,r}}{v}$.
   - $\Gamma^t_{t\theta} = \Gamma^t_{\theta t} = \frac{1}{2}g^{tt}(g_{\theta t,t} + g_{tt,\theta} - g_{t\theta,t}) = 0$.
   - $\Gamma^t_{t\phi} = \Gamma^t_{\phi t} = \frac{1}{2}g^{tt}(g_{\phi t,t} + g_{tt,\phi} - g_{t\phi,t}) = 0$.
   - $\Gamma^t_{rr} = \frac{1}{2}g^{tt}(2g_{rt,r} - g_{rr,t}) = \frac{1}{2}\frac{1}{v^2}(2uu_{,t}) = \frac{uu_{,t}}{v^2}$.
   - $\Gamma^t_{r\theta} = \Gamma^t_{\theta r} = \frac{1}{2}g^{tt}(g_{\theta t,r} + g_{rt,\theta} - g_{r\theta,t}) = 0$.
   - $\Gamma^t_{r\phi} = \Gamma^t_{\phi r} = \frac{1}{2}g^{tt}(g_{\phi t,r} + g_{rt,\phi} - g_{r\phi,t}) = 0$.
   - $\Gamma^t_{\theta\theta} = \frac{1}{2}g^{tt}(2g_{\theta t,\theta} - g_{\theta\theta,t}) = 0$.
   - $\Gamma^t_{\theta\phi} = \Gamma^t_{\phi\theta} = \frac{1}{2}g^{tt}(g_{\phi t,\theta} + g_{\theta t,\phi} - g_{\theta\phi,t}) = 0$.
   - $\Gamma^t_{\phi\phi} = \frac{1}{2}g^{tt}(2g_{\phi t,\phi} - g_{\phi\phi,t}) = 0$.

   Therefore we have

$$\Gamma^t = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} \frac{v_{,t}}{v} & \frac{v_{,r}}{v} & 0 & 0 \\ \frac{v_{,r}}{v} & \frac{uu_{,t}}{v^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \text{(A.1.2)}$$

2. The computation of Γ^r:

- $\Gamma^r_{tt} = \frac{1}{2}g^{rr}(2g_{tr,t} - g_{tt,r}) = \frac{1}{2}\frac{1}{u^2}(2vv_{,r}) = \frac{vv_{,r}}{u^2}$.
- $\Gamma^r_{tr} = \Gamma^r_{rt} = \frac{1}{2}g^{rr}(g_{rr,t} + g_{tr,r} - g_{tr,r}) = \frac{1}{2}\frac{1}{u^2}(2uu_{,t}) = \frac{u_{,t}}{u}$.
- $\Gamma^r_{t\theta} = \Gamma^r_{\theta t} = \frac{1}{2}g^{rr}(g_{\theta r,t} + g_{tr,\theta} - g_{t\theta,r}) = 0$.
- $\Gamma^r_{t\phi} = \Gamma^r_{\phi t} = \frac{1}{2}g^{rr}(g_{\phi r,t} + g_{tr,\phi} - g_{t\phi,r}) = 0$.
- $\Gamma^r_{rr} = \frac{1}{2}g^{rr}g_{rr,r} = \frac{1}{2}\frac{1}{u^2}(2uu_{,r}) = \frac{u_{,r}}{u}$.
- $\Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \frac{1}{2}g^{rr}(g_{\theta r,r} + g_{rr,\theta} - g_{r\theta,r}) = 0$.
- $\Gamma^r_{r\phi} = \Gamma^r_{\phi r} = \frac{1}{2}g^{rr}(g_{\phi r,r} + g_{rr,\phi} - g_{r\phi,r}) = 0$.
- $\Gamma^r_{\theta\theta} = \frac{1}{2}(2g_{\theta r,\theta} - g_{\theta\theta,r}) = \frac{1}{2}\frac{1}{u^2}(-2r) = -\frac{r}{u^2}$.
- $\Gamma^r_{\theta\phi} = \Gamma^r_{\phi\theta} = \frac{1}{2}g^{rr}(g_{\phi r,\theta} + g_{\theta r,\phi} - g_{\theta\phi,r}) = 0$.
- $\Gamma^r_{\phi\phi} = \frac{1}{2}g^{rr}(2g_{\phi r,\phi} - g_{\phi\phi,r}) = \frac{1}{2}\frac{1}{u^2}(-2r\sin^2\theta) = -\frac{r\sin^2\theta}{u^2}$.

Therefore we have

$$\Gamma^r = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} \frac{vv_{,r}}{u^2} & \frac{u_{,t}}{u} & 0 & 0 \\ \frac{u_{,t}}{u} & \frac{u_{,r}}{u} & 0 & 0 \\ 0 & 0 & -\frac{r}{u^2} & 0 \\ 0 & 0 & 0 & -\frac{r\sin^2\theta}{u^2} \end{pmatrix} \end{array} \tag{A.1.3}$$

3. The computation of Γ^θ:

- $\Gamma^\theta_{tt} = \frac{1}{2}g^{\theta\theta}(2g_{t\theta,t} - g_{tt,\theta}) = 0$.
- $\Gamma^\theta_{tr} = \Gamma^\theta_{rt} = \frac{1}{2}g^{\theta\theta}(g_{t\theta,r} + g_{r\theta,t} - g_{tt,\theta}) = 0$.
- $\Gamma^\theta_{t\theta} = \Gamma^\theta_{\theta t} = \frac{1}{2}g^{\theta\theta}(g_{\theta\theta,t} + g_{t\theta,\theta} - g_{t\theta,\theta}) = 0$.
- $\Gamma^\theta_{t\phi} = \Gamma^\theta_{\phi t} = \frac{1}{2}g^{\theta\theta}(g_{\phi\theta,t} + g_{t\theta,\phi} - g_{t\phi,\theta}) = 0$.

- $\Gamma^{\theta}_{rr} = \frac{1}{2}g^{\theta\theta}(2g_{r\theta,r} - g_{rr,\theta}) = 0.$

- $\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{2}g^{\theta\theta}(g_{\theta\theta,r} + g_{r\theta,\theta} - g_{r\theta,\theta}) = \frac{1}{2}\frac{1}{r^2}2r = \frac{1}{r}.$

- $\Gamma^{\theta}_{r\phi} = \Gamma^{\theta}_{\phi r} = \frac{1}{2}g^{\theta\theta}(g_{\phi\theta,r} + g_{r\theta,\phi} - g_{r\phi,\theta}) = 0.$

- $\Gamma^{\theta}_{\theta\theta} = \frac{1}{2}g^{\theta\theta}g_{\theta\theta,\theta} = 0.$

- $\Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = \frac{1}{2}g^{\theta\theta}(g_{\phi\theta,\theta} + g_{\theta\theta,\phi} - g_{\theta\phi,\theta}) = 0.$

- $\Gamma^{\theta}_{\phi\phi} = \frac{1}{2}g^{\theta\theta}(2g_{\phi\theta,\phi} - g_{\phi\phi,\theta}) = \frac{1}{2}\frac{1}{r^2}(-2r^2\sin\theta\cos\theta) = -sin\theta\cos\theta.$

Therefore we have

$$\Gamma^{\theta} = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix} \end{array} \tag{A.1.4}$$

4. The computation of Γ^{ϕ}:

- $\Gamma^{\phi}_{tt} = \frac{1}{2}g^{\phi\phi}(2g_{t\phi,t} - g_{tt,\phi}) = 0.$

- $\Gamma^{\phi}_{tr} = \Gamma^{\phi}_{rt} = \frac{1}{2}g^{\phi\phi}(g_{r\phi,t} + g_{t\phi,r} - g_{tr,\phi}) = 0.$

- $\Gamma^{\phi}_{t\theta} = \Gamma^{\phi}_{\theta t} = \frac{1}{2}g^{\phi\phi}(g_{\theta\phi,t} + g_{t\phi,\theta} - g_{t\theta,\phi}) = 0.$

- $\Gamma^{\phi}_{t\phi} = \Gamma^{\phi}_{\phi t} = \frac{1}{2}g^{\phi\phi}(g_{\phi\phi,t} + g_{t\phi,\phi} - g_{t\phi,\phi}) = 0.$

- $\Gamma^{\phi}_{rr} = \frac{1}{2}g^{\phi\phi}(2g_{r\phi,r} - g_{rr,\phi}) = 0.$

- $\Gamma^{\phi}_{r\theta} = \Gamma^{\phi}_{\theta r} = \frac{1}{2}g^{\phi\phi}(g_{\theta\phi,r} + g_{r\phi,\theta} - g_{r\theta,\phi}) = 0.$

- $\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{2}g^{\phi\phi}(g_{\phi\phi,r} + g_{r\phi,\phi} - g_{r\phi,\phi}) = \frac{1}{2}\frac{1}{r^2\sin^2\theta}2r\sin^2\theta = \frac{1}{r}.$

- $\Gamma^{\phi}_{\theta\theta} = \frac{1}{2}g^{\phi\phi}(2g_{\theta\phi,\theta} - g_{\theta\theta,\phi}) = 0.$

- $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{1}{2}g^{\phi\phi}(g_{\phi\phi,\theta} + g_{\theta\phi,\phi} - g_{\theta\phi,\phi}) = \frac{1}{2}\frac{1}{r^2\sin^2\theta}r^2 2\sin\theta\cos\theta = \frac{\cos\theta}{\sin\theta}.$

- $\Gamma^{\phi}_{\phi\phi} = \frac{1}{2} g^{\phi\phi} g_{\phi\phi,\phi} = 0.$

Therefore we have

$$\Gamma^{\phi} = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos\theta}{\sin\theta} \\ 0 & \frac{1}{r} & \frac{\cos\theta}{\sin\theta} & 0 \end{pmatrix} \end{array} \tag{A.1.5}$$

Now we compute each of the components of the Ricci curvature with respect to the basis $\{\partial_t, \partial_r, \partial_\theta, \partial_\phi\}$.

1.

$$\begin{aligned} \mathrm{Ric}_{tt} &= \mathrm{Ric}(\partial_t, \partial_t) \\ &= \Gamma^r_{tt,r} - \Gamma^r_{tr,t} + \Gamma^r_{rt}\Gamma^t_{tt} + \Gamma^r_{rr}\Gamma^r_{tt} - \Gamma^r_{tt}\Gamma^t_{tr} - \Gamma^r_{tr}\Gamma^r_{tr} + \Gamma^\theta_{\theta r}\Gamma^r_{tt} + \Gamma^\phi_{\phi r}\Gamma^r_{tt} \\ &= \left(\frac{vv_{,r}}{u^2}\right)_{,r} - \left(\frac{u_{,t}}{u}\right)_{,t} + \frac{u_{,t}}{u}\frac{v_{,t}}{v} + \frac{u_{,r}}{u}\frac{vv_{,r}}{u^2} - \frac{vv_{,r}}{u^2}\frac{v_{,r}}{v} - \frac{u_{,t}}{u}\frac{u_{,t}}{u} + \frac{2}{r}\frac{vv_{,r}}{u^2} \\ &= \frac{1}{u^2}\left(vv_{,rr} + \frac{2}{r}vv_{,r}\right) - \frac{u_{,r}}{u^3}vv_{,r} + \frac{1}{u}\left(\frac{u_{,t}v_{,t}}{v} - u_{,tt}\right) \end{aligned}$$

2.

$$\begin{aligned} \mathrm{Ric}_{tr} &= \mathrm{Ric}(\partial_t, \partial_r) = \Gamma^t_{tr,t} - \Gamma^t_{tt,r} + \Gamma^t_{tr}\Gamma^r_{tr} - \Gamma^t_{rr}\Gamma^r_{tt} + \Gamma^\theta_{\theta r}\Gamma^r_{tr} + \Gamma^\phi_{\phi r}\Gamma^r_{tr} \\ &= \left(\frac{v_{,r}}{v}\right)_{,t} - \left(\frac{v_{,t}}{v}\right)_{,r} + \frac{v_{,r}}{v}\frac{u_{,t}}{u} - \frac{uu_{,t}}{v^2}\frac{vv_{,r}}{u^2} + \frac{1}{r}\frac{u_{,t}}{u} + \frac{1}{r}\frac{u_{,t}}{u} \\ &= \frac{2}{r}\frac{u_{,t}}{u} \end{aligned}$$

3.

$$\mathrm{Ric}_{t\theta} = \mathrm{Ric}_{\theta t} = -\Gamma^t_{tt,\theta} - \Gamma^r_{tr,\theta} = 0$$

4.

$$\mathrm{Ric}_{t\phi} = \mathrm{Ric}_{\phi t} = -\Gamma^t_{tt,\phi} - \Gamma^r_{tr,\phi} = 0$$

5.

$$\begin{aligned}\mathrm{Ric}_{rr} &= \Gamma^t_{rr,t} - \Gamma^t_{rt,r} + \Gamma^t_{tt}\Gamma^t_{rr} + \Gamma^t_{tr}\Gamma^r_{rr} - \Gamma^t_{rt}\Gamma^t_{rt} - \Gamma^t_{rr}\Gamma^r_{rt} - \Gamma^\theta_{r\theta,r} + \Gamma^\theta_{\theta r}\Gamma^r_{rr} - \Gamma^\theta_{r\theta}\Gamma^\theta_{r\theta} \\ &\quad - \Gamma^\phi_{r\phi,r} + \Gamma^\phi_{\phi r}\Gamma^r_{rr} - \Gamma^\phi_{r\phi}\Gamma^\phi_{r\phi} \\ &= \left(\frac{uu_{,t}}{v^2}\right)_{,t} - \left(\frac{v_{,r}}{v}\right)_{,r} + \frac{v_{,t}}{v}\frac{uu_{,t}}{v^2} + \frac{v_{,r}}{v}\frac{u_{,r}}{u} - \frac{v_{,r}}{v}\frac{v_{,r}}{v} - \frac{uu_{,t}}{v^2}\frac{u_{,t}}{u} + \frac{2}{r}\frac{u_{,r}}{u} \\ &= -\frac{v_{,rr}}{v} + \frac{2}{r}\frac{u_{,r}}{u} - \frac{v_{,t}}{v^3}uu_{,t} + \frac{1}{v}\left(\frac{uu_{,tt}}{v} + \frac{v_{,r}u_{,r}}{u}\right)\end{aligned}$$

6.

$$\mathrm{Ric}_{r\theta} = \mathrm{Ric}_{\theta r} = 0$$

7.

$$\mathrm{Ric}_{r\phi} = \mathrm{Ric}_{\phi r} = 0$$

8.

$$\begin{aligned}\mathrm{Ric}_{\theta\theta} &= \Gamma^t_{tr}\Gamma^r_{\theta\theta} + \Gamma^r_{\theta\theta,r} + \Gamma^r_{rr}\Gamma^r_{\theta\theta} - \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r} - \Gamma^\phi_{\theta\phi,\theta} + \Gamma^\phi_{\phi r}\Gamma^r_{\theta\theta} - \Gamma^\phi_{\theta\phi}\Gamma^\phi_{\theta\phi} \\ &= \frac{v_{,r}}{v}\frac{-r}{u^2} + \left(\frac{-r}{u^2}\right)_{,r} + \frac{u_{,r}}{u}\frac{-r}{u^2} - \frac{-r}{u^2}\frac{1}{r} - \left(\frac{\cos\theta}{\sin\theta}\right)_{,\theta} + \frac{1}{r}\frac{-r}{u^2} - \left(\frac{\cos\theta}{\sin\theta}\right)^2 \\ &= \left(1 - \frac{1}{u^2}\right) + r\frac{u_{,r}}{u^3} - r\frac{v_{,r}}{v}\frac{1}{u^2}\end{aligned}$$

9.

$$\mathrm{Ric}_{\theta\phi} = \mathrm{Ric}_{\phi\theta} = 0$$

10.

$$\begin{aligned}\mathrm{Ric}_{\phi\phi} &= \Gamma^t_{tr}\Gamma^r_{\phi\phi} + \Gamma^r_{\phi\phi,r} + \Gamma^r_{rr}\Gamma^r_{\phi\phi} - \Gamma^r_{\phi\phi}\Gamma^\phi_{\phi r} + \Gamma^\theta_{\phi\phi,\theta} + \Gamma^\theta_{\theta r}\Gamma^r_{\phi\phi} - \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\phi\theta} \\ &= \frac{v_{,r}}{v}\frac{-r\sin^2\theta}{u^2} + \left(\frac{-r\sin^2\theta}{u^2}\right)_{,r} + \frac{u_{,r}}{u}\frac{-r\sin^2\theta}{u^2} + (-\sin\theta\cos\theta)_{,\theta} + \sin\theta\cos\theta\frac{\cos\theta}{\sin\theta} \\ &= \left(1 - \frac{1}{u^2}\right)\sin^2\theta + r\sin^2\theta\frac{u_{,r}}{u^3} - r\sin^2\theta\frac{v_{,r}}{v}\frac{1}{u^2}\end{aligned}$$

Now we can compute the scalar curvature R of this metric as follows:

$$R = \mathrm{tr}_g \mathrm{Ric} = g^{tt}\mathrm{Ric}_{tt} + g^{rr}\mathrm{Ric}_{rr} + g^{\theta\theta}\mathrm{Ric}_{\theta\theta} + g^{\phi\phi}\mathrm{Ric}_{\phi\phi} \qquad (g \text{ is diagonal})$$

$$= \frac{-1}{v^2}\left[\frac{1}{u^2}\left(vv_{,rr} + \frac{2}{r}vv_{,r}\right) - \frac{u_{,r}}{u^3}vv_{,r} + \frac{v_{,t}}{v}\frac{u_{,t}}{u} - \frac{u_{,tt}}{u}\right]$$

$$+ \frac{1}{u^2}\left[-\frac{v_{,rr}}{v} + \frac{2}{r}\frac{u_{,r}}{u} - \frac{v_{,t}}{v^3}uu_{,t} + \frac{1}{v}\left(\frac{uu_{,tt}}{v} + \frac{v_{,r}u_{,r}}{u}\right)\right]$$

$$+ \frac{2}{r^2}\left[\left(1 - \frac{1}{u^2}\right) + r\frac{u_{,r}}{u^3} - r\frac{v_{,r}}{v}\frac{1}{u^2}\right]$$

$$= -\frac{1}{u^2}\frac{v_{,rr}}{v} - \frac{2}{r}\frac{1}{u^2}\frac{v_{,r}}{v} + \frac{u_{,r}}{u^3}\frac{v_{,r}}{v} - \frac{u_{,t}}{u}\frac{v_{,t}}{v^3} + \frac{u_{,tt}}{u}\frac{1}{v^2} - \frac{1}{u^2}\frac{v_{,rr}}{v} + \frac{2}{r}\frac{u_{,r}}{u^3} + \frac{u_{,r}}{u^3}\frac{v_{,r}}{v}$$

$$- \frac{u_{,t}}{u}\frac{v_{,t}}{v^3} + \frac{u_{,tt}}{u}\frac{1}{v^2} + \frac{2}{r^2}\left(1 - \frac{1}{u^2}\right) + \frac{2}{r}\frac{u_{,r}}{u^3} - \frac{2}{r}\frac{1}{u^2}\frac{v_{,r}}{v}$$

$$= -\frac{2}{u^2}\frac{v_{,rr}}{v} + 2\frac{u_{,r}}{u^3}\frac{v_{,r}}{v} - 2\frac{u_{,t}}{u}\frac{v_{,t}}{v^3} + 2\frac{u_{,tt}}{u}\frac{1}{v^2} + \frac{4}{r}\frac{u_{,r}}{u^3} - \frac{4}{r}\frac{1}{u^2}\frac{v_{,r}}{v} + \frac{2}{r^2}\left(1 - \frac{1}{u^2}\right)$$

That is

$$R = -\frac{2}{u^2}\frac{v_{,rr}}{v} + 2\frac{u_{,r}}{u^3}\frac{v_{,r}}{v} - 2\frac{u_{,t}}{u}\frac{v_{,t}}{v^3} + 2\frac{u_{,tt}}{u}\frac{1}{v^2} + \frac{4}{r}\frac{u_{,r}}{u^3} - \frac{4}{r}\frac{1}{u^2}\frac{v_{,r}}{v} + \frac{2}{r^2}\left(1 - \frac{1}{u^2}\right). \quad \text{(A.1.6)}$$

The Einstein curvature tensor G is given by

$$G = \mathrm{Ric} - \frac{1}{2}Rg. \quad \text{(A.1.7)}$$

In our case of the spacetime metric $(A.1.1)$, we compute the components of G with respect to the basis $\{\partial_t, \partial_r, \partial_\theta, \partial_\phi\}$:

1.

$$G_{tt} = G(\partial_t, \partial_t) = \mathrm{Ric}_{tt} - \frac{1}{2}Rg_{tt} = \frac{1}{u^2}\left(vv_{,rr} + \frac{2}{r}vv_{,r}\right) - \frac{u_{,r}}{u^3}vv_{,r}$$

$$+ \frac{1}{u}\left(\frac{u_{,t}v_{,t}}{v} - u_{,tt}\right) + \frac{v^2}{2}R$$

$$= \frac{2}{r}\frac{u_{,r}}{u^3}v^2 + \frac{1}{r^2}v^2\left(1 - \frac{1}{u^2}\right) \quad \text{(A.1.8)}$$

2.

$$G_{tr} = G_{rt} = \text{Ric}_{tr} - \frac{1}{2}Rg_{tr} = \text{Ric}_{tr} = \frac{2}{r}\frac{u_{,t}}{u} \tag{A.1.9}$$

3.

$$G_{t\theta} = G_{\theta t} = \text{Ric}_{t\theta} - \frac{1}{2}Rg_{t\theta} = \text{Ric}_{t\theta} = 0$$

4.

$$G_{t\phi} = G_{\phi t} = \text{Ric}_{t\phi} - \frac{1}{2}Rg_{t\phi} = \text{Ric}_{t\phi} = 0$$

5.

$$\begin{aligned} G_{rr} = \text{Ric}_{rr} - \frac{1}{2}Rg_{rr} &= -\frac{v_{,rr}}{v} + \frac{2}{r}\frac{u_{,r}}{u} - \frac{v_{,t}}{v^3}uu_{,t} + \frac{1}{v}\left(\frac{uu_{,tt}}{v} - \frac{v_{,r}u_{,r}}{u}\right) - \frac{u^2}{2}R \\ &= \frac{2}{r}\frac{v_{,r}}{v} - \frac{u^2}{r^2} + \frac{1}{r^2} \end{aligned} \tag{A.1.10}$$

6.

$$G_{r\theta} = G_{\theta r} = \text{Ric}_{r\theta} - \frac{1}{2}Rg_{r\theta} = \text{Ric}_{r\theta} = 0$$

7.

$$G_{r\phi} = G_{\phi r} = \text{Ric}_{r\phi} - \frac{1}{2}Rg_{r\phi} = \text{Ric}_{\phi} = 0$$

8.

$$\begin{aligned} G_{\theta\theta} = \text{Ric}_{\theta\theta} - \frac{1}{2}Rg_{\theta\theta} &= \left(1 - \frac{1}{u^2}\right) + r\frac{u_{,r}}{u^3} - r\frac{v_{,r}}{v}\frac{1}{u^2} - \frac{r^2}{2}R \\ &= \frac{r^2}{u^2}\frac{v_{,rr}}{v} - r^2\frac{u_{,r}}{u^3}\frac{v_{,r}}{v} + r^2\frac{u_{,t}}{u}\frac{v_{,t}}{v^3} - r^2\frac{u_{,tt}}{u}\frac{1}{v^2} - r\frac{u_{,r}}{u^3} + r\frac{1}{u^2}\frac{v_{,r}}{v} \end{aligned} \tag{A.1.11}$$

9.

$$\begin{aligned} G_{\phi\phi} = \text{Ric}_{\phi\phi} - \frac{1}{2}Rg_{\phi\phi} &= \sin^2\theta \cdot \text{Ric}_{\theta\theta} - \frac{1}{2}Rr^2\sin^2\theta = \sin^2\theta \cdot G_{\theta\theta} \\ &= \sin^2\theta\left(\frac{r^2}{u^2}\frac{v_{,rr}}{v} - r^2\frac{u_{,r}}{u^3}\frac{v_{,r}}{v} + r^2\frac{u_{,t}}{u}\frac{v_{,t}}{v^3} - r^2\frac{u_{,tt}}{u}\frac{1}{v^2} - r\frac{u_{,r}}{u^3} + r\frac{1}{u^2}\frac{v_{,r}}{v}\right) \end{aligned} \tag{A.1.12}$$

A.2 Calculations in Inverse Mean Curvature Vector Flow Coordinates

A.2.1 Determinant of the Spacetime Metric and its Inverse in Inverse Mean Curvature Vector Flow Coordinates

Given a matrix A, its (i, j)th cofactor is $C_{ij} := (-1)^{i+j} M_{ij}$, where M_{ij} is determinant of the matrix obtained by deleting the ith row and the jth column of A. The adjoint matrix $\mathrm{adj}(A)$ is defined as $\mathrm{adj}(A)_{ij} := C_{ji} = (-1)^{j+i} M_{ji}$. If A is invertible, then $A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$.

Let A be the matrix representation of the spacetime metric $\overline{g}$ as in (4.2.1).

$$\mathrm{adj}(A)_{11} = C_{11} = M_{11} = u^2(ab - c^2) \tag{A.2.1}$$

$$\mathrm{adj}(A)_{12} = C_{21} = -M_{21} = -d(ab - c^2) \tag{A.2.2}$$

$$\mathrm{adj}(A)_{13} = C_{31} = M_{31} = u^2(cf - be) \tag{A.2.3}$$

$$\mathrm{adj}(A)_{14} = C_{41} = -M_{41} = u^2(ce - af) \tag{A.2.4}$$

$$\mathrm{adj}(A)_{22} = C_{22} = M_{22} = -v^2(ab - c^2) + f(ce - af) + e(cf - be) \tag{A.2.5}$$

$$\mathrm{adj}(A)_{23} = C_{32} = -M_{32} = -d(cf - be) \tag{A.2.6}$$

$$\mathrm{adj}(A)_{24} = C_{42} = M_{42} = -d(ce - af) \tag{A.2.7}$$

$$\mathrm{adj}(A)_{33} = C_{33} = M_{33} = -u^2v^2b - u^2f^2 - bd^2 \tag{A.2.8}$$

$$\mathrm{adj}(A)_{34} = C_{43} = -M_{43} = u^2v^2c + u^2ef + cd^2 \tag{A.2.9}$$

$$\mathrm{adj}(A)_{44} = C_{44} = M_{44} = -u^2v^2a - u^2e^2 - ad^2 \tag{A.2.10}$$

$$\begin{aligned}\det(\overline{g}) &= A_{21}C_{21} + A_{22}C_{22} = -dM_{21} + u^2 M_{22} \\ &= -d^2(ab - c^2) + u^2\big[-v^2(ab - c^2) + f(ce - af) + e(cf - be)\big] \\ &= (-u^2v^2 - d^2)(ab - c^2) + eu^2(cf - be) + fu^2(ce - af)\end{aligned} \tag{A.2.11}$$

These prove Lemma 4.5.

A.2.2 Computation of $\langle \boldsymbol{n}, \boldsymbol{n} \rangle$

Lemma A.1.

$$\langle \boldsymbol{n}, \boldsymbol{n} \rangle = \frac{det(\overline{g})}{u^2(ab-c^2)} = \frac{det(\overline{g})}{u^2 det(g_S)} =: \frac{|\overline{g}|}{u^2|g_S|}, \tag{A.2.12}$$

where we set $|g_S| := det(g_S)$*.*

Proof. We simply compute that

$$\langle \mathbf{n}, \mathbf{n} \rangle = \Big\langle \frac{\partial}{\partial t} + \frac{-d}{u^2}\frac{\partial}{\partial r} + \frac{cf-be}{ab-c^2}\frac{\partial}{\partial \theta} + \frac{ce-af}{ab-c^2}\frac{\partial}{\partial \phi}, \frac{\partial}{\partial t} + \frac{-d}{u^2}\frac{\partial}{\partial r} + \frac{cf-be}{ab-c^2}\frac{\partial}{\partial \theta}$$

$$+ \frac{ce-af}{ab-c^2}\frac{\partial}{\partial \phi}\Big\rangle$$

$$= \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle + \frac{-d}{u^2}\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial r} \rangle + \frac{cf-be}{ab-c^2}\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta} \rangle + \frac{ce-af}{ab-c^2}\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial \phi} \rangle$$

$$+ \frac{-d}{u^2}\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial t} \rangle + \frac{d^2}{u^4}\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle + 0 + 0$$

$$+ \frac{cf-be}{ab-c^2}\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial t} \rangle + 0 + \left(\frac{cf-be}{ab-c^2}\right)^2 \langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \rangle + \frac{(cf-be)(ce-af)}{(ab-c^2)^2}\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \rangle$$

$$+ \frac{ce-af}{ab-c^2}\langle \frac{\partial}{\partial \phi}, \frac{\partial}{\partial t} \rangle + 0 + \frac{(ce-af)(cf-be)}{(ab-c^2)^2}\langle \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta} \rangle + \left(\frac{ce-af}{ab-c^2}\right)^2 \langle \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi} \rangle$$

$$= -v^2 + \frac{-d^2}{u^2} + \frac{1}{ab-c^2}(2(cf-be)e + 2(ce-af)f) + \frac{-d^2}{u^2} + \frac{d^2}{u^2}$$

$$+ 2\frac{(ce-af)(cf-be)c}{(ab-c^2)^2} + \left(\frac{cf-be}{ab-c^2}\right)^2 a + \left(\frac{ce-af}{ab-c^2}\right)^2 b$$

$$=: -v^2 + \frac{-d^2}{u^2} + \frac{1}{ab-c^2}(2cef - 2be^2 + 2cef - 2af^2) + (A)$$

where (A) is

$$A = 2\frac{(ce-af)(cf-be)c}{(ab-c^2)^2} + \left(\frac{cf-be}{ab-c^2}\right)^2 a + \left(\frac{ce-af}{ab-c^2}\right)^2 b$$

$$= \frac{1}{(ab-c^2)^2}\Big(2c^3ef - 2bc^2e^2 - 2ac^2f^2 + 2abcef + ac^2f^2 + ab^2e^2 - 2abcef + bc^2e^2$$

$$+ a^2bf^2 - 2abcef\Big)$$

$$= \frac{1}{(ab-c^2)^2}\Big(2c^3ef - 2abcef - bc^2e^2 - ac^2f^2 + ab^2e^2 + a^2bf^2\Big)$$

$$= \frac{1}{(ab-c^2)^2}\Big(2cef(c^2-ab) - c^2(af^2+be^2) + ab(be^2+af^2)\Big)$$

$$= \frac{1}{(ab-c^2)^2}\Big(-2cef(ab-c^2) + (af^2+be^2)(ab-c^2)\Big)$$

$$= \frac{1}{ab-c^2}(af^2+be^2-2cef).$$

Plug (A) back into the above, we get:

$$\langle \mathbf{n}, \mathbf{n}\rangle = -v^2 + \frac{-d^2}{u^2} + \frac{1}{ab-c^2}(2cef - 2be^2 + 2cef - 2af^2)$$

$$+ \frac{1}{ab-c^2}(af^2+be^2-2cef)$$

$$= -v^2 + \frac{-d^2}{u^2} + \frac{1}{ab-c^2}(2cef - 2be^2 + 2cef - 2af^2 + af^2 + be^2 - 2cef)$$

$$= \frac{-u^2v^2-d^2}{u^2} + \frac{1}{ab-c^2}(2cef - be^2 - af^2)$$

$$= \frac{1}{u^2(ab-c^2)}\Big((-u^2v^2-d^2)(ab-c^2) + u^2(2cef-be^2-af^2)\Big) \qquad \text{(A.2.13)}$$

$$= \frac{|\overline{g}|}{u^2|g_S|}, \qquad \text{(A.2.14)}$$

by Equation (4.2.2). □

A.2.3 Laplacian along $S_{t,r}$

Note that the laplacian of d on the surface $(S_{t,r}, g_S)$ is:

$$\Delta_{g_S} d = \mathrm{div}_{g_S}(\nabla_{g_S} d) = \frac{1}{\sqrt{|g_S|}} \sum_{i=1}^{2} \frac{\partial}{\partial x^i} \left((\nabla_{g_S} d)^i \sqrt{|g_S|} \right) = \frac{1}{\sqrt{|g_S|}} \sum_{i=1}^{2} \frac{\partial}{\partial x^i} \left((g_S^{ij} d_{,j}) \sqrt{|g_S|} \right)$$

$$= \frac{1}{r^2 \sin\theta} \left(\frac{\partial}{\partial \theta} \left((g_S^{\theta\theta} d_{,\theta} + g_S^{\theta\phi} d_{,\phi}) r^2 \sin\theta \right) + \frac{\partial}{\partial \phi} \left((g_S^{\phi\theta} d_{,\theta} + g_S^{\phi\phi} d_{,\phi}) r^2 \sin\theta \right) \right)$$

$$= \frac{1}{r^2 \sin\theta} \Big((g_{S,\theta}^{\theta\theta} d_{,\theta} + g_S^{\theta\theta} d_{,\theta\theta} + g_{S,\theta}^{\theta\phi} d_{,\phi} + g_S^{\theta\phi} d_{,\phi\theta}) r^2 \sin\theta + (g_S^{\theta\theta} d_{,\theta} + g_S^{\theta\phi} d_{,\phi}) r^2 \cos\theta$$

$$+ (g_{S,\phi}^{\phi\theta} d_{,\theta} + g_S^{\phi\theta} d_{,\theta\phi} + g_{S,\phi}^{\phi\phi} d_{,\phi} + g_S^{\phi\phi} d_{,\phi\phi}) r^2 \sin\theta + 0 \Big)$$

$$= (g_S^{\theta\theta} d_{,\theta\theta} + 2 g_S^{\theta\phi} d_{,\phi\theta} + g_S^{\phi\phi} d_{,\phi\phi}) \qquad \text{(second order derivative in } d\text{)}$$

$$+ (g_{S,\theta}^{\theta\theta} d_{,\theta} + g_{S,\theta}^{\theta\phi} d_{,\phi} + g_{S,\phi}^{\phi\theta} d_{,\theta} + g_{S,\phi}^{\phi\phi} d_{,\phi}) + (g_S^{\theta\theta} d_{,\theta} + g_S^{\theta\phi} d_{,\phi}) \cot\theta \qquad \text{(A.2.15)}$$

That is:

$$\Delta_{g_S} d = (g_S^{\theta\theta} d_{,\theta\theta} + 2 g_S^{\theta\phi} d_{,\phi\theta} + g_S^{\phi\phi} d_{,\phi\phi}) + (g_{S,\theta}^{\theta\theta} + g_{S,\phi}^{\phi\theta} + g_S^{\theta\theta} \cot\theta) d_{,\theta}$$

$$+ (g_{S,\theta}^{\theta\phi} + g_{S,\phi}^{\phi\phi} + g_S^{\theta\phi} \cot\theta) d_{,\phi}$$

$$= \frac{1}{|g_S|} \Big((b d_{,\theta\theta} - 2c d_{,\phi\theta} + a d_{,\phi\phi}) + (b_{,\theta} - 2b \cot\theta - c_{,\phi} + b \cot\theta) d_{,\theta}$$

$$+ (-c_{,\theta} + 2c \cot\theta + a_{,\phi} - c \cot\theta) d_{,\phi} \Big)$$

$$= \frac{1}{|g_S|} \Big((b d_{,\theta\theta} - 2c d_{,\phi\theta} + a d_{,\phi\phi}) + (b_{,\theta} - b \cot\theta - c_{,\phi}) d_{,\theta} + (-c_{,\theta} + c \cot\theta + a_{,\phi}) d_{,\phi} \Big) \qquad \text{(A.2.16)}$$

For the second to the last equality above, we have used the following computations:

$$|g_S|_{,\theta} = (r^4 \sin^2\theta)_{,\theta} = r^4 2 \sin\theta \cos\theta = 2r^4 \sin^2\theta \cot\theta = 2|g_S| \cot\theta. \qquad \text{(A.2.17)}$$

and

$$g^{\theta\theta}_{S,\theta} = \left(\frac{b}{|g_S|}\right)_\theta = \frac{b_{,\theta}|g_S| - b|g_S|_{,\theta}}{|g_S|^2} = \frac{1}{|g_S|}(b_{,\theta} - 2b\cot\theta); \quad g^{\theta\theta}_{S,\phi} = \left(\frac{b}{|g_S|}\right)_\phi = \frac{b_{,\phi}}{|g_S|}. \tag{A.2.18}$$

Similar for the other derivatives of the inverse of g_S.

A.3 First Variation of Area

Let Σ^{n-1} be an embedded closed (compact without boundary) hypersurface in a Riemannian manifold $(M^n, g, \overline{\nabla})$. Endow Σ with the induced metric. We consider a variation of Σ as follows:

$$F : \Sigma \times (-\delta, \delta) \longrightarrow M, \quad \delta > 0, \tag{A.3.1}$$

such that for all $x \in \Sigma_t := F(\Sigma, t)$, and $t \in (-\delta, \delta)$,

$$\frac{\partial}{\partial t} F(x, t) = \eta(x, t)\nu(x, t), \tag{A.3.2}$$

where η is a smooth function $\eta \in C^\infty(\Sigma \times (-\delta, \delta))$, and $\nu(x, t)$ is the unit outward normal vector to Σ_t at (x, t). Therefore the variational vector fields along each surface Σ_t is $\frac{\partial}{\partial t} = \eta_t \cdot \nu_t$. Let g_t be the induced metric on Σ_t, and let ∇^t be the associated Levi-Civita connection. Let $d\sigma_t$ be the corresponding $(n-1)$-volume form on Σ_t, and A_t the $(n-1)$-volume. Let V_t be the n-volume enclosed by Σ_t. We shall refer to A_t as the *area* of Σ_t, and V_t the *volume*, in analogy to the case where Σ_t are surfaces in a 3-dimensional manifold. Let II_t and $H_t := \mathrm{tr}_{g_t}\mathrm{II}$ be the second fundamental form and the mean curvature of Σ with respect to $\nu(x, t)$ respectively. We first compute the variation of $d\sigma_t$. Let $\{U; x_1, x_2, \cdots, x_{n-1}\}$ be a local coordinate chart of Σ, then Σ_t can be locally parametrized as $\{x_1, x_2, \cdots, x_{n-1}, t\}$ with each fixed t. Let

$g_t = (g_t)_{ij} dx^i dx^j$ be the local representation of the metric on Σ_t, $i, j = 1, 2, \cdots, n-1$.

$$\begin{aligned}
\frac{\partial}{\partial t} d\sigma_t &= \frac{\partial}{\partial t}\sqrt{\det(g_t)}dx^1 \wedge dx^2 \cdots \wedge dx^{n-1} \\
&= \frac{1}{2}\frac{1}{\sqrt{\det(g_t)}}\det(g_t) \cdot \operatorname{trace}\left(g_t^{-1}\frac{\partial}{\partial t}g_t\right) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n-1} \qquad \text{(A.3.3)} \\
&= \frac{1}{2}\sqrt{\det(g_t)}\operatorname{trace}\left(g_t^{-1}\frac{\partial}{\partial t}g_t\right) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n-1}
\end{aligned}$$

where equation $(A.3.3)$ follows from the identity

$$\frac{d}{dt}\det(A) = \det(A)\operatorname{trace}\left(A^{-1}\frac{d}{dt}A\right), \qquad \text{(A.3.4)}$$

for any square matrix A with entries functions of t. Now

$$\begin{aligned}
\frac{\partial}{\partial t}(g_t)_{ij} &= \frac{\partial}{\partial t}\langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle = \langle\overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle + \langle\frac{\partial}{\partial x^i}, \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^j}\rangle \\
&= \langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial t}, \frac{\partial}{\partial x^j}\rangle + \langle\frac{\partial}{\partial x^i}, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial t}\rangle \qquad (\overline{\nabla} \text{ is torsion free}) \\
&= \langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\eta\nu, \frac{\partial}{\partial x^j}\rangle + \langle\frac{\partial}{\partial x^i}, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\eta\nu\rangle \\
&= 2\eta \mathrm{II}_t\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right). \qquad \text{(A.3.5)}
\end{aligned}$$

where the last identity follows from the fact that ν is a normal to the surfaces. Notice that Equation (A.3.5) implies that the first derivative of the metric (along the variational vector fields) is given by the second fundamental form. Now plug (A.3.5) into (A.3.3):

$$\frac{\partial}{\partial t}d\sigma_t = \frac{1}{2}\sqrt{\det(g_t)}\operatorname{trace}\left(g_t^{-1} \cdot 2\eta \cdot \mathrm{II}_t(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\right) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n-1} = H_t \eta d\sigma_t \qquad \text{(A.3.6)}$$

Therefore

$$\frac{\partial}{\partial t}A_t = \int_{\Sigma_t}\frac{\partial}{\partial t}d\sigma_t = \int_{\Sigma_t} H_t(x)\eta(x,t)\, d\sigma_t(x). \qquad \text{(A.3.7)}$$

A.4 Second Variation of Area

Now we compute the first variation of mean curvature H_t, which gives rise to the second derivative of area. Recall that $H_t = g_t^{ij}(\mathrm{II}_t)_{ij}$ in local coordinates, $i, j = 1, 2, \cdots, n-1$. Thus

$$\frac{\partial}{\partial t}H_t = \frac{\partial}{\partial t}g_t^{ij}(\mathrm{II}_t)_{ij} + g_t^{ij}\frac{\partial}{\partial t}(\mathrm{II}_t)_{ij} \tag{A.4.1}$$

Since $0 = \frac{\partial}{\partial t}(g_t g_t^{-1}) = (\frac{\partial}{\partial t}g_t)g_t^{-1} + g_t(\frac{\partial}{\partial t}g_t^{-1})$, we have $\frac{\partial}{\partial t}g_t^{-1} = -g_t^{-1}(\frac{\partial}{\partial t}g_t)g_t^{-1}$. Thus the first term in the above becomes

$$\frac{\partial}{\partial t}g_t^{ij}(\mathrm{II}_t)_{ij} = -g_t^{ik}\left(\frac{\partial}{\partial t}(g_t)_{kl}\right)g_t^{lj}(\mathrm{II}_t)_{ij} = -g_t^{ik}2\eta(\mathrm{II}_t)_{kl}g_t^{lj}(\mathrm{II}_t)_{ij} = -2\eta||\mathrm{II}_t||^2 \tag{A.4.2}$$

We now compute the derivative of the second fundamental form.

$$\frac{\partial}{\partial t}(\mathrm{II}_t)_{ij} = \frac{\partial}{\partial t}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \frac{\partial}{\partial x^j}\rangle$$

$$= \langle\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \frac{\partial}{\partial x^j}\rangle + \langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^j}\rangle$$

$$= \langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\overline{\nabla}_{\frac{\partial}{\partial t}}\nu, \frac{\partial}{\partial x^j}\rangle + \langle(\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial x^i}} - \overline{\nabla}_{\frac{\partial}{\partial x^i}}\overline{\nabla}_{\frac{\partial}{\partial t}} - \overline{\nabla}_{[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t}]})\nu, \frac{\partial}{\partial x^j}\rangle + \langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \overline{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^j}\rangle$$

$$= \langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\overline{\nabla}_{\frac{\partial}{\partial t}}\nu, \frac{\partial}{\partial x^j}\rangle + \langle R_g(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i})\nu, \frac{\partial}{\partial x^j}\rangle + \langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial t}\rangle$$

$$= \langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\left(-\nabla_{\Sigma_t}\eta\right), \frac{\partial}{\partial x^j}\rangle + \eta\langle R(\nu, \frac{\partial}{\partial x^i})\nu, \frac{\partial}{\partial x^j}\rangle + \eta\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\nu\rangle$$

where we have used two lemmas, which will be proved below:

Lemma A.2. $\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu$ *is tangential, i.e.,* $\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \nu\rangle = 0$.

Lemma A.3. $\overline{\nabla}_{\frac{\partial}{\partial t}}\nu = -\nabla_{\Sigma_t}\eta$, *where* ∇_{Σ_t} *is the surface gradient on* Σ_t.

Therefore

$$
\begin{aligned}
g_t^{ij}\frac{\partial}{\partial t}(\mathrm{II}_t)_{ij} &= -\Delta_{\Sigma_t}\eta - \eta g_t^{ij}\langle R(\nu,\frac{\partial}{\partial x^i}),\frac{\partial}{\partial x^j},\nu\rangle + \eta||\mathrm{II}_t||^2 \\
&= -\Delta_{\Sigma_t}\eta - \eta g^{ij}\langle R(\nu,\frac{\partial}{\partial x^i}),\frac{\partial}{\partial x^j},\nu\rangle + \eta||\mathrm{II}_t||^2 \quad \text{(ambient metric } g \text{ trace)} \\
&= -\Delta_{\Sigma_t}\eta - \eta \mathrm{Ric}_g(\nu,\nu) + \eta||\mathrm{II}_t||^2.
\end{aligned}
$$

where we used

Lemma A.4. $g_t^{ij}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\left(\nabla_{\Sigma_t}\eta\right), \frac{\partial}{\partial x^j}\rangle = \Delta_{\Sigma_t}\eta$.

and

Lemma A.5. $g_t^{ij}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\nu\rangle = ||II_t||^2$.

Combining above, we get

$$
\frac{\partial}{\partial t}H_t = -\Delta_{\Sigma_t}\eta - \eta\left(\mathrm{Ric}_g(\nu,\nu) + ||\mathrm{II}_t||^2\right) =: L_{\Sigma_t}, \tag{A.4.3}
$$

where L_{Σ_t} is called the *stability operator* of Σ_t. The second variation of area is then given by:

$$
\frac{\partial^2}{\partial t^2}A_t = \int_\Sigma \eta(x,t)(L_{\Sigma_t}\eta)(x,t)\,d\sigma_t(x) + H_t(x)\left(\frac{\partial}{\partial t}\eta(x,t)\right)d\sigma_t(x) + H_t^2\eta(x,t)^2\,d\sigma_t(x). \tag{A.4.4}
$$

Now we verity the above lemmas.

Proof of Lemma A.2. Since ν is the unit outward normal vector field, we have

$$
0 = \frac{\partial}{\partial x^i}\langle\nu,\nu\rangle = 2\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu,\nu\rangle. \tag{A.4.5}
$$

Thus $\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu$ is tangential. Similarly, $\overline{\nabla}_{\frac{\partial}{\partial t}}\nu$ is also tangential. □

Proof of Lemma A.3. Recall that the gradient of a smooth function η along the surface Σ_t is defined as

$$\nabla_{\Sigma_t}\eta := \nabla\eta - \langle\nabla\eta, \nu\rangle\nu, \tag{A.4.6}$$

that is, the tangential component of the gradient with respect to the ambient metric. For any point $p \in \Sigma_t$, choose geodesic normal coordinates $\{U; e_1, e_2, \cdots, e_n\}$ around p such that $e_1, e_2, \cdots, e_{n-1}$ span $T_p\Sigma_t$, and $e_n = \nu$. Since $\overline{\nabla}_{e_i}\nu$ is tangential, it suffices to show that $\langle\overline{\nabla}_{\frac{\partial}{\partial t}}\nu, e_i\rangle(p) = \langle-\nabla_{\Sigma_t}\eta, e_i\rangle(p)$, for $i = 1, 2, \cdots, n-1$. Indeed:

$$\begin{aligned}
\langle\overline{\nabla}_{\frac{\partial}{\partial t}}\nu, e_i\rangle(p) &= -\langle\nu, \overline{\nabla}_{\frac{\partial}{\partial t}}e_i\rangle(p) = -\langle\nu, \overline{\nabla}_{e_i}\frac{\partial}{\partial t}\rangle(p) && (\overline{\nabla} \text{ is torsion free})\\
&= -e_i(\eta)\langle\nu, \nu\rangle(p) - \eta\langle\nu, \overline{\nabla}_{e_i}\nu\rangle(p)\\
&= -e_i(\eta)(p) && (\overline{\nabla}_{e_i}\nu \text{ is tangential})\\
&= -\langle\sum_{j=1}^{n-1} e_j(\eta)e_j, e_i)\rangle(p)\\
&= \langle-\nabla_{\Sigma_t}\eta, e_i\rangle(p).
\end{aligned}$$

Since p is arbitrary, $\overline{\nabla}_{\frac{\partial}{\partial t}}\nu = -\nabla_{\Sigma_t}\eta$, as desired. □

Proof of Lemma A.4. First note that $g_t^{ij}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}(\nabla_{\Sigma_t}\eta), \frac{\partial}{\partial x^j}\rangle = g_t^{ij}\langle\nabla^t_{\frac{\partial}{\partial x^i}}(\nabla_{\Sigma_t}\eta), \frac{\partial}{\partial x^j}\rangle$. But

$$g_t^{ij}\langle\nabla^t_{\frac{\partial}{\partial x^i}}(\nabla_{\Sigma_t}\eta), \frac{\partial}{\partial x^j}\rangle = tr_{g_t}(\nabla^t(\nabla_{\Sigma_t}\eta)) = \mathrm{div}_{g_t}(\nabla_{\Sigma_t}\eta) = \Delta_{\Sigma_t}\eta, \tag{A.4.7}$$

where $\nabla^t(\nabla_{\Sigma_t}\eta)$ is the covariant derivative of the vector field $\nabla_{\Sigma_t}\eta$, hence is a $(1,1)$-tensor field. □

Proof of Lemma A.5. Define vector fields $X(i) := \overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu$ and $Y(j) := \overline{\nabla}_{\frac{\partial}{\partial x^j}}\nu$. Using local coordinates, we can also write $X(i) = \sum_{k=1}^{n-1} X(i)^k\frac{\partial}{\partial x^k}$ and $Y(j) = \sum_{l=1}^{n-1} Y(j)^l\frac{\partial}{\partial x^l}$.

Then

$$||\mathrm{II}_t||^2 = g_t^{ij}g_t^{kl}(\mathrm{II}_t)_{ik}(\mathrm{II}_t)_{jl} = g_t^{ij}g_t^{kl}\left\langle X(i), \frac{\partial}{\partial x^k}\right\rangle\left\langle Y(j), \frac{\partial}{\partial x^l}\right\rangle$$
$$= g_t^{ij}g_t^{kl}X(i)^\alpha(g_t)_{\alpha k}Y(j)^\beta(g_t)_{\beta l}$$
$$= g_t^{ij}(g_t)_{\alpha\beta}X(i)^\alpha Y(j)^\beta$$
$$= g_t^{ij}\langle X(i), Y(j)\rangle$$
$$= g_t^{ij}\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\nu, \overline{\nabla}_{\frac{\partial}{\partial x^j}}\nu\rangle$$

as desired. □

A.5 Transformation Formulae of Ricci and Scalar Metric under Conformal Change of Metrics

Given Riemannian manifold (M^n, g), recall that the Riemann curvature operator R acts on vector fields $X, Y, Z \in \Gamma(TM)$ as follows:

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z. \tag{A.5.1}$$

We can thus define a $(1,3)$ tensor field $R = R_{lij}^k dx^l \otimes dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}$, such that

$$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^l} =: R_{lij}^k \frac{\partial}{\partial x^k}.$$

One can verify that

$$R_{lij}^k = \frac{\partial\Gamma_{jl}^k}{\partial x^i} - \frac{\Gamma_{il}^k}{\partial x^j} + \Gamma_{im}^k\Gamma_{jl}^m - \Gamma_{jm}^k\Gamma_{il}^m. \tag{A.5.2}$$

Using the metric g, we can define a new $(0,4)$ tensor field $Rm := R_{klij}dx^k \otimes dx^l \otimes dx^i \otimes dx^j$ with components $R_{klij} := g_{km}R_{lij}^m$. One verifies that

$$R_{klij} = \langle R(\partial_i, \partial_j)\partial_l, \partial_k\rangle.$$

Rm is called the Riemannian curvature tensor with respect to the metric g. The Ricci curvature in the direction $X \in T_pM$ is given by

$$\mathrm{Ric}(X, X) := g^{jl}\langle R(X, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^l}, X\rangle. \tag{A.5.3}$$

Therefore the Ricci curvature tensor is given by the (2,4)-contraction of the Riemann curvature tensor, i.e.,

$$\begin{aligned}\mathrm{Ric}_{ij} &= \mathrm{Ric}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = g^{kl}\langle R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k})\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^j}\rangle = g^{kl}Rjlik = g^{kl}R_{ikjl} = -R^l_{ijl} \\ &= \frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{ik}}{\partial x^j} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{jl}\Gamma^l_{ik}.\end{aligned} \tag{A.5.4}$$

The scalar curvature is

$$R := g^{ij}\mathrm{Ric}_{ij}. \tag{A.5.5}$$

Proposition A.6. *Let (M^n, g) be a Riemannian manifold of dimension n, and $\rho > 0$ is a smooth function on M. Consider the new metric $\widetilde{g} := \rho g$, i.e., $\widetilde{g}$ is a conformal change of g. Then the corresponding Ricci curvatures changes in the following way:*

$$\widetilde{R}_{ij} = R_{ij} - \frac{n-2}{2}(\log\rho)_{,ij} + \frac{n-2}{4}(\log\rho)_{,i}(\log\rho)_{,j} - \frac{1}{2}g_{ij}\left[\Delta_g \log\rho + \frac{n-2}{2}|\nabla \log\rho|^2_g\right]. \tag{A.5.6}$$

Proof. We will verity equation $(A.5.6)$ at the center $p \in M$ of a geodesic normal coordinate neighborhood. Then at p, we have $g_{ij} = \delta_{ij}$ and $\Gamma^k_{ij} \equiv 0$, and therefore

$$g_{ij,k} = \partial_k\langle\partial_i, \partial_j\rangle = \langle\Gamma^m_{ki}\partial_m, \partial_j\rangle + \langle\partial_i, \Gamma^n_{kj}\partial_n\rangle = 0 \text{ at } p. \tag{A.5.7}$$

By $(A.5.4)$, the Ricci curvature of $\widetilde{g}$ is given by:

$$\widetilde{R}_{ij} = \widetilde{\Gamma}^k_{ij,k} - \widetilde{\Gamma}^k_{ik,j} + \widetilde{\Gamma}^s_{ij}\widetilde{\Gamma}^k_{sk} - \widetilde{\Gamma}^k_{ik}\widetilde{\Gamma}^k_{sj}. \tag{A.5.8}$$

Thus at p, we have:

$$
\begin{aligned}
\widetilde{R}_{ij} = & \Big(\Gamma^k_{ij,k} + \frac{1}{2}\delta_{ik}(\log\rho)_{,jk} + \frac{1}{2}\delta_{jk}(\log\rho)_{,ik} - \frac{1}{2}g_{ij,k}g^{kl}(\log\rho)_{,l} - \frac{1}{2}g_{ij}g^{kl}_{,k}(\log\rho)_{,l} \\
& - \frac{1}{2}g_{ij}g^{kl}(\log\rho)_{,lk}\Big) - \Big(\Gamma^k_{ik,j} + \frac{1}{2}\delta_{ik}(\log\rho)_{,kj} + \frac{1}{2}\delta_{kk}(\log\rho)_{,ij} - \frac{1}{2}g_{ik,j}g^{kl}(\log\rho)_{,l} \\
& - \frac{1}{2}g_{ik}g^{kl}_{,j}(\log\rho)_{,l} - \frac{1}{2}g_{ik}g^{kl}(\log\rho)_{,ij}\Big) \\
& + \Big(\Gamma^s_{ij} + \frac{1}{2}\delta_{is}(\log\rho)_{,j} + \frac{1}{2}\delta_{js}(\log\rho)_{,i} - \frac{1}{2}g_{ij}g^{sl}(\log\rho)_{,l}\Big) \times \\
& \Big(\Gamma^k_{sk} + \frac{1}{2}\delta_{sk}(\log\rho)_{,k} + \frac{1}{2}\delta_{kk}(\log\rho)_{,s} - \frac{1}{2}g_{sk}g^{kl}(\log\rho)_{,l}\Big) \\
& - \Big(\Gamma^s_{ik} + \frac{1}{2}\delta_{is}(\log\rho)_{,k} + \frac{1}{2}\delta_{ks}(\log\rho)_{,i} - \frac{1}{2}g_{ik}g^{sl}(\log\rho)_{,l}\Big) \times \\
& \Big(\Gamma^k_{sj} + \frac{1}{2}\delta_{sk}(\log\rho)_{,j} + \frac{1}{2}\delta_{jk}(\log\rho)_{,s} - \frac{1}{2}g_{sj}g^{kl}(\log\rho)_{,l}\Big) \\
& = R_{ij} + \frac{2-n}{2}(\log\rho)_{,ij} - \frac{1}{2}g_{ij}(\log\rho)_{,kk} + (\text{product terms}), \qquad \text{(A.5.9)}
\end{aligned}
$$

where the red terms cancel, green terms cancel and the blue terms vanish at p.

$$
\begin{aligned}
&(\text{product terms}) = \frac{1}{4}\delta_{js}\delta_{kk}(\log\rho)_{,i}(\log\rho)_{,s} - \frac{1}{4}\delta_{ks}\delta_{jk}(\log\rho)_{,i}(\log\rho)_{,s} \\
&+ \frac{1}{4}g_{ij}g_{sk}g^{sl}g^{kl}(\log\rho)_{,l}(\log\rho)_{,l} - \frac{1}{4}g_{ik}g_{sj}g^{sl}g^{kl}(\log\rho)_{,l}(\log\rho)_{,l} \\
&- \frac{1}{4}g_{jk}g^{kl}(\log\rho)_{,l}(\log\rho)_{,i} - \frac{n}{4}g_{ij}g^{sj}(\log\rho)_{,s}(\log\rho)_{,l} \\
&+ g_{jk}g^{kl}(\log\rho)_{,l}(\log\rho)_{,i} + \frac{1}{4}g_{ij}g^{sl}(\log\rho)_{,s}(\log\rho)_{,l} \\
&= \frac{n-1}{4}(\log\rho)_{,i}(\log\rho)_{,j} + \frac{1-n}{4}g_{ij}(\log\rho)_{,l}(\log\rho)_{,l} + \frac{1}{4}g_{ij}\delta_{lk}\delta_{lk}(\log\rho)_{,l}(\log\rho)_{,l} \\
&- \frac{1}{4}\delta_{il}\delta_{jl}(\log\rho)_{,l}(\log\rho)_{,l} \\
&= \frac{n-1}{4}(\log\rho)_{,i}(\log\rho)_{,j} - \frac{n-1}{4}g_{ij}(\log\rho)_{,l}(\log\rho)_{,l} + \frac{1}{4}g_{ij}(\log\rho)_{,k}(\log\rho)_{,k} \\
&- \frac{1}{4}(\log\rho)_{,i}(\log\rho)_{,j} \\
&= \frac{n-2}{4}(\log\rho)_{,i}(\log\rho)_{,j} + \frac{2-n}{4}g_{ij}(\log\rho)_{,l}(\log\rho)_{,l} \qquad \text{(A.5.10)}
\end{aligned}
$$

Where the red terms cancel. Therefore

$$
\begin{aligned}
&\widetilde{R}_{ij} = R_{ij} + \frac{2-n}{2}(\log\rho)_{,ij} - \frac{1}{2}g_{ij}(\log\rho)_{,kk} + \frac{n-2}{4}(\log\rho)_{,i}(\log\rho)_{,j} \\
&+ \frac{2-n}{4}g_{ij}(\log\rho)_{,l}(\log\rho)_{,l} \\
&= R_{ij} - \frac{n-2}{2}(\log\rho)_{,ij} + \frac{n-2}{4}(\log\rho)_{,i}(\log\rho)_{,j} - \frac{1}{2}\left(\Delta_g((\log\rho)) + \frac{n-2}{2}|\nabla(\log\rho)|_g^2\right) \qquad \text{(A.5.11)}
\end{aligned}
$$

□

Corollary A.7. *Continue from Proposition (A.6), then the scalar curvature of $\widetilde{g}$ is given by:*

$$\widetilde{R} = \frac{1}{\rho}R - \frac{n-1}{\rho^2}\Delta_g\rho - \frac{(n-1)(n-6)}{4\rho^3}|\nabla\rho|_g^2. \qquad \text{(A.5.12)}$$

Proof. We again compute at p, which is the center of a geodesic normal coordinate neighborhood. The result then follows from a simple calculation:

$$\widetilde{R} := \widetilde{g}^{ij}\widetilde{R}_{ij} = \frac{1}{\rho}g^{ij}\bigg(R_{ij} - \frac{n-2}{2}(\log\rho)_{,ij} + \frac{n-2}{4}(\log\rho)_{,i}(\log\rho)_{,j}$$

$$-\frac{1}{2}\Big(\Delta_g(\log\rho) + \frac{n-2}{2}|\nabla(\log\rho)|_g^2\Big)\bigg)$$

$$= \frac{1}{\rho}R - \frac{n-2}{2\rho}\Delta_g(\log\rho) + \frac{n-2}{4\rho}g^{ij}(\log\rho)_{,i}(\log\rho)_{,j} - \frac{n}{2\rho}\left[\Delta_g(\log\rho) + \frac{n-2}{2}|\nabla\log\rho|_g^2\right]$$

$$= \frac{1}{\rho}R - \left(\frac{n-2}{2\rho} + \frac{n}{2\rho}\right)\Delta_g(\log\rho) + \frac{n-2}{4\rho}|\nabla\log\rho|_g^2 - \frac{n(n-2)}{4\rho}|\nabla\log\rho|_g^2$$

$$= \frac{1}{\rho}R - \frac{2(n-1)}{2\rho}\Delta_g(\log\rho) + \frac{-n^2+3n-2}{4\rho}|\nabla\log\rho|_g^2$$

$$= \frac{1}{\rho}R - \frac{n-1}{\rho}\left(\frac{1}{\rho}\rho_{,i}\right)_{,i} + \frac{-n^2+3n-2}{4\rho}\left(\frac{1}{\rho^2}\rho_{,i}\rho_{,i}\right)$$

$$= \frac{1}{\rho}R - \frac{n-1}{\rho}\left(-\frac{1}{\rho^2}\rho_{,i}\rho_{,i} + \frac{1}{\rho}\rho_{,ii}\right) + \frac{-n^2+3n-2}{4\rho^3}|\nabla\rho|_g^2$$

$$= \frac{1}{\rho}R - \frac{n-1}{\rho^2}\Delta_g\rho + \frac{4n-4-n^2+3n-2}{4\rho^3}|\nabla\rho|_g^2$$

$$= \frac{1}{\rho}R - \frac{n-1}{\rho^2}\Delta_g\rho - \frac{(n-1)(n-6)}{4\rho^3}|\nabla\rho|_g^2,$$

as desired. □

We want to eliminate the gradient term, and for that purpose we need to distinguish the following two cases.

Corollary A.8. *Suppose the manifold is of dimension* $n = 2$*. Let* $\rho := e^{2u}$*, where* $u > 0$ *is a smooth function on* M*, then*

$$\widetilde{R} = e^{-2u}\left(R - 2\Delta_g u\right). \tag{A.5.13}$$

Proof. We plug $\rho = e^{2u}$ into equation $(A.5.12)$, and again, we will be computing at $p \in M$ in a geodesic normal coordinate neighborhood:

$$
\begin{aligned}
\widetilde{R} &= \rho^{-1}R - \rho^{-2}\Delta_g\rho + \rho^{-3}|\nabla\rho|^2 \\
&= e^{-2u}R - e^{-4u}\left(e^{2u}2u_{,i}\right)_{,i} + e^{-6u}(e^{2u}2u_{,i})^2 \\
&= e^{-2u}R - e^{-4u}\left(e^{2u}2u_{,i}2u_{,i} + e^{2u}2u_{,ii}\right) + e^{-6u}4e^{4u}u_{,i}u_{,i} \\
&= e^{-2u}\left(R - 2\Delta_g u\right),
\end{aligned}
$$

where the red terms cancel. □

Corollary A.9. *Now suppose manifold is of dimension $n \geqslant 3$, we set $\rho := u^{\frac{4}{n-2}}$, for u a positive function on M. Then equation $(A.5.12)$ becomes:*

$$
\widetilde{R} = u^{-\frac{n+2}{n-2}}\left(Ru - \frac{4(n-1)}{n-2}\Delta_g u\right). \qquad \text{(A.5.14)}
$$

Proof. The result follows from plugging $\rho = u^{\frac{4}{n-2}}$ into equation $(A.5.12)$, again we will be computing at p:

$$
\begin{aligned}
\widetilde{R} &= u^{-\frac{4}{n-2}}R - (n-1)u^{-\frac{8}{n-2}}\Delta_g(u^{\frac{4}{n-2}}) - \frac{(n-1)(n-6)}{4}u^{-\frac{12}{n-2}}|\nabla(u^{\frac{4}{n-2}})|_g^2 \\
&= u^{-\frac{4}{n-2}}R - (n-1)u^{-\frac{8}{n-2}}\left(\frac{4}{n-2}u^{\frac{6-n}{n-2}}u_{,i}\right)_{,i} - \frac{(n-1)(n-6)}{4}u^{-\frac{12}{n-2}}\left(\frac{4}{n-2}\right)^2 \\
&\cdot u^{\frac{2(6-n)}{n-2}}u_{,i}u_{,i} \\
&= u^{-\frac{4}{n-2}}R - (n-1)u^{-\frac{8}{n-2}}\frac{4}{n-2}\left(\frac{6-n}{n-2}u^{\frac{8-2n}{n-2}}u_{,i}u_{,i} + u^{\frac{6-n}{n-2}}u_{,ii}\right) \\
&- \frac{(n-1)(n-6)}{4}\left(\frac{4}{n-2}\right)^2 u^{\frac{-12+12-2n}{n-2}}u_{,i}u_{,i} \\
&= u^{-\frac{4}{n-2}}R - (n-1)\frac{4}{n-2}u^{-\frac{n+2}{n-2}}\Delta_g u \\
&- |\nabla u|_g^2\left((n-1)\frac{4}{n-2}\frac{6-n}{n-2}u^{-\frac{2n}{n-2}} + \frac{(n-1)(n-6)}{4}\left(\frac{4}{n-2}\right)^2 u^{-\frac{2n}{n-2}}\right) \\
&= u^{-\frac{4}{n-2}}R - \frac{4(n-1)}{(n-2)}u^{-\frac{n+2}{n-2}}\Delta_g u - |\nabla u|_g^2\left(\frac{-4(n-1)(n-6)}{(n-2)^2} + \frac{(n-1)(n-6)4^2}{4(n-2)^2}\right) \\
&\cdot u^{-\frac{2n}{n-2}} \\
&= u^{-\frac{4}{n-2}}R - \frac{4(n-1)}{(n-2)}u^{-\frac{n+2}{n-2}}\Delta_g u \\
&= u^{-\frac{n+2}{n-2}}\left(Ru - \frac{4(n-1)}{n-2}\Delta_g u\right), \qquad \text{(A.5.15)}
\end{aligned}
$$

which is desired. □

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Biography

Hangjun Xu was born in August, 1987 in Hangzhou, China. In 2005, He went to Zhejiang Unviersity in China and obtained a Bachelor of Science degree in Mathematics and Applied Mathematics with the *Chu Kochen Honors Program* certificate in June, 2009. After graduation, he went to Duke University to pursue a Ph.D. degree in Mathematics under the supervision of Professor Hubert L. Bray. His field of research has been differential geometry and geometric analysis. He developed a side interest in computational geometry, and since fall 2011, he started pursuing a Master of Science degree in Computer Science under the supervision of Professor Pankaj K. Agarwal, en route to his Ph.D. program. During his stay at Duke, he has taught 8 undergraduate courses as an instructor, including calculus I and II, linear algebra, ordinary and partial differential equations. In 2012, he received the *Graduate School Research Fellowship* for five thousand dollars. Starting June 2014, he will be a senior software engineer of Oracle at Santa Clara, California. His dream is to become an animation and visual effects artist.