Managing Material and Financial Flows in

Supply Chains

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Business Administration
in the Graduate School of Duke University
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Abstract

This dissertation studies the integration of material and financial flows in supply chains, with the goal of examining the impact of cash flows on the individual firm's decision making and the overall supply chain efficiency. We develop analytical models to provide effective policy recommendations and derive managerial insights.

We first consider a credit-constrained firm that orders inventory to satisfy stochastic demand in a finite horizon. The firm provides trade credit to the customer and receives it from the supplier. A default penalty is incurred on the unfulfilled payment to the supplier. We utilize an accounting concept of working capital to obtain optimal and near-optimal inventory policies. The model enables us to suggest an acceptable purchasing price offered in the supplier's trade credit contract, and to demonstrate how liquidity provision can mitigate the bullwhip effect.

We then study a joint inventory and cash management problem for a multi-divisional supply chain. We consider different levels of cash concentration: cash pooling and transfer pricing. We develop the optimal joint inventory replenishment and cash retention policy for the cash pooling model, and construct cost lower bounds for the transfer pricing model. The comparison between these two models shows the value of cash pooling, although a big portion of this benefit may be recovered through optimal transfer pricing schemes.

Finally, we build a supply chain model to investigate the material flow variability without cash constraint. Our analytical results provide conditions under which the material bullwhip effect exists. These results can be extended to explain the similar effect when financial flows are involved.

In sum, this dissertation demonstrates the importance of working capital and financial integration in supply chain management.

To My Family

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List of Abbreviations and Symbols

Symbols

Notations for Chapter 2

a'	left-control threshold in the $(\mathbf{d}, \mathbf{a}, S)$ policy.
a''	right-control threshold in the $(\mathbf{d}, \mathbf{a}, S)$ policy.
A'	left-adjustment in the three-piece linear approximation.
A''	right-adjustment in the three-piece linear approximation.
b	backorder penalty cost rate.
$\mathcal{B}$	band in which inventory does not exceed base-stock level.
c	unit procurement cost.
$\bar{C}_2$	cost of the $(\mathbf{d}, \mathbf{a}, S)$ heuristic.
C_3	cost of the three-piece lower bound.
$\bar{C}_3$	cost of the (d, S) heuristic.
C_L	cost of lower bound under negative demand shocks.
C_U	cost of heuristic under negative demand shocks.
d	default threshold.
$\bar{d}$	expected default threshold.
D	random customer demand.
e	column vector of ones.
E	expectation.
f	probability density function (p.d.f.) of D.

F	cumulative distribution function (c.d.f.) of D.
$\bar{F}$	complementary cumulative distribution function (c.c.d.f.) of D.
$\hat{F}$	loss function of D.
g	default-free single-period cost function.
G	single-period cost function.
h	inventory holding cost rate.
H	single-period holding and backorder cost function.
L	default penalty linear function.
m	payment period.
M	default penalty cost function.
M^{-}	two-piece linear approximation of M.
$\bar{M}$	three-piece linear approximation of M.
n	collection period.
p	default penalty rate (per unit inventory).
$\bar{p}$	expected default penalty rate (per unit inventory).
p'	default penalty rate (per dollar).
P	accounts payable.
$\mathbf{P}$	vector of accounts payable.
$\mathbf{Pr}$	probability.
r	unit revenue retained for operations (working capital requirement).
R	accounts receivable.
$\mathbf{R}$	vector of accounts receivable.
$\bar{R}$	doubtful receivable.
S	default-free base-stock level.
t	time variable.
T	time horizon.

u	default quantity.
v	unconstrained myopic minimization function.
V_t	minimum expected cost function over t to $T + 1$.
w	working capital level (in inventory unit).
$\bar{w}$	expected working capital level.
w'	net cash level.
W	decoupled cost-to-go function of working capital.
x	net inventory level.
y	inventory position.
z	order quantity.
α	discount factor.
Γ	linear function in the three-piece approximation.
δ	estimated proportion of collectible receivable.
ϵ	random variable representing the payment default uncertainty.
θ	$r/c - 1$.
λ	minimum of the payment period and the collection period.
μ	mean of demand D.
$\bar{\mu}$	mean of ϵ.
ρ	r/c.
σ	standard deviation of demand D.
$\bar{\sigma}$	standard deviation of ϵ.
$*$	optimal.
$\hat{}$	pre-transformation function.
$\bar{}$	three-piece lower bound function.
$\leqslant_{st}$	stochastically increasing.
$\wedge$	minimum.
$\vee$	maximum.

Notations for Chapter 3

For stage $i = 1, 2$, echelon $j = 1, 2, 3, 4$, and subsystem $k = 1, 2$

b	backorder penalty cost rate.
B	constant in the flow conservation equation.
c	unit procurement cost from the outside vendor.
C_L	cost of the integrated lower bound.
C_O	cost of the optimal pricing model.
C_R	cost of the constraint relaxation bound.
C_S	cost of the sample path bound.
C^*	optimal cost of the cash pooling model.
$d(\omega)$	demand realization given sample path ω.
D	random customer demand.
E	expectation.
$f_{j,t}$	expected optimal cost for echelon j over t to $T + 1$.
g_j	single-period cost function for echelon j (without transaction cost).
G	single-period cost function.
G^k	single-period cost function for subsystem k.
h_i	inventory holding cost rate of stage i.
H_j	single-period holding and backorder cost function of echelon j.
H_1^k	single-period holding and backorder cost function of echelon 1 for subsystem k.
J_t	expected cost function over t to $T + 1$.
K	upper-limit of asset that can be liquidated to assist operations.
l^*	lower threshold in the cash retention policy.
L	lower threshold linear function.
p_1	unit selling price to the end customer.

p_2	unit transfer price.
r	working capital position in the cash pooling model.
r_1	working capital position of stage 1 in the transfer pricing model.
r_f	risk-free rate.
R	return rate for external investment.
S	constraint set.
S^d	constraint set under deterministic demand.
t	time variable.
T	time horizon.
u^*	upper threshold in the cash retention policy.
U	upper threshold linear function.
v	amount of cash transferred into the operating account.
V_t	minimum expected cost function over t to $T+1$.
V_t^k	minimum expected cost function over t to $T+1$ for subsystem k.
V_t^d	minimum expected cost function over t to $T+1$ for subsystem 2, given demand sample path.
w	net working capital level in the cash pooling model.
w_i	echelon working capital level of stage i in the transfer pricing model.
w'	cash balance in the pooled account.
w_i'	cash balance in stage i's account.
x_i	echelon i net inventory level.
x_1'	net inventory level at stage 1.
x_2'	on-hand inventory level at stage 2.
y_j	inventory position for echelon j.
z_i	order quantity for stage i.
α	discount factor.

β_i unit transaction cost on cash transferred into the operating account.

β_o unit transaction cost on cash transferred out of the operating account.

Γ_2 induced penalty cost from echelon 1 to echelon 2.

Γ_3 induced penalty cost from echelon 2 to echelon 3.

η cash holding cost rate of the pooled account.

η_i cash holding cost rate of stage i.

θ $p_1/c - 1$.

Λ_2 induced penalty cost from echelon 3 to echelon 2.

Λ_3 self-induced penalty cost at echelon 3.

ρ p_2/c.

ω demand sample path.

$*$ optimal.

$'$ local; e.g.,

       $\eta' = local$ cash holding cost,

       $\eta = echelon$ cash holding cost.

$\hat{\ }$ pre-transformation function.

$\wedge$ minimum.

$\vee$ maximum.

Notations for Chapter 4

$j = 1, 2$

B_j local backorders of stage j.

D random customer demand.

E expectation.

IL_j local net inventory level of stage j.

M_j shipment released to stage j from its upstream supplier.

M_0	realized sales to the end customer.
O_j	order quantity from stage j to its upstream supplier.
s_j	local base-stock level of stage j.
t	time variable.
$U(0, d)$	uniform distribution with support $[0, d]$.
var	variance.
τ	cycle length.

Abbreviations

CCC	cash conversion cycle.
A/P	accounts payable.
A/R	accounts receivable.
LC	letter of credit.
CP	cash pooling.
TP	transfer pricing.
OP	optimal pricing.
CR	constraint relaxation.
SP	sample path.

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1

Introduction

With the worldwide economic crisis rendering bank financing increasingly difficult to secure, coordination between material and financial flows has taken on added importance. Great opportunities and challenges lie ahead in managing the financial flows in supply chains. More specifically, in a logistics-integrated supply chain, the trading partners jointly determine inventory replenishment according to the demand information without considering each partner's financial condition. This makes sense if the financial market is perfect (Modigliani and Miller 1958). However, when there are market frictions, the supply chain partners have to jointly maintain a healthy financial ecosystem in order to drive operational efficiencies.

Nevertheless, the literature on the integration of material and financial flows is relatively sparse, even though these two flows are closely related and affect each other. This dissertation constructs a modeling framework to explicitly incorporate financial flows into the inventory system of a single firm, and a multi-divisional supply chain. It seeks to address the following research questions. From a single firm's perspective, what is the impact of upstream and downstream payment terms on firm's optimal replenishment decisions? What is the right cash conversion cycle for a firm that faces various demand patterns and working capital requirements? From a supply

chain's perspective, what is the value of cash pooling? How much of this value can be recovered through advanced internal transfer pricing scheme? And how does financial flow affect the material bullwhip effect in the supply chain? The answers to these research questions can be found in the following three chapters.

Chapter 2 considers a single firm that orders inventory periodically to satisfy random customer demand in a finite horizon. The firm provides trade credit to its customer while receiving trade credit from its supplier. The trade credit is in the form of a one-part contract, i.e., the payment is due within a specific period of time following the invoice. A default penalty cost is incurred on the unfulfilled payment to the supplier. The objective is to obtain an inventory policy that minimizes the total inventory related and default penalty cost. Utilizing an accounting concept of work capital (which incorporates cash, inventory, accounts receivable and accounts payable), we prove that a myopic policy is optimal when the sales collection period is longer than or equal to the purchases payment period. The myopic policy has a simple structure – an order is placed to achieve a target base-stock level that depends on the firm's working capital. When the payment period is longer than the collection period, we derive a lower bound to the optimal cost and propose an effective heuristic that has a generalized form of the above structured policy. These policies resemble practical *working capital management* under which firms decide inventory policies according to their working capital status. The policy parameters have a closed-form expression, which shows the impact of demand variability on the inventory decision and the tradeoff between cost parameters. The model enables us to suggest an acceptable purchasing price offered in the supplier's trade credit contract, and to demonstrate how liquidity provision can mitigate the bullwhip effect.

Chapter 3 develops a centralized supply chain model that aims to assess the value of cash pooling. The supply chain is owned by a single corporation with two divisions, where the downstream division (headquarter), facing random customer demand, replenishes materials from the upstream one. The downstream division receives cash

payments from customers and determines a system-wide inventory replenishment and cash retention policy. We consider two cash management systems that represent different levels of cash concentration. For cash pooling, the supply chain adopts a financial services platform which allows the headquarter to create a corporate master account that aggregates the divisions' cash. For transfer pricing, on the other hand, each division owns its cash and pays for the ordered material according to a fixed price. Comparing both systems yields the value of adopting such financial services. We prove that the optimal policy for the cash pooling model has a surprisingly simple structure – both divisions implement a base-stock policy for material control; the headquarter monitors the corporate working capital and implements a two-threshold policy for cash retention. Solving the transfer pricing model is more involved. We derive a lower bound on the optimal cost by connecting the model to an assembly system. Our results show that the value of cash pooling can be very significant when demand is increasing (stationary) and the markup for the upstream division is small (high). Nevertheless, a big portion of the pooling benefit may be recovered if the headquarter can decide the optimal transfer price and the lead time is short.

Chapter 4 focuses on the material bullwhip effect in supply chains, a phenomenon that the variability of shipment is amplified when moving upstream the supply chain. Economists have observed this phenomenon in empirical studies. However, this observation appears to be counter-intuitive as they would expect the opposite - the "production smoothing" effect (smaller shipment variability at the upstream stage). We provide an analytical model to show that it is possible to observe both bullwhip and variability dampening in supply chains. These results can be extended to explain the similar effect when inventory replenishment is subject to the cash constraint, hence providing analytical support for the findings in Chapter 2 and Chapter 3.

The Appendix contains all the proofs.

Inventory Systems with Trade Credit

This chapter considers a firm that orders inventory periodically to satisfy random customer demand in a finite horizon. The firm provides trade credit to its customer while receiving trade credit from its supplier. The trade credit is in the form of a one-part contract, i.e., the payment is due within a specific period of time following the invoice. A default penalty cost is incurred on the unfulfilled payment to the supplier. The objective is to obtain an inventory policy that minimizes the total inventory related and default penalty cost. Utilizing an accounting concept of work capital (which incorporates cash, inventory, accounts receivable and accounts payable), we prove that a myopic policy is optimal when the sales collection period is longer than or equal to the purchases payment period. The myopic policy has a simple structure – an order is placed to achieve a target base-stock level that depends on the firm's working capital. When the payment period is longer than the collection period, we derive a lower bound to the optimal cost and propose an effective heuristic that has a generalized form of the above structured policy. These policies resemble practical *working capital management* under which firms decide inventory policies according to their working capital status. The policy parameters have a closed-form expression, which shows the impact of demand variability on the inventory decision and the

tradeoff between cost parameters. The model enables us to suggest an acceptable purchasing price offered in the supplier's trade credit contract, and to demonstrate how liquidity provision can mitigate the bullwhip effect.

2.1 Introduction

Trade credit is widely used for business transactions in supply chains, and is the single most important source of external finance for firms (Petersen and Rajan, 1997). It appears on every balance sheet and accounts for about one half of the short-term debt in two samples of UK and US firms (Cunat, 2007). In the finance literature, there have been various theories to explain the existence of trade credit despite its high cost. Our paper takes trade credit as a premise and aims to investigate the impact of trade credit on a firm's inventory policy and operating cost. We consider a firm that orders inventory periodically to satisfy stochastic customer demand in a finite horizon. The firm grants trade credit to its customer while obtaining it from its supplier. (We do not consider bank financing in the model.) The trade credit we consider is a one-part contract, that is, the payment is due within a certain time period after the invoice is issued[1]. Thus, the firm pays for the ordered inventory after a deferral period following the delivery of goods, and receives sales revenue after a collection period following the demand. In the 1998 NSSBF survey, however, 46% of the firms declared that they had made some payments to their suppliers after the due date. These delayed payments often do not carry an explicit penalty for the customers (Cunat, 2007). Nevertheless, payment default will hurt a firm's credit record, making it hard to finance in the future[2]. In light of this intangible cost, we introduce a default penalty cost incurred upon the unfulfilled payment to the firm's supplier. The objective is to obtain an inventory ordering policy that minimizes the

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[1] According to the 1998 National Survey of Small Business Finance (NSSBF), 49% of the trade credit contracts are one-part.

[2] For example, Dun and Bradstreet keeps credit records and provides credit reports of small businesses.

total discounted system cost, which consists of the inventory purchasing cost, holding and backorder costs, as well as the default penalty cost.

Our research question is related to a broader issue of *working capital management*. Working capital refers to the difference between current assets and current liabilities (assets and liabilities with maturities of less than one year). On the balance sheet, current assets include cash, short-term investments, accounts receivable (A/R), and inventory, while current liabilities include accounts payable (A/P) and short-term loans. In our model, the deferred payment is recorded as A/P and the delayed sales collection as A/R. The goal of working capital management is to increase the profitability of a firm and to ensure that it has sufficient liquidity to meet short-term operations so to continue in business (Pass and Pike, 1984). Profitability and liquidity are conflicting goals as investments in current assets usually lead to a smaller return. Thus, a firm needs to decide a working capital policy, which budgets how much revenue received to be invested in working capital. In general, a firm can adopt three types of working capital strategies: aggressive, moderate, and conservative (Gallagher and Andrew, 2007). (With aggressive strategy firms choose to operate with low cash, inventory, and trade receivables.) In this paper, we assume a given working capital strategy and aim to study the optimal inventory policy when trade credits are present.

Most inventory models in the literature do not explicitly consider the interrelationship between the inventory decision and the accounts payable/receivable because traditionally the former is a function of an operations manager and the latter a treasurer or controller. We find it important to study this interrelationship for the following reasons. First, today's inventory order decision will directly affect the future cash payment. If a firm orders too much, it is not only to incur a higher purchase cost and possibly inventory holding cost, but also to increase the chance of payment default as the future cash balance may not be enough to pay off the current inventory order due to demand/sales uncertainty. On the other hand, if a firm orders too little, it is more likely to incur a higher backorder cost. Thus, there is a clear tradeoff

between these system costs when making an inventory decision. Second, a firm often wishes to extend the payment period and shorten the collection period so that its cash conversion cycle (cash collection periods + on-hand inventory in periods - inventory payment periods) can be reduced. However, extending the payment period may lead to an increase of the unit wholesale price, and thus may not be ideal for the firm. Therefore, it would be useful to provide a decision support tool that characterizes the tradeoff between a longer payment period and a higher purchase cost.

We formulate the inventory system with trade credit into a multi-state dynamic program that keeps track of inventory level, cash balance, as well as different ages of accounts payable and accounts receivable within the payment and collection periods, respectively. This dynamic program is hard to solve because the state space is high-dimensional. We borrow a concept in accounting called working capital (= cash + inventory + accounts receivable - accounts payable) to redefine the state and simplify the original dynamic program. We consider three cases regarding the different lengths of payment and collection periods. When the payment period is equal to or less than the collection period, we prove that a myopic policy is optimal when the demand is non-decreasing. The optimal policy is operated under two control parameters (d, S), $d \leqslant S$: the firm reviews its inventory-equivalent working capital level (i.e., working capital divided by the unit purchase cost) and inventory position at the beginning of each period; if the working capital is lower (higher) than $d(S)$, the firm places an order to bring its inventory position up to $d(S)$; if the working capital level is between d and S, the firm orders up to the working capital level. When the payment period is longer than the collection period, the firm's future payment depends on the future cash inflow, which in turn depends on the random demand. Consequently, it is difficult to characterize the optimal policy. Nevertheless, we develop a lower bound to the optimal cost and propose an effective heuristic. The heuristic policy, referred to as the $(\mathbf{d}, \mathbf{a}, S)$ policy, is operated under five control parameters and can be viewed as a generalization of the (d, S) policy. In a numerical study, we show that the heuris-

tic is near optimal. The optimal and heuristic policy parameters have a closed-form expression which allows us to investigate the impact of demand variability on the inventory decision as well as the tradeoffs between inventory holding cost, backorder cost, and payment default cost. Finally, we also test our optimal and heuristic policies under different non-stationary demand forms, and the performance remains satisfactory. Thus, the suggested policies can comfortably be applied to systems with general demand patterns. Notice that the heuristics resemble practical working capital management under which firms determine the inventory policy according to the working capital level. Consequently, we can use them to gain insights.

We summarize the main contributions and key insights obtained from this study. First, managers are hindered from integrating accounts payable/receivable into the inventory policy due to the typical organizational structure of the firm. These two functions need to be aligned in order to improve the firm's profit. Our model captures the dynamics between inventory decision as well as accounts payable/receivable resulted from trade credit terms and provide a simple and implementable inventory policy. Second, the optimal policy suggests that a firm should consider working capital when making inventory decisions. This result naturally connects operations to accounting. Also, the closed-form expression for the optimal policy suggests that a firm would possibly choose to default on the payment to its supplier if its current working capital level is low and the backorder penalty is higher than the default penalty. This result echoes the NSSBF survey that 46% of firms experience payment defaults. Third, we provide a decision support tool in trade credit contract negotiation by quantifying the impact of payment periods on the firm's total operating cost. We find that increasing demand and high backorder cost justify the usage of trade credit despite its high cost. In addition, our numerical study shows that firms with a shorter cash conversion cycle have more incentive to extend credit periods with suppliers, which predicts a positive correlation between the firm's upstream and downstream credit periods. Finally, we show that customer payment default drives the bullwhip effect.

The bullwhip ratio increases with the downstream default volatility and the upstream default penalty cost. This suggests that the supplier could effectively mitigate the bullwhip effect through liquidity provision.

2.2 Literature Review

Our paper is related to inventory systems with trade credit contracts. This literature can be categorized based on whether a single- or multi-period problem is considered. For the single-period model, Zhou and Groenevelt (2008) consider the impact of financial collaboration in a third-party supply chain. They find that the total supply chain profit with bank financing is slightly higher than that with open account (trade credit) financing. Yang and Birge (2012) study how different priority rules of order repayment influence trade credit usage.

As for the multi-period model, this literature can be further categorized based on how trade credits are modeled. One category is to characterize the impact of trade credit on the inventory holding cost. Beranek (1967) uses a lot-size model to illustrate how a firm's inventory holding cost should be adjusted according to the firm's actual financial arrangements. Maddah et al. (2004) investigate the effect of permissible delay in payments on ordering policies in a periodic review (s, S) inventory model with stochastic demand. They develop heuristic approaches to approximate inventory control parameters. Gupta and Wang (2009) consider a stochastic inventory system where trade credit term is modeled as a non-decreasing holding cost rate according to an item's shelf age. Under the assumption that the full payment is made when the item is sold, they prove that a base-stock policy is optimal. Huh et al. (2010) and Federgruen and Wang (2010) generalize the results of Gupta and Wang. Song and Tong (2012) consider an inventory system where a base-stock policy is implemented. They investigate how the holding cost rate is affected by the different payment and collection periods.

Another category, which is more related to our model, is to explicitly characterize

cash flow dynamics resulted from the trade credit terms. Haley and Higgins (1973) expand Beranek's model and consider a problem of jointly optimizing inventory decision and payment times when demand is deterministic and inventory is financed with trade credit. Schiff and Lieber (1974) consider a problem of optimizing inventory and trade credit policy for a firm where the demand is deterministic but depends on the credit term and inventory level. Bendavid et al. (2012) study a self-financing firm whose replenishment decisions are constrained by the working capital requirement. Their model is similar to ours in the sense that they also consider how inventory replenishment is affected by the payment and the collection periods. However, their model considers i.i.d demand and implements a base-stock policy with inventory ordering subject to a hard constraint - the working capital requirement. Thus, no defaults are allowed. They characterize the dynamics of system variables and obtain the optimal base-stock level via a simulation approach.

There has been an emerging research stream that aims to jointly model financial and operational decisions without explicitly considering the trade credit. Xu and Birge (2004) analyze the interactions between a firm's production and financing decisions as a tradeoff between the tax benefits of debt and financial distress costs. Li et al. (2013) study a dynamic model in which inventory and financial decisions are made simultaneously in order to maximize the expected present value of dividends net of capital subscriptions. Xu and Birge (2006) propose an integrated corporate planning model, which extends the forecasting-based discount dividend pricing method into an optimization-based valuation framework to make production and financial decisions simultaneously for a firm facing market uncertainty. Chao et al. (2008) consider a self-financed retailer who replenishes inventory in a finite horizon. Luo and Shang (2012) integrates material flow and cash flow in a supply chain. They characterize the optimal joint policy and investigate the value of payment flexibility. Tanrisever et al. (2012) explore the tradeoff between investment in process development and reservation of cash in order to avoid bankruptcy for a start-up firm. They provide

managerial insights by characterizing how to create operational hedges against the bankruptcy risk. Other noteworthy examples include Babich and Sobel (2004), Buzacott and Zhang (2004), Ding et al. (2007), Dada and Hu (2008), Kouvelis and Zhao (2009), Caldentey and Chen (2010). The research questions in these papers are quite different from ours.

The motivation and the assumption of our model are related to the following empirical finance literature. Petersen and Rajan (1997) find that there is a greater extension of credit by firms with negative income and negative sales growth. They suggest that trade credit can be used as a signal of financial health of a firm. Wilson and Summers (2002) provide reasons why suppliers still maintain business relationships with retailers who default on their payments. This finding also supports our payment default setup in a finite horizon model. Cunat (2007) provides an empirical evidence that suppliers serve a role as liquidity providers insuring against liquidity shocks that could endanger the survival of their customer relationships. Thus, the high cost of trade credit can be interpreted as the insurance and default premium. The paper provides an empirical support of the default penalty assumed in our model. It also suggests that there is a big portion of firms that use one-part trade credit contracts. Boissay and Gropp (2007) investigate liquidity shocks for small-sized French firms. They find that the payment default in a supply chain stops when it reaches firms that are large and have access to financial markets. Guedes and Mateus (2009) examine the trade credit linkages on the propagation of liquidity shocks in supply chains. It is a common practice that firms often provide trade credit to its customer while receiving trade credit from its supplier. In a similar spirit, we also study the relationship between trade credit and materials bullwhip effect in our model.

Finally, our model is related to two streams of inventory problems. The on-hand cash in our model resembles a capacity constraint on inventory ordering. However, we allow payment defaults (i.e., order more than the on-hand cash level) and the cash balance is endogenously determined by the inventory decision. We refer the reader

to Tayur (1997) for a review and Levi et al. (2008) for recent developments. The other stream is inventory systems with advance demand information. The incoming and outgoing cash flows in accounts payable and receivable can be viewed as advance cash flow information. For the research of advanced demand models, see Ozer and Wei (2004) and references therein.

The rest of this chapter is organized as follows. §2.3 describes the model and formulates the corresponding dynamic program. §2.4 focuses on the model with balanced payment and collection periods and proves the optimal policy. §2.5 (§2.6) considers the model with longer payment (collection) periods. §3.5 examines the effectiveness of the heuristics, and discusses the qualitative insights through a numerical study. §3.6 concludes. Proofs are provided in Appendix A.1. Throughout this chapter, we define $x^+ = \max(x, 0)$, $x^- = -\min(x, 0)$, $a \vee b = \max(a, b)$, and $a \wedge b = \min(a, b)$.

2.3   The Model

We consider a finite-horizon, periodic-review inventory system where a firm orders from its supplier and sells to its customer. Trade credit is employed for transactions at both upstream and downstream and is in the form of a one-part contract, that is, the firm pays its supplier after a payment period following the delivery of goods, and receives cash from its customer after a collection period following the demand. In accounting, the inventory payment period (sales collection period) is also referred to as the *payables (receivables) conversion period* or *days purchases (sales) outstanding*. The payment and collection periods jointly affect the *cash conversion cycle* (CCC). Figure 2.1 illustrates the referenced times of four events associated with buying and selling a discrete batch of inventories: R, inventory order received; S, inventory sold; P, cash paid to the supplier, and C, cash collected from the customer. The CCC has three components: the payment period represented by the time interval between R and P; the *inventory conversion period* (or *days in inventory*) represented by the time interval between R and S; the collection period represented by the time interval

between S and C. In practice, CCC is calculated as follows:

$$\text{CCC} = \text{Inventory conversion period} + \text{Collection period} - \text{Payment period.}$$

Transactions based on the trade credit will affect a firm's accounts payable (A/P) and accounts receivable (A/R). Table 2.1 lists the four events and the corresponding changes in inventory and cash flow, as well as accounts payable and receivable.

FIGURE 2.1: The cash conversion cycle
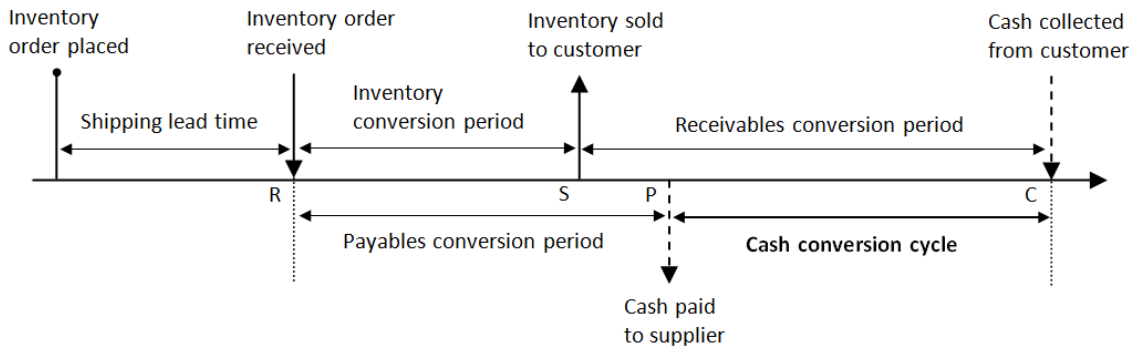
We now formalize the above description into our model. Since the focus is on cash and inventory dynamics under trade credit, for simplicity and without loss of generality, we assume that lead time is zero. Let m be the payment period and n be the collection period. The sequence of events is as follows: At the beginning of period t, (1) inventory order decision is made; (2) shipment arrives; (3) payment due in this period (corresponding to the inventory ordered in period $t - m$) is made to the supplier; (4) default penalty is incurred in case of insufficient payment; (5) customer payment due in this period (corresponding to the sales in period $t - n$) is collected. During the period, demand is realized. At the end of the period, all inventory related costs and default penalty cost are calculated. The objective is to minimize the firm's total discounted cost over the entire horizon of T periods.

Customer demand in period t is modeled as a nonnegative random variable D_t with probability density function (p.d.f.) f_t, cumulative distribution function (c.d.f.) F_t, mean μ_t and variance σ_t^2. The demand is independent from period to period but the

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Table 2.1: Events and accounting variables associated with the cash conversion cycle

Label	Transaction	Inventory/Cash flow	Accounting
R	Receiving X units of inventory	Inventory ↑ X	A/P ↑ \$X
S	Selling Y units of inventory	Inventory ↓ Y	A/R ↑ \$Y
P	Paying \$X to the supplier	Cash ↓ \$X	A/P ↓ \$X
C	Collecting \$Y from the customer	Cash ↑ \$Y	A/R ↓ \$Y

distribution could be non-stationary. We assume that the unsatisfied demand is fully backlogged. Figure 2.2 takes a snapshot of the system in period t with the material and cash flows in solid and dashed arrows, respectively. We count the time forward. As shown, P_{t-i} and R_{t-j} denote the accounts payable and accounts receivable made in period $t-i$ and $t-j$, respectively, for $i = 0, 1, ..., m$ and $j = 0, 1, ..., n$. So P_{t-m} and R_{t-n} are the most aged A/P and A/R, respectively.

FIGURE 2.2: The base model with material and cash flows
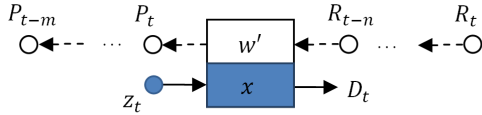
Let us now define the state and decision variables at the beginning of period t:

$$z_t = \text{order quantity made in Event (1);}$$

$$x_t = \text{net inventory level before Event (2);}$$

$$w'_t = \text{net cash level before Event (3);}$$

$$\mathbf{P}_t = (P_{t-m}, ..., P_{t-1})\text{: } m\text{-dimensional vector of accounts payable;}$$

$$\mathbf{R}_t = (R_{t-n}, ..., R_{t-1})\text{: } n\text{-dimensional vector of accounts receivable.}$$

Denote $\mathbf{P}_t^i$ as the vector consisting of the first i elements of $\mathbf{P}_t$, and $\mathbf{P}_t^{-i}$ as vector $\mathbf{P}_t$ without the first i elements. Let r be the unit revenue retained for operations[3]

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[3] Firms usually determine an operations budget as a percentage of sales revenue during an inte-

and c be the unit procurement cost in period t. According to Table 2.1, the dynamics of states between two periods are:

$$x_{t+1} = x_t + z_t - D_t, \tag{2.1}$$

$$w'_{t+1} = w'_t - P_{t-m} + R_{t-n}, \tag{2.2}$$

$$\mathbf{P}_{t+1} = (\mathbf{P}_t^{-1}, cz_t), \tag{2.3}$$

$$\mathbf{R}_{t+1} = (\mathbf{R}_t^{-1}, rD_t). \tag{2.4}$$

In the cash dynamics (2.2) and (2.4) we assume that the firm is guaranteed to receive a full payment from the customer in n periods after the trade. This assumption will be relaxed in the model extension where customer default is considered; see §2.7.3.

We introduce the cost parameters. Denote h as the holding cost per unit inventory per period, and b the backorder cost per unit backorder per period. If the inventory position at the beginning of the period is y, then the holding and backorder cost of the period can be expressed as

$$H_t(y) = \mathsf{E}[h(y - D_t)^+ + b(y - D_t)^-].$$

Here and in the sequel, the expectation is taken over D_t, unless otherwise specified.

Let p' denote the default penalty per dollar per period. In practice, this penalty cost may include two parts: a monetary part equal to the interest charged by the supplier upon overdue payment and a non-monetary part representing the intangible consequence of defaults, such as loss of credibility. This interpretation of default penalty resembles the backorder cost incurred due to failure of fulfilling the demand. Although we mainly consider illiquidity default, our model can incorporate bankruptcy default by setting the penalty cost p' sufficiently large. Now let us write the single-period cost function:

$$\hat{G}_t(x_t, z_t, w'_t, P_{t-m}) = H_t(x_t + z_t) + p'(w'_t - P_{t-m})^- + \alpha^m cz_t, \tag{2.5}$$

grated operations/sales planning process (sales and operations planning.). Here, r reflects the firm's policy on the working capital requirement

where the first term represents the expected inventory holding and backorder cost; the second term is the default penalty cost for not being able to make the payment in full at the end of the payment period; the last term is the procurement cost, which is realized in m periods later when payment is due. We charge this cost in the current period.

We make a remark here. Although we include inventory holding, backorder and default penalty costs in the periodic cost function in (2.5), these costs do not appear in the cash dynamics in (2.2). This is because the non-monetary part of these costs do not correspond to real cash flows. In addition, the monetary part, mainly including the physical holding cost and the interest charged on payment default[4], is minimal and not reflected in periodic cash dynamics.

Denote $\hat{V}_t(x_t, w'_t, \mathbf{P}_t, \mathbf{R}_t)$ as the minimum expected cost over period t to $T+1$, and over all feasible decisions. Let α be the single-period discount rate. The dynamic program is

$$\hat{V}_t(x_t, w'_t, \mathbf{P}_t, \mathbf{R}_t) = \min_{0 \leqslant z_t} \left\{ \begin{array}{l} \hat{G}_t(x_t, z_t, w'_t, P_{t-m}) \\ +\alpha \mathsf{E} \hat{V}_{t+1}(x_{t+1}, w'_{t+1}, \mathbf{P}_{t+1}, \mathbf{R}_{t+1}), \end{array} \right\} \qquad (2.6)$$

$$\hat{V}_{T+1}(x_{T+1}, w'_{T+1}, \mathbf{P}_{T+1}, \mathbf{R}_{T+1}) = -\alpha^m c x_{T+1} + \sum_{s=1}^{m+1} \alpha^{s-1} \mathsf{E} p'(w'_{T+s} - P_{T+s-m})^-, \quad (2.7)$$

where the expressions of w'_{T+s} follow the dynamics shown in (2.2).

Here, two assumptions are made regarding the terminal cost function. First, we assume that the end-of-horizon inventory has the unit salvage value c, and backlogged demand has to be satisfied. This can be interpreted by setting $z_{T+1} = -x_{T+1}$. That is, the supplier will buy back the left-over inventory x^+_{T+1} at the unit price c or the firm has to make a final order of x^-_{T+1} to fulfill the unsatisfied demand. Correspondingly, we have $P_{T+1} = -cx_{T+1}$. This payment is realized after m periods, and we charge the resulting cost $-\alpha^m c x_{T+1}$ to the terminating period $T+1$. This explains the term

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[4] According to the 1998 NSSBF sample, 43% of the trade credit contracts do not carry any explicit penalty. The median penalty rate for the contracts with explicit penalty is an annual rate of 29.7%, or monthly rate of 2.19%. See Boissay and Gropp (2007).

in equation (2.7).

Second, we assume that the default penalty applies from time $T + 1$ to $T + m + 1$. When $m > n$, the derivation of w'_{T+s} $(s = 1, ..., m + 1)$ needs the additional sales from the next planning horizon. This explains the second part of the terminal cost, in which the expectation is taken over the sum of demands that are necessary to derive the net cash level w'_{T+s}.

The base model formulated in (2.6) and (2.7) is difficult to solve. One can show the joint convexity of $\hat{V}_t(\cdot)$ and $\hat{G}_t(\cdot)$, thus deriving a state-dependent global optimal solution. However, the computing is quite hard due to the curse of dimensionality (state space has $m + n + 2$ dimensions). In the next three sections, we introduce new system variables to reduce the state space, and provide optimal solution or simple heuristics for the base model.

2.4   Balanced Credit Periods

This section considers the system with equal payment and collection periods, i.e., $m = n$. We shall prove that the base model can be solved by redefining a new state variable.

2.4.1   State Space Reduction

When $m = n = \lambda$, for any given period t, the firm knows how much cash it will receive from the customer, i.e., R_{t-i} and how much cash it will pay to its supplier, i.e., P_{t-i}, $i = 1, ..., \lambda$. Thus, the firm has complete information about cash dynamics in each of the incoming λ periods. We now reduce the state space by introducing new system variables. Let e_λ be the λ-dimensional column vector of ones. Define

$$y = x + z, \ \ w = x + (w' - \mathbf{P}e_\lambda + \mathbf{R}e_\lambda)/c.$$

We refer to y as the inventory position and w as the *working capital level* measured in inventory units, at the beginning of the period t. This is consistent with the

accounting definition of *net working capital*, which equals current assets (inventory, cash, and A/R) minus current liabilities (A/P). Furthermore, let $p = cp'$, $\rho = r/c$, and $\theta = \rho - 1$. Applying dynamics (2.1)-(2.4) repeatedly, we have

$$\hat{V}_t(x_t, w'_t, \mathbf{P}_t, \mathbf{R}_t) = p'(w'_t - P_{t-\lambda})^- + \alpha p'(w'_t - P_{t-\lambda} + R_{t-\lambda} - P_{t-\lambda+1})^- + \cdots$$

$$+ \alpha^{\lambda-1} p'(w'_t - P_{t-\lambda} - \cdots - P_{t-2} + R_{t-\lambda} + \cdots + R_{t-2} - P_{t-1})^-$$

$$+ V_t(x_t, x_t + (w'_t - \mathbf{P}_t e_\lambda + \mathbf{R}_t e_\lambda)/c), \tag{2.8}$$

and that V_t satisfies the functional equation

$$V_t(x, w) = \min_{x \leqslant y} \left\{ G_t(x, w, y) + \alpha \mathsf{E} V_{t+1}(y - D_t, w + \theta D_t) \right\}, \tag{2.9}$$

where the one-period cost function can be shown as

$$G_t(x, w, y) = H_t(y) + \alpha^\lambda p(y - w)^+ + \alpha^\lambda c(y - x). \tag{2.10}$$

Since $z_{T+1} = -x_{T+1}$ is equivalent to $y_{T+1} = 0$, the terminal cost function becomes

$$V_{T+1}(x, w) = -\alpha^\lambda cx + \alpha^\lambda pw^-. \tag{2.11}$$

From the dynamic program (2.8)-(2.11), it is clear to see that the optimal inventory decision is determined by a functional equation $V_t(x, w)$, which is defined in (2.9)-(2.11). We defined this as transformed dynamic program. Intuitively, the penalty cost incurred by the current cash level w'_t, the accounts payable vector $(P_{t-\lambda}, ..., P_{t-1})$ and accounts receivable vector $(P_{t-\lambda}, ..., P_{t-1})$ can be viewed as a sunk cost, which does not affect the inventory decision. This the sum of the default payment costs shown in (2.8).

Examining the single-period cost function in (2.9) in the transformed dynamic program, it is clear that we charge the payment default penalty cost and the inventory purchase cost occurred in period $t + \lambda - 1$ to the current period t. It is this cost shift scheme that makes us define a new state variable as working capital. To see this for the default penalty term, notice that

$$p'(w'_t - \mathbf{P}_t e_\lambda + \mathbf{R}_t e_\lambda - P_t)^- = p'(cx_t + w'_t - \mathbf{P}_t e_\lambda + \mathbf{R}_t e_\lambda - cx_t - cz_t)^- = p(y_t - w_t)^+.$$

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In a classical single-stage inventory problem, Karlin and Scarf (1958) introduce the notion of inventory position that transforms the original multi-state dynamic program into a single-state problem and proves that a base-stock policy is optimal. Here, we derive a similar result for the inventory model with two-level trade credit contracts. More specifically, by introducing the notation of working capital w, we show that the optimal inventory decision can be determined by the transformed dynamic program in (2.9)-(2.11), referred to as the transformed λ-model. Nevertheless, this model is more complicated than the classical inventory problem as it has two state variables: working capital w and inventory level x. The complexity comes from the fact that the inventory decision y in the single-period cost function G_t depends on both w and x. In the next subsection we proceed to show how to derive the optimal policy by further decoupling the states.

2.4.2   The Optimal Policy

It is difficult to characterize the exact optimal policy for the dynamic problem in (2.9)-(2.11). Nonetheless, we shall show that a myopic policy is optimal when demand is non-decreasing. This myopic policy has a simple structure that can reveal insights and be implemented easily. As we shall see, the myopic policy remains very effective for the general demand case.

We first explain the myopic policy, which includes two control parameters (d, S) in each period. The policy is operated as follows: the firm monitors its inventory level x and working capital w at the beginning of each period. If $w \leqslant d$, the firm orders inventory up to d; if $d < w \leqslant S$, the firm uses up all cash and orders inventory up to w; if $w > S$, the firm orders inventory up to S. Denoting y^* as the resulting optimal base-stock level, the (d, S) policy can be mathematically states as

$$y^*(w) = (d \vee w) \wedge S. \tag{2.12}$$

We next illustrate how these optimal control parameters are obtained. For fixed

w, the unconstrained myopic minimization problem at period t can be written as

$$v_t(w) = \min_y \left\{ g_t(y) + \alpha^\lambda p(y-w)^+ \right\}, \tag{2.13}$$

where $g_t(y) = H_t(y) + \alpha^\lambda(1-\alpha)cy$. Here, $H_t(y) + \alpha^\lambda cy$ is the single-period purchase cost and inventory related cost; $-\alpha^{\lambda+1}cy$ represents the fact that the myopic system allows returns with a refund of c for any unsold unit.

Let S_t be the optimal base-stock level without considering the default penalty cost, i.e.,

$$S_t = \arg\min_y \left\{ g_t(y) \right\}. \tag{2.14}$$

We term S_t the *default-free base-stock level*. We define the *default threshold* as follows:

$$d_t = \sup \left\{ y : \frac{\partial}{\partial y} g_t(y) \leqslant -\alpha^\lambda p \right\}. \tag{2.15}$$

To solve the problem in (2.13), we consider three cases.

Case 1. When $w \leqslant d_t$, the system's working capital is lower than the default threshold d_t. In this case, the firm has an incentive to order up to d_t as the marginal backorder cost outweighs the marginal holding and default penalty cost. Thus, we have $v_t(w) = L_t(w) = -\alpha^\lambda p(w - d_t) + g_t(d_t)$.

Case 2. When $d_t < w \leqslant S_t$, the system is working capital constrained. Now it is optimal to order up to w as ordering either less or more will lead to a higher cost than $g_t(w)$. Thus, $v_t(w) = g_t(w)$.

Case 3. When $S_t < w$, the system has ample working capital and orders up to the target base stock S_t. In this case, there is extra cash left after ordering, and $v_t(w) = g_t(S_t)$.

We summarize the above three cases into the following proposition.

Proposition 1. *The (d, S) policy in (2.12) is optimal for the myopic problem in (2.13).*

As a result, equation (2.13) becomes

$$v_t(w) = \begin{cases} L_t(w), & \text{if } w \leqslant d_t \\ g_t(w), & \text{if } d_t < w \leqslant S_t \\ g_t(S_t), & \text{if } S_t < w \end{cases}, \qquad (2.16)$$

and the critical ratios of the control parameters can be found in the following lemma.

Lemma 1. *The control parameters of the myopic policy in (2.12) satisfy*

$$F_t(d_t) = \frac{b - \alpha^\lambda p - \alpha^\lambda (1-\alpha)c}{h+b}, \quad F_t(S_t) = \frac{b - \alpha^\lambda (1-\alpha)c}{h+b}. \qquad (2.17)$$

Here, the condition for $d_t > -\infty$ is $p \leqslant \alpha^{-\lambda} b - (1-\alpha)c$. When $\alpha = 1$, this condition becomes $p \leqslant b$, i.e., the firm will not default as long as the late payment penalty is greater than the backorder cost. As we shall see in the next section, the above statement will be generalized.

Figure 2.3(a) depicts functions $g(\cdot)$, $L(\cdot)$ and $v(\cdot)$ while solving the myopic minimization problem. The default threshold is obtained as the tangent point of curve $g(\cdot)$ and a line with slope $-\alpha^\lambda p$. Function $v(\cdot)$ is shown as the bold convex curve connected by three different functions (from the left to the right): the linear function $L(\cdot)$, the convex function $g(\cdot)$, and the horizontal line.

FIGURE 2.3: The optimal solution of the transformed λ-model

(a) The myopic minimization and the optimal control parameters

(b) The optimal (d, S) policy
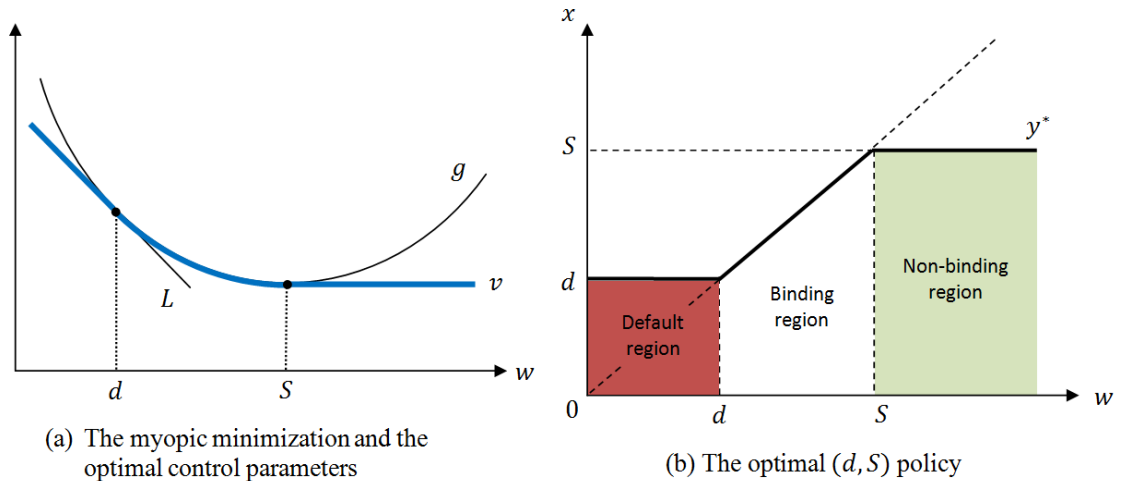
Next, we show the optimality of the myopic policy. We find it convenient to define the following region, commonly referred to as the "band" at time t:

$$\mathcal{B}_t = \left\{ (x_t, w_t) \in \Re^2 \mid x_t \leqslant y_t^*(w_t) \right\}. \tag{2.18}$$

This band establishes the region where inventory does not exceed the base-stock level. Figure 2.3(b) depicts the piecewise linear function $y^*(w)$. By definition, band $\mathcal{B}$ covers the area below $y^*(w)$ on the x-w plain. The following proposition shows the optimality results through state decomposition.

Proposition 2. *If D_t is stochastically increasing in t, then we have:*

(a) *The control parameters d_t and S_t are non-decreasing in t and $d_t \leqslant S_t$ for all t;*

(b) *$V_t(x, w) = -\alpha^\lambda cx + W_t(w)$ for all t and $(x, w) \in \mathcal{B}_t$, where*

$$W_t(w) = v_t(w) + \alpha \mathsf{E} W_{t+1}(w + \theta D_t),$$

and $W_{T+1}(w) = \alpha^\lambda p w^-$; $W_t(w)$ is convex in w;

(c) *The (d, S) policy is optimal for the transformed λ-model.*

Proposition 2(a) is a direct result of the assumption $D_t \leqslant_{st} D_{t+1}$. Proposition 2(c) shows the optimality of the (d, S) policy. As illustrated in Figure 2.3(b), the band is divided into three sub-regions. When $w \leqslant d$, the firm falls into the default region where the optimal order policy will lead to negative cash and late payment; when $d \leqslant w < S$, the firm will hold zero cash after ordering, i.e., cash working capital constraint is binding; when $S \leqslant w$, the firm has sufficient cash and orders up to the default-free base-stock. Consequently, the working captial constraint is non-binding. To formally characterize the firm's order strategy under default risk, we define the *optimal default quantity* as $u^*(w) = y^*(w) - w$. Figure 2.3(b) implies that $u^*(w)$ is decreasing in w, in other words, the firm will default less if there is more working capital. This optimal behavior is consistent with the empirical findings that the operational decisions of smaller firms are more aggressive and thus induce higher default risks.

2.5 Longer Payment Period $(m > n)$

When the payment period is longer than the collection period, i.e., $m > n = \lambda$, the firm has complete cash flow information up to λ periods, and it is exposed to uncertain cash inflows from period $t + \lambda$ to $t + m$. As we shall see, this uncertainty leads to the complication for the analysis.

Recall the base model in Equations (2.6) and (2.7). We conduct a similar analysis as in §2.4 to derive the transformed dynamic program. Keeping the same notation as before without confusion, we define the working capital level measured in inventory units as

$$w = x + (w' - \mathbf{P}e_m + \mathbf{R}e_\lambda)/c.$$

In addition, we define the aggregated demand as $D_t^m = D_t + ... + D_{t+m-1}$ $(D_t^0 = 0)$. Let F_t^m, f_t^m, μ_t^m, and $(\sigma_t^m)^2$ be the c.d.f., the p.d.f., mean, and variance of the random variable of D_t^m, respectively. Moreover, denote $\bar{F}^m$ and $\hat{F}^m$ as the complementary cumulative distribution function (c.c.d.f.) and the loss function of random variable D^m. That is, $\hat{F}^m(x) = \int_x^\infty \bar{F}^m(y)dy$.

After some algebra, the transformed dynamic program can be shown as

$$V_t(x, w) = \min_{x \leqslant y} \{G_t(x, w, y) + \alpha \mathsf{E} V_{t+1}(y - D_t, w + \theta D_t)\}, \tag{2.19}$$

where the single-period cost function is

$$G_t(x, w, y) = H_t(y) + \alpha^m \mathsf{E} p(y - w - \rho D_{t+\lambda}^{m-\lambda})^+ + \alpha^m c(y - x), \tag{2.20}$$

the expectation is taken over $D_{t+\lambda}^{m-\lambda}$. The terminal cost function is modified to

$$V_{T+1}(x, w) = -\alpha^m cx + \alpha^m \mathsf{E} p(w + \rho D_{T+1+\lambda}^{m-\lambda})^-. \tag{2.21}$$

Notice that λ is the number of periods of the known cash flow and will not affect the policy structure. Thus, for ease of exposition, we shall omit λ and reformulate the model with $\lambda = 0$. Let $u = y - w$, then the default penalty cost can be rewritten as

$$M_t(u) = \mathsf{E}_{D_t^m} p(u - \rho D_t^m)^+. \tag{2.22}$$

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And the dynamic program becomes

$$V_t(x, w) = \min_{x \leqslant y} \left\{ H_t(y) + \alpha^m M_t(y - w) + \alpha^m c(y - x) + \alpha \mathsf{E} V_{t+1}(y - D_t, w + \theta D_t) \right\},$$

(2.23)

$$V_{T+1}(x, w) = -\alpha^m cx + \alpha^m \mathsf{E}_{D_{T+1}^m} p(w + \rho D_{T+1}^m)^-.$$

(2.24)

We refer to (2.23) and (2.24) as the transformed m-model. Unlike the λ-model, it is difficult to characterize the exact optimal policy because the default penalty cost function M_t is a general convex function (instead of a two-piece linear function in the λ-model), so an optimal policy, if existed, would be a general state-dependent policy. Below we provide simple heuristics and cost lower bounds based on linear approximations.

2.5.1 Linear Approximation

We propose two types of piecewise linear functions to approximate the convex function M. As we shall see, each of the linear functions will lead to a lower bound and a heuristic for the m-model. Here and in the sequel, we suppress the time subscript without confusion.

Two-piece linear approximation

The first piece-wise linear approximation is generated by replacing the random variable D_t^m with the mean value μ^m in the M_t function. More specifically, define

$$M^-(u) = p(u - \rho \mu^m)^+.$$

(2.25)

We have the following relationship between function M^- and M.

Lemma 2. *For all u, $M^-(u) \leqslant M(u)$ holds. Moreover, $\lim_{u \to \infty}(M(u) - M^-(u)) = 0$.*

Lemma 2 shows that the two-piece linear function M^- is a lower bound of the convex function M, and both functions have asymptotic slope p. See Figure 2.4(a). In fact, M becomes M^- if aggregated demand D^m is deterministic.

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Three-piece linear approximation

The above two-piece linear approximation only characterizes the first moment of random variable D^m. Here, we further develop an approximation based on a three-piece linear function. This approximation, while more complicated to generate, takes into account the demand variability.

We demonstrate how to construct the three-piece approximation. We generate a linear function Γ by constructing a tangent line to the convex curve M at point $(\rho\mu^m, M(\rho\mu^m))$; See Figure 2.4(a). Let $\bar{p}$ be the slope of Γ. It can be shown that $\bar{p} = pF^m(\mu^m)$ and $M(\rho\mu^m) = p\rho\hat{F}^m(\mu^m)$. Thus,

$$\Gamma(u) = \bar{p}(u - \rho\mu^m) + M(\rho\mu^m)$$

$$= pF^m(\mu^m)(u - \rho\mu^m) + p\rho\hat{F}^m(\mu^m).$$

We term $\bar{p}$ the *expected default penalty cost*, which is the marginal cost rate when $u = \rho\mu^m$.

Let $(\rho A', 0)$ be the intersection point of Γ and the u-axis, and $(\rho A'', M^-(\rho A''))$ the intersection point of Γ and M^-. Now, define the three-piece linear function

$$\bar{M}(u) = \max\left\{M^-(u), \Gamma(u)\right\} = \left\{\begin{array}{ll} 0, & \text{if } u \leqslant \rho A' \\ \Gamma(u), & \text{if } \rho A' < u \leqslant \rho A'' \\ M^-(u), & \text{if } \rho A'' < u \end{array}\right\}, \qquad (2.26)$$

where A' and A'' can be shown as

$$A' = \mu^m - \frac{\hat{F}^m(\mu^m)}{F^m(\mu^m)}, \qquad A'' = \mu^m + \frac{\hat{F}^m(\mu^m)}{\bar{F}^m(\mu^m)}. \qquad (2.27)$$

To see this linear approximation takes into account the variability of the aggregated demand, notice that $a' = \rho\mu^m - \rho A'$ and $a'' = \rho A'' - \rho\mu^m$, i.e.,

$$a' = \frac{\rho\hat{F}^m(\mu^m)}{F^m(\mu^m)}, \quad a'' = \frac{\rho\hat{F}^m(\mu^m)}{\bar{F}^m(\mu^m)}. \qquad (2.28)$$

Furthermore, for most unimodal distribution functions[5], we can show that

$$\hat{F}^m(\mu^m) = (\sigma^m)^2 f(\mu^m). \qquad (2.29)$$

---

[5] The demand functions tested include, but not limited to, Poisson, Geometric, Negative-Binomial, Exponential, Gamma, and Normal, etc.

Thus, when the aggregated demand is more variable, a' and a'' will be bigger (or equivalently, A' will be smaller and A'' will be bigger). Figure 2.4(a) depicts A', A'', a', a'', two-piece linear function M^-, and three-piece linear function $\bar{M}$. The following proposition formally establishes the lower bound systems.

Lemma 3. *By replacing M_t with M_t^- (or $\bar{M}_t$) in (2.23), the optimal cost of the resulting model forms a lower bound to the optimal cost of the transformed m-model.*

We refer to the resulting model with M_t^- ($\bar{M}_t$) as the two-piece (three-piece) lower bound. Clearly, the two-piece lower bound becomes the exact system if D^m is deterministic. The following lemma shows the same result for the three-piece lower bound if D^m follows a two-point distribution.

Lemma 4. *If demand $\bar{D}$ follows a two-point distribution with probability mass $\mathbf{Pr}\{\bar{D} = \rho A'\} = \bar{p}/p$ and $\mathbf{Pr}\{\bar{D} = \rho A''\} = 1 - \bar{p}/p$, then the three-piece linear function becomes the exact cost function, i.e., $M(u) = \mathsf{E}_{\bar{D}}p(u - \rho\bar{D})^+ = \bar{M}(u)$.*

FIGURE 2.4: Linear approximations and optimal control policies

(a) Piecewise linear approximations   (b) The optimal (d, S) and $(\mathbf{d}, \mathbf{a}, S)$ policies
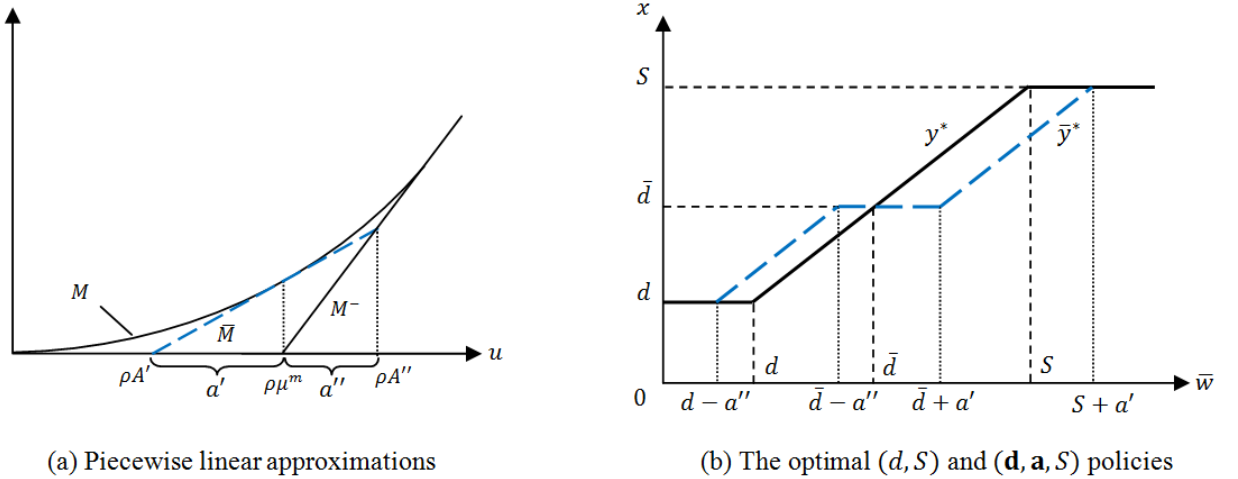
2.5.2   Lower Bound Solutions

To establish the lower bound solutions, it is opportune to define

$$\bar{w} = w + \rho\mathsf{E}_{D^m}(\mathbf{R}e_m) = w + \rho\mu^m.$$

We refer to $\bar{w}$ as the *expected working capital level,* which takes into account the expected A/R during the m periods.

First, we derive the optimal solution to the two-piece lower bound. By replacing M_t with M_t^- and substituting $\bar{w}$, the default penalty becomes $\alpha^m p(y - \bar{w})^+$, which shares the same structure as in (2.10). Therefore, under the same assumptions as in Proposition 2, the (d, S) policy is optimal for the two-piece lower bound. The solid function in Figure 2.4(b) depicts the optimal base-stock level of this policy. Equivalently,

$$y^*(\bar{w}) = (d \vee \bar{w}) \wedge S. \tag{2.30}$$

Next, we develop the optimal policy of the three-piece lower bound. By replacing M_t with $\bar{M}_t$ and w with $\bar{w}$, the transformed m-model in (2.23) and (2.24) becomes

$$\bar{V}_t(x, \bar{w}) = \min_{x \leqslant y} \left\{ \begin{array}{l} H_t(y) + \alpha^m \bar{M}_t(y - \bar{w} + \rho\mu_t^m) + \alpha^m c(y - x) \\ + \alpha\mathsf{E}\bar{V}_{t+1}(y - D_t, \bar{w} + \theta D_t + \rho\mu_{t+m} - \rho\mu_t) \end{array} \right\}, \tag{2.31}$$

$$\bar{V}_{T+1}(x, \bar{w}) = -\alpha^m cx + \alpha^m \mathsf{E}p(\bar{w} + \rho D_{T+1}^m - \rho\mu_{T+1}^m)^-. \tag{2.32}$$

We first state the myopic policy. Let $\mathbf{d} = (d, \bar{d})$ and $\mathbf{a} = (a', a'')$, then the optimal policy consists of five control parameters $(\mathbf{d}, \mathbf{a}, S)$. The firm implements a base-stock policy with the optimal base-stock level dependent on the expected working capital $\bar{w}$ (in inventory units). More specifically, let $\bar{y}^*(\bar{w})$ be the optimal base-stock, then,

$$\bar{y}^*(\bar{w}) = \left\{ \begin{array}{ll} d, & \text{if } \bar{w} \leqslant d - a'' \\ \bar{w} + a'', & \text{if } d - a'' < \bar{w} \leqslant \bar{d} - a'' \\ \bar{d}, & \text{if } \bar{d} - a'' < \bar{w} \leqslant \bar{d} + a' \\ \bar{w} - a', & \text{if } \bar{d} + a' < \bar{w} \leqslant S + a' \\ S, & \text{if } S + a' < \bar{w} \end{array} \right\}. \tag{2.33}$$

We next illustrate how these optimal control parameters are obtained. For fixed $\bar{w}$, the unconstrained myopic minimization problem at period t can be written as

$$\bar{v}_t(\bar{w}) = \min_y \left\{ \bar{g}_t(y) + \alpha^m \bar{M}_t(y - \bar{w} + \rho\mu_t^m) \right\}, \tag{2.34}$$

where $\bar{g}_t(y) = H_t(y) + \alpha^m(1 - \alpha)cy.$

The optimal control parameters a' and a'' are derived in (2.28); The default-free base-stock S_t and default threshold d_t can be derived from (2.14) and (2.15) by replacing $g_t(\cdot)$ with $\bar{g}_t(\cdot)$ and λ with m; We refer to $\bar{d}_t$ as the *expected default threshold*, which can be obtained from

$$\bar{d}_t = \sup\left\{ y : \frac{\partial}{\partial y}\bar{g}_t(y) \leqslant -\alpha^m \bar{p}_t \right\}. \tag{2.35}$$

Let us define $a_t = a'_t + a''_t$. To solve the problem in (2.34), we consider five cases.

Case 1. When $\bar{w} \leqslant d_t - a''_t$, the system's expected working capital is lower than $d_t - a''_t$. Now it is optimal to order up to threshold d_t, and $\bar{v}_t(\bar{w}) = \bar{L}_t(\bar{w} + a_t) + \alpha^m \bar{p}_t a_t$, where $\bar{L}_t(\bar{w}) = -\alpha^m p(\bar{w} - d_t - a'_t) + \bar{g}_t(d_t + a'_t)$.

Case 2. When if $d_t - a''_t < \bar{w} \leqslant \bar{d}_t - a''_t$, the system's expected working capital is lower than $\bar{d}_t - a''_t$. Now it is optimal to default by a''_t in expectation, and $\bar{v}_t(\bar{w}) = \bar{g}_t(\bar{w} + a_t) + \alpha^m \bar{p}_t a_t$.

Case 3. When $\bar{d}_t - a''_t < \bar{w} \leqslant \bar{d}_t + a'_t$, the system's expected working capital is lower than $\bar{d}_t + a'_t$. Now it is optimal to order up to threshold $\bar{d}_t$, and $\bar{v}_t(\bar{w}) = \bar{\bar{L}}_t(\bar{w}) = -\alpha^m \bar{p}_t(\bar{w} - \bar{d}_t - a'_t) + \bar{g}_t(d_t + a'_t)$.

Case 4. When $\bar{d}_t + a'_t < \bar{w} \leqslant S_t + a'_t$, the system is working capital constrained. Now it is optimal to order up to $\bar{w} - a'_t$ and leave no cash on hand in expectation. Thus, $\bar{v}_t(\bar{w}) = \bar{g}_t(\bar{w} - a'_t)$.

Case 5. When $S_t + a'_t < \bar{w}$, the system has ample working capital and orders up to the target base stock S_t. In this case, the expected cash balance will be nonnegative, and $\bar{v}_t(\bar{w}) = \bar{g}_t(S_t)$.

We summarize the above five cases into the following proposition.

Proposition 3. *The $(\mathbf{d}, \mathbf{a}, S)$ policy in (2.30) is optimal for the myopic problem in (2.34).*

As a result, equation (2.34) becomes

$$\bar{v}_t(\bar{w}) = \begin{cases} \bar{L}_t(\bar{w} + a_t) + \alpha^m \bar{p}_t a_t, & \text{if } \bar{w} \leqslant d_t - a_t'' \\ \bar{g}_t(\bar{w} + a_t) + \alpha^m \bar{p}_t a_t, & \text{if } d_t - a_t'' < \bar{w} \leqslant \bar{d}_t - a_t'' \\ \bar{\bar{L}}_t(\bar{w}), & \text{if } \bar{d}_t - a_t'' < \bar{w} \leqslant \bar{d}_t + a_t' \\ \bar{g}_t(\bar{w} - a_t'), & \text{if } \bar{d}_t + a_t' < \bar{w} \leqslant S_t + a_t' \\ \bar{g}_t(S_t), & \text{if } S_t + a_t' < \bar{w} \end{cases}, \tag{2.36}$$

and the critical ratio of the control parameter $\bar{d}_t$ can be found in the following lemma.

Lemma 5. *The expected default threshold $\bar{d}_t$ satisfies*

$$F_t(\bar{d}_t) = \frac{b - \alpha^m p F_t^m(\mu_t^m) - \alpha^m (1 - \alpha)c}{h + b}. \tag{2.37}$$

Figure 2.5(a) depicts functions $\bar{g}(\cdot)$, $\bar{L}(\cdot)$, $\bar{\bar{L}}(\cdot)$ and $\bar{v}(\cdot)$ while solving the myopic problem in (2.34). The slopes of the linear functions $\bar{L}(\cdot)$ and $\bar{\bar{L}}(\cdot)$ are $-\alpha^m p$ and $-\alpha^m \bar{p}$, respectively. Function $\bar{v}(\cdot)$ is shown as the bold convex curve connected by five different functions (from the left to the right): the shifted linear function $\bar{L}(\cdot)$, the shifted convex function $\bar{g}(\cdot)$, linear function $\bar{\bar{L}}(\cdot)$, convex function $\bar{g}(\cdot)$, and the horizontal line.

Now, we show the optimality of the myopic policy. Similar to §2.4.2, we define the "band" as
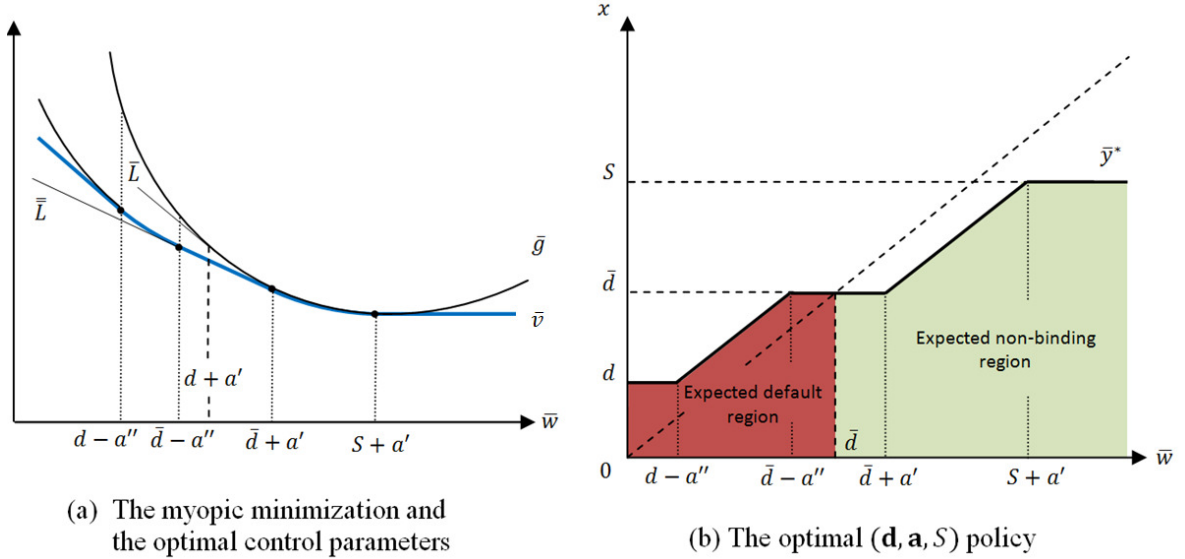
$$\bar{\mathcal{B}}_t = \left\{ (x_t, \bar{w}_t) \in \Re^2 \mid x_t \leqslant \bar{y}_t^*(\bar{w}_t) \right\}.$$

Figure 2.5(b) depicts the optimal base-stock $\bar{y}^*$ and band $\bar{\mathcal{B}}$, which is the area below $\bar{y}^*$.

We next explain how to derive the optimal policy. First, it is convenient to define $A_t = F_t^m(\mu_t^m)$ as a measure of asymmetry of demand D_t^m. In addition, recall the definitions of A' and A'' in (2.27). In analogy to §2.4.2, the following proposition shows the optimality through decoupling.

Proposition 4. *Assume that (1) D_t is stochastically increasing; (2) A_t is decreasing in t; (3) both A_t' and A_t'' are increasing in t. Then we have:*

FIGURE 2.5: The optimal solution of the three-piece lower bound

(a) The myopic minimization and
the optimal control parameters

(b) The optimal $(\mathbf{d}, \mathbf{a}, S)$ policy

(a) The control parameters d_t, $\bar{d}_t$ and S_t are non-decreasing in t and $d_t \leqslant \bar{d}_t \leqslant S_t$ for all t;

(b) $\bar{V}_t(x, \bar{w}) = -\alpha^m cx + \bar{W}_t(\bar{w})$ for all t and $(x, \bar{w}) \in \bar{\mathcal{B}}_t$, where

$$\bar{W}_t(\bar{w}) = \bar{v}_t(\bar{w}) + \alpha \mathsf{E}\bar{W}_{t+1}(\bar{w} + \theta D_t + \rho\mu_{t+m} - \rho\mu_t),$$

and $\bar{W}_{T+1}(\bar{w}) = \alpha^m \mathsf{E}p(\bar{w} + \rho D^m_{T+1} - \rho\mu^m_{T+1})^-$; $\bar{W}_t(\bar{w})$ is convex in $\bar{w}$;

(c) The $(\mathbf{d}, \mathbf{a}, S)$ policy is optimal for the three-piece lower bound of the transformed m-model.

Assumption (1) in Proposition 4 is similar to that in Proposition 2. Assumption (2) requires, typically but not necessarily, that the aggregated demand D^m_t is less right-skewed when t gets larger. Note that most of the real life demand functions, such as Poisson(λ) and Gamma($k, 1$), are right-skewed and become more symmetric under larger mean values (λ and k), hence satisfying Assumption (2). For zero-skewed (or symmetric) distributions, such as Normal, the following lemma guarantees Assumption (2) and (3). Moreover, most asymmetric demand distributions (Poisson, Gamma, etc.) can be shown or tested to satisfy Assumption (3).

Lemma 6. *If A_t is constant over t, then both A'_t and A''_t are increasing in t.*

Proposition 4(c) shows the optimality of the $(\mathbf{d}, \mathbf{a}, S)$ policy. As illustrated in Figure 2.5(b), the band is divided into two sub-regions by the expected default threshold $\bar{d}$: if the expected working capital level $\bar{w} < \bar{d}$, the firm will order more than its expected working capital (i.e., $\bar{y}^*(\bar{w}) \geqslant \bar{w}$). We call this *over-order* region. In the over-order region, the firm takes advantage of the cash flow volatility and order more aggressively. On the other hand, if $\bar{w} > \bar{d}$, the firm will order less than its expected working capital level and hold hold extra cash on expectation. We call this *under-order* region. in the under-order region, the firm tries to avoid the cash flow risks by ordering more conservatively. The over-order (under-order) deviation amount depends on a'' (a'), which is proportional to the variance of the aggregated demand under the same mean value. Notice that the binding region in the (d, S) policy reduces to a single point in the $(\mathbf{d}, \mathbf{a}, S)$ policy where $\bar{w} = \bar{d}$.

Similarly as in §2.4.2, we define the *expected optimal default quantity* as $\bar{u}^*(\bar{w}) = \bar{y}^*(\bar{w}) - \bar{w}$. The optimal $(\mathbf{d}, \mathbf{a}, S)$ remains the property that $\bar{u}^*(\bar{w})$ is increasing with $\bar{w}$, implying that lower (higher) working level leads to more aggressive (conservative) inventory ordering decisions. This is consistent with the (d, S) policy. However, the optimal base-stock in the $(\mathbf{d}, \mathbf{a}, S)$ policy deviates from that of the (d, S) policy in different directions, due to the volatility of the stochastic cash inflow ρD^m.

2.5.3 Heuristics

We develop two heuristic policies for the transformed m-model, basing on the two-piece and three-piece lower bound systems. We refer to it as the (d, S) and $(\mathbf{d}, \mathbf{a}, S)$ heuristic, respectively. The control parameters can be obtained from (2.28), Lemma 1 and 5. There are three steps to implement the heuristic policy: first, observe w and compute $\bar{w}$; second, derive the optimal base-stock y^* from (2.30) for the (d, S) policy, and $\bar{y}^*$ from (2.33) for the $(\mathbf{d}, \mathbf{a}, S)$ policy; third, order inventory up to the base-stock, or as close as possible.

To this end, we shall expect that the three-piece heuristic works better than the two-piece heuristic, although the latter involves less control parameters, and thus is easier to implement. The performance gap between these two heuristic policies gets bigger when aggregated demand is more volatile. In practice, the two-piece heuristic could serve as a simple substitute for the three-piece policy if the demand is less variable.

2.6   Longer Collection Period $(m < n)$

When the collection period is longer than the payment period, i.e., $\lambda = m < n$, the firm has complete cash flow information up to λ periods plus known cash inflow information from period $t + \lambda$ to $t + n$. Unlike §2.4 and §2.5, here we define the working capital level (excluding the extra receivables beyond $t + \lambda$) at the beginning of period t as

$$w = x + (w' - \mathbf{P}e_\lambda + \mathbf{R}^\lambda e_\lambda)/c,$$

where $\mathbf{R}_t^\lambda = (R_{t-n}, ..., R_{t-n+\lambda-1})$. Define the extra known accounts receivable as $\mathbf{R}_t^{-\lambda} = (R_{t-n+\lambda}, ..., R_{t-1})$.

With these definitions and a similar analysis as in the previous sections, the inventory decision for the base model in (2.6)-(2.7) can be determined by the following dynamic program:

$$V_t(x, w, \mathbf{R}_t^{-\lambda}) = \min_{x \leqslant y} \left\{ G_t(x, w, y) + \alpha \mathsf{E} V_{t+1}(y - D_t, w + R_{t-n+\lambda}/c - D_t, \mathbf{R}_{t+1}^{-\lambda}) \right\},$$
$$(2.38)$$

$$V_{T+1}(x, w, \mathbf{R}_{T+1}^{-\lambda}) = -\alpha^\lambda cx + \alpha^\lambda p w^-,\qquad (2.39)$$

where the single-period cost function $G_t(x, w, y)$ is the same as in (2.10), and the dynamic of $\mathbf{R}_t^{-\lambda}$ is $\mathbf{R}_{t+1}^{-\lambda} = (R_{t-n+1}, ..., R_{t-n+\lambda-1}, rD_t)$.

We refer to (2.38) and (2.39) as the transformed n-model. Denoting band $\mathcal{B}_t$ the same as in (2.18), we show the optimal policy in the following proposition.

Proposition 5. *If D_t is stochastically increasing, then we have:*

(a) The control parameters d_t and S_t are non-decreasing in t and $d_t \leqslant S_t$ for all t;

(b) $V_t(x, w, \mathbf{R}_t^{-\lambda})) = -\alpha^\lambda cx + W_t(w, \mathbf{R}_t^{-\lambda}))$ for all t and $(x, w) \in \mathcal{B}_t$, where W_t is joint convex;

(c) The (d, S) policy is optimal for the transformed n-model.

Proposition 5 suggests that the extra cash flow information $\mathbf{R}_t^{-\lambda}$ does not affect the inventory decisions. Therefore, the optimal policy is the same as that for the transformed λ-model.

2.7 Numerical Study

§2.7.1 examines the effectiveness of the heuristics. §2.7.2 discusses the impact of payment periods on the system's total cost. §2.7.3 measures the bullwhip effect driven by the customer payment default.

2.7.1 Effectiveness of the Heuristics

The Heuristics for the Model with a Longer Payment Period

Here we test the performance of the (d, S) and $(\mathbf{d}, \mathbf{a}, S)$ heuristics in §2.5.3. Both heuristics are compared to the three-piece lower bound of the transformed m-model developed in §2.5.1. We conduct an individual test for $m = 1, 2, 3$, and summarize the overall performances. In each test, let $\bar{C}_2$ be the cost of the (d, S) heuristic based on the two-piece linear approximation, and $\bar{C}_3$ be the cost of the $(\mathbf{d}, \mathbf{a}, S)$ heuristic based on the three-piece linear approximation. Furthermore, let C_3 be the cost of the three-piece lower bound, then the percentage errors are defined as

$$\% \text{ error-2} = \frac{\bar{C}_2 - C_3}{C_3} \times 100\%, \quad \% \text{ error-3} = \frac{\bar{C}_3 - C_3}{C_3} \times 100\%.$$

We consider two demand forms. For the i.i.d. demand case, D_t is Normal distributed with mean $\mu_t = 10$ for all t; for the increasing demand case, D_t is Normal

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distributed with the first period mean $\mu_1 = 10$ and μ_t increasing at a rate of 10% per period. We fix the working capital requirement policy with r taking two values: $r = (1, 1.1)$. Moreover, the initial working capital level can also vary between two values: $w_1 = (20 - 10m, 10 - 10m)$, i.e., the leftover A/P from the last planning horizon is considered at period 1. In addition, we fix parameters $c = 1$, $\alpha = 1$, and vary the other parameters with each taking two values: $h = (0.05, 0.2)$, $b = (1, 4)$, $p = (0.5, 2)$. The total number of instances generated for each heuristic is 384. The average (maximum, minimum) performance error for the (d, S) heuristic is 2.12% (21.23%, 0.00%), and that for the $(\mathbf{d}, \mathbf{a}, S)$ is 1.46% (13.21%, 0.00%). As expected, the $(\mathbf{d}, \mathbf{a}, S)$ heuristic performs well in general, and the performance difference between these two heuristic policies is higher when demand becomes more variable.

Negative Demand Shocks

Recall from §2.4.2 and §2.6 that the (d, S) policy is shown to be optimal for the transformed λ-model and the transformed n-model, respectively. In addition, §2.5.2 shows that the $(\mathbf{d}, \mathbf{a}, S)$ policy is optimal for the three-piece lower bound of the transformed m-model. All these results were proved under the assumption of stochastically increasing demand. With the existence of negative demand shocks, the above optimal policies become heuristics. In what follows we test the effectiveness of these heuristics under different non-stationary demand forms with negative shocks.

We conduct an individual test for each of the three models mentioned above and summarize the overall heuristic performance. In each test, let C_U be the cost of the heuristic which serves as an upper bound cost for the underlined model. We compare C_U with a lower bound cost C_L obtained by relaxing the constraint $x \leqslant y$ in each period, i.e., the feasibility of the optimal base-stock level is guaranteed by allowing inventory return at the purchasing cost. To evaluate the heuristic, we define the percentage error as

$$\% \text{ error} = \frac{C_U - C_L}{C_L} \times 100\%.$$

Our numerical study starts with a test bed which has the time horizon of 12 periods. We fix parameters $\lambda = 0$, $m = 1$, $n = 1$. In addition, we consider two demand forms with negative shocks: seasonal demand and product life cycle demand. In each demand form, D_t is normally distributed with mean μ_t shown in Table 2.2. In our text bed, the demand coefficient of variation (c.v.) can vary by taking two values: c.v.= $(0.15, 0.3)$. We also consider two working capital requirement policies. For the fixed policy, we set $r_t = 1$ for all t; for the responsive policy, r_t is non-stationary with values set according to Table 2.2. Finally, we assume the initial on-hand inventory $x_1 = 10$ and vary the initial working capital level between two values: $w_1 = (10, 40)$. The values of other parameters remain the same as in the test bed of "the heuristics for the model with a longer payment period". In total, we generate 384 instances. The combination of these parameters covers a wide range of different system characteristics.

Table 2.2: Demand mean and responsive working capital requirement

Period (t)	1	2	3	4	5	6	7	8	9	10	11	12
μ_t - seasonal demand	10	12	20	60	20	12	10	12	20	60	20	12
r_t - seasonal demand	1.2	1.2	1.5	0.8	0.8	1.0	1.2	1.2	1.5	0.8	0.8	1.0
μ_t - life cycle demand	10	12	14	18	22	38	50	56	60	52	36	8
r_t - life cycle demand	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.8	0.8	0.8	0.8

The average (maximum, minimum) performance error for the test bed instances is 0.36% (4.63%, 0.00%). When negative shocks exist in the demand sequence, the underlined heuristics perform well in general. Nevertheless, the heuristics perform less effectively when both the c.v. and the backorder cost are large. To see this, recall from Proposition 2 that the myopic policy is optimal under the condition that the states will stay within the band if they are already in the band. When demand is expected to drop, the optimal base-stock level will decrease accordingly. Therefore, it is probable that a small demand realization will cause the states traverse outside the

band, making the heuristic less effective. This is more likely when demand is more variable and there is a bigger decrease in base-stock levels. The latter is mainly due to a higher backorder cost.

2.7.2 Impact of Payment Periods

Although the firm may benefit from a longer purchases payment period, the supplier will usually quote a higher unit wholesale price to compensate for the postponed cash inflow. This price increase can be regarded as the implicit cost of trade credit for the firm. To fully understand this tradeoff, the firm needs to analyze the cost saving from extending the payment period. In this subsection we conduct a numerical study to illustrate the impact of payment periods on the firm's total cost, thus providing a decision support tool in the trade credit contract negotiation.
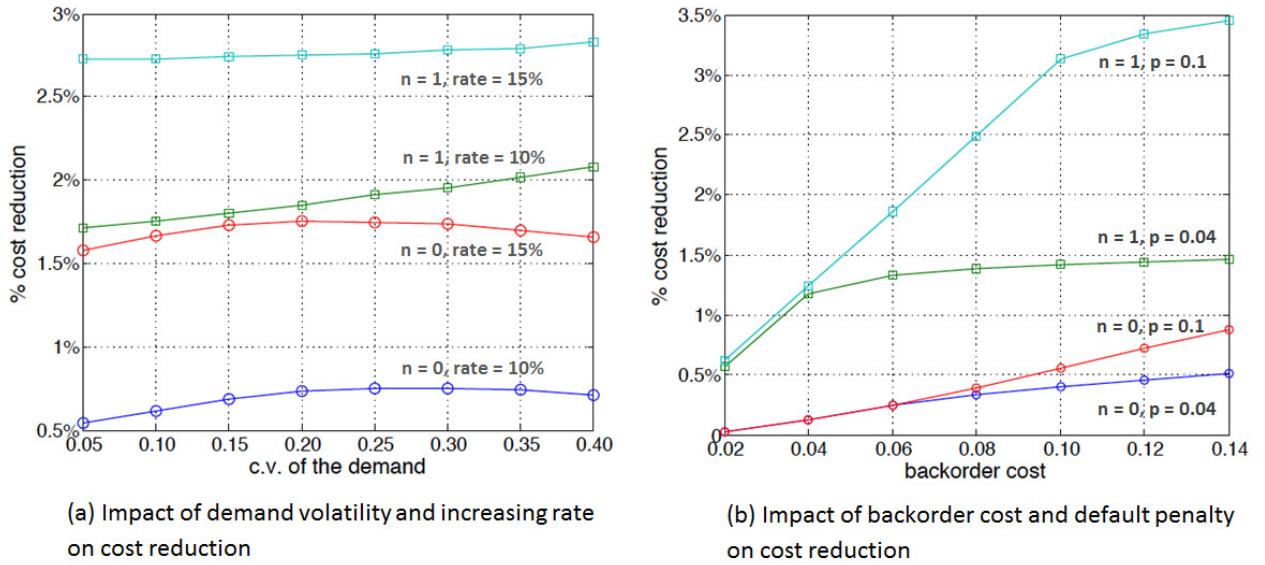
In our numerical study, we compute the percentage cost reduction achieved by the model with $m = 1$ over that with $m = 0$. To obtain the true value of this cost reduction, we keep the unit purchasing cost c unchanged. The underlined model and policy differ with respect to the length of the collection period. When $n = 0$, we use the $(\mathbf{d}, \mathbf{a}, S)$ heuristic of the transformed m-model to compute the cost with $m = 1$, and the (d, S) policy of the transformed λ-model ($\lambda = 0$) for the cost with $m = 0$; when $n = 1$, we use the (d, S) policy of the transformed n-model for the cost with $m = 0$, and the same policy of the transformed λ-model ($\lambda = 1$) for the cost with $m = 1$.

We consider a planning horizon of 12 periods with the initial states $x_1 = 10$ and $w_1 = 20$.[6] Demand is Normal distributed with mean in the first period $\mu_1 = 10$ and μ_t is increasing at a constant rate. In addition, we set $r = 1$, $\alpha = 1$, and $h = 0.05$. We study the impact of payment delays on cost reduction under different system parameters and summarize the results in Figure 2.6. In particular, Figure 2.6(a) plots the percentage cost reduction curve with respect to the c.v. of the demand, and

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[6] Similar to the last section, in the case of $m = 1$, w_1 needs to be adjusted to consider the leftover A/P from the last planning horizon.

for all combinations of $n = (0, 1)$ and *demand increasing rate* $= (10\%, 15\%)$ while fixing $b = 0.2$ and $p = 0.05$; Figure 2.6(b) plots the same curve with respect to b, and for all combinations of $n = (0, 1)$ and $p = (0.04, 0.1)$ while fixing c.v. $= 0.2$ and the demand increasing rate at 10%.

FIGURE 2.6: Impact of system parameters on the cost reduction through payment period extension

(a) Impact of demand volatility and increasing rate on cost reduction

(b) Impact of backorder cost and default penalty on cost reduction

As shown in both figures, the cost savings of extending m from 0 to 1 are higher when $n = 1$ than $n = 0$. This suggests that, all else being equal, firms with more negative cash conversion cycle have more incentive to extend credit periods with suppliers. Based on this we shall expect that firms' upstream and downstream credit periods are positively correlated, which is consistent with the empirical findings in Guedes and Mateus (2009). We provide two reasons for this. First, the firm with a longer collection period is in a worse place in generating cash flows to catch up with the increasing demand. Therefore, the marginal benefit of extending payment period is larger. Second, when moving from balanced credit periods to imbalanced ones, the firm is forced to commit to order decisions with uncertain cash inflows during the payment period. Hence, the best it could do is to follow the $(\mathbf{d}, \mathbf{a}, S)$ heuristic, which compromises the marginal benefit.

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Figure 2.6 also shows that the cost reduction increases with the backorder cost, the default penalty, and the demand increasing rate. This is because the higher these parameters are, the more cost firms will incur by not being able to catch up with the increasing demand due to negative cash conversion cycles. Therefore, there is more incentive to enhance the cash flow by extending the payment period. Interestingly, the cost reduction is not monotone in demand volatility when $n = 0$, as shown in Figure 2.6(a). To see this, note that a higher c.v. will lead to higher control thresholds, and possibly to a higher base-stock level, which will increase the potential benefit of a longer payment period. However, the demand volatility also makes it harder to make inventory order decision when $m > n$ due to the uncertainty of future cash inflows. When c.v. is large enough, the latter effect will dominate the former one, making it less attractive to extend the payment period.

2.7.3 Bullwhip Effect

Bullwhip effect is a phenomenon that the order variability amplifies when moving along the supply chain from downstream to upstream (Lee et al., 1997). In this stream of literature it has been shown that imposing finite capacity to the system does not cause bullwhip effect. Two different types of capacities are considered: for the shelf capacity, order variability equals to the demand variability[7]; for the production/order capacity, order variability is less than the demand variability, i.e., the smoothing effect (Chen and Lee, 2012). In our original model, the on-hand cash can be regarded as a random and endogenous capacity. After the transformation, the working capital level becomes an exogenous shelf capacity. When customers do not default on trade credit, we should not be able to observe bullwhip effect under stationary demand and fixed working capital requirement $r = 1$. However, when there is a customer payment default, the working capital level becomes more variable, which may amplify the order variability and cause the bullwhip effect.

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[7] In this case, the order sequence will replicate the demand sequence.

To further understand this driver of bullwhip effect, we extend our model to incorporate customer payment defaults. More specifically, we model the *doubtful receivable* of age $t - n$ as follows

$$\bar{R}_{t-n} = (\delta_t R_{t-n} + \epsilon_t)^+, \tag{2.40}$$

where R_{t-n} is the expected full payment according to (2.4), $\delta_t \in [0, 1]$ is the estimated proportion of collectible amount, and ϵ_t is a random variable representing the payment default uncertainty. We assume ϵ_t is independent from period to period with mean $\bar{\mu}_t$ and variance $\bar{\sigma}_t^2$. To measure the bullwhip effect, we conduct a simulation study with $m = n = 0$ over 36 periods. The demand is Normal distributed with stationary mean $\mu = 10$ and standard deviation $\sigma = 2$. In addition, we assume that the default noise ϵ_t also follows an i.i.d Normal distribution with $\bar{\mu} = 0$ and $\bar{\sigma}$ varying from 0 to 4. Finally, we set $\delta = 1$, $r = 1$, $\alpha = 1$, $x_1 = w_1 = 10$, $b = 0.4$ and vary p to take values at 0, 0.2, and 0.4. This set of system parameters allow us to study the impact of customer default volatility and default penalty cost on the bullwhip ratio, which is defined as the ratio between order variance and demand variance.

FIGURE 2.7: Customer payment default and bullwhip effect

(a) Working capital evolvement with $p = 0.2$ and $\bar{\sigma} = 1$

(b) Working capital evolvement with $p = 0.4$ and $\bar{\sigma} = 2$

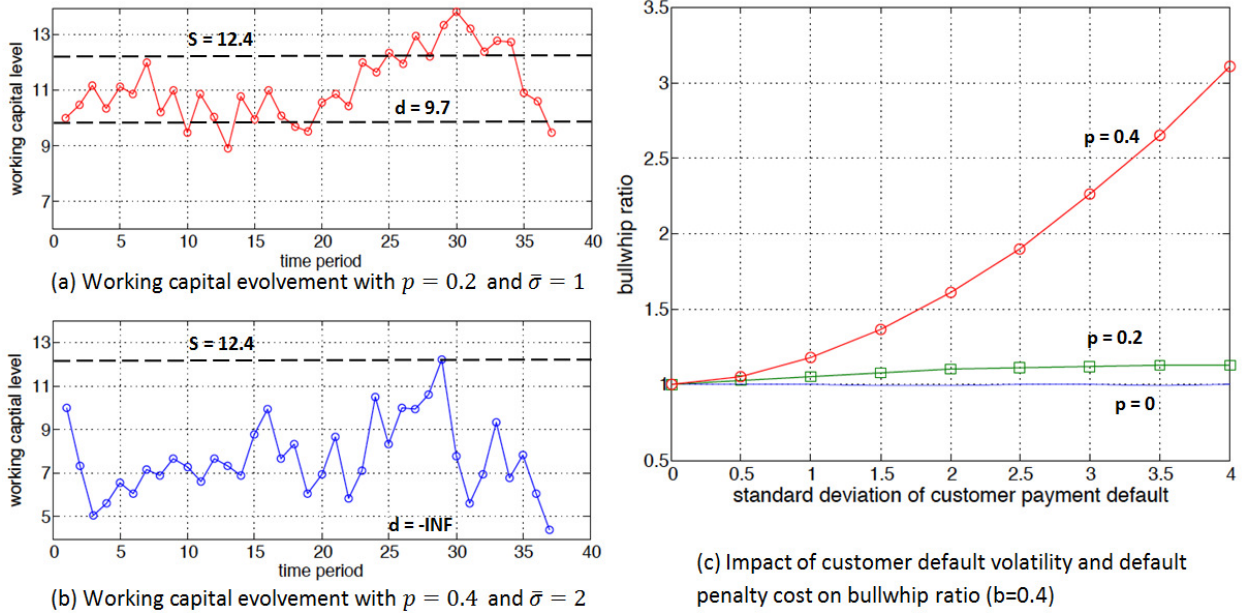(c) Impact of customer default volatility and default penalty cost on bullwhip ratio (b=0.4)

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Figure 2.7(a) and (b) plot the sample path of working capital level under the parameter combination $(p, \bar{\sigma}) = (0.2, 1)$ and $(0.4, 2)$, respectively. Following the (d, S) policy, the optimal base-stock level in period t equals to w_t if $d < w_t \leqslant S$. As w_t varies due to customer payment defaults, the order sequence will no longer replicate the demand sequence. Rather, the order quantity will become more variable, making the bullwhip ratio larger than 1. When w_t travels below d (above S), the optimal base-stock is fixed at d (S). And consequently, the order variance will reduce to the demand variance. Therefore, the area between d and S is where the customer payment defaults can effectively drive the bullwhip effect. As shown in 2.7(c), the bullwhip ratio increases in both the customer default volatility $\bar{\sigma}$ and the default penalty cost p. As p approaches to b, d approaches to negative infinity, in which case the bullwhip effect is the most significant.

As illustrated in Figure 2.7(c), a smaller default penalty cost p leads to a lower bullwhip ratio. When p decreases, the firm will default more, which implies that the supplier could effectively mitigate the bullwhip effect through liquidity provision. In this sense, we suggest another rationale for suppliers to provide trade credit when banks do not.

2.8 Conclusion

This chapter studies the impact of trade credit on a firm's inventory decision. The firm provides and receives one-part trade credit contracts. We introduce a notion of working capital that simplifies the computation and characterizes the optimal and near-optimal policies. This result naturally connects operations and accounting. Our analysis reveals insights on the relationship between the payment period and the resulting procurement cost and on how supplier's liquidity provision can mitigate the bullwhip effect.

<div align="right">**3**</div>

Joint Inventory and Cash Management

This chapter develops a centralized supply chain model that aims to assess the value of cash pooling. The supply chain is owned by a single corporation with two divisions, where the downstream division (headquarter), facing random customer demand, replenishes materials from the upstream one. The downstream division receives cash payments from customers and determines a system-wide inventory replenishment and cash retention policy. We consider two cash management systems that represent different levels of cash concentration. For cash pooling, the supply chain adopts a financial services platform which allows the headquarter to create a corporate master account that aggregates the divisions' cash. For transfer pricing, on the other hand, each division owns its cash and pays for the ordered material according to a fixed price. Comparing both systems yields the value of adopting such financial services. We prove that the optimal policy for the cash pooling model has a surprisingly simple structure – both divisions implement a base-stock policy for material control; the headquarter monitors the corporate working capital and implements a two-threshold policy for cash retention. Solving the transfer pricing model is more involved. We derive a lower bound on the optimal cost by connecting the model to an assembly system. Our results show that the value of cash pooling can be very significant when

demand is increasing (stationary) and the markup for the upstream division is small (high). Nevertheless, a big portion of the pooling benefit may be recovered if the headquarter can decide the optimal transfer price and the lead time is short.

3.1 Introduction

The fundamental objective of supply chain management is to efficiently coordinate material, information, and financial flows so as to reduce mismatches between demand and supply. When financial markets are efficient, i.e., external funding is plentiful and relatively inexpensive, the financial decisions may be decoupled from the logistics decisions (Modigliani and Miller, 1958). In such case, a downstream party pays material it orders from an upstream one so financial flow becomes an output of logistics decisions. This perspective may explain why the supply chain literature has largely focused on the integration of material and information flows. Nonetheless, with the recent global financial crisis limiting the availability of external funding, many multinational, multi-divisional corporations in their "hunt for cash" have witnessed a significant increase on their intragroup financial transactions (Rogers et al., 2009). The reason is simple: these multinationals realize that they can concentrate the intragroup liquidity for centralized planning to receive most benefit. For example, Hewlett-Packard and General Electric transfer funds from their overseas divisions to their domestic ones by considering the benefit of the entire company group. (Linebaugh, 2013). One of the common practices of cash concentration is cash pooling (Polak and Klusacek, 2010). Under cash pooling, the headquarter creates a corporate master account that aggregates division's cash on a daily basis (Jansen, 2011). While the value of cash pooling has been studied in the finance literature, there is little study that assesses the value from a supply chain perspective. Indeed, the discussion of integrating financial flows into supply chain models is relatively sparse in the supply chain literature. The objective of this paper is to fill this gap.

We consider a corporation that owns a supply chain consisting of two divisions

(can be generalized to N divisions). The downstream division, division 1, replenishes inventory from an upstream division, division 2, which further replenishes from an outside ample vendor. There are positive delivery lead times for both divisions. Division 1 is the headquarter that faces stochastic customer demand. The demands are independent between periods but not necessarily identical. Division 1 receives cash payment from customers who order the material. Similarly, division 2 pays to the outside vendor for the material shipped to the division. To make the consideration of financial flow relevant, we assume that there is no external borrowing. The decision is centralized and the headquarter has to decide a system-wide inventory replenishment and cash retention policy. The cash retention policy refers to how much cash the supply chain should hold for operations, i.e., inventory payments in our context. Typically, firms do not wish to hold excess cash as it loses the potential benefit from external investments; on the other hand, liquidating invested assets into cash for operations incurs transaction costs or may not be feasible (Baumol, 1952; Miller and Orr, 1966). Thus, the headquarter has to find a balance of amount of cash between retained for internal operations and invested for external assets.

We consider two cash management systems that represent different levels of cash concentration. For the *cash pooling* system, the supply chain adopts a financial services platform, so the entire supply chain is operated under a single account for conducting financial transactions with customers and the outside vendor. For the *transfer pricing* system, no such platform is installed so each division maintains its own cash and division 1 pays exactly what it orders to division 2 according to a fixed internal transfer price, i.e., the price that a selling division charges for a product or service supplied to a buying division of the same corporation (Abdallah, 1989). We assume that the transfer price is pre-determined according to a market price (Martini, 2011). There are linear holding and backorder costs related to the inventory. In addition, there is an opportunity cost for holding cash, which represents the opportunity cost of holding cash for internal operations. The objective is to find a joint

inventory replenishment and cash retention policy such that the total supply chain cost is minimized within a finite horizon under each of the cash management systems.

The logistics system of the considered supply chain is a seminal model proposed by Clark and Scarf (1960). We incorporate cash flows into this classical model. The transfer pricing model represents a traditional supply chain in the sense that cash payment is driven by the inventory decision (constrained by the available cash). Thus, cash may not be efficiently distributed, leading to a less effective inventory and cash retention policy. For example, cash shortage of the upstream division will affect its normal operations, which, in turn, affects the material supply to the downstream one. This inefficiency can be mitigated under cash pooling because the headquarter can consolidate the cash within the supply chain for a better usage. In practice, there are physical pooling and notional (virtual) pooling (i.e., funds are not physically transferred but managed as if they were in a single account). In any case, cash pooling usually involves financial and legal services provided by a third party and requires installing a costly system-wide technology platform, such as treasury management system, for transferring funds from divisions to the headquarter (Camerinelli, 2010). Thus, our study of comparing these two systems can be used to justify the value of adopting such financial services.

We first formulate a dynamic program for the cash pooling model which includes two inventory states and one cash state that represents the corporate master account. To be consistent with the inventory literature, we name division and stage interchangeably. The problem is difficult to solve as one cannot directly prove a structured joint optimal policy. Nevertheless, by redefining the state variables into *echelon* terms, we can transform the original two-stage system into a three-echelon system, under which the optimal joint policy can be characterized. The optimal policy is surprisingly simple. The inventory policy has the same structure as that for the traditional multi-echelon system (cf. Clark and Scarf, 1960): each stage reviews the echelon inventory position at the beginning of a period and orders up to a target

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echelon base-stock level. For the cash retention policy, stage 1 (or the headquarter) reviews the entire *system working capital* (= inventory on hand at both stages + inventory in transit - backorders at stage 1 + inventory-equivalent total system cash) at the beginning of each period and retain the cash holding within an interval. A technical contribution for this model is that we simplify the computation by decoupling the original dynamic program with three states into three separate dynamic programs, each with one state variable. Thus, the optimal policy parameters can be easily found. The decoupling result is based on a set of penalty cost functions, some appearing to be new in the literature. We also provide economic meanings for these penalty functions.

Solving the transfer pricing model is more involved. Simply speaking, the problem is similar to a serial capacitated system (cf. Parker and Kapuscinski, 2004) in the sense that the on-hand cash level at each stage can be viewed as a budgetary constraint that restricts the amount of inventory ordering. However, the major difference between the traditional capacitated system and ours is that the cash constraint is *endogenously* determined by the inventory and cash retention decisions. Although we are not able to characterize the optimal policy, we provide a lower bound to the optimal cost by connecting the transfer pricing model to an assembly system (cf. Rosling, 1989) with two component flows – one is stage 1's cash flow and the other is the system's material flow.

We obtain several insights from the above analysis. First, the optimal policy of the cash pooling model suggests that the inventory decision can be made separately from the cash retention decision; however, making the cash retention decision has to take into account the entire supply chain inventory. That is, monitoring system working capital level is key to ensuring the system efficiency. In most firms, cash payment is managed by a treasurer in the accounting department, and replenishment decision is made by an inventory manager in the operations department. A implication of our finding is that a close inter-departmental collaboration is crucial. Second, comparing

the optimal cost of the cash pooling model and the lower bound cost of the transfer pricing model renders the (conservative) value of cash pooling, or equivalently, installing the financial services platform. Our numerical result suggests that the value of cash pooling can be very significant when the profit margin of the upstream stage is low and the demand is increasing or when the profit margin is high and the demand tends to be stationary. For the former case, the upstream division tends to have cash shortage in the transfer pricing model. Lacking cash at the upstream stage restricts the order quantity, which affects the material supply for the downstream stage. On the other hand, for the latter case, there is excess cash accumulated in the upstream division. Pooling cash together will facilitate the headquarter to invest the excess cash to external opportunities. Third, we compare the optimal cost of the cash pooling model to that obtained from the Clark-Scarf algorithm. We find that ignoring the impact of financial flow can be significantly suboptimal.

The above comparison leads to an interesting question: If the headquarter can determine the internal transfer price for the divisions, how much benefit can be recovered by employing the optimal transfer price? Determining transfer prices is one of the most controversial topics for multi-divisional firms in the finance literature. When an inventory manager attempts to determine the optimal flows of products among divisions, the price of a product is almost always considered a given parameter, as the setting in the transfer pricing model. However, this is not the case in real multi-divisional firms since the transfer price is inherently subjective and the headquarter can determine it with some degree of flexibility through *advance pricing agreements* (Vidal and Goetschalckx, 2001; Lakhal, 2006; Perron et al., 2010; Martini, 2011)[1]. Thus, one can treat transfer pricing as a tool of re-distributing cash between

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[1] Our focus is to determine the system-wide optimal transfer price. Certainly, an optimal transfer price may not be aligned with each division's best interest, so there is a separate issue regarding how to implement the optimal transfer prices for the divisions. However, if implementing the optimal transfer price decreases the supply chain's cost, the headquarter can capture this benefit and design a incentive compatible compensation, such as side payment, that induces the division to accept the price.

divisions (Stewart, 1977). We call the system with optimized internal transfer price as the *optimal pricing* system. The optimal transfer price can be determined if we obtain the optimal order quantity and the *optimal cash payment* between these two divisions in each period (i.e., transfer price is equal to cash payment divided by order quantity)[2]. In other words, we need to obtain an optimal joint inventory replenishment, cash payment, and cash retention policy for the supply chain. Interestingly, we find that this joint optimal policy can be obtained by extending the solution approach for the cash pooling model. More specifically, the inventory replenishment and cash retention policy structure remains the same as those in the cash pooling system; for the payment policy between divisions, division 2 monitors its echelon working capital level and receives the cash payment up to a target level.

The optimal pricing model allows us to gain additional insights. While the cash pooling system certainly dominates the optimal pricing one, we find that the supply chain can recover a big portion of the cash pooling benefit by optimizing the transfer price. The benefit of re-distributing liquidity through the optimal transfer price can be clearly demonstrated in a product life cycle example in §3.5: during the introduction and growth stages, the upstream division is normally short of cash so cash subsidy from the downstream stage is valuable. On the other hand, during the mature and decline stage, division 2 has accumulated sufficient fund for the decreased demand, so a reduction of cash payment from division 1 is beneficial. Finally, we investigate material and cash bullwhip effects (i.e., shipment and payment variability) under the optimal pricing and transfer pricing systems. We find that the variability of cash payment from division 1 to the division 2 is larger (smaller) than that from the division 2 to the outside vendor under the optimal pricing (transfer pricing) system. On the other hand, the variability of inventory shipment grows when moving upstream under both schemes. Thus, the material and financial bullwhip effects may

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[2] In our context, the optimal cash payment between two divisions can broadly include intragroup loans or financial subsidies.

not amplify in the same direction in a fully integrated supply chain. Comparing the material bullwhip between these systems, we find that the material bullwhip effect is less significant under the optimal pricing system. This implies that an effective transfer price policy can be a useful tool to mitigate the material bullwhip effect.

3.2 Literature Review

Our work is related to four streams of research in the literature: cash management, multi-echelon inventory models, capacitated inventory models, and inventory model with financial issues.

For cash management in single firms, most papers treat cash as inventory and use inventory control tools to find the optimal cash balance for firms. Baumol (1952) studied the optimal cash level for a firm that uses cash either for paying transactions or for investment. We have a similar setup for the headquarter in our model. This line of research was further extended by Tobin (1956) and Miller and Orr (1966). For dynamic, periodic-review cash balance problems, Girgis (1968) modeled the selection of a cash level in anticipation of future net expenses as a single-product, multi-period inventory system. Heyman (1973) presented a model to minimize the average cash balance subject to a constraint on the probability of stock-out. The difference between these studies and ours is that we specifically model the cash and inventory dynamics as two inter-related flows.

For cash management in multi-divisional corporations, our model is related to resource allocation from a centralized planning perspective. This literature can be categorized into two groups. The first group is related to cash pooling. Eijie and Westerman (2002) suggested that the reduction of financial imperfections in transferring cash in the euro zone diminishes the need for separate local cash holdings and facilitates the cash concentration and headquarter's financial control. We refer the reader to Jansen (2011) for a detailed discussion on cash pooling concepts and practices surveyed in fifteen countries. The second group concerns obtaining transfer

prices to maximize the profit for a multi-divisional corporation, e.g., Merville and Petty (1978), Vidal and Goetschalckx (2001), Gjerdrum et al. (2002), Lakhal (2006), Villegas and Ouenniche (2008), and Perron et al. (2010). Our model is different from these papers in that we consider a supply chain setting in a finite horizon and characterize the optimal control policy. There is another stream of research regarding how to design transfer prices to coordinate *decentralized* divisions (i.e., each division has high autonomy and is treated as a profit center). Since the focus is different from our paper, we refer the interested reader to Ronen and McKinney (1970) and Yeom et al. (2000).

Our research is also related to the multi-echelon literature. In particular, our model incorporates cash flows into the seminal supply chain model developed by Clark and Scarf (1960), who proved that an echelon base-stock policy is optimal. Furthermore, they showed that the problem can be decoupled into a series of one-dimensional dynamic programs by introducing the notion of echelon inventories. Federgruen and Zipkin (1984) and Chen and Zheng (1994) streamlined the analysis by considering an infinite horizon model. Recently, Angelus (2011) considered a multi-echelon model which allows each stage to dispose excess inventory to a secondary market. He introduced a class of heuristic policies, called disposal saturation policies, which can be obtained by using the Clark-Scarf decomposition.

The capacitated inventory problem is related to our model since the cash constraint on inventory replenishment can be viewed as the supply capacity. For single-stage systems, Federgruen and Zipkin (1986) showed that the modified base-stock policy is optimal. Angelus and Porteus (2002) derived the optimal joint capacity adjustment and production plan with and without carryover of unsold inventory units. Their capacity adjustment decision is similar to our cash investment decision, but our cash capacity is also affected by payment decisions and random sales. For serial systems, Parker and Kapuscinski (2004) demonstrated that a modified echelon base-stock policy is optimal in a two-stage system where there is a smaller capacity at the

downstream facility. Glasserman and Tayur (1995) and Huh et al. (2010) studied the stability issue of the system. The main difference between the serial capacitated models and ours is that the cash constraint is endogenously determined by the inventory and cash decisions.

Finally, there have been several recent studies to incorporate financial decisions or budget constraints into inventory models. Most of these papers are based on single-stage systems. Buzacott and Zhang (2004) incorporated asset-based financing into production decisions. They demonstrated the importance of joint consideration of production and financing decisions to capital constrained firms. Li et al. (2013) studied a dynamic model in which inventory and financial decisions are made simultaneously in the presence of uncertain demand. The objective is to maximize the expected present value of dividends. The authors proved that a myopic policy is optimal. Ding et al. (2007) studied an integrated operational and financial hedging decision faced by a global firm which sells to both home and foreign markets. Chao et al. (2008) considered a self-financing retailer who replenishes inventory under a cash budget constraint. They characterized the optimal inventory control policy. Gupta and Wang (2009) presented a discrete-time inventory model with trade credit and showed that the problem can be converted into a single-stage system model with refined holding cost rates. Babich (2010) studied a manufacturer's joint inventory and financial subsidy decisions when facing a supplier whose financial state is governed by a firm-value model. He showed that an order-up-to policy and subsidize-up-to policy are optimal for the manufacturer. Yang and Birge (2011) modeled a Stackelberg game between a retailer and a supplier with the use of a trade credit contract. They demonstrated that an effective trade credit contract can enhance supply chain efficiency. Bendavid et al. (2012) analyzed the material management practices of a self-financing firm under working capital requirement. Tanrisever et al. (2012) built a two-period model to study a start-up firm's trade-off between process investment and survival. For multi-echelon models, Hu and Sobel (2007) studied a

serial inventory model with the objective of optimizing the expected present value of dividends. They showed that there is no optimal echelon base-stock policy if there are financial constraints. Protopappa-Sieke and Seifert (2010) conducted a simulation study on a two-stage supply chain to reveal qualitative insights on the allocation of working capital between the supply chain partners. Chou et al. (2013) studied a one-warehouse-multi-retailer system with trade credits.

The rest of this paper is organized as follows. §3.3 studies the cash pooling model and formulates the corresponding dynamic program. §3.4 focuses on the transfer pricing model. We provide lower bounds to the optimal cost. §3.5 discusses the qualitative insights through a numerical study. §3.6 concludes. Appendix provides proofs. Throughout this paper, we define $x^+ = \max(x, 0)$, $x^- = -\min(x, 0)$, $a \vee b = \max(a, b)$, and $a \wedge b = \min(a, b)$.

3.3 Cash Pooling System

We consider a periodic-review, two-stage serial supply chain where stage 1 orders from stage 2, which orders from an outside ample vendor. The supply chain is owned by a single corporation, with stage 1 being the headquarter and stage 2 the subsidiary. Stage 1 faces a stochastic customer demand D_t in period t. The demands are independent between periods, but the demand distributions may differ from period to period. We assume that unsatisfied demand is fully backlogged, and the material lead time is one period for both stages (without loss of generality).

This section focuses on the cash pooling (CP) system, in which the headquarter (stage 1) creates a corporate master account that aggregates the divisions' cash and pays for the outside vendor. Here and in the sequel, we use prime to indicate local (stage specific) variables and parameters. To model the opportunity cost of holding cash, let r_f be the risk-free rate of investment, and R the overall return rate for the headquarter's investment activities, where $R \geqslant r_f$. Assuming risk neutrality, the opportunity cost of holding cash can be expressed as $\eta' = R - r_f$. We refer to η'

as *cash holding cost rate.* Moreover, we assume the headquarter can liquidate its assets to assist inventory payment, if necessary. Nevertheless, how much cash can flow into the pooled account depends on an exogenous market condition described by a limit $K'(\geqslant 0)$ in each period[3]. Let β'_i and β'_o denote the unit transaction cost charged on the cash transferred to and from the pooled cash account, respectively. In practice, these transaction costs can be regarded as brokerage fees. Here, β'_i, β'_o and K' represent the market friction in this model. Figure 3.1 shows the material and cash flows in solid and dashed arrows, respectively. The circle in Figure 3.1 represents the investment portfolio; the top white rectangle represents the pooled cash balance, or the *operating account.*

FIGURE 3.1: The two-stage cash pooling model with material and cash flows
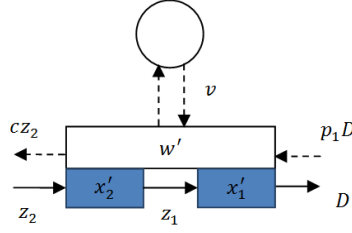
We now introduce the other cost parameters. Following the inventory literature, we charge a linear local holding cost h'_i for each unit of inventory held at stage i in each period, and a backorder cost b for each unit of backorder incurred at stage 1 in each period. Here, we assume that $h'_1 > h'_2 > \eta' c$, i.e., holding a unit of inventory at downstream is more costly than that at upstream, and holding an unit of inventory is more costly than holding the same value amount of cash. The later is generally true since inventory holding cost consists of both the financial opportunity cost and the physical shelf cost.

The inventory replenishment and cash retention decision is made centrally by the headquarter. The sequence of events in a period as follows: At the beginning of

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[3] The sequence of K'_t can be generalized to a Markov chain that captures the stochastic liquidity level according to the market condition.

the period, (1) shipments are received at both stages; (2) payment is made to the outside vendor; (3) cash retention decision is made; (4) orders are placed at both stages. During the period, demand is realized and sales revenue is collected. At the end of the period, all inventory and cash related costs are calculated. The planning horizon is T periods, and the objective is to minimize the supply chain's total expected discounted cost within the entire horizon.

We now define state and decision variables. For stage $i = 1, 2$ and period t, let

$x_{1,t}' =$ net inventory level at stage 1 after Event (1);

$x_{2,t}' =$ on-hand inventory level at stage 2 after Event (1);

$w_t' =$ cash balance in the pooled account after Event (2);

$v_t =$ amount of cash transferred into the pooled account in Event (3);

$z_{i,t} =$ order quantity for stage i made in Event (4);

Note that v_t^+ is the cash amount that flows into the pooled account and v_t^- is the cash amount that flows out for investment. Clearly, v_t cannot exceed K'. Let p_1 be the unit selling price to the end customer and c be the unit procurement cost from the outside vendor. We assume $c < p_1$ to ensure profitability. The system dynamics are shown below:

$$x_{1,t+1}' = x_{1,t}' + z_{1,t} - D_t, \tag{3.1}$$

$$x_{2,t+1}' = x_{2,t}' + z_{2,t} - z_{1,t}, \tag{3.2}$$

$$w_{t+1}' = w_t' + v_t - cz_{2,t} + p_1 D_t. \tag{3.3}$$

We assume that the actual payment transaction occurs upon the receipt of shipments. That is, the outside vendor will not receive the payment determined in period t until period $t + 1$, when stage 2 receives the shipment (placed in period t). This payment practice is similar to a Letter of Credit (LC). In other words, we can view that there is a one-period lead time for the cash payment.

For the cash dynamic in (3.3), we assume that the customer will pay at the order

epoch. This assumption is reasonable as all demand will filled under the backorder model. It is also commonly seen in practice (such as iPhone) and in the dynamic pricing literature, e.g., Federgruen and Heching (1999). We do not include inventory holding and backorder costs in (3.3) because inventory holding cost is usually not incurred in the periodic cash transactions, and backorder cost usually represents loss of goodwill, which is a non-monetary cost.

Define $\mathbf{x}' = (x_1', x_2')$, and $\mathbf{z} = (z_1, z_2)$. The constraint set in each period is

$$\hat{S}(x_2', w') = \left\{ \mathbf{z}, v \mid 0 \leqslant z_1 \leqslant x_2', \, 0 \leqslant z_2 \leqslant (w' + v)/c, \, v \leqslant K' \right\}.$$

The first constraint states that stage 1's order quantity cannot exceed stage 2's on-hand inventory; the second constraint states that stage 2's order quantity is constrained by the cash balance in the pooled account, which also implies that the investment amount in each period cannot exceed its on-hand cash level, i.e., $v \geqslant -w'$. Finally, the last constraint imposes a limit K' on the amount of cash that can be injected into the pooled cash account.

The single-period expected cost function is

$$\hat{G}_t(\mathbf{x}', w', z_2, v) = \mathsf{E}_{D_t} \left[h_1'(x_1' - D_t)^+ + b(x_1' - D_t)^- \right] + h_2' x_2' + cz_2$$

$$+ \eta' \mathsf{E}_{D_t} \left(w' + v + p_1 D_t \right) + \beta_i' v^+ + \beta_o' v^-. \tag{3.4}$$

The first line in the cost function is the inventory-related cost, which includes inventory holding, backlogging and procurement costs. By convention, we charge h_2' to the pipeline inventory so $h_2' x_2'$ is the cost for the inventories held at stage 2 plus those in the pipeline. The second line is the cash-related cost, which includes cash holding and transaction costs. As shown, we charge η' for $w' + v + p_1 D_t$ because the inventory payment to the outside vendor is held until the receipt of goods.

Let α be the single-period discount rate. Denote $\hat{J}_t(\mathbf{x}', w', \mathbf{z}, v)$ as the expected cost over period t to $T + 1$, given states and decisions $(\mathbf{x}', w', \mathbf{z}, v)$. Denote $\hat{V}_t(\mathbf{x}', w')$ as the minimum expected cost over period t to $T + 1$ over all feasible decisions. The

dynamic program is

$$\hat{J}_t(\mathbf{x}', w', \mathbf{z}, v) = \hat{G}_t(\mathbf{x}', w', z_2, v)$$

$$+ \alpha \mathsf{E}_{D_t} \left[\hat{V}_{t+1}(x_1' + z_1 - D_t, \, x_2' + z_2 - z_1, \, w' + v - cz_2 + p_1 D_t) \right], \quad (3.5)$$

$$\hat{V}_t(\mathbf{x}', w') = \min_{\mathbf{z}, v \in \hat{S}(x_2', w')} \hat{J}_t(\mathbf{x}', w', \mathbf{z}, v), \quad (3.6)$$

with $\hat{V}_{T+1}(\mathbf{x}', w') = 0$. Here we assume a zero terminating cost for simplicity. In the sequel, we omit the terminating cost from the dynamics program if it equals to zero.

The local formulation in (3.5) and (3.6) is difficult to solve. Specifically, one can show the joint convexity of $\hat{J}_t(\cdot)$ and derive a state-dependent global minimum solution. However, computing the solution is quite hard due to the curse of dimensionality. In the next section, we transform the original problem into a new system, from which the exact optimal joint policy can be shown to have a surprisingly simple structure.

3.3.1 Echelon Formulation

We transform the original two-stage system into a three-stage serial model by introducing new system variables. First, define the following *echelon* variables:

$$x_1 = x_1', \quad x_2 = x_1' + x_2', \quad w = x_1' + x_2' + w'/c.$$

Let $\mathbf{x} = (x_1, x_2)$. We refer to $\mathbf{x}$ as the *echelon net inventory level*, and w as the *net working capital level* measured in inventory unit, which is obtained by converting cash to inventory at the value of c. This state transformation explicitly treats cash as inventory. More specifically, the financial flow in the system can be seen as an extension of the material flow after "flipping" the corporate master account to upstream. We define the corresponding echelon decision variables:

$$y_1 = x_1' + z_1, \quad y_2 = x_1' + x_2' + z_2, \quad r = x_1' + x_2' + (w' + v)/c.$$

Let $\mathbf{y} = (y_1, y_2)$. Figure 3.2 shows the transformed CP system. With this transformation, the cash account becomes stage 3 in the new system, directly supplying stage 2. We hereby call echelon 3 (with state variable w) as the system working capital.

FIGURE 3.2: The three-stage transformed cash pooling system
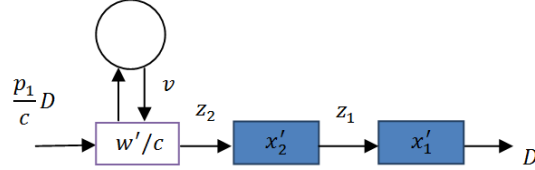
Similar to the multi-echelon inventory model, we derive the echelon holding cost rate as follows: $\eta = \eta' c$, $h_2 = h_2' - \eta' c$, and $h_1 = h_1' - h_2'$. Since $h_1' > h_2' > \eta' c$ by assumption, we have $h_1 > 0$ and $h_2 > 0$. Furthermore, let $\beta_i = \beta_i' c$, $\beta_o = \beta_o' c$, $\theta = p_1/c - 1 > 0$, and $K = K'/c$. With these echelon terms, the state dynamics in (3.1)-(3.3) become

$$x_{1,t+1} = y_{1,t} - D_t, \quad x_{2,t+1} = y_{2,t} - D_t, \quad w_{t+1} = r_t + \theta D_t,$$

and the constraint set becomes

$$S(\mathbf{x}, w) = \{\mathbf{y}, r \,|\, x_1 \leqslant y_1 \leqslant x_2 \leqslant y_2 \leqslant r \leqslant w + K\}.$$

We further specify the holding and backorder cost associated with each echelon:

$$H_{1,t}(x_1) = \mathsf{E}_{D_t}\big[(h_1 + h_2 + \eta + b)(D_t - x_1)^+ + h_1(x_1 - D_t)\big],$$

$$H_{2,t}(x_2) = \mathsf{E}_{D_t} h_2(x_2 - D_t),$$

$$H_{3,t}(r) = \mathsf{E}_{D_t} \eta(r + \theta D_t).$$

Then, we can rewrite the dynamic program in (3.5) and (3.6) as follows:

$$J_t(\mathbf{x}, w, \mathbf{y}, r) = G_t(\mathbf{x}, w, y_2, r) + \alpha \mathsf{E}_{D_t} V_{t+1}(y_1 - D_t, y_2 - D_t, r + \theta D_t), \tag{3.7}$$

$$V_t(\mathbf{x}, w) = \min_{\mathbf{y}, r \in S(\mathbf{x}, w)} J_t(\mathbf{x}, w, \mathbf{y}, r), \tag{3.8}$$

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where the single-period cost function can be shown as

$$G_t(\mathbf{x}, w, y_2, r) = H_{1,t}(x_1) + H_{2,t}(x_2) + H_{3,t}(r)$$

$$+ c(y_2 - x_2) + \beta_i(r - w)^+ + \beta_o(r - w)^-.$$

We refer to (3.7) and (3.8) as the echelon formulation of the CP model.

3.3.2   The Optimal Policy

We first state the optimal joint policy for the CP model, which includes two types of decisions made through four control parameters (y_1^*, y_2^*, l^*, u^*) in each period. For the inventory ordering decisions, each stage implements an echelon base-stock policy. That is, stage i reviews its x_i at the beginning of each period. If $x_i < y_i^*$, it orders up to y_i^* or as close as possible if its upstream does not have sufficient stock; otherwise, it does not order. For the cash retention decision, stage 1 reviews w: if $w > u^*$, it disposes cash down to the maximum of u^* and x_2; if $w < l^*$, it retrieves cash up to l^* or as close as possible (due to the upper bound K); otherwise, it does not transfer cash.

For the traditional multi-echelon inventory model, there exists an equivalence result between echelon and local base-stock policies. Namely, each stage will generate exactly the same inventory orders based on the local and echelon policies[4]; see, e.g., Chapter 8 of Zipkin 2000. This result can dramatically simplify the implementation of the optimal policy as each stage can monitor its local information to execute the optimal policy. We have a similar result here: the optimal echelon policy (y_1^*, y_2^*, l^*, u^*) can be converted back to the local term $(y_1^{'*}, y_2^{'*}, l^{'*}, u^{'*})$, where $y_1^{'*} = y_1^*$, $y_2^{'*} = y_2^* - y_1^*$, $l^{'*} = l^* - y_2^*$, and $u^{'*} = u^* - y_2^*$. In this way, the procurement department of stage i can implement a local base-stock policy based on its local inventory level x_i'; the

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[4] The equivalence holds when each stage places an order in each period. It is possible, although very rare, that no order is placed under the echelon policy while the corresponding local policy suggests ordering. In such a case, the echelon and local policy are not equivalent. This can be easily fixed by modifying the rule of placing an order. That is, under the local policy, the upstream stage places an order only when it receives an order from its downstream stage and when its inventory state is lower than the target level.

accounting department of the headquarter can implement a local two-threshold policy based on the master account cash position w'.

We next explain how the optimal policy is derived and how to calculate these policy parameters. This is done by transforming a three-state dynamic program into three, single-dimensional dynamic programs. We summarize the main result in the following proposition.

Proposition 6. *For all t and $(\mathbf{x}, w)$, $V_t(\mathbf{x}, w) = f_{1,t}(x_1) + f_{2,t}(x_2) + f_{3,t}(w)$, where $f_{i,t}(\cdot)$ is convex.*

We define $f_{i,t}(\cdot)$ as the expected optimal cost for echelon i in period t. Starting from echelon 1, we have

$$f_{1,t}(x_1) = H_{1,t}(x_1) + \min_{x_1 \leqslant y_1} \big\{ \alpha \mathsf{E}_{D_t} f_{1,t+1}(y_1 - D_t) \big\}. \tag{3.9}$$

Let $g_{1,t}(y_1) = \alpha \mathsf{E}_{D_t} f_{1,t+1}(y_1 - D_t)$. Then, the optimal control parameter $y_{1,t}^*$ can be obtained by solving the minimization problem:

$$y_{1,t}^* = \arg\min_{y_1} \big\{ g_{1,t}(y_1) \big\}.$$

Now, we express the expected optimal cost functions of echelon 2 as follows:

$$f_{2,t}(x_2) = H_{2,t}(x_2) + \Gamma_{2,t}(x_2) + \Lambda_{2,t}(x_2) + \min_{x_2 \leqslant y_2} \big\{ c(y_2 - x_2) + \alpha \mathsf{E}_{D_t} f_{2,t+1}(y_2 - D_t) \big\}. \tag{3.10}$$

Similar to echelon 1, let $g_{2,t}(y_2) = cy_2 + \alpha \mathsf{E}_{D_t} f_{2,t+1}(y_2 - D_t)$ and $y_{2,t}^* = \arg\min_{y_2} \big\{ g_{2,t}(y_2) \big\}$. For echelon 3,

$$f_{3,t}(w) = \Lambda_{3,t}(w) + \begin{cases} L_t(w), & \text{if } w \leqslant l_t^* \\ H_{3,t}(w) + \Gamma_{3,t}(w) + \alpha \mathsf{E}_{D_t} f_{3,t+1}(w + \theta D_t), & \text{if } l_t^* < w \leqslant u_t^* \\ U_t(w), & \text{if } u_t^* < w \end{cases}. \tag{3.11}$$

Note that $\Gamma_{2,t}(\cdot)$ and $\Gamma_{3,t}(\cdot)$ are the so-called induced penalty cost functions defined

in Clark and Scarf (1960), i.e.,

$$\Gamma_{2,t}(x_2) = \begin{cases} \alpha\mathsf{E}_{D_t}\left[f_{1,t+1}(x_2 - D_t) - f_{1,t+1}(y_{1,t}^* - D_t)\right], & x_2 \leqslant y_{1,t}^*, \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

$$\Gamma_{3,t}(r) = \begin{cases} c(r - y_{2,t}^*) + \alpha\mathsf{E}_{D_t}\left[f_{2,t+1}(r - D_t) - f_{2,t+1}(y_{2,t}^* - D_t)\right], & r \leqslant y_{2,t}^*, \\ 0, & \text{otherwise.} \end{cases} \tag{3.13}$$

Here, $\Gamma_{2,t}(\cdot)$ represents the penalty cost charged to echelon 2 if stage 2 cannot ship up to stage 1's target base-stock level $y_{1,t}^*$. Although bearing the same structure, $\Gamma_{3,t}(\cdot)$ has a different economic meaning: it represents the penalty cost charged to the headquarter's accounting department (which manages the master account), if it fails to hold sufficient cash to pay for the inventory procurement up to the target echelon base-stock level $y_{2,t}^*$.

There are *new* penalty cost functions $\Lambda_{2,t}(\cdot)$ and $\Lambda_{3,t}(\cdot)$ in (3.10) and (3.11). To illustrate their meanings, we define

$$g_{3,t}(w) = H_{3,t}(w) + \Gamma_{3,t}(w) + \alpha\mathsf{E}_{D_t}f_{3,t+1}(w + \theta D_t), \tag{3.14}$$

$$L_t(w) = -\beta_i(w - l_t^*) + g_{3,t}(l_t^*), \tag{3.15}$$

$$U_t(w) = \beta_o(w - u_t^*) + g_{3,t}(u_t^*). \tag{3.16}$$

One can view $g_{3,t}(w)$ as the optimal cost for echelon 3 when the system working capital w is in $[l_t^*, u_t^*]$. Under the optimal policy, when $w < l_t^*$, stage 1 should retrieve cash until w reaches l_t^*. Thus, $L_t(w)$ can be viewed as the optimal cost when $w < l_t^*$. Similarly, $U_t(w)$ can be viewed as the optimal cost when $w > u_t^*$ because in this case stage 1 should dispose cash down to u_t^*. With these explanations, the two new penalty cost functions can be defined as follows:

$$\Lambda_{2,t}(x_2) = \begin{cases} 0, & \text{if } x_2 \leqslant u_t^*, \\ g_{3,t}(x_2) - U_t(x_2), & \text{otherwise,} \end{cases} \tag{3.17}$$

$$\Lambda_{3,t}(w) = \begin{cases} g_{3,t}(w + K) + \beta_i K - L_t(w), & w \leqslant l_t^* - K, \\ 0, & \text{otherwise.} \end{cases} \tag{3.18}$$

Let us first consider $\Lambda_{2,t}(x_2)$ in (3.17). This is a penalty cost charged to echelon 2 if the system carries too much inventory. Intuitively, if echelon inventory x_2 is less

than or equal to u_t^*, echelon 3 (or stage 1 cash department) can always maintain a system working capital between l_t^* and u_t^*. However, if $x_2 > u_t^*$, the best that stage 1 can do is to dispose all cash on hand, making $w = x_2$. In such case, the extra cost $g_{3,t}(x_2) - U_t(x_2)$ incurred at echelon 3 should be charged to echelon 2 due to its excess inventory. For this reason, we call $\Lambda_{2,t}(x_2)$ the *excess inventory penalty*. (Recall that $\Gamma_{2,t}(x_2)$ is the penalty cost charged to echelon 2 due to insufficient inventory holding.) The cash retention control thresholds can be obtained from the following equations:

$$l_t^* = \sup\left\{ r : \frac{\partial}{\partial r} g_{3,t} \leqslant -\beta_i \right\}, \quad u_t^* = \sup\left\{ r : \frac{\partial}{\partial r} g_{3,t}(r) \leqslant \beta_o \right\}.$$

With a similar logic, $\Lambda_{3,t}(w)$ in (3.18) can be explained: this is a self-induced penalty cost charged to echelon 3 if the system working capital w is less than $l_t^* - K$ due to too much cash disposal in the previous period. In such a case, stage 1 is penalized with the extra cost $g_{3,t}(w + K) + \beta_i K - L_t(w)$ for over-disposing cash.

FIGURE 3.3: Induced penalty functions of the cash pooling model

(a) Induced penalty functions at echelon 2 and 3.

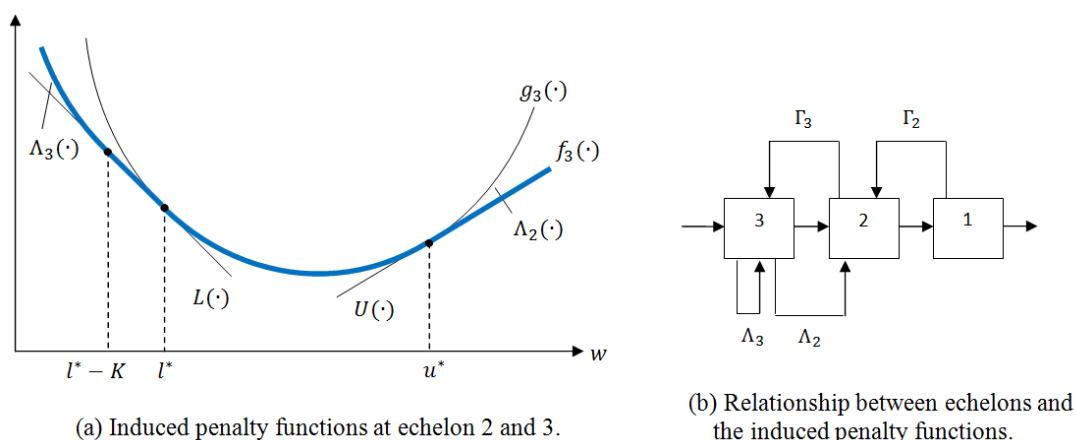(b) Relationship between echelons and the induced penalty functions.

Figure 3.3(a) depicts functions $L(\cdot)$, $U(\cdot)$, $g_3(\cdot)$, $f_3(\cdot)$, as well as induced penalty functions $\Lambda_2(\cdot)$ and $\Lambda_3(\cdot)$ created while decoupling echelon 2 and 3 (with time subscripts suppressed). The optimal control threshold l^* (u^*) derived as the tangent point of curve $g_{3,t}(\cdot)$ and a line with slope $-\beta_i$ (β_o). Function $f_3(\cdot)$ is shown as the bold convex curve connected by four different functions, which are, from the right to

the left, the linear function $U(\cdot)$, the convex function $g_3(\cdot)$, the linear function $L(\cdot)$, and the convex function g_3 shifted from point $(l^*, g_3(l^*))$ to point $(l^* - K, L(l^* - K))$; the induced penalty function $\Lambda_3(w)$ is the difference between $f_3(w)$ and $L(w)$ to the left of $l^* - K$; the induced penalty function $\Lambda_2(x_2)$ is the difference between $g_3(x_2)$ and $U(x_2)$ to the right of u^*. Figure 3.3(b) illustrates the relationship between four echelons and five penalty cost functions in our problem. The direction of the arrow indicates to which echelon that the penalty cost is charged.

3.4 Transfer Pricing System

Let us now consider the transfer pricing (TP) system. In this setting, stage i holds its own, separate cash account w_i', and stage 1 pays stage 2 for the ordered material according to a fixed transfer price p_2. The investment function is held at stage 1 (the headquarter). Thus, the cash retention decision directly affects the dynamics of stage 1's cash balance w_1'. Similar to the cash pooling scheme, we attach cash holding cost η_i' to stage i's cash account. We make no ex ante assumption on the order of η_1' and η_2'. The rest of the notation remains the same as that in §3.3. Figure 3.4(a) shows the material and financial flows of the TP model.

The inventory dynamics of the TP model are identical to the CP model, as in equation (3.1) and (3.2). Due to separate accounts, the cash dynamics of the TP model become

$$w_{2,t+1}' = w_{2,t}' + p_2 z_{1,t} - c z_{2,t}, \tag{3.19}$$

$$w_{1,t+1}' = w_{1,t}' + v_t - p_2 z_{1,t} + p_1 D_t. \tag{3.20}$$

Note that we again assume the payment to stage 2 occurs upon the receipt of shipment. Define $\mathbf{w}' = (w_2', w_1')$. The constraint set for the TP model is

$$\hat{S}(x_2', \mathbf{w}') = \left\{ \mathbf{z}, v \mid 0 \leqslant z_1 \leqslant \min\left(\frac{w_1' + v}{p_2}, x_2' \right), 0 \leqslant z_2 \leqslant w_2'/c, v \leqslant K' \right\}. \tag{3.21}$$

As shown in the first inequality, $p_2 z_1$ cannot exceed the available cash $w_1' + v$.

The single-period expected cost function is

$$\hat{G}_t(\mathbf{x}', \mathbf{w}', z_2, v) = \mathsf{E}_{D_t}\left[h_1'(x_1' - D_t)^+ + b(x_1' - D_t)^-\right] + h_2' x_2' + c z_2$$

$$+ \eta_2' w_2' + \eta_1' \mathsf{E}_{D_t}\left(w_1' + v + p_1 D_t\right) + \beta_i' v^+ + \beta_o' v^-. \tag{3.22}$$

The dynamic program of the TP model can be expressed as follows:

$$\hat{J}_t(\mathbf{x}', \mathbf{w}', \mathbf{z}, v) = \hat{G}_t(\mathbf{x}', \mathbf{w}', z_2, v) + \alpha \mathsf{E}_{D_t}\left[\hat{V}_{t+1}(x_1' + z_1 - D_t, x_2' + z_2 - z_1,\right.$$

$$\left. w_2' + p_2 z_1 - c z_2, w_1' + v - p_2 z_1 + p_1 D_t)\right], \tag{3.23}$$

$$\hat{V}_t(\mathbf{x}', \mathbf{w}') = \min_{\mathbf{z}, v \in \hat{S}(x_2', \mathbf{w}')} \hat{J}_t(\mathbf{x}', \mathbf{w}', \mathbf{z}, v). \tag{3.24}$$

The TP model is essentially a serial inventory problem with capacities (in the form of cash constraints) at both stages. However, these constraints are random and *endogenous*, which are different from those assumed in the traditional capacitated inventory model (e.g. Parker and Kapuscinski, 2004).

FIGURE 3.4: Transformation of the transfer pricing model

(a) The two-stage TP model.

(b) The transformed TP model.
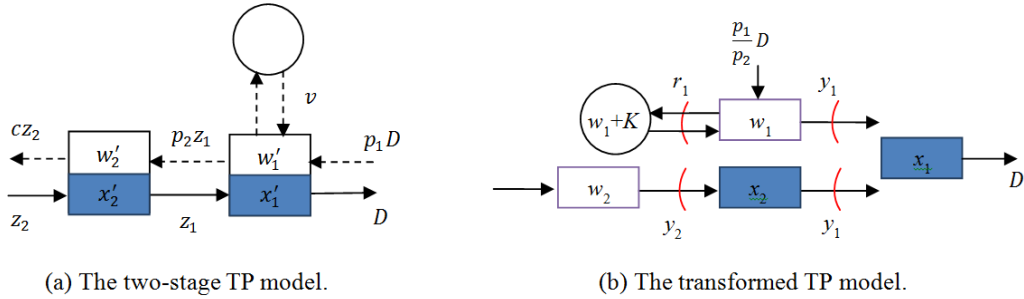
We are not able to obtain the exact optimal joint policy for the TP model. Nonetheless, we can obtain a lower bound to the optimal cost of the TP model. In the subsequent sections, we shall introduce a different echelon notion from that of the CP model. From this new echelon formulation, we can connect the TP problem to an assembly system from which the lower bound cost is obtained.

3.4.1 Echelon Formulation

We shall create a different echelon transformation scheme for the TP model. Define

$$x_1 = x_1', \ y_1 = x_1' + z_1, \ x_2 = x_1' + x_2', \ y_2 = x_1' + x_2' + z_2,$$

$$w_1 = x_1' + w_1'/p_2, \ r_1 = x_1' + (w_1' + v)/p_2, \ w_2 = x_1' + x_2' + w_2'/c.$$

Here, $\mathbf{x}$ and $\mathbf{y}$ are the same as in the CP model; w_1 is defined to be stage 1's working capital (in inventory units); w_2 is defined as stage 2's echelon working capital, which includes inventory at both stages and stage 2's cash balance (in inventory units). With these state transformations, we redefine the echelon holding cost parameters for the TP model: $\eta_2 = \eta_2' c$, $h_2 = h_2' - \eta_2' c$, $\eta_1 = \eta_1' p_2$, and $h_1 = h_1' - h_2' - \eta_1' p_2$. Also redefine $\beta_i = p_2 \beta_i'$, $\beta_o = p_2 \beta_o'$, $\theta = p_1/p_2 - 1 > 0$, $K = K'/p_2$, and finally $\rho = p_2/c$. With the new echelon terms, the feasible set becomes

$$S(\mathbf{x}, \mathbf{w}) = \{\mathbf{y}, r_1 \mid x_1 \leqslant y_1 \leqslant r_1 \leqslant w_1 + K, \ x_1 \leqslant y_1 \leqslant x_2 \leqslant y_2 \leqslant w_2\}.$$

As shown in Figure 3.4(b), the transformed TP system is similar to an *assembly* system. We further redefine the holding and backorder cost associated with each echelon as

$$H_{1,t}(x_1) = \mathsf{E}_{D_t}\big[(h_1 + h_2 + \eta_2 + \eta_1 + b)(D_t - x_1)^+ + h_1(x_1 - D_t)\big],$$

$$H_{2,t}(x_2) = \mathsf{E}_{D_t} h_2(x_2 - D_t), \quad H_{3,t}(w_2) = \mathsf{E}_{D_t} \eta_2(w_2 - D_t), \quad H_{4,t}(r_1) = \mathsf{E}_{D_t} \eta_1(r_1 + \theta D_t).$$

The echelon formulation of the TP model becomes

$$J_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, r_1) = G_t(\mathbf{x}, \mathbf{w}, y_2, r_1) + \mathsf{E}_{D_t} V_{t+1}(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t, r_1 + \theta D_t),$$

$$\tag{3.25}$$

$$V_t(\mathbf{x}, \mathbf{w}) = \min_{\mathbf{y}, r_1 \in S(\mathbf{x}, \mathbf{w})} J_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, r_1), \tag{3.26}$$

where the single-period cost function can be shown as

$$G_t(\mathbf{x}, \mathbf{w}, y_2, r_1) = H_{1,t}(x_1) + H_{2,t}(x_2) + H_{3,t}(w_2) + H_{4,t}(r_1)$$

$$+ c(y_2 - x_2) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^-. \tag{3.27}$$

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After the new transformation, some of the complexities caused by the endogenous constraints disappear. More specifically, the dynamics of the new echelon variable w_1 no longer depend on z_1. However, the dynamics of echelon w_2 still depend on the decision $y_1 - x_1$ associated with echelon 1, as shown in (3.25). This unique property undermines the decomposition structure in the CP model and differentiates the TP model from the traditional assembly system (Rosling, 1989). Below, we derive lower bounds to the optimal cost of the TP model.

3.4.2   Lower Bounds

This subsection establishes two lower bounds to the optimal cost for the TP model. Recall that the TP model is similar to an assembly system. The main idea of constructing these lower bounds is to decompose this assembly system. Specifically, the expression of $S(\mathbf{x}, \mathbf{w})$ indicates that stage 1's decision y_1 is subject to two constraints: one is $y_1 \leqslant r_1 \leqslant w_1 + K$, which represents the cash constraint on the order quantity; the other is $y_1 \leqslant x_2 \leqslant y_2 \leqslant w_2$, which can be viewed as a material order constraint in a two-stage system with an endogenous, random capacity w_2 at the upstream stage 2. Figure 3.5(a) shows these two sets of constraints.

Now, imagine that the final product sold at stage 1 consists of two components: a physical component (depicted by triangles) supplied from stage 2's stock, and a "cash" component (depicted by circles) supplied from stage 1's operating account. The constraint $0 \leqslant z_1 \leqslant \min\{(w_1' + v)/p_2, x_2'\}$ in (3.21) (or, equivalently, $x_1 \leqslant y_1 \leqslant \min\{r_1, x_2\}$) implies a similar structure to an assembly system: the same amount of inventory and cash equivalent are matched through replenishment at stage 1.

To derive a lower bound to the optimal cost, we relax the above matching constraint by assuming that the components can be ordered and sold separately. As a result, the original system is decoupled into two independent subsystems as shown in Figure 3.5(b) – Subsystem 1 represents the cash flows; Subsystem 2 represents the material flow. The sum of the minimum costs of subsystems is a lower bound on the

minimum cost of the original system.

FIGURE 3.5: Decomposition of the transfer pricing system

(a) The Original TP system   (b) Subsystem 1 (top) and Subsystem 2 (bottom)
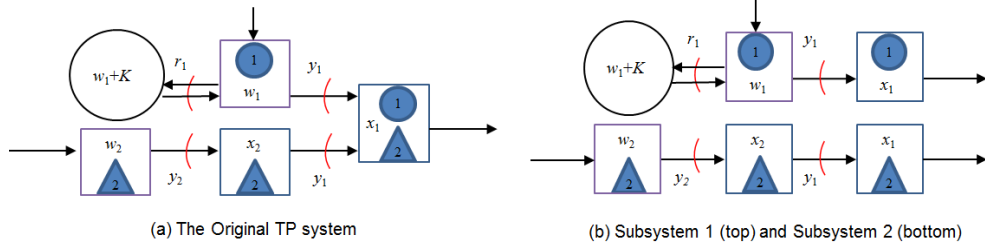
We specify the total cost function for each of the subsystems. Let h_1^1 and h_1^2 be the inventory holding cost for Subsystem 1 and 2, respectively, where $h_1^1 + h_1^2 = h_1$. Let b_1 and b_2 be the backorder cost for Subsystem 1 and 2, respectively, where $b_1 + b_2 = b$.

$$H_{1,t}^1(x_1) = \mathsf{E}_{D_t}\left[(h_1^1 + \eta_1 + b_1)(D_t - x_1)^+ + h_1^1(x_1 - D_t)\right], \tag{3.28}$$

$$H_{1,t}^2(x_1) = \mathsf{E}_{D_t}\left[(h_1^2 + h_2 + \eta_2 + b_2)(D_t - x_1)^+ + h_1^2(x_1 - D_t)\right]. \tag{3.29}$$

Now, let us define

$$G_t^1(x_1, w_1, r_1) = H_{1,t}^1(x_1) + H_{4,t}(r_1) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^-, \tag{3.30}$$

$$G_t^2(x_1, x_2, w_2, y_2) = H_{1,t}^2(x_1) + H_{2,t}(x_2) + H_{3,t}(w_2) + c(y_2 - x_2). \tag{3.31}$$

Note that $H_{1,t}(x_1) = H_{1,t}^1(x_1) + H_{1,t}^2(x_1)$, hence $G_t^1(x_1, w_1, r_1) + G_t^2(x_1, x_2, w_2, y_2) = G_t(\mathbf{x}, \mathbf{w}, y_2, r_1)$. With this cost allocation, the dynamic program for Subsystem 1 can be expressed as

$$V_t^1(x_1, w_1) = \min_{x_1 \leqslant y_1 \leqslant r_1 \leqslant w_1 + K} \left\{G_t^1(x_1, w_1, r_1) + \alpha \mathsf{E}_{D_t} V_{t+1}^1(y_1 - D_t, r_1 + \theta D_t)\right\}. \tag{3.32}$$

And the dynamic program for Subsystem 2 is

$$V_t^2(x_1, x_2, w_2) = \min_{x_1 \leqslant y_1 \leqslant x_2 \leqslant y_2 \leqslant w_2} \left\{G_t^2(x_1, x_2, w_2, y_2)\right.$$

$$\left. + \alpha \mathsf{E}_{D_t} V_{t+1}^2(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t)\right\}. \tag{3.33}$$

Proposition 7. $V_t(\mathbf{x}, \mathbf{w}) \geqslant V_t^1(x_1, w_1) + V_t^2(x_1, x_2, w_2)$ for all $(\mathbf{x}, \mathbf{w})$ and t.

Proposition 7 shows that for any combination of (h_1^1, h_1^2) and (b_1, b_2), the sum of the two subsystems forms a cost lower bound to the original system. Maximizing expected cost over all parameter combinations yields the best lower bound.

The remaining question is how to find the optimal cost of these subsystems. A careful examination of Subsystem 1 described in (3.32) reveals that it is the echelon transformation for a single-stage system with a joint inventory and cash retention decision. Thus, we can characterize the optimal joint policy, i.e., using the base-stock policy to control the inventory replenishment and the two-threshold policy to monitor the working capital level.

Solving Subsystem 2 is more difficult. The dynamic problem described in (3.33) is the echelon expression of a two-stage inventory model with random, *endogenous* capacity at the upstream stage. There exists no known optimal policy for this model. Thus, we provide two approaches to further develop a lower bound to the optimal cost for Subsystem 2.

Constraint Relaxation (CR) Bound

We form the lower bound by relaxing the constraint $y_2 \leqslant w_2$ at stage 2. Once w_2 is removed from the constraint set, it only appears in the expected cost function of each period. The following lemma characterizes the expected value of w_2 through the flow conservation.

Lemma 7. *Given the initial states $w_{2,1}$ and $x_{1,1}$, for any policy we have*

$$\mathop{\mathsf{E}}_{D_1,\ldots,D_{t-1}} w_{2,t} = \rho \cdot \mathop{\mathsf{E}}_{D_1,\ldots,D_{t-1}} x_{1,t} + B_t,$$

where $B_t = (\rho - 1) \sum_{s=1}^{t-1} \mu_s + w_{2,1} - \rho x_{1,1}$.

Recall that in the single-period cost function (3.31), the function $H_{3,t}(w_2)$ is a linear function of w_2. Therefore, by using Lemma 7, we can replace $H_{3,t}(w_2)$ with $H_{3,t}(\rho x_1 + B_t)$ without affecting the optimal decision in each period. With this construction, w_2 can be replaced by x_1 and Subsystem 2 becomes a classical two-stage

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serial system in which Clark and Scarf's algorithm can be applied to find the optimal echelon base-stock levels for both stages. The CR bound generally works well when the constraint $y_2 \leqslant w_2$ is not binding, i.e., when stage 2 holds sufficient cash. This occurs if the stage 2's markup $(p_2/c_2 - 1)$ is high and demand tends to be stationary. However, under increasing demand, it is optimal for stage 2 to order more in anticipation of future demand uprise. In such case, stage 2's cash constraint could become binding, especially if its markup is low. Thus, we need another lower bound to complement the performance of the CR bound.

Sample Path (SP) Bound

The difficulty of solving Subsystem 2 comes from keeping track of the state $w_{2,t}$. As stated earlier, the current period's $w_{2,t}$ depends on the previous period's demand and order quantity. However, if we consider a specific demand sample path, $w_{2,t}$ can be fully characterized by flow conservation.

Lemma 8. *Let $d_t(\omega)$ represent the demand realization in period t given a demand sample path ω. With initial states $w_{2,1}$ and $x_{1,1}$, we have $w_{2,t} = \rho x_{1,t} + B_t(\omega)$, where*

$$B_t(\omega) = (\rho - 1) \sum_{s=1}^{t-1} d_s(\omega) + w_{2,1} - \rho x_{1,1}.$$

The proof of Lemma 8 is similar to that of Lemma 1, and thus omitted. Given the initial states and a demand sample path, $B_t(\omega)$ is a constant. If we replace $w_{2,t}$ (according to Lemma 8) in both the constraint set and the periodic cost function, Subsystem 2 can be reduced to a two-stage serial system with deterministic demand subject to the following constraint (at time t):

$$S_t^d(x_1, x_2 \mid \omega) = \big\{ y_1, y_2 \mid x_1 \leqslant y_1 \leqslant x_2 \leqslant y_2 \leqslant \rho x_1 + B_t(\omega) \big\}.$$

The constraints state that stage 1's order decision y_1 is affected by stage 2's echelon inventory level x_2; stage 2's order decision y_2 is affected by a linear function of stage

1's inventory level x_1. The optimal y_1^* and y_2^* can be obtained by solving a two-dimensional convex program in each period. To facilitate the computation, we prove that this problem can be decoupled into two one-dimensional convex programs. Let $V_t^d(x_1, x_2 \mid \omega)$ represent the optimal cost for Subsystem 2 for any demand sample path ω after $w_{2,t}$ is substituted with $\rho x_{1,t} + B_t(\omega)$. The following proposition shows the decoupling result.

Proposition 8. $V_t^d(x_1, x_2 \mid \omega) = v_t^1(x_1 \mid \omega) + v_t^2(x_2 \mid \omega)$, *where* $v_t^i(x_i \mid \omega)$ *is a convex function.*

We refer the reader to the proof for the detailed formulation of v_t^1 and v_t^2 functions. A lower bound to the optimal cost of the Subsystem 2 under the SP approach can be found by averaging total costs over all demand sample paths. In summary, we are able to generate two lower bounds – the sum of the optimal cost obtained from Subsystem 1 and the optimal cost obtained from either the CR approach or the SP approach.

3.4.3 Optimal Transfer Pricing Model

For some multi-divisional corporations with a powerful headquarter, it is possible that the headquarter can determine transfer price to efficiently distribute liquidity. This section extends the TP model to optimize the transfer price between the divisions. Notice that the optimal transfer price can be obtained by the optimal order quantity and the optimal cash payment in each period. Thus, we modify the transfer pricing model to incorporate the inter-division cash payment decision. For period t, define

m_t = amount of cash payment paid from stage 1 to stage 2 before the demand occurs.

Then, the optimal transfer pricing (OP) model can be obtained by replacing $p_2 z_{1,t}$ with m_t in the TP model, as shown in Figure 3.6(a).

Interestingly, solving the OP model is no harder than solving the CP model. More specifically, we can follow the same logic in the CP model, and define a set

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of new echelon variables and cost parameters, making the original two-stage model transformed into a four-stage serial system. Figure 3.6(b) shows the transformed system with division 2's and division 1's cash account being stage 3 and stage 4, respectively. Similarly, we can decompose the resulting four-state dynamic program into four separable, single-state dynamic program. We refer the reader to Luo and Shang (2012) for the detailed analysis. In summary, we can obtain the exact optimal joint inventory, cash payment and retention policy for the OP model. The optimal transfer price is equal to the optimal cash payment divided by the optimal order quantity.

Proposition 9. *The optimal policy for the OP model can be described as follows. For inventory replenishment, both stages implement an echelon base-stock policy; for cash payment, stage 2 monitors its echelon working capital $(x'_1 + x'_2 + w'_2/c)$ and receives payment up to a target level; for cash retention, stage 1 monitors the system working capital and maintains it within an interval.*

FIGURE 3.6: The transformed optimal pricing model

(a) The two-stage OP model. (b) The four-stage transformed OP model.
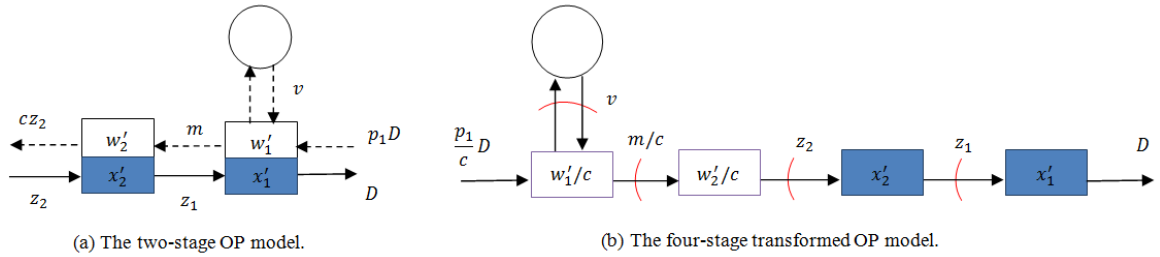
3.5   Numerical Study

We assess the value of cash pooling in §3.5.1, present the insights from the optimal transfer pricing in §3.5.2, and discuss the impact of cash management systems on material bullwhip effect in §3.5.3.

3.5.1 Value of Cash Pooling

We assess the value of cash pooling by comparing the optimal cost of the CP model, C^*, with the lower bound cost of the TP model, $C_L = \max\{C_R, C_S\}$, where C_R and C_S represent the cost of the constraint relaxation bound and the sample path bound, respectively. Note that the value we obtain is a lower bound to the actual value. We define the value of cash pooling as

$$\% \text{ value} = \frac{C_L - C^*}{C_L} \times 100\%.$$

This represents the percentage of cost reduction of the TP model if cash pooling is implemented.

We conduct a numerical study by starting with a test bed which has the time horizon of 10 periods. We fix parameters $\alpha = 0.95$, $c = 1$, $\eta'_1 = 0.05$, $h'_1 = 1$, and vary the other parameters with each taking two values: $p_2 = (1.2, 2)$, $p_1 = (2.5, 4)$, $b = (5, 10)$, $\eta'_2 = (0.05, 0.2)$, $h'_2 = (0.25, 0.75)$, $\beta'_o = (0.05, 0.15)$, and $\beta'_i = (0.05, 0.2)$. In addition, two demand forms are considered. For the i.i.d. demand case, D_t is Poisson distributed with mean $\mu_t = 10$ for all t; for the increasing demand case, D_t is Poisson distributed with the first period mean $\mu_1 = 10$ and μ_t increasing at a rate of 1.2 per period. In both demand cases, we fix the liquidity level $K'_t = \mu_t$. For each demand form, we generate 128 instances. The total number of instances in the test bed is 256. The combination of these parameters covers a wide range of different system characteristics. For example, when $(p_1, p_2) = (2.5, 2)$ $((4, 1.2)$, respectively) the transfer price is high (low, respectively) compared to the retail price. For all cases we assume the initial on-hand inventory and cash level $(x'_{1,1}, x'_{2,1}, w'_{2,1}, w'_{1,1}) = (16, 10, 10, 10)$, roughly equal to the steady-state inventory/cash level under the i.i.d. demand. When computing C_S, we run a simulation of 1000 iterations for each instance. When computing C^*, we assume $\eta' = \min\{\eta'_1, \eta'_2\}$, and set the initial balance of the cash pool as $w'_1 = w'_{1,1} + w'_{2,1}$.

In this test bed of 256 cases, the average cost reduction of adopting cash pooling is 29.29%. More specifically, the cost reduction is 13.62% when demand is i.i.d and 44.96% when demand is increasing. Table 3.1 (left) summarizes the value of cash pooling under the i.i.d. demand (128 cases). The results are further aggregated into 4 quadrants, each displaying the average value of 32 cases with the same p_2 and η_2'. As shown in Table 3.1 (left), cash pooling does not add much value if the transfer price is low (e.g., $p_2 = 1.2$). This is because under the i.i.d. demand, division 2 has to purchase inventory in each period to cover the (stationary) order received from division 1. With a low transfer price, division 2's average inventory procurement cost per period will be close to the average payment received per period. Thus, division 1 will not accumulate too much cash that leads to system inefficiency (hence the value of cash pooling is small). On the other hand, with a high transfer price (e.g., $p_2 = 2$), cash pooling will then play a significant role – it will be better off to allocate more cash to division 1 so less cash will be accumulated at division 2. The value of cash pooling is more significant when division 2 cash holding cost η_2' increases as the excess cash will be charged with a higher rate.

Table 3.1: Value of cash pooling - i.i.d. demand (left) and increasing demand (right)

Cash holding cost η_2'	Transfer price p_2		Backorder cost b	Transfer price p_2	
	1.2	2		1.2	2
0.05	4.68%	9.56%	5	66.14%	16.29%
0.2	12.87%	27.38%	10	80.25%	17.16%

Under the increasing demand, the TP scheme will make the system perform poorly when the transfer price is low. More specifically, as division 1 order size increases with the demand, ideally division 2 should in turn increase its inventory stocking to prepare for the future increasing order sizes. However, under fixed transfer pricing, division 2 might not have sufficient cash to do so due to its low markup ($p_2 = 1.2$). This vicious circle will make the supply chain very inefficient. Table 1 (right) demonstrates this inefficiency. As shown, when the transfer price is low and demand is increasing, the

value of cash pooling can be very significant. Interestingly, this finding is consistent with Caterpillar's strategy: After the financial crisis, many suppliers of Caterpillar have difficulty getting funds from external banks to stock up the raw material for the expected soaring demand. Thus, Caterpillar took a more proactive role to subsidize their suppliers (Aeppel, 2010). The value of cash pooling is clearly higher when backorder cost is larger. Figure 3.7(a) summarizes the conditions under which cash pooling has a significant value[5].

FIGURE 3.7: Value of cash pooling

(a) Demand form, transfer price, and CP value.
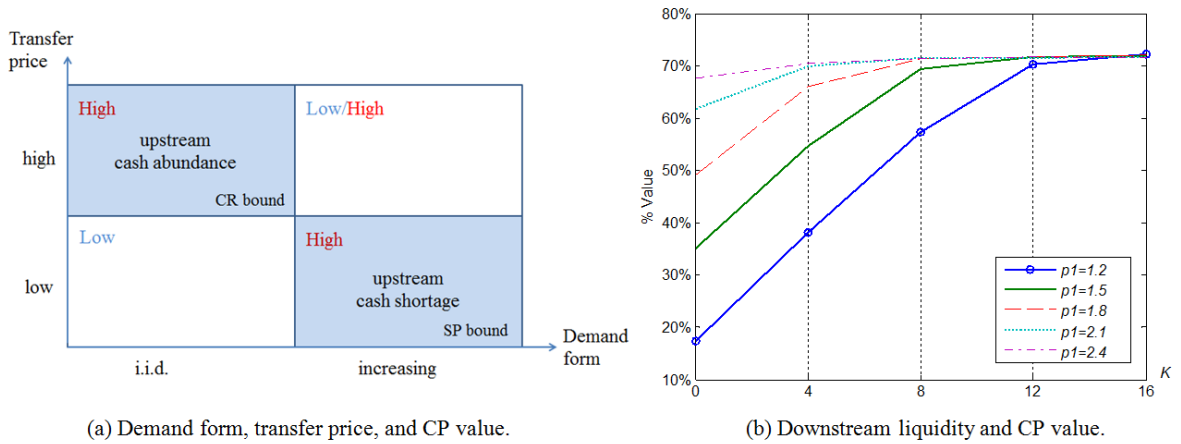
(b) Downstream liquidity and CP value.

Figure 3.7(b) illustrates the impact of downstream liquidity level K on the value of cash pooling when demand is increasing under different selling prices p_1. Here, we set the transfer price $p_2 = 1.05$, $b = 5$, $\eta_2' = 0.2$, $h_2' = 0.25$, $\beta_o' = \beta_i' = 0.05$ and the other parameters are the same as in the test bed. For a fixed selling price, the value of cash pooling is increasing in K, but the *marginal* benefit of cash pooling decreases in K. In addition, we find that p_1 and K complement each other's role as a liquidity source. For example, $(K, p_1) = (8, 1.2)$ and $(4, 1.5)$ yield a similar CP value.

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[5] Figure 3.7(a) also shows that CR (SP) bound performs better in the upper left (lower right) quadrant.

3.5.2 Optimal Transfer Pricing

While we have demonstrated significant value of cash pooling, one interesting question to investigate is that how much benefit can be recovered by optimizing the transfer price. To answer this question, we compare the optimal cost from the CP model, C^*, with the optimal cost from the OP model, C_O. We define the percentage cost reduction as $(C_O - C^*)/C_O \times 100\%$. We test the same 256 instances. The average (maximum, minimum) percentage cost reduction of adopting cash pooling is 6.38% (9.98%, 3.34%) for the i.i.d. demand and 9.33% (14.75%, 4.65%) for the increasing demand case. The cost reduction is more significant when the cash holding cost η_2' or the backorder cost b is high. Two reasons lead to this cost difference. First, cash pooling can consolidate the entire system cash to a single account that has a smaller cash holding cost rate. (This explains why the cost saving is more significant when η_2' is larger.) Second, cash pooling eliminates the lead time for the payment and allows cash to move bi-directionally to upstream or downstream, making the supply chain more responsive and leading to a smaller number of backorders. (Thus, the benefit of cash pooling is more significant when the backorder cost is high.)

Compared with the cost reduction with the fixed transfer price tested in the previous section, the percentage cost reduction is relatively small when the transfer price is optimized. This suggests that optimizing the transfer price can retain a big portion of the benefit achieved by adopting cash pooling. This indeed is useful for firms if adopting cash pooling is not possible due to legal issues or cash shortage for investing in the costly financial services platform.

It is interesting to see how the optimized transfer price helps to re-distribute the supply chain cash between these two divisions for firms facing a product life cycle demand. More specifically, we consider a time horizon of 22 periods with Poisson demand in each period. The demand mean starts from 6, ramps up at a peak of 36, declines to 14 and remains there for the last 5 periods. These demand rates

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represent introduction, growth, maturity, and decline stages in a product life cycle (see Figure 3.8). To illustrate the transfer pricing dynamics, we consider a instance with $p_2 = 1.25$, $p_1 = 1.5$, $b = 55$, $\eta_2' = 0.15$, $h_2' = 0.2$, $\beta_o' = \beta_i' = 0.05$, $K' = 0$, and the other fixed parameters in the test bed. We obtain the optimal transfer price as the optimal cash payment, m, divided by the optimal order quantity, z_1 in a simulation study and plot the average optimal transfer price in each period. Figure 3.9 shows the dynamics of the corresponding optimal transfer price from period to period. Notice that if the optimal transfer price p_2^* is larger than $p_1 = 1.5$, the price difference can be viewed as financial subsidy offered by stage 1; on the other hand, if p_2^* is smaller than the purchase cost $c = 1$, the price difference can be viewed as delayed payment made by stage 1.
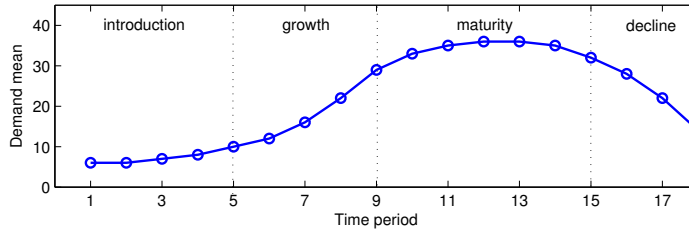
FIGURE 3.8: Product life cycle demand

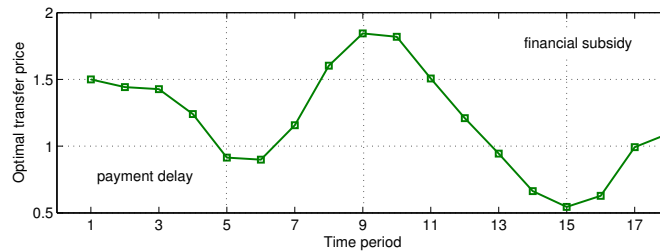FIGURE 3.9: Optimal transfer price under product life cycle demand

Figure 3.9 provides an interesting insight on how to set up the optimal transfer price. Recall that this is a case with $K' = 0$, i.e., the source of stage 1's liquidity is completely from the sales revenue. From the figure, we find that during the introduction stage, the transfer price should be set to a value close to the selling price p_1. This

implies that division 1's cash should be moved to division 2, as the latter needs to spare more cash for material ordering. The transfer price declines slightly but ramps up quickly during the growth stage, reflecting the fact that division 1 should even subsidize division 2 for material ordering. Finally, during the maturity and decline stages, division 2 does not need to reserve excess cash for material ordering, so the transfer price declines gradually. During the periods 13 to 17, the transfer price can be lower than the purchase cost c. We can view this as a payment reduction received by division 1 to compensate the cash subsidy offered to division 2 during the growth stage.

3.5.3   Bullwhip Effect

Bullwhip effect is a phenomenon that the order variability amplifies when moving along the supply chain from downstream toward upstream stages (Lee et al., 1997). It describes a phenomenon of order information distortion. In the empirical literature, material bullwhip effect, i.e., the amplification of shipments, is often used as a proxy for the order bullwhip effect, and observed in practical supply chains (e.g., Blanchard, 1983; Cachon et al., 2007). Interestingly, in the finance literature, a similar phenomenon called "financial contagion" has been observed in practice (e.g., Allen and Gale, 2000). The financial contagion describes that the risk of financial payment defaults amplifies when moving toward upstream in a supply chain. One reason that causes the financial contagion is the material bullwhip effect. This is because the payment amount is usually associated with the shipment size via transfer price. When the shipment variability amplifies, an upstream stage requires more capital for its inventory payments. If the upstream stage has weaker cash liquidity, which is usually observed in supply chains, this will result in a higher payment default risk. In our TP model, we observe that both material and financial bullwhip effect amplify in the same direction with the same scale – in a simulation study based on the test bed of the i.i.d. demand instances, we find that the average coefficient of variation (c.v.) of

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the shipment received by (or the payment paid by) stage 1 and stage 2 is 0.28 and 0.32, respectively. Interestingly, by optimizing the transfer price, one can reverse the direction of amplification for the payment variability. More specifically, for the OP model with the same test bed, the average c.v. of the shipment received by stage 1 and stage 2 is 0.26 and 0.28, respectively, whereas the average c.v. of the payment paid by stage 1 and stage 2 is 0.32 and 0.28, respectively. Our numerical finding suggests that (1) the material and financial bullwhip effects may not amplify in the same direction in an integrated supply chain, and (2) the optimal transfer pricing is a useful tool to smoothen the payment variability to the outside vendor, which mitigates the material bullwhip effect in the system.

3.6   Concluding Remarks

The paper studies the benefit of cash pooling for a corporation that owns a supply chain with two divisions. We quantity the value of cash pooling by comparing two cash management systems, representing different level of cash concentration. We prove the exact optimal inventory and cash retention policy for the cash pooling model and construct a lower bound to the optimal cost for the transfer pricing model. We quantity the conditions under which investing in a financial services platform that achieves cash pooling provides the most benefit. Our study suggests that monitoring the entire supply chain working capital through an inter-departmental collaboration between accounting and operations is crucial to ensure system efficiency. Our results can be extended to the system with general lead times and number of stages, as well as Markov modulated system parameters.

The focus of this paper is to derive a centralized solution for supply chains with different cash management systems. This perspective is appropriate for a single-owner supply chain. Nonetheless, there are decentralized supply chains in which the entities are individual firms, each with its own interests. An important question for the decentralized supply chain is to design an incentive scheme such that each individual

firm would choose the first best solution. The centralized solution we obtain can be viewed as the first best solution for this purpose. We leave this decentralized control issue for the future research.

4

Material Flow Variability

4.1   Introduction

We study the material bullwhip effect in supply chains, a phenomenon that the variability of shipment is amplified when moving upstream the supply chain. Economists have observed this phenomenon in empirical studies. However, this observation appears to be counter-intuitive as they would expect the opposite - the "production smoothing" effect (smaller shipment variability at the upstream stage). We provide an analytical model to show that it is possible to observe both bullwhip and reverse bullwhip effects in supply chains. These results can be extended to explain the material bullwhip effect when inventory replenishment is subject to the cash constraint, hence providing analytical support for the findings in Chapter 2 and Chapter 3.

4.2   Main Results

Let us consider a periodic-review, two-stage serial inventory system where stage 1 orders from stage 2, and stage 2 orders from an outside vendor with ample supply. The system is subject to i.i.d. customer demand D_t and full backlog at both stages. The optimal ordering policy for such a system is known static echelon base-stock policy,

from which an equivalent local base-stock policy can be derived (see Clark and Scarf 1960, Rosling 1989). Figure 4.1 shows the two-stage model with the material and information flows in opposite directions. The material flow is composed of shipments and sales, while the information flow includes orders and customer demand.
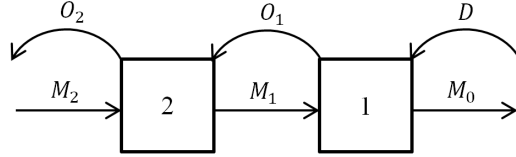

FIGURE 4.1: The two-stage supply chain model with material and information flows

Under the optimal local base-stock policy, it is clear that $O_{2,t} = O_{1,t} = D_{t-1}$. Thus, we have $\text{var}(O_{2,t}) = \text{var}(O_{1,t}) = \text{var}(D_{t-1})$, i.e., the information flow has no bullwhip effect.

Now let us examine the material flow in the two-stage supply chain. For this purpose, we need to describe the shipment and sales in the system. We assume that the lead time is one period for both stages. We formalize the sequence of events as follows: At the beginning of the period, (1) both stages receive shipments released in the last period; (2) both stages make an order decision; (3) inventory levels and backorders at both stages are updated; During the period, demand occurs and sales are realized; At the end of the period, all costs are calculated. We first describe the notation. For stage $j = 1, 2$ and period t, define

$$M_{j,t} = \text{shipment released to stage } j \text{ from its upstream supplier}$$

$$M_{0,t} = \text{realized sales to the end customer}$$

$$O_{j,t} = \text{order quantity from stage } j \text{ to its upstream supplier}$$

$$B_{j,t} = \text{local backorders after making the order decision}$$

$$s_j = \text{local base-stock level}$$

Next, we express M_2, M_1 and M_0 in terms of demand and local backorders, which are in turn expressed in terms of demand and local base-stock levels. Note from the

system dynamic of stage 2, we have $B_{2,t} = B_{2,t-1} + D_{t-1} - M_{1,t}$, i.e., current period's backorder is the difference between the accumulated backorder $B_{2,t-1} + D_{t-1}$ and the released shipment $M_{1,t}$. The dynamic of stage 1 is the same except that the realized sales are shifted one period backwards, due to the zero information lead time from stage 1 to the end customer. We summarize the results in the following lemma.

Lemma 1. *For all t, we have $M_{2,t} = D_{t-1}$, and*

$$M_{1,t} = B_{2,t-1} - B_{2,t} + D_{t-1}, \tag{4.1}$$

$$M_{0,t-1} = B_{1,t-1} - B_{1,t} + D_{t-1}, \tag{4.2}$$

where the local backorders can be expressed as

$$B_{2,t} = -\min\{0, s_2 - D_{t-1}\}, \tag{4.3}$$

$$B_{1,t} = -\min\{0, s_1 - D_{t-1} + \min\{0, s_2 - D_{t-2}\}\}. \tag{4.4}$$

Remark Equation (4.2) presents the exact expression of sales in a multi-period model with backorders. In inventory literature, however, an alternative expression has been commonly used to approximate sales - the minimum of demand and on-hand inventory. As shown here, this approximation is accurate only when $B_{1,t-1} = 0$, i.e., stage 1 has cleared its backlogs in period $t - 1$.

Given Lemma 1, we define the material bullwhip ratio at stage 2 as $\text{var}(M_{2,t})/\text{var}(M_{1,t})$, and that at stage 1 as $\text{var}(M_{1,t})/\text{var}(M_{0,t-1})$. The reverse material bullwhip occurs whenever the material bullwhip ratio is less than 1.

Let us first consider the material flow from stage 2 to stage 1. Note from (4.3) that when $s_2 \geqslant 0$, $B_{2,t}$ visits the zero point infinitely many times. Let us examine a cycle that spans $\tau + 1$ periods between two consecutive zero-point visits, i.e., $B_{2,t} = 0$, $B_{2,t+1} > 0$, $B_{2,t+2} > 0$, ..., $B_{2,t+\tau} > 0$, and $B_{2,t+\tau+1} = 0$. In this cycle, the shipment from stage 2 to stage 1 is given by

$$M_{1,t+i} = \left\{ \begin{array}{ll} s_2, & \text{if } i = 1, \\ D_{t+i-2}, & \text{if } 2 \leqslant i \leqslant \tau, \\ D_{t+\tau-1} + D_{t+\tau} - s_2, & \text{if } i = \tau + 1. \end{array} \right\} \tag{4.5}$$

It turns out that we can show that $\sum_{i=1}^{\tau+1} M_{2,t+i}^2 = \sum_{i=1}^{\tau+1} D_{t+i-1}^2 \geqslant \sum_{i=1}^{\tau+1} M_{1,t+i}^2$ (see the proof given in the appendix). Thus, the following proposition immediately follows from the ergodic theorem.

Proposition 10. *For any $s_2 \geqslant 0$, we have $var(M_{2,t}) = var(D_{t-1}) \geqslant var(M_{1,t})$.*

Next, we consider the material flow from stage 1 to the end customer. We identify the cycle in a similar way as in the proof of Proposition 10. When $s_1 \geqslant 0$, $B_{1,t}$ visits the zero point infinitely many times. We examine a cycle that spans $\tau + 1$ periods between two consecutive zero-point visits, i.e., $B_{1,t} = 0$, $B_{1,t+1} > 0$, $B_{1,t+2} > 0$, ..., $B_{1,t+\tau} > 0$, and $B_{1,t+\tau+1} = 0$. In this cycle, the sales from stage 1 to the customer are given by

$$M_{0,t+i-1} = \begin{cases} s_1, & \text{if } i = 1, \\ M_{1,t+i-1}, & \text{if } 2 \leqslant i \leqslant \tau, \\ D_{t+\tau-1} + D_{t+\tau} - \min\{0, s_2 - D_{t+\tau-2}\} - s_1, & \text{if } i = \tau + 1. \end{cases} \tag{4.6}$$

It turns out that we can leverage the above structure to show that $\sum_{i=1}^{\tau+1} M_{1,t+i}^2 \geqslant \sum_{i=1}^{\tau+1} M_{0,t+i-1}^2$ under the condition of $s_2 \geqslant s_1$ (see the proof given in the appendix). Again from the ergodic theorem we have the following proposition.

Proposition 11. *For any $s_2 \geqslant s_1 \geqslant 0$, we have $var(M_{1,t}) \geqslant var(M_{0,t-1})$.*

When $s_2 < s_1$, the analysis becomes more complicated. And in fact, the relationship between $var(M_{1,t})$ and $var(M_{0,t-1})$ depends on s_1, s_2, and the demand. To see this, we assume D_t follows a Uniform distribution with support $[0, d]$, denoted as $U(0, d)$. Note that $var(D_t) = d^2/12$. We first express $var(M_{1,t})$ in terms of s_2 and d.

Lemma 2. *Assuming D_t follows $U(0, d)$, we have*

$$var(M_{1,t}) = \begin{cases} \dfrac{d^2}{12} - \dfrac{s_2^2}{2}\left(1 - \dfrac{s_2}{d}\right)^2, & \text{if } s_2 \leqslant d, \\ \dfrac{d^2}{12}, & \text{if } s_2 > d. \end{cases} \tag{4.7}$$

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Taking derivative of $\text{var}(M_{1,t})$ with respect to s_2 in the region of $s_2 \leqslant d$, we have

$$\frac{\partial \text{var}(M_{1,t})}{\partial s_2} = -s_2 \left(1 - \frac{s_2}{d}\right)\left(1 - \frac{2s_2}{d}\right). \qquad (4.8)$$

Therefore, as s_2 increases from 0 to d, $\text{var}(M_{1,t})$ will first decrease then increase. Figure 4.2 plots $\text{var}(M_{1,t})$ as function of s_2 when demand follows $U(0, 12)$. The corresponding bullwhip ratio is plotted in Figure 4.3, which shows that the material bullwhip effect at stage 2 is most significant when $s_2 = d/2$, with a bullwhip ratio of 1.6.
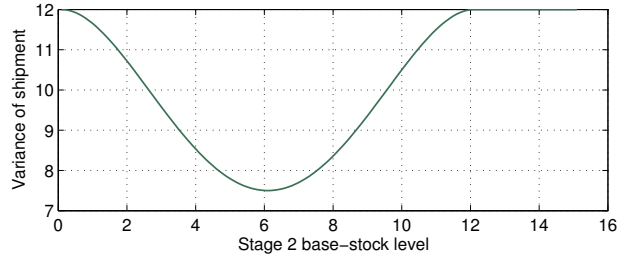

FIGURE 4.2: Variance of shipment $\text{var}(M_{1,t})$ vs. base-stock s_2 under $U(0, 12)$
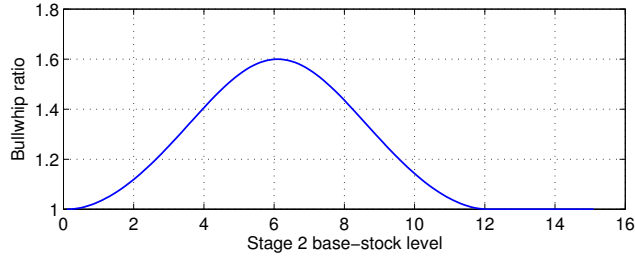

FIGURE 4.3: Bullwhip ratio $\text{var}(M_{2,t})/\text{var}(M_{1,t})$ vs. base-stock s_2 under $U(0, 12)$

Next, we derive $\text{var}(M_{0,t-1})$ under uniformly distributed demand. Note that in most supply chains, the unit backorder cost is usually much higher than the inventory holding cost at the downstream stage. In our current model with one period lead time, this indicates that the optimal base-stock s_1 will be greater than d, the mean of the sum of two independent uniformly distributed demand. Hence in what follows, we focus on the case with $d < s_1$.

Lemma 3. *Assuming D_t follows $U(0, d)$, we have*

$$var(M_{0,t-1}) = \begin{cases} d^2/12, & \text{if } d < s_2 < s_1, \\ d^2/12, & \text{if } s_2 \leqslant d < s_1 \text{ and } 2d - s_2 < s_1, \\ \dfrac{d^2}{12} - \dfrac{1}{2d} \displaystyle\int_{s_1+s_2-d}^{d} y^2 \left(1 - \dfrac{y}{d}\right)^2 dy, & \text{if } s_2 \leqslant d \leqslant s_1 \leqslant 2d - s_2. \end{cases}$$

$$(4.9)$$

Now, we can combine Lemma 2 and Lemma 3 to compare $var(M_{1,t})$ and $var(M_{0,t-1})$. We discuss the results in three regions: (1) when $d < s_2 < s_1$, $var(M_{2,t}) = var(M_{1,t}) = var(M_{0,t-1})$, i.e., no material bullwhip at either stage; (2) when $s_2 \leqslant d < s_1$ and $2d - s_2 < s_1$, $var(M_{2,t}) = var(M_{0,t-1}) \geqslant var(M_{1,t})$, i.e., bullwhip at stage 2 and reverse bullwhip at stage 1; (3) when $s_2 \leqslant d \leqslant s_1 \leqslant 2d - s_2$, there is still bullwhip at stage 2, but the relationship between $var(M_{1,t})$ and $var(M_{0,t-1})$ could be either way. To further explore this, let us define

$$var(\bar{M}_{0,t-1}) = \frac{d^2}{12} - \frac{1}{2d} \int_{s_2}^{d} y^2 \left(1 - \frac{y}{d}\right)^2 dy, \qquad (4.10)$$

which is obtained from $var(M_{0,t-1})$ by making $s_1 = d$. Note that $var(M_{0,t-1})$ depends on both s_1 and s_2, while $var(\bar{M}_{0,t-1})$ only depends on s_2. Furthermore, it can be seen from (4.10) that

$$var(M_{0,t-1})(s_2, s_1) = var(\bar{M}_{0,t-1})(s_2 + s_1 - d). \qquad (4.11)$$

It turns out that $var(\bar{M}_{0,t-1})$ can help us determine the threshold in comparing $var(M_{1,t})$ and $var(M_{0,t-1})$. As shown in Figure 4.4, threshold $\bar{s}_2$ is obtained as the intersection of curve $var(M_{1,t})$ and $var(\bar{M}_{0,t-1})$. When $s_2 < \bar{s}_2$, there exists a threshold $\bar{s}_1(s_2)$ that satisfies

$$s_2 + \bar{s}_1(s_2) - d = \text{arg } var(\bar{M}_{0,t-1}) \left(var(M_{1,t})(s_2)\right). \qquad (4.12)$$

It can be shown that $var(M_{1,t}) \geqslant var(M_{0,t-1})$ if $s_1 \leqslant \bar{s}_1$, and the sign is flipped otherwise. When $s_2 > \bar{s}_2$, $var(M_{1,t}) < var(M_{0,t-1})$ for all s_1. We formalize this result in Proposition 12.
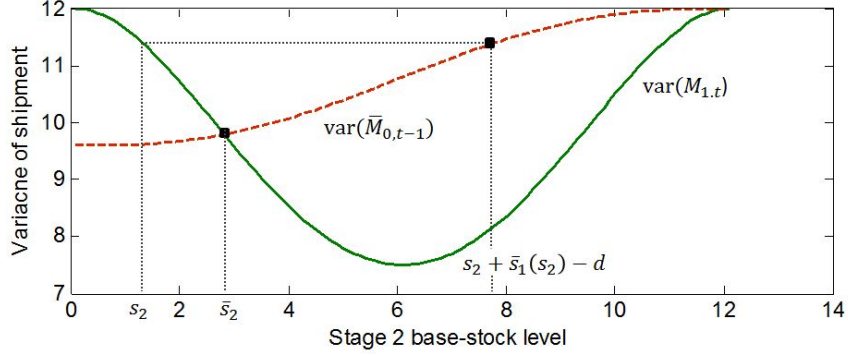
FIGURE 4.4: $\text{var}(M_{1,t})$ and $\text{var}(\bar{M}_{0,t-1})$ as a function of base-stock s_2 under $U(0,12)$

Proposition 12. *Assume D_t follows $U(0,d)$ and consider the region $\mathbf{R} = \{s_1, s_2 | s_2 \leqslant d \leqslant s_1 \leqslant 2d - s_2\}$. Then, there exists a threshold $\bar{s}_2 \in (0, d/2)$, such that for $(s_1, s_2) \in$*

R

1. *When $s_2 \leqslant \bar{s}_2$, there exists a threshold $\bar{s}_1$ such that $\text{var}(M_{1,t}) \geqslant \text{var}(M_{0,t-1})$ if $s_1 \leqslant \bar{s}_1$, and $\text{var}(M_{1,t}) < \text{var}(M_{0,t-1})$ if $s_1 > \bar{s}_1$;*

2. *When $s_2 > \bar{s}_2$, $\text{var}(M_{1,t}) < \text{var}(M_{0,t-1})$ for all s_1.*

Appendix A

Proof of Results

A.1 Proof of Results in Chapter 2

Proposition 1.

Proof. We define $\pi_t(y, w) = g_t(y) + \alpha^\lambda p(y - w)^+$ and take the partial derivative with respect to y:

$$\frac{\partial}{\partial y}\pi_t(y, w) = \frac{\partial}{\partial y}g_t(y) + \left\{ \begin{array}{ll} 0, & \text{if } y < w \\ \alpha^\lambda p, & \text{if } w < y \end{array} \right\}. \tag{A.1}$$

Now, let us consider the three cases in the (d, S) policy. For Case 1, i.e., $w \leqslant d_t$, it can be shown from (2.15) that for small positive ϵ, $\frac{\partial}{\partial y}\pi_t(d_t-\epsilon, w) < 0$ and $\frac{\partial}{\partial y}\pi_t(d_t+\epsilon, w) > 0$. Since $\pi_t(y, w)$ is convex in y, we have $y_t^*(w) = d_t$ when $w \leqslant d_t$. The other two cases can be similarly proved. $\qquad\square$

Lemma 1.

Proof. Taking the first derivative of $g_t(y)$, we have

$$\frac{\partial}{\partial y}g_t(y) = (h + b)F_t(y) - b + \alpha^\lambda(1 - \alpha)cy. \tag{A.2}$$

Then, the expressions in (2.17) can be obtained by solving $\frac{\partial}{\partial y}g_t(y) = 0$ and $\frac{\partial}{\partial y}g_t(y) = -\alpha^\lambda p$. $\qquad\square$

Proposition 2.

Proof. (a) can be directly obtained from Lemma 1 and the definition of the usual stochastic order. We prove (b) and (c) by induction. As shown in (2.11), the claim trivially holds for $t = T + 1$. Assume $V_{t+1}(x, w) = -\alpha^\lambda cx + W_{t+1}(w)$ for all $(x, w) \in \mathcal{B}_{t+1}$, then

$$V_t(x, w) = -\alpha^\lambda cx + \min_{x \leqslant y} \{J_t(y, w)\}, \tag{A.3}$$

where $J_t(y, w) = H_t(y) + \alpha^\lambda p(y - w)^+ + \alpha^\lambda cy + \alpha \mathsf{E} V_{t+1}(y - D_t, w + \theta D_t)$. To solve the problem in (A.3), we consider the following three cases.

Case 1: $w_t \leqslant d_t$. To see that d_t is a minimizer of J_t, note from (a) and demand non-negativity that $x_{t+1} = d_t - D_t \leqslant d_{t+1}$, i.e., $(x_{t+1}, w_{t+1}) \in \mathcal{B}_{t+1}$. From induction and Proposition 1, it can be shown that $y_t^*(w_t) = d_t$. If $x_t \leqslant d_t$, i.e., $(x_t, w_t) \in \mathcal{B}_t$, the base-stock is achievable, then

$$W_t(w_t) = \min_{x_t \leqslant y} \{J_t(y, w_t)\} = \alpha \mathsf{E} W_{t+1}(w_t + \theta D_t) + L_t(w_t).$$

Case 2: $d_t < w_t \leqslant S_t$. To see that w_t is a minimizer of J_t, note from (a) and demand non-negativity that $x_{t+1} = w_t - D_t \leqslant S_t \leqslant S_{t+1}$ and $x_{t+1} = w_t - D_t \leqslant w_t + \theta D_t = w_{t+1}$. Therefore, $x_{t+1} \leqslant S_{t+1} \wedge w_{t+1}$, i.e., $(x_{t+1}, w_{t+1}) \in \mathcal{B}_{t+1}$. From induction and Proposition 1, it can be shown that $y_t^*(w_t) = w_t$. If $x_t \leqslant w_t$, i.e., $(x_t, w_t) \in \mathcal{B}_t$, the base-stock is achievable, then

$$W_t(w_t) = \min_{x_t \leqslant y} \{J_t(y, w_t)\} = \alpha \mathsf{E} W_{t+1}(w_t + \theta D_t) + g_t(w_t).$$

Case 3: $S_t < w_t$. To see that S_t is a minimizer of J_t, note from (a) and demand non-negativity that $x_{t+1} = S_t - D_t \leqslant S_{t+1}$ and $x_{t+1} = S_t - D_t < w_t + \theta D_t = w_{t+1}$. Therefore, $x_{t+1} \leqslant S_{t+1} \wedge w_{t+1}$, i.e., $(x_{t+1}, w_{t+1}) \in \mathcal{B}_{t+1}$. From induction and Proposition 1, it can be shown that $y_t^*(w_t) = S_t$. If $x_t \leqslant S_t$, i.e., $(x_t, w_t) \in \mathcal{B}_t$, the base-stock is achievable, then

$$W_t(w_t) = \min_{x_t \leqslant y} \{J_t(y, w_t)\} = \alpha \mathsf{E} W_{t+1}(w_t + \theta D_t) + g_t(S_t).$$

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Summarizing the above three cases, we prove the optimality of the (d, S) policy and the decomposition of $V_t(x, w)$. Moreover, since $W_{t+1}(\cdot)$ is convex from induction, $W_t(\cdot)$ is also convex. $\qquad\square$

Lemma 2.

Proof. From (2.22) and (2.25), $M^-(u) \leqslant M(u)$ can be directly obtained by applying Jensen's inequality. Furthermore, when $u \to \infty$, $M^-(u) = M(u) = p(u - \rho\mu^m)$. $\qquad\square$

Lemma 3.

Proof. Lemma 2 shows that $M^-(u) \leqslant M(u)$ for all u. From the construction of $\Gamma(u)$ and convexity of $M(u)$, we have $\Gamma(u) \leqslant M(u)$. Then from (2.26), $\bar{M}(u) \leqslant M(u)$ holds for all u. Now, we replace M_t with M_t^- and prove the result by induction. Let $V_t^-(x, w)$ be the minimum expected cost function by replacing M_t with M_t^-. It can be easily shown that $V_{T+1}^-(x, w) = V_{T+1}(x, w)$. Assume $V_{t+1}^-(x, w) \leqslant V_{t+1}(x, w)$ for all (x, w), then

$$V_t^-(x, w) = \min_{x \leqslant y} \left\{ H_t(y) + \alpha^m M_t^-(y - w) + \alpha^m c(y - x) + \alpha \mathsf{E} V_{t+1}^-(y - D_t, w + \theta D_t) \right\}$$

$$\leqslant \min_{x \leqslant y} \left\{ H_t(y) + \alpha^m M_t(y - w) + \alpha^m c(y - x) + \alpha \mathsf{E} V_{t+1}(y - D_t, w + \theta D_t) \right\}$$

$$= V_t(x, w).$$

Similar proof applies to the statement with $\bar{M}_t$. $\qquad\square$

Lemma 4. The proof is straightforward algebra and hence omitted here.

Proposition 3.

Proof. We define $\bar{\pi}_t(y, \bar{w}) = \bar{g}_t(y) + \alpha^m \bar{M}_t(y - \bar{w} + \rho\mu_t^m)$ and take derivative with respect to y:

$$\frac{\partial}{\partial y} \bar{\pi}_t(y, \bar{w}) = \frac{\partial}{\partial y} \bar{g}_t(y) + \left\{ \begin{array}{ll} 0, & \text{if } y < \bar{w} - a_t' \\ \alpha^m \bar{p}_t, & \text{if } \bar{w} - a_t' < y < \bar{w} + a_t'' \\ \alpha^m p, & \text{if } \bar{w} + a_t'' < y \end{array} \right\}. \qquad \text{(A.4)}$$

Now, let us consider the five cases in the $(\mathbf{d}, \mathbf{a}, S)$ policy. For Case 1, i.e., $\bar{w} \leqslant d_t - a_t''$, it can be shown from (2.35) that for small positive ϵ, $\frac{\partial}{\partial y}\bar{\pi}_t(d_t - \epsilon, \bar{w}) < 0$ and $\frac{\partial}{\partial y}\bar{\pi}_t(d_t + \epsilon, \bar{w}) > 0$. Since $\bar{\pi}_t(y, \bar{w})$ is convex in y, we have $\bar{y}_t^*(\bar{w}) = d_t$ when $\bar{w} \leqslant d_t - a_t''$. The proofs of other cases are similar. $\qquad\square$

Lemma 5.

Proof. Same as the proof of Lemma 1 by replacing $g_t(\cdot)$ with $\bar{g}_t(\cdot)$, and $-\alpha^\lambda p$ with $-\alpha^m \bar{p}_t$. $\qquad\square$

Proposition 4.

Proof. (a) can be directly obtained from Lemma 1, Lemma 5, and the definition of the usual stochastic order. We now prove (b) and (c) by induction. First, we derive $y^*(w)$ from (2.33) and the definition of $\bar{w}$ as follows:

$$
y^*(w) = \left\{
\begin{array}{ll}
d, & \text{if } w + \rho A'' \leqslant d \\
w + \rho A'', & \text{if } d < w + \rho A'' \leqslant \bar{d} \\
\bar{d}, & \text{if } w + \rho A' \leqslant \bar{d} < w + \rho A'' \\
w + \rho A', & \text{if } \bar{d} < w + \rho A' \leqslant S \\
S, & \text{if } S < w + \rho A'
\end{array}
\right\}. \tag{A.5}
$$

Thus, $x_t \leqslant y_t^*(w_t)$ is equivalent to $x_t \leqslant \bar{y}_t^*(\bar{w}_t)$, i.e., $(x_t, \bar{w}_t) \in \bar{\mathcal{B}}_t$. As shown in (2.32), the claim trivially holds for $t = T + 1$. Assume $\bar{V}_{t+1}(x, \bar{w}) = -\alpha^m cx + \bar{W}_{t+1}(\bar{w})$ for all $(x, \bar{w}) \in \bar{\mathcal{B}}_{t+1}$, then

$$
\bar{V}_t(x, \bar{w}) = -\alpha^m cx + \min_{x \leqslant y}\left\{\bar{J}_t(y, \bar{w})\right\}, \tag{A.6}
$$

where $\bar{J}_t(y, \bar{w}) = H_t(y) + \alpha^m \bar{M}_t(y - \bar{w} + \rho\mu_t^m) + \alpha^m cy + \alpha \mathsf{E}\bar{V}_{t+1}(y - D_t, \bar{w} + \theta D_t + \rho\mu_{t+m} - \rho\mu_t)$. To solve the problem in (A.6), we consider the following five cases.

Case 1: $\bar{w}_t \leqslant d_t - a_t''$, i.e., $w_t + \rho A_t'' \leqslant d_t$. To see that d_t is a minimizer of $\bar{J}_t$, note from (a) and demand non-negativity that $x_{t+1} = d_t - D_t \leqslant d_{t+1}$. Therefore, we have $(x_{t+1}, \bar{w}_{t+1}) \in \bar{\mathcal{B}}_{t+1}$. From induction and Proposition 3, it can be shown that $\bar{y}_t^*(\bar{w}_t) = d_t$. If $x_t \leqslant d_t$, i.e., $(x_t, \bar{w}_t) \in \bar{\mathcal{B}}_t$, the base-stock is achievable, the rest of the proof is similar to Proposition 2.

Case 2: $d_t - a''_t < \bar{w}_t \leqslant \bar{d}_t - a''_t$, i.e., $d_t < w_t + \rho A''_t \leqslant \bar{d}_t$. To see that $\bar{w}_t + a''_t$ is a minimizer of $\bar{J}_t$, note from (a) and demand non-negativity that $x_{t+1} = w_t + \rho A''_t - D_t \leqslant \bar{d}_{t+1}$ and $x_{t+1} = w_t + \rho A''_t - D_t \leqslant w_t + \theta D_t + \rho A''_t \leqslant w_{t+1} + \rho A''_{t+1}$. Therefore, we have $(x_{t+1}, \bar{w}_{t+1}) \in \bar{\mathcal{B}}_{t+1}$. The rest of the proof is similar to Case 1.

Case 3: $\bar{d}_t - a''_t < \bar{w}_t \leqslant \bar{d}_t + a'_t$, i.e., $w_t + \rho A'_t \leqslant \bar{d}_t < w_t + \rho A''_t$. To see that $\bar{d}_t$ is a minimizer of $\bar{J}_t$, note from (a) and demand non-negativity that $x_{t+1} = \bar{d}_t - D_t \leqslant \bar{d}_{t+1}$ and $x_{t+1} = \bar{d}_t - D_t < w_t + \theta D_t + \rho A''_t \leqslant w_{t+1} + \rho A''_{t+1}$. Therefore, we have $(x_{t+1}, \bar{w}_{t+1}) \in \bar{\mathcal{B}}_{t+1}$. The rest of the proof is similar to Case 1.

Case 4: $\bar{d}_t + a'_t < \bar{w}_t \leqslant S_t + a'_t$, i.e., $\bar{d}_t < w_t + \rho A'_t \leqslant S_t$. To see that $\bar{w}_t - a'_t$ is a minimizer of $\bar{J}_t$, note from (a) and demand non-negativity that $x_{t+1} = w_t + \rho A'_t - D_t \leqslant S_{t+1}$ and $x_{t+1} = w_t + \rho A'_t - D_t \leqslant w_t + \theta D_t + \rho A'_t \leqslant w_{t+1} + \rho A'_{t+1}$. Therefore, we have $(x_{t+1}, \bar{w}_{t+1}) \in \bar{\mathcal{B}}_{t+1}$. The rest of the proof is similar to Case 1.

Case 5: $S_t + a'_t < \bar{w}_t$, i.e., $S_t < w_t + \rho A'_t$. To see that S_t is a minimizer of $\bar{J}_t$, note from (a) and demand non-negativity that $x_{t+1} = S_t - D_t \leqslant S_{t+1}$ and $x_{t+1} = S_t - D_t \leqslant w_t + \theta D_t + \rho A'_t \leqslant w_{t+1} + \rho A'_{t+1}$. Therefore, from (A.5) we have $x_{t+1} \leqslant y^*_t(w_{t+1})$, thus, $(x_{t+1}, \bar{w}_{t+1}) \in \bar{\mathcal{B}}_{t+1}$. The rest of the proof is similar to Case 1.

Summarizing the above three cases, we prove the optimality of the $(\mathbf{d}, \mathbf{a}, S)$ policy and the decomposition of $\bar{V}_t(x, \bar{w})$. Since $\bar{W}_{t+1}(\cdot)$ is convex from induction, $\bar{W}_t(\cdot)$ is also convex. $\qquad\square$

Lemma 6.

Proof. To simplify the notation, we assume $m = 1$ and drop the superscript without loss of generalization. By definition of loss function, we have

$$\hat{F}_t(\mu_t) = \int_{\mu_t}^{\infty} \bar{F}_t(y)dy = \int_0^{\mu_t} F_t(y)dy = \int_0^{F_t(\mu_t)} (\mu_t - F_t^{-1}(y))dy,$$

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where $F^{-1}(\cdot)$ is the inverse function of $F(\cdot)$. From (2.27) we have

$$A'_t = (\mu_t F_t(\mu_t) - \hat{F}_t(\mu_t))/F_t(\mu_t) = \int_0^{F_t(\mu_t)} F_t^{-1}(y) dy / F_t(\mu_t).$$

Similarly, it can be shown that $A''_t = \int_{F_t(\mu_t)}^1 F_t^{-1}(y) dy / \bar{F}_t(\mu_t)$. Due to usual stochastic order, $F_t^{-1}(y)$ is increasing in t for any $y \in [0,1]$. Given that $A_t = F_t(\mu_t) = 1 - \bar{F}_t(\mu_t)$ is constant over t, we have A'_t and A''_t are increasing in t. □

Proposition 5. Similar to the proof of Proposition 2 and hence omitted here.

A.2   Proof of Results in Chapter 3

Lemma 9 (Karush, 1959) shows the additive separation of a function value.

Lemma 9. *If a function $f(y)$ is convex on $(-\infty, \infty)$ and attains its minimum at y^*, then*

$$\min_{a \leqslant y \leqslant b} f(y) = f_L(a) + f_U(b),$$

where $f_L(a) = \min_{a \leqslant y} f(y)$ is convex non-decreasing in a, and $f_U(b) = f(b) - f(b \vee y^*)$ is convex non-increasing in b.

Proposition 6.

Proof. We prove by induction. The claim trivially holds for $t = T + 1$. Assume $V_{t+1}(\mathbf{x}, w) = f_{1,t+1}(x_1) + f_{2,t+1}(x_2) + f_{3,t+1}(w)$, then

$$V_t(\mathbf{x}, w) = \min_{\mathbf{y}, r \in S(\mathbf{x}, w)} \left\{ \begin{array}{l} H_{1,t}(x_1) + \alpha \mathsf{E}_{D_t} f_{1,t+1}(y_1 - D_t) \\ + H_{2,t}(x_2) + c(y_2 - x_2) + \alpha \mathsf{E}_{D_t} f_{2,t+1}(y_2 - D_t) \\ + H_{3,t}(r) + \beta_i (r - w)^+ + \beta_o (r - w)^- + \alpha \mathsf{E}_{D_t} f_{3,t+1}(r + \theta D_t) \end{array} \right\}.$$
(A.7)

Let $g_{1,t}(y_1) = \alpha \mathsf{E}_{D_t} f_{1,t+1}(y_1 - D_t)$, and $g_{2,t}(y_2) = cy_2 + \alpha \mathsf{E}_{D_t} f_{2,t+1}(y_2 - D_t)$. Since $f_{i,t+1}(\cdot)$ is convex (from induction), by Lemma 9 we can decompose the cost functions:

$$\min_{x_1 \leqslant y_1 \leqslant x_2} g_{1,t}(y_1) = \min_{x_1 \leqslant y_1} \left\{ \alpha \mathsf{E}_{D_t} f_{1,t+1}(y_1 - D_t) \right\} + \Gamma_{2,t}(x_2),$$

$$\min_{x_2 \leqslant y_2 \leqslant r} g_{2,t}(y_2) = \min_{x_2 \leqslant y_2} \left\{ cy_2 + \alpha \mathsf{E}_{D_t} f_{2,t+1}(y_2 - D_t) \right\} + \Gamma_{3,t}(r),$$

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where the induced penalty functions $\Gamma_{2,t}(x_2)$ and $\Gamma_{3,t}(r)$ are expressed in (3.12) and (3.13), respectively.

Now, let us define $f_{1,t}(x_1)$ as in (3.9), and

$$g_{3,t}(r) = H_{3,t}(r) + \Gamma_{3,t}(r) + \alpha \mathsf{E}_{D_t} f_{3,t+1}(r + \theta D_t).$$

Plugging the expressions of $f_{1,t}(x_1)$ and $g_{3,t}(r)$ in (A.7), we have

$$V_t(\mathbf{x}, w) = f_{1,t}(x_1) + H_{2,t}(x_2) + \Gamma_{2,t}(x_2) + \min_{x_2 \leqslant y_2} \{\alpha \mathsf{E}_{D_t} f_{2,t+1}(y_2 - D_t)\}$$

$$+ \min_{x_2 \leqslant r \leqslant w+K} \{g_{3,t}(r) + \beta_i(r - w)^+ + \beta_o(r - w)^-\}. \tag{A.8}$$

Let $\tilde{r}_t = \arg\min_r \{g_{3,t}(r)\}$, and $r_t^* = \arg\min_r \{g_{3,t}(r) + \beta_i(r - w)^+ + \beta_o(r - w)^-\}$. The convexity of $g_{3,t}(r)$ implies the existence of the one-sided derivative $\partial g_{3,t}(r)/\partial r$. Define

$$l_t^* = \sup\{r : \frac{\partial g_{3,t}(r)}{\partial r} \leqslant -\beta_i\}, \quad u_t^* = \sup\{r : \frac{\partial g_{3,t}(r)}{\partial r} \leqslant \beta_o\}.$$

The monotonicity of $\partial g_{3,t}(r)/\partial r$ implies $l_t^* \leqslant \tilde{r}_t \leqslant u_t^*$. Using Proposition B-7 in Heyman and Sobel (1984), we have

$$r_t^* = \left\{ \begin{array}{ll} l_t^*, & \text{if } w \leqslant l_t^*, \\ w, & \text{if } l_t^* < w \leqslant u_t^*, \\ u_t^*, & \text{if } u_t^* < w. \end{array} \right\} \tag{A.9}$$

Define $L_t(w) = -\beta_i(w - l_t^*) + g_{3,t}(l_t^*)$, $U_t(w) = \beta_o(w - u_t^*) + g_{3,t}(u_t^*)$, and let

$$W_t(w) = \left\{ \begin{array}{ll} L_t(w), & \text{if } w \leqslant l_t^*, \\ g_{3,t}(w), & \text{if } l_t^* < w \leqslant u_t^*, \\ U_t(w), & \text{if } u_t^* < w. \end{array} \right\} \tag{A.10}$$

From (A.9) it can be easily shown that

$$W_t(w) = \min_r \{g_{3,t}(r) + \beta_i(r - w)^+ + \beta_o(r - w)^-\}.$$

Now, we impose the constraint $x_2 \leqslant r \leqslant w + K$. First, let

$$r_t^{**} = \arg\min_{x_2 \leqslant r \leqslant w+K} \{g_{3,t}(r) + \beta_i(r - w)^+ + \beta_o(r - w)^-\}, \tag{A.11}$$

and define the induced penalty functions $\Lambda_{2,t}(x_2)$ and $\Lambda_{3,t}(w)$ according to (3.17) and (3.18), respectively. Then, we define $f_{3,t}(w)$ as in (3.11). The convexity of $f_{3,t}(w_1)$ can be easily proved by showing that $\partial f_{3,t}(w)/\partial w$ is non-decreasing in w. Next, we prove the decomposition of echelon 3:

$$\min_{x_2 \leqslant r \leqslant w+K} \left\{ g_{3,t}(r) + \beta_i(r-w)^+ + \beta_o(r-w)^- \right\} = f_{3,t}(w) + \Lambda_{2,t}(x_2). \qquad (A.12)$$

The echelon system dynamics and constraint guarantee that $x_2 \leqslant w$ holds for all periods. To prove (A.12), we consider all possible relationships between x_2, w, l_t^* and u_t^*, as extensively described in the four cases below.

Case 1. When $w \leqslant l_t^* - K$, we have $r_t^{**} = w + K \leqslant l_t^*$, $f_{3,t}(w) = g_{3,t}(w+K) + \beta_i K$, and $\Lambda_{2,t}(x_2) = 0$. Thus, $f_{3,t}(w) + \Lambda_{2,t}(x_2) = g_{3,t}(w+K) + \beta_i K = g_{3,t}(r_t^{**}) + \beta_i(r_t^{**} - w)^+ + \beta_o(r_t^{**} - w)^-$, i.e., (A.12) holds.

Case 2. When $l_t^* - K < w \leqslant l_t^*$, we have $r_t^{**} = r_t^* = l_t^*$, $f_{3,t}(w) = L_t(w)$ and $\Lambda_{2,t}(x_2) = 0$. Clearly, (A.12) holds.

Case 3. When $l_t^* < w$, and $x_2 \leqslant u_t^*$, we have $r_t^{**} = r_t^* = \tilde{r}_t$, $f_{3,t}(w) = g_{3,t}(w)$, and $\Lambda_{2,t}(x_2) = 0$. Clearly, (A.12) holds.

Case 4. When $u_t^* < x_2 \leqslant w$, we have $r_t^{**} = x_2$, $f_{3,t}(w) = U_t(w)$ and $\Lambda_{2,t}(x_2) = g_{3,t}(x_2) - U_t(x_2)$. Thus, $f_{3,t}(w) + \Lambda_{2,t}(x_2) = g_{3,t}(x_2) + U_t(w) - U_t(x_2) = g_{3,t}(x_2) + \beta_i(w - x_2) = g_{3,t}(r_t^{**}) + \beta_i(r_t^{**} - w)^+ + \beta_o(r_t^{**} - w)^-$, i.e., (A.12) holds.

Therefore, we verified that (A.12) holds in all cases. Substituting (A.12) into (A.8), and defining $f_{2,t}(x_2)$ as in (3.10), we complete the induction $V_t(\mathbf{x}, w) = f_{1,t}(x_1) + f_{2,t}(x_2) + f_{3,t}(w)$. Using Lemma 9, all induced penalty functions are convex, thus, $f_{i,t}(\cdot)$ is convex, $(i = 1, 2, 3)$. $\qquad \square$

Proposition 7.

Proof. We prove by induction. $V_{T+1}(\mathbf{x}, \mathbf{w}) = 0 = V_{T+1}^1(x_1, w_1) + V_{T+1}^2(x_1, x_2, w_2)$.

Suppose $V_{t+1}(\mathbf{x}, \mathbf{w}) \geqslant V_{t+1}^1(x_1, w_1) + V_{t+1}^2(x_1, x_2, w_2)$ for all $(\mathbf{x}, \mathbf{w})$, then

$$V_t(\mathbf{x}, \mathbf{w}) = \min_{\mathbf{y}, r_1 \in S(\mathbf{x}, \mathbf{w})} \{G_t(\mathbf{x}, \mathbf{w}, y_2, r_1) \tag{A.13}$$

$$+ \alpha \mathsf{E}_{D_t} V_{t+1}(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t, r_1 + \theta D_t)\}$$

$$\geqslant \min_{\mathbf{y}, r_1 \in S(\mathbf{x}, \mathbf{w})} \{G_t(\mathbf{x}, \mathbf{w}, y_2, r_1) \tag{A.14}$$

$$+ \alpha \mathsf{E}_{D_t} V_{t+1}^1(y_1 - D_t, r_1 + \theta D_t) + \alpha \mathsf{E}_{D_t} V_{t+1}^2(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t)\}$$

$$\geqslant \min_{\mathbf{y}, r_1 \in S(\mathbf{x}, \mathbf{w})} \{G_t^1(x_1, w_1, r_1) + \alpha \mathsf{E}_{D_t} V_{t+1}^1(y_1 - D_t, r_1 + \theta D_t)\} \tag{A.15}$$

$$+ \min_{\mathbf{y}, r_1 \in S(\mathbf{x}, \mathbf{w})} \{G_t^2(\mathbf{x}, w_2, y_2) + \alpha \mathsf{E}_{D_t} V_{t+1}^2(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t)\}$$

$$\geqslant \min_{x_1 \leqslant y_1 \leqslant r_1 \leqslant w_1 + K} \{G_t^1(x_1, w_1, r_1) + \alpha \mathsf{E}_{D_t} V_{t+1}^1(y_1 - D_t, r_1 + \theta D_t)\} \tag{A.16}$$

$$+ \min_{x_1 \leqslant y_1 \leqslant x_2 \leqslant y_2 \leqslant w_2} \{G_t^2(\mathbf{x}, w_2, y_2) + \alpha \mathsf{E}_{D_t} V_{t+1}^2(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t)\}$$

$$= V_t^1(x_1, w_1) + V_t^2(x_1, x_2, w_2). \tag{A.17}$$

The inequality in (A.14) and (A.16) are due to induction and constraint relaxation, respectively. The above relationship holds for all $(\mathbf{x}, \mathbf{w})$ in period t, completing the induction. $\qquad \square$

Lemma 7.

Proof. We write out the flow conservation of w_2 and x_1 from $s = 1$ to $s = t - 1$.

$$\mathsf{E}_{D_1, \ldots, D_{t-1}} w_{2,t} - w_{2,1} = \sum_{s=1}^{t-1} \rho z_{1,s} - \sum_{s=1}^{t-1} \mu_s, \tag{A.18}$$

$$\mathsf{E}_{D_1, \ldots, D_{t-1}} x_{1,t} - x_{1,1} = \sum_{s=1}^{t-1} z_{1,s} - \sum_{s=1}^{t-1} \mu_s. \tag{A.19}$$

The result is shown by subtracting $\rho \times$ (A.19) from (A.18). $\qquad \square$

Proposition 8.

Proof. We first specify the cost functions when demand is deterministic. Define

$$H_{1,t}^{2d}(x_1) = (h_1^2 + h_2 + \eta_2 + b_2)(d_t(\omega) - x_1)^+ + h_1^2(x_1 - d_t(\omega)),$$

$$H_{2,t}^d(x_2) = h_2(x_2 - d_t(\omega)), \quad H_{3,t}^d(a) = \eta_2(a - d_t(\omega)).$$

We then prove by induction. The claim trivially holds for $t = T+1$. Now, we assume $V_{t+1}^d(x_1, x_2 \mid \omega) = v_{t+1}^1(x_1 \mid \omega) + v_{t+1}^2(x_2 \mid \omega)$. Let $g_{1,t}^d = \alpha v_{t+1}^i(y_1 - d(\omega) \mid \omega)$ and $g_{2,t}^d = c_2 y_2 + \alpha v_{t+1}^i(y_2 - d(\omega) \mid \omega)$. From the convexity of $v_{t+1}^i(\cdot)$ and Lemma 9, we can decompose the cost functions of echelon 1 and 2 as follows:

$$\min_{x_1 \leqslant y_1 \leqslant x_2} g_{1,t}^d(y_1) = \min_{x_1 \leqslant y_1} \left\{ \alpha v_{t+1}^1(y_1 - d(\omega) \mid \omega) \right\} + \Gamma_{2,t}^d(x_2),$$

$$\min_{x_2 \leqslant y_2 \leqslant a} g_{2,t}^d(y_2) = \min_{x_2 \leqslant y_2} \left\{ c y_2 + \alpha v_{t+1}^2(y_2 - d(\omega) \mid \omega) \right\} + \Gamma_{1,t}^d(a),$$

where $a = \rho x_1 + B_t(\omega)$. Let $y_{1,t}^*$ minimize $g_{i,t}^d(y_i)$, the induced penalty functions are

$$\Gamma_{2,t}^d(x_2) = \left\{ \begin{array}{ll} \alpha \left[v_{t+1}^1(x_2 - d_t(\omega) \mid \omega) - v_{t+1}^1(y_{1,t}^* - d_t(\omega) \mid \omega) \right], & x_2 \leqslant y_{1,t}^*, \\ 0, & \text{otherwise,} \end{array} \right\}$$

$$\Gamma_{1,t}^d(x_1) = \left\{ \begin{array}{ll} c(a - y_{2,t}^*) + \alpha \left[v_{t+1}^2(a - d_t(\omega) \mid \omega) - v_{t+1}^2(y_{2,t}^* - d_t(\omega) \mid \omega) \right], & a \leqslant y_{2,t}^*, \\ 0, & \text{otherwise.} \end{array} \right\}$$

From Lemma 9, the functions above are convex. Therefore, the following functions are convex:

$$v_t^1(x_1 \mid \omega) = H_{1,t}^{2d}(x_1) + H_{3,t}^d(a) + \Gamma_{1,t}^d(a) + \min_{x_1 \leqslant y_1} \left\{ \alpha v_{t+1}^1(y_1 - d_t(\omega) \mid \omega) \right\},$$

$$v_t^2(x_2 \mid \omega) = H_{2,t}^d(x_2) + \Gamma_{2,t}^d(x_2) + \min_{x_2 \leqslant y_2} \left\{ c(y_2 - x_2) + \alpha v_{t+1}^2(y_2 - d_t(\omega) \mid \omega) \right\}.$$

Furthermore, $V_t^d(x_1, x_2 \mid \omega) = v_t^1(x_1 \mid \omega) + v_t^2(x_2 \mid \omega)$, completing the proof. $\qquad \square$

A.3 Proof of Results in Chapter 4

Lemma 1.

Proof. Since stage 2 orders from an ample supply, it can be easily shown that $M_{2,t} = D_{t-1}$. Equation (4.1) and (4.2) can be directly obtained from the system dynamics $B_{2,t} = B_{2,t-1} + D_{t-1} - M_{1,t}$, and $B_{1,t} = B_{1,t-1} + D_{t-1} - M_{0,t-1}$, respectively. To derive local backorders $B_{2,t}$ and $B_{1,t}$, we need the dynamics of local net inventory levels. For stage $j = 1, 2$ and period t, define

$$
\begin{aligned}
IL_{j,t} &= \text{local net inventory level after receiving the shipment} \\
&= s_j - D_{t-1} - B_{j+1,t-1}
\end{aligned}
$$

where $B_{3,t} \equiv 0$ due to the ample supply. Plugging the above relationship into $B_{j,t} = -\min\{0, IL_{j,t}\}$, we can immediately get equation (4.3) and (4.4). Below we write out the full expressions of $M_{1,t}$ and $M_{0,t-1}$ for future reference.

$$M_{1,t} = \min\{0, s_2 - D_{t-1}\} - \min\{0, s_2 - D_{t-2}\} + D_{t-1}, \qquad (A.20)$$

$$M_{0,t-1} = \min\{0, s_1 - D_{t-1} + \min\{0, s_2 - D_{t-2}\}\}$$

$$- \min\{0, s_1 - D_{t-2} + \min\{0, s_2 - D_{t-3}\}\} + D_{t-1}, \qquad (A.21)$$

completing the proof. □

Proposition 10.

Proof. For any cycle such that $B_{2,t} = 0$, $B_{2,t+1} > 0$, $B_{2,t+2} > 0$, ..., $B_{2,t+\tau} > 0$, $B_{2,t+\tau+1} = 0$, we have $D_{t-1} \leqslant s_2$, $D_t > s_2$, $D_{t+1} > s_2$, ..., $D_{t+\tau-1} > s_2$, $D_{t+\tau} \leqslant s_2$. Recall from Lemma 1 that $M_{2,t} = D_{t-1}$. Applying (A.20) repeatedly, we have

$$\sum_{i=1}^{\tau+1} M_{2,t+i}^2 - \sum_{i=1}^{\tau+1} M_{1,t+i}^2$$

$$= \sum_{i=1}^{\tau+1} D_{t+i-1}^2 - \left[s_2^2 + \sum_{i=1}^{\tau-1} D_{t+i-1}^2 + (D_{t+\tau-1} + D_{t+\tau} - s_2)^2 \right]$$

$$= 2s_2 D_{t+\tau-1} + 2s_2 D_{t+\tau} - 2D_{t+\tau-1} D_{t+\tau} - 2s_2^2$$

$$= 2(D_{t+\tau-1} - s_2)(s_2 - D_{t+\tau}) \geqslant 0.$$

Thus, we conclude that $\sum_{i=1}^{\tau+1} M_{2,t+i}^2 \geqslant \sum_{i=1}^{\tau+1} M_{1,t+i}^2$. Note that $M_{1,t} = D_{t-1} = M_{2,t}$ when $B_{2,t-1} = B_{2,t} = 0$, i.e., when period t is not included in any cycle. Also, it is straightforward to verify that $\sum_{i=1}^{\tau+1} M_{2,t+i} = \sum_{i=1}^{\tau+1} M_{1,t+i}$. Because $B_{2,t}$ is a stationary process, by the ergodic theorem, we conclude that $E[M_{2,t}^2] \geqslant E[M_{1,t}^2]$ and $E[M_{2,t}] = E[M_{1,t}]$. Therefore, $\text{var}(M_{2,t}) \geqslant \text{var}(M_{1,t})$, completing the proof. □

Proposition 11.

Proof. For any cycle such that $B_{1,t} = 0$, $B_{1,t+1} > 0$, $B_{1,t+2} > 0$, ..., $B_{1,t+\tau} > 0$, $B_{1,t+\tau+1} = 0$, we calculate $M_{0,t+i-1}$ for $i = 1, ..., \tau + 1$ in the following three steps.

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(a) For $i = 1$, since $B_{1,t} = 0$ and $B_{1,t+1} > 0$, we have $D_{t-1} \leqslant s_1 + \min\{0, s_2 - D_{t-2}\} \leqslant s_1 \leqslant s_2$, hence, $M_{0,t} = s_1 + \min\{0, s_2 - D_{t-1}\} = s_1$.

(b) For $i = 2, ..., \tau$, since $B_{1,t+i-1} > 0$ and $B_{1,t+i} > 0$, we have $M_{0,t+i-1} = \min\{0, s_2 - D_{t+i-2}\} - \min\{0, s_2 - D_{t+i-3}\} + D_{t+i-2} = M_{1,t+i-1}$.

(c) For $i = \tau+1$, since $B_{1,t+\tau} > 0$ and $B_{1,t+\tau+1} = 0$, we have $D_{t+\tau} \leqslant s_1 + \min\{0, s_2 - D_{t+\tau-1}\} \leqslant s_1 \leqslant s_2$, and $M_{0,t+\tau} = D_{t+\tau-1} + D_{t+\tau} - \min\{0, s_2 - D_{t+\tau-2}\} - s_1$.

Combining results (a) and (b), we have

$$\sum_{i=1}^{\tau+1} M_{1,t+i}^2 - \sum_{i=1}^{\tau+1} M_{0,t+i-1}^2 = M_{1,t+\tau}^2 + M_{1,t+\tau+1}^2 - \left[s_1^2 + M_{0,t+\tau}^2\right], \qquad \text{(A.22)}$$

where from Lemma 1 and result (c) above, we have

$$M_{1,t+\tau} = \min\{0, s_2 - D_{t+\tau-1}\} - \min\{0, s_2 - D_{t+\tau-2}\} + D_{t+\tau-1},$$

$$M_{1,t+\tau+1} = D_{t+\tau} - \min\{0, s_2 - D_{t+\tau-1}\},$$

$$M_{0,t+\tau} = D_{t+\tau-1} + D_{t+\tau} - \min\{0, s_2 - D_{t+\tau-2}\} - s_1.$$

As shown above, it is not immediately clear whether $M_{1,t}$ is more variable than $M_{0,t-1}$. To further simplify (A.22), we discuss the following four cases.

Case 1. When $D_{t+\tau-2} \leqslant s_2$ and $D_{t+\tau-1} \leqslant s_2$, from $B_{1,t+\tau} > 0$ we have $D_{t+\tau-1} > s_1 + \min\{0, s_2 - D_{t+\tau-2}\} = s_1$. Together with $D_{t+\tau} \leqslant s_1$ as in result (c), equation (A.22) becomes

$$\sum_{i=1}^{\tau+1} M_{1,t+i}^2 - \sum_{i=1}^{\tau+1} M_{0,t+i-1}^2 = D_{t+\tau-1}^2 + D_{t+\tau}^2 - \left[s_1^2 + (D_{t+\tau-1} + D_{t+\tau} - s_1)^2\right]$$

$$= 2(D_{t+\tau-1} - s_1)(s_1 - D_{t+\tau}) \geqslant 0.$$

Case 2. When $D_{t+\tau-2} \leqslant s_2$ and $D_{t+\tau-1} > s_2$, from $B_{1,t+\tau+1} = 0$ we have $D_{t+\tau} \leqslant s_1 + \min\{0, s_2 - D_{t+\tau-1}\} = s_1 + s_2 - D_{t+\tau-1}$. Therefore, equation (A.22) becomes

$$\sum_{i=1}^{\tau+1} M_{1,t+i}^2 - \sum_{i=1}^{\tau+1} M_{0,t+i-1}^2 = s_2^2 + (D_{t+\tau-1} + D_{t+\tau} - s_2)^2 - \left[s_1^2 + (D_{t+\tau-1} + D_{t+\tau} - s_1)^2\right]$$

$$= 2(s_2 - s_1)(s_2 + s_1 - D_{t+\tau-1} - D_{t+\tau}) \geqslant 0.$$

Case 3. When $D_{t+\tau-2} > s_2$ and $D_{t+\tau-1} \leqslant s_2$, from $B_{1,t+\tau} > 0$ we have $D_{t+\tau-1} +$ $D_{t+\tau-2} > s_1 + s_2$. Together with $D_{t+\tau} \leqslant s_1$ as in result (c), equation (A.22) becomes

$$\sum_{i=1}^{\tau+1} M_{1,t+i}^2 - \sum_{i=1}^{\tau+1} M_{0,t+i-1}^2 = (D_{t+\tau-2} + D_{t+\tau-1} - s_2)^2 + D_{t+\tau}^2$$

$$- \left[s_1^2 + (D_{t+\tau-2} + D_{t+\tau-1} + D_{t+\tau} - s_2 - s_1)^2 \right]$$

$$= 2(s_1 - D_{t+\tau})(D_{t+\tau-2} + D_{t+\tau-1} - s_2 - s_1) \geqslant 0.$$

Case 4. When $D_{t+\tau-2} > s_2$ and $D_{t+\tau-1} > s_2$, from $B_{1.t+\tau+1} = 0$ we have $D_{t+\tau} \leqslant$ $s_1 + \min\{0, s_2 - D_{t+\tau-1}\} = s_1 + s_2 - D_{t+\tau-1}$. Together with $D_{t+\tau-2} > s_2 \geqslant s_1$, equation (A.22) becomes

$$\sum_{i=1}^{\tau+1} M_{1,t+i}^2 - \sum_{i=1}^{\tau+1} M_{0,t+i-1}^2 = D_{t+\tau-2}^2 + (D_{t+\tau-1} + D_{t+\tau} - s_2)^2$$

$$- \left[s_1^2 + (D_{t+\tau-2} + D_{t+\tau-1} + D_{t+\tau} - s_2 - s_1)^2 \right]$$

$$= 2(D_{t+\tau-2} - s_1)(s_2 + s_1 - D_{t+\tau-1} - D_{t+\tau}) \geqslant 0.$$

Since the above four cases include all possible demand sample paths, we can conclude that $\sum_{i=1}^{\tau+1} M_{1,t+i}^2 \geqslant \sum_{i=1}^{\tau+1} M_{0,t+i-1}^2$. Note that when $B_{1,t-1} = B_{1,t} = 0$, we have $D_{t-1} \leqslant s_1 \leqslant s_2$ and $D_{t-2} \leqslant s_1 \leqslant s_2$. Hence $M_{1,t} = M_{0,t-1} = D_{t-1}$ when period t is not included in any cycle. Also, it is straightforward to verify that $\sum_{i=1}^{\tau+1} M_{1,t+i} = \sum_{i=1}^{\tau+1} M_{0,t+i-1}$. Because $B_{1,t}$ is a stationary process, by the ergodic theorem, we conclude that $E[M_{1,t}^2] \geqslant E[M_{0,t-1}^2]$ and $E[M_{1,t}] = E[M_{0,t-1}]$. Therefore, $\text{var}(M_{1,t}) \geqslant \text{var}(M_{0,t-1})$, completing the proof. $\qquad\square$

Lemma 2.

Proof. The proof is trivial when $s_2 > d$. We derive $\text{var}(M_{1,t})$ when $s_2 \leqslant d$. From (A.20) we have

$$M_{1,t} = \min\{0, s_2 - D_{t-1}\} - \min\{0, s_2 - D_{t-2}\} + D_{t-1}$$

$$= \min\{s_2, D_{t-1}\} + \max\{s_2, D_{t-2}\} - s_2.$$

Let $X = \min\{s_2, D_{t-1}\}$. Since D_t follows $U(0, d)$, we have

$$E(X) = \frac{1}{d}\int_0^{s_2} x dx + \frac{1}{d}\int_{s_2}^d s_2 dx = s_2 - \frac{s_2^2}{2d}, \tag{A.23}$$

$$E(X^2) = \frac{1}{d}\int_0^{s_2} x^2 dx + \frac{1}{d}\int_{s_2}^d s_2^2 dx = s_2^2 - \frac{2s_2^3}{3d}, \tag{A.24}$$

Similarly, define $Y = \max\{s_2, D_{t-2}\}$, and we have

$$E(Y) = \frac{1}{d}\int_0^{s_2} s_2 dx + \frac{1}{d}\int_{s_2}^d x dx = \frac{d}{2} + \frac{s_2^2}{2d}, \tag{A.25}$$

$$E(Y^2) = \frac{1}{d}\int_0^{s_2} s_2^2 dx + \frac{1}{d}\int_{s_2}^d x^2 dx = \frac{d^2}{3} + \frac{2s_2^3}{3d}. \tag{A.26}$$

Since X and Y are independent, we have

$$\text{var}(M_{1,t}) = \text{var}(X) + \text{var}(Y)$$

$$= E(X^2) - [E(X)]^2 + E(Y^2) - [E(Y)]^2$$

$$= \frac{s_2^3}{3d} - \frac{s_2^4}{4d^2} + \left(\frac{2s_2^3}{3d} - \frac{s_2^4}{4d^2} + \frac{d^2}{12} - \frac{s_2^2}{2}\right)$$

$$= \frac{d^2}{12} - \frac{s_2^2}{2}\left(1 - \frac{s_2}{d}\right)^2, \tag{A.27}$$

completing the proof. $\qquad\square$

Lemma 3.

Proof. The proof is trivial when $d < s_2 < s_1$. We derive $\text{var}(M_{0,t-1})$ for the other two cases. First, when $s_2 \leqslant d < s_1$ and $2d - s_2 < s_1$, from (A.21) we have

$$M_{0,t-1} = \min\{s_2, D_{t-2}, D_{t-2} + D_{t-1} - s_1\} + \max\{s_2, D_{t-3}, s_1 + s_2 - D_{t-2}\} - s_2 \tag{A.28}$$

$$= \min\{s_2, D_{t-2} + D_{t-1} - s_1\} + \max\{D_{t-3}, s_1 + s_2 - D_{t-2}\} - s_2 \tag{A.29}$$

$$= D_{t-1}, \tag{A.30}$$

where (A.28) is simple algebra, (A.29) is due to $d < s_1$, and (A.30) is due to $2d - s_2 < s_1$. Therefore, $\text{var}(M_{0,t-1}) = \text{var}(D_{t-1}) = d^2/12$.

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Next, we show the case $s_2 \leqslant d \leqslant s_1 \leqslant 2d - s_2$. Let us rewrite (A.29) as

$$M_{0,t-1} = \min\{s_1 + s_2 - D_{t-2}, D_{t-1}\} + \max\{s_1 + s_2 - D_{t-2}, D_{t-3}\} + D_{t-2} - s_1 - s_2. \tag{A.31}$$

Let $D_{t-2} = x$. When $x \leqslant s_1 + s_2 - d$, i.e., $s_1 + s_2 - x \geqslant d$, we have

$$E(M_{0,t-1} \mid x) = E(D_{t-1} \mid x) = E(D_{t-1}) = d/2,$$

$$\text{var}(M_{0,t-1} \mid x) = \text{var}(D_{t-1} \mid x) = \text{var}(D_{t-1}) = d^2/12.$$

When $x > s_1 + s_2 - d$, replacing s_2 with $s_1 + s_2 - x$ in (A.23), (A.25), and (A.27), we have

$$E(M_{0,t-1} \mid x) = s_1 + s_2 - x - \frac{(s_1 + s_2 - x)^2}{2d} + \left(\frac{d}{2} + \frac{(s_1 + s_2 - x)^2}{2d} \right) + x - s_1 - s_2 = \frac{d}{2},$$

$$\text{var}(M_{0,t-1} \mid x) = \frac{d^2}{12} - \frac{(s_1 + s_2 - x)^2}{2} \left(1 - \frac{s_1 + s_2 - x}{d} \right)^2.$$

Therefore, $\text{var}(E(M_{0,t-1} \mid x)) = 0$. Denoting $y = s_1 + s_2 - x$, we have

$$\text{var}(M_{0,t-1}) = 0 + E_x(\text{var}(M_{0,t-1} \mid x))$$

$$= \frac{1}{d} \int_0^{s_1+s_2-d} \frac{d^2}{12} dx + \frac{1}{d} \int_{s_1+s_2-d}^{d} \left[\frac{d^2}{12} - \frac{(s_1 + s_2 - x)^2}{2} \left(1 - \frac{s_1 + s_2 - x}{d} \right)^2 \right] dx$$

$$= \frac{d^2}{12} - \frac{1}{2d} \int_{s_1+s_2-d}^{d} y^2 \left(1 - \frac{y}{d} \right)^2 dy.$$

$\square$

Proposition 12.

Proof. Solving the integration in (4.10), we have

$$\text{var}(\bar{M}_{0,t-1}) = \frac{d^2}{15} + \frac{s_2^3}{6d} - \frac{s_2^4}{4d^2} + \frac{s_2^5}{10d^3}. \tag{A.32}$$

Taking derivative of $\text{var}(\bar{M}_{0,t-1})$ with respect to s_2, we have

$$\frac{\partial \text{var}(\bar{M}_{0,t-1})}{\partial s_2} = \frac{s_2^2}{2d} \left(1 - \frac{s_2}{d} \right)^2 \geqslant 0, \tag{A.33}$$

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i.e., $\mathrm{var}(\bar{M}_{0,t-1})$ is increasing throughout $[0, d]$.

Now, let us first examine $s_2 \in [0, d/2]$, in which $\mathrm{var}(M_{1,t})$ is decreasing according to (4.8). Moreover, it can be verified that $\mathrm{var}(M_{1,t}) < \mathrm{var}(\bar{M}_{0,t-1})$ when $s_2 = 0$, and $\mathrm{var}(M_{1,t}) > \mathrm{var}(\bar{M}_{0,t-1})$ when $s_2 = d/2$. Therefore, there exists a point $\bar{s}_2 \in (0, d/2)$ such that

$$\mathrm{var}(M_{1,t})(\bar{s}_2) = \mathrm{var}(\bar{M}_{0,t-1})(\bar{s}_2).$$

Next, we examine $s_2 \in (d/2, d]$. Note that for any $y > s_2 \in (d/2, d)$, we have $y^2(1 - y/d)^2 < s_2^2(1 - s_2/d)^2$. Therefore from (4.10) we have for any $s_2 \in (d/2, d)$

$$\mathrm{var}(\bar{M}_{0,t-1}) > \frac{d^2}{12} - \frac{1}{2(d - s_2)} \int_{s_2}^{d} s_2^2 \left(1 - \frac{s_2}{d}\right)^2 dy = \mathrm{var}(M_{1,t}).$$

In addition, $\mathrm{var}(\bar{M}_{0,t-1}) = \mathrm{var}(M_{1,t})$ when $s_2 = d$. Thus, $\mathrm{var}(M_{1,t}) < \mathrm{var}(\bar{M}_{0,t-1})$ if $s_2 \in [0, d/2)$, and the sign is flipped if $s_2 \in (d/2, d)$. Then, the results can be verified using (4.11) and the continuity of $\mathrm{var}(\bar{M}_{0,t-1})$ and $\mathrm{var}(M_{1,t})$.

$\square$

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