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REGULAR ARTICLE

Inference in a structural heteroskedastic calibration model

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Abstract The main goal of this paper is to study inference in an heteroskedastic calibration model. We embrace a multivariate structural model with known diagonal covariance error matrices, which is a common setup when different measurement methods are compared. Maximum likelihood estimates are computed numerically via the EM algorithm. Consistent estimation of the asymptotic variance of the maximum likelihood estimators and a graphical device for model checking are also discussed. Test statistics are proposed for testing hypotheses of interest with the asymptotic chi-square distribution which guarantees correct asymptotic significance levels. Results of simulations comprising point estimation, interval estimation, and hypothesis testing are reported. An application to a real data set is given. Up to best of our knowledge, topics such as model checking and hypotheses testing have received only scarce attention in the literature on calibration models.

Keywords EM algorithm · Calibration · Estimation · Hypotheses testing · Maximum likelihood · Measurement error models · Structural models

Mathematics Subject Classification 62J05 · 62J99

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1 Introduction

Measurement error models (MEM), also known as errors-in-variables models, are useful for describing different phenomena in many disciplines. MEM establish functional relationships among variables observed subject to random measurement errors. Comprehensive accounts of MEM can be found, for example, in the books by Fuller (1987), Cheng and Van Ness (1999), Carroll et al. (2006), and Buonaccorsi (2010). This work is mainly concentrated on a multiple regression model with measurement error, that is, when the predictors are contaminated with measurement errors. This model corresponds to an extension of the classical multiple linear regression model. Recently, heteroskedastic measurement error models have received attention in the literature. Kulathinal et al. (2002) and Cheng and Riu (2006), for example, present a univariate structural heteroskedastic model, but they concentrated solely on estimation. Our endeavor aims to investigate estimation as well as hypotheses testing in a multiple model.

In this paper we extend the homoskedastic calibration model introduced by Vilca-Labra et al. (2011) to a class of heteroskedastic normal calibration models, meaning that the true covariate is distributed according to a normal distribution and the measurement error variances change across observations. Additionally, we suppose that the measurement errors are uncorrelated and their variances are known and greater than 0. This constitutes a common setup in many application areas such as Analytical Chemistry and Epidemiology, as can be seen in Cheng and Riu (2006).

It is nowadays widely recognized that assessing the adequacy of the postulated model plays a prominent role in a statistical analysis. By reviewing the measurement error models literature, we realize that inference and influence assessment are the subjects of many published works, whereas goodness of fit has received only scarce attention. In order to shorten this gap, we describe a graphical tool for model checking based on the simulated envelope presented by Atkinson (1985).

General results for structural heteroskedastic models are presented by Patriota et al. (2009), Patriota et al. (2011), and Melo et al. (2013), among others. Instead, in our work, as in Vilca-Labra et al. (2011), we take the calibration model on its own grounds and explore specific inferential problems. In particular, we deal with interval estimation and power of statistical tests. The hypotheses testing procedures have applications in the methods comparison problem through a calibration model.

An outline of the paper is as follows. Section 2 covers model formulation and parameter estimation. In Sect. 3 we propose statistics to test hypotheses of interest. Results of a simulation study and an application to a real data set are reported in Sect. 4 and 5, respectively. In Sect. 6 we conclude bringing some general remarks.

2 Model and parameter estimation

Let n be the sample size; X_i, the observed value of the covariate in unit i; Y_{ij}, the j-th observed response in unit i and x_i, the unobserved (true) covariate value for unit i. Relating these variables, as in Shyr and Gleser (1986), we postulate the model

$$X_i = x_i + u_i \quad \text{and} \quad Y_{ij} = \alpha_j + \beta_j x_i + e_{ij}, \tag{1}$$

$j = 1, \ldots, r$ and $i = 1, \ldots, n$. As a motivation, this model is applicable to the comparison of measurement methods problem (Ripley and Thompson 1987; Riu and Rius 1996; Galea-Rojas et al. 2003; de Castro et al. 2004). In this case, r is the number of 'new' methods to be compared to a reference one, whereas α_j and β_j correspond to the additive and multiplicative biases of method j with respect to the reference method.

Letting $\boldsymbol{e}_i = (e_{i1}, \ldots, e_{ir})^\top$ and $\boldsymbol{r}_i = (u_i, \boldsymbol{e}_i^\top)^\top$, we assume that

$$\begin{aligned} &\boldsymbol{r}_i \text{ and } x_j \text{ are independent}, \quad j = 1, \ldots, n, \\ &\boldsymbol{r}_i \overset{\text{indep.}}{\sim} \mathrm{N}_{r+1}(\mathbf{0}, \boldsymbol{\Phi}_i), \\ &\text{and } x_i \overset{\text{iid}}{\sim} \mathrm{N}(\mu, \phi), \quad i = 1, \ldots, n, \end{aligned} \tag{2}$$

where $\boldsymbol{\Phi}_i = \boldsymbol{D}(\kappa_i, \boldsymbol{\lambda}_i)$ stands for a diagonal matrix with $\kappa_i, \lambda_{i1}, \ldots, \lambda_{ir}$ in the diagonal. Variances κ_i and $\boldsymbol{\lambda}_i$ are supposedly known and greater than 0, $i = 1, \ldots, n$.

The model defined by equations (1) can be written as $\boldsymbol{Z}_i = (X_i, \boldsymbol{Y}_i^\top)^\top = \boldsymbol{a} + x_i\boldsymbol{b} + (u_i, \boldsymbol{e}_i^\top)^\top$, where $\boldsymbol{Y}_i = (Y_{i1}, \ldots, Y_{ir})^\top$, $\boldsymbol{a} = (0, \boldsymbol{\alpha}^\top)^\top$, and $\boldsymbol{b} = (1, \boldsymbol{\beta}^\top)^\top$. Then, under the assumptions in (2), it follows that

$$\boldsymbol{Z}_i \overset{\text{indep.}}{\sim} \mathrm{N}_{r+1}(\boldsymbol{a} + \boldsymbol{b}\mu, \boldsymbol{V}_i), \tag{3}$$

where $\boldsymbol{V}_i = \phi\boldsymbol{b}\boldsymbol{b}^\top + \boldsymbol{\Phi}_i$, $i = 1, \ldots, n$. From (3), it is true that

$$q_i = (\boldsymbol{Z}_i - \boldsymbol{a} - \mu\boldsymbol{b})^\top \boldsymbol{V}_i^{-1}(\boldsymbol{Z}_i - \boldsymbol{a} - \mu\boldsymbol{b}) \overset{\text{iid}}{\sim} \chi^2_{r+1}, \quad i = 1, \ldots, n. \tag{4}$$

This distributional result enables us to assess the adequacy of the model, as we will explore in Sect. 5.

Let $\boldsymbol{\theta} = (\boldsymbol{\alpha}^\top, \boldsymbol{\beta}^\top, \mu, \phi)^\top$ be the $(2r + 2) \times 1$ parameter vector. The log-likelihood function corresponding to the model defined by (3) can be written as

$$l(\boldsymbol{\theta}) = \sum_{i=1}^{n} l_i(\boldsymbol{\theta}), \quad l_i(\boldsymbol{\theta}) = \text{const.} - \frac{1}{2}\log|\boldsymbol{V}_i| - \frac{1}{2}q_i, \tag{5}$$

where q_i in (4) can be written as $q_i = q_{i1} - c_i q_{i2}^2$, with

$$\begin{aligned} q_{i1} &= (\boldsymbol{Z}_i - \boldsymbol{a} - \mu\boldsymbol{b})^\top \boldsymbol{\Phi}_i^{-1}(\boldsymbol{Z}_i - \boldsymbol{a} - \mu\boldsymbol{b}) \\ &= \kappa_i^{-1}(X_i - \mu)^2 + (\boldsymbol{Y}_i - \boldsymbol{\alpha} - \mu\boldsymbol{\beta})^\top \boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)(\boldsymbol{Y}_i - \boldsymbol{\alpha} - \mu\boldsymbol{\beta}), \end{aligned}$$

$c_i = \phi(1 + \phi\boldsymbol{b}^\top\boldsymbol{\Phi}_i^{-1}\boldsymbol{b})^{-1}$ and $q_{i2} = \kappa_i^{-1}(X_i - \mu) + \boldsymbol{\beta}^\top\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)(\boldsymbol{Y}_i - \boldsymbol{\alpha} - \mu\boldsymbol{\beta})$, for $i = 1, \ldots, n$. The determinant and the inverse of $\boldsymbol{V}_i$ are $|\boldsymbol{V}_i| = |\boldsymbol{\Phi}_i|c_i^{-1}\phi$ and

$\boldsymbol{V}_i^{-1} = \boldsymbol{\Phi}_i^{-1} - c_i \boldsymbol{\Phi}_i^{-1} \boldsymbol{b}\boldsymbol{b}^\top \boldsymbol{\Phi}_i^{-1}$, $i = 1, \ldots, n$. The maximization of the log-likelihood function (5) is quite involved. So, maximum likelihood estimates are more easily computed with the EM algorithm (Dempster et al. 1977). Let $\boldsymbol{W}_i = (x_i, \boldsymbol{Z}_i^\top)^\top$ be the vectors of complete data. The complete data log-likelihood function is denoted by $l_c(\boldsymbol{\theta})$. Under the assumptions in (2), it follows that

$$\boldsymbol{W}_i \overset{\text{indep.}}{\sim} \mathrm{N}_{r+2}(\boldsymbol{\mu}_w, \boldsymbol{\Sigma}_{wi}), \tag{6}$$

where $\boldsymbol{\mu}_w = \boldsymbol{a}_w + \mu \boldsymbol{b}_w$ and $\boldsymbol{\Sigma}_{wi} = \phi \boldsymbol{b}_w \boldsymbol{b}_w^\top + \boldsymbol{\Phi}_{ci}$, with $\boldsymbol{a}_w = (0, \boldsymbol{a}^\top)^\top$, $\boldsymbol{b}_w = (1, \boldsymbol{b}^\top)^\top$, and $\boldsymbol{\Phi}_{ci} = \boldsymbol{D}(0, \kappa_i, \boldsymbol{\lambda}_i)$, $i = 1, \ldots, n$. Taking into account that

$$|\boldsymbol{\Sigma}_{wi}| = \phi |\boldsymbol{\Phi}_i| \quad \text{and} \quad \boldsymbol{\Sigma}_{wi}^{-1} = \begin{bmatrix} c_i^{-1} & -\boldsymbol{b}^\top \boldsymbol{\Phi}_i^{-1} \\ -\boldsymbol{\Phi}_i^{-1} \boldsymbol{b} & \boldsymbol{\Phi}_i^{-1} \end{bmatrix},$$

the complete data log-likelihood function is given by

$$l_c(\boldsymbol{\theta}) = \sum_{i=1}^{n} l_{ci}(\boldsymbol{\theta}), \tag{7}$$

where

$$l_{ci}(\boldsymbol{\theta}) = \text{const.} - \frac{1}{2} \log |\boldsymbol{\Sigma}_{wi}| - \frac{1}{2} q_{wi},$$

with

$$\begin{aligned} q_{wi} = (\boldsymbol{W}_i - \boldsymbol{\mu}_w)^\top \boldsymbol{\Sigma}_{wi}^{-1} (\boldsymbol{W}_i - \boldsymbol{\mu}_w) &= c_i^{-1}(x_i - \mu)^2 - 2(\boldsymbol{Z}_i - \boldsymbol{a} - \boldsymbol{b}\mu)^\top \boldsymbol{\Phi}_i^{-1} \boldsymbol{b}(x_i - \mu) \\ &+ (\boldsymbol{Z}_i - \boldsymbol{a} - \mu \boldsymbol{b})^\top \boldsymbol{\Phi}_i^{-1} (\boldsymbol{Z}_i - \boldsymbol{a} - \mu \boldsymbol{b}), \end{aligned}$$

$i = 1, \ldots, n$. With the current estimate of $\boldsymbol{\theta}$, in the E step the expectation $\mathrm{E}[l_c(\boldsymbol{\theta}) | \boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n]$ is computed. Owing to the distribution in (6), the E step requires

$$\widehat{x_i} = \mathrm{E}[x_i | \boldsymbol{Z}_i; \boldsymbol{\theta}] = c_i \{\phi^{-1} \mu + \boldsymbol{b}^\top \boldsymbol{\Phi}_i^{-1} (\boldsymbol{Z}_i - \boldsymbol{a})\} \quad \text{and} \tag{8}$$

$$\widehat{x_i^2} = \mathrm{E}[x_i^2 | \boldsymbol{Z}_i; \boldsymbol{\theta}] = c_i + \widehat{x_i}^2, \tag{9}$$

$i = 1, \ldots, n$. In the M step the function in (7) with x_i and x_i^2 replaced by (8) and (9), respectively, is maximized. After computing $\partial l_c(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}$ and solving $\partial l_c(\boldsymbol{\theta}) / \partial \boldsymbol{\theta} = \boldsymbol{0}$ we arrive at

$$\beta_j = \frac{\sum_{i=1}^{n} \lambda_{ij}^{-1} (Y_{ij} - \overline{Y}_{*j}) x_i}{\sum_{i=1}^{n} \lambda_{ij}^{-1} x_i^2 - \overline{x}_{*j}^2 \sum_{i=1}^{n} \lambda_{ij}^{-1}} \quad \text{and} \quad \alpha_j = \overline{Y}_{*j} - \overline{x}_{*j} \beta_j, \tag{10}$$

with

$$\overline{Y}_{*j} = \frac{\sum_{i=1}^{n} \lambda_{ij}^{-1} Y_{ij}}{\sum_{i=1}^{n} \lambda_{ij}^{-1}}, \ \overline{x}_{*j} = \frac{\sum_{i=1}^{n} \lambda_{ij}^{-1} x_i}{\sum_{i=1}^{n} \lambda_{ij}^{-1}}, \quad j = 1, \ldots, r,$$

$$\mu = \overline{x}, \quad \text{and} \quad \phi = n^{-1} \sum_{i=1}^{n} x_i^2 - \overline{x}^2. \tag{11}$$

Maximum likelihood estimates of $\boldsymbol{\theta}$ are computed by cycling from (8) through (11) until convergence. Our stopping rule is based on the relative differences between estimates in two successive iterations, as in (24). Starting values for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be taken from the estimates in the functional model (de Castro et al. 2004) or from the corrected score method (de Castro et al. 2006). de Castro et al. (2004) also provide predictors for $x_i, i = 1, \ldots, n$, from which initial estimates of μ and ϕ can be obtained.

2.1 Score and information matrices

After some algebraic manipulations we get from (5) the elements of the score vector $\boldsymbol{U}(\boldsymbol{\theta})$; namely,

$$\boldsymbol{U}(\boldsymbol{\theta}) = \left(\boldsymbol{U}_{\alpha}(\boldsymbol{\theta})^{\top}, \boldsymbol{U}_{\beta}(\boldsymbol{\theta})^{\top}, U_{\mu}(\boldsymbol{\theta}), U_{\phi}(\boldsymbol{\theta})\right)^{\top} = \sum_{i=1}^{n} \frac{\partial l_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$$
$$= \sum_{i=1}^{n} \boldsymbol{U}_i(\boldsymbol{\theta}), \quad \boldsymbol{U}_i(\boldsymbol{\theta}) = (\boldsymbol{U}_{i\alpha}^{\top}, \boldsymbol{U}_{i\beta}^{\top}, U_{i\mu}, U_{i\phi})^{\top},$$

where

$$\boldsymbol{U}_{i\gamma} = -\frac{1}{2}\frac{\partial \log |\boldsymbol{V}_i|}{\partial \boldsymbol{\gamma}} - \frac{1}{2}\frac{\partial q_i}{\partial \boldsymbol{\gamma}}, \tag{12}$$

for $\boldsymbol{\gamma} = \boldsymbol{\alpha}, \boldsymbol{\beta}, \mu, \phi$, whose elements can be found in Appendix A.

After lengthy algebraic manipulations we get from (12) the elements of the observed information matrix $\boldsymbol{J}(\boldsymbol{\theta})$,

$$\boldsymbol{J}(\boldsymbol{\theta}) = -\sum_{i=1}^{n} \frac{\partial^2 l_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^{\top}} = -\sum_{i=1}^{n} \boldsymbol{L}_i(\boldsymbol{\theta}), \tag{13}$$

whose presentation is postponed to Appendix B. The elements of the expected information matrix $\boldsymbol{F}(\boldsymbol{\theta})$ are

$$\boldsymbol{F}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \mathrm{E}\left[-\frac{\partial^2 l_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^{\top}}\right] = \sum_{i=1}^{n} \boldsymbol{F}_i(\boldsymbol{\theta}) \tag{14}$$

and can be found in Appendix B. The results in Sect. 4 and 5 were obtained using the expected information matrix.

Confidence regions for the parameters can be constructed from asymptotic results. Fahrmeir (1988) handles the lack of i.i.d. observations in maximum likelihood inference. Under some regularity conditions, it can be shown that the approximate distribution of $\widehat{\boldsymbol{\theta}}$ is $\mathrm{N}_{2r+2}\big(\boldsymbol{\theta},\ \boldsymbol{F}(\boldsymbol{\theta})^{-1}\big)$. With respect to the regularity conditions, we assume that the measurement error variances are bounded and $n^{-1}\sum_{i=1}^{n}\kappa_i$ has a finite and greater than 0 limit when $n\to\infty$.

3 Hypotheses testing

As it is usual in the context of regression analysis, now we deal with the problem of testing the general linear hypothesis

$$\mathrm{H}_0 : \boldsymbol{R\theta} = \boldsymbol{d}, \tag{15}$$

where $\boldsymbol{R}$ is a $r_R \times (2r+2)$ matrix of rank r_R and $\boldsymbol{d}$ is a r_R-dimensional vector, $\boldsymbol{R}$ and $\boldsymbol{d}$ known. Hypothesis (15) can be tested using the statistics

$$\begin{aligned} \text{Likelihood ratio:}\quad & LR = -2\{l(\widehat{\boldsymbol{\theta}}_0) - l(\widehat{\boldsymbol{\theta}})\}, \\ \text{Wald:}\quad & W = (\boldsymbol{R}\widehat{\boldsymbol{\theta}} - \boldsymbol{d})^\top\{\boldsymbol{R}\boldsymbol{F}(\widehat{\boldsymbol{\theta}})^{-1}\boldsymbol{R}^\top\}^{-1}(\boldsymbol{R}\widehat{\boldsymbol{\theta}} - \boldsymbol{d}), \quad \text{and} \\ \text{Score:}\quad & S = \boldsymbol{U}(\widehat{\boldsymbol{\theta}}_0)^\top\boldsymbol{R}^\top\{\boldsymbol{R}\boldsymbol{F}(\widehat{\boldsymbol{\theta}}_0)\boldsymbol{R}^\top\}^{-1}\boldsymbol{R}\boldsymbol{U}(\widehat{\boldsymbol{\theta}}_0), \end{aligned} \tag{16}$$

where $\widehat{\boldsymbol{\theta}}_0$ denotes the maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$ restricted to H_0 in (15). Under some suitable regularity conditions (Fahrmeir 1988), when H_0 is true we have that LR, W, and $S \xrightarrow{\mathrm{d}} \chi^2(r_R)$, as $n\to\infty$.

Of particular interest in many applications is the hypothesis

$$\mathrm{H}_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_{10}, \tag{17}$$

where $\boldsymbol{\theta}_1$ is a subset of $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^\top, \boldsymbol{\theta}_2^\top)^\top$. As in the homoskedastic model by Vilca-Labra et al. (2011), some examples of hypotheses involving the biases that might be of interest are

$$\begin{aligned} &\mathrm{H}_{01} : \alpha_1 = \cdots = \alpha_r = 0, \beta_1 = \cdots = \beta_r = 1, \\ &\mathrm{H}_{02} : \alpha_1 = \cdots = \alpha_r = 0, \quad \text{and} \quad \mathrm{H}_{03} : \beta_1 = \cdots = \beta_r = 1. \end{aligned} \tag{18}$$

The score vector, the observed information matrix, and its inverse are partitioned accordingly, resulting in $\boldsymbol{U}(\boldsymbol{\theta}) = (\boldsymbol{U}_1(\boldsymbol{\theta})^\top, \boldsymbol{U}_2(\boldsymbol{\theta})^\top)^\top$,

$$\boldsymbol{F}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{F}_{11}(\boldsymbol{\theta}) & \boldsymbol{F}_{12}(\boldsymbol{\theta}) \\ \boldsymbol{F}_{21}(\boldsymbol{\theta}) & \boldsymbol{F}_{22}(\boldsymbol{\theta}) \end{bmatrix}, \quad \text{and} \quad \boldsymbol{F}(\boldsymbol{\theta})^{-1} = \begin{bmatrix} \boldsymbol{F}^{11}(\boldsymbol{\theta}) & \boldsymbol{F}^{12}(\boldsymbol{\theta}) \\ \boldsymbol{F}^{21}(\boldsymbol{\theta}) & \boldsymbol{F}^{22}(\boldsymbol{\theta}) \end{bmatrix},$$

so that the Wald and score statistics become

$$\text{Wald:}\quad W=(\widehat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}_{10})^\top\{\boldsymbol{F}_{11}(\widehat{\boldsymbol{\theta}})-\boldsymbol{F}_{12}(\widehat{\boldsymbol{\theta}})\boldsymbol{F}_{22}(\widehat{\boldsymbol{\theta}})^{-1}\boldsymbol{F}_{21}(\widehat{\boldsymbol{\theta}})\}(\widehat{\boldsymbol{\theta}}_1-\boldsymbol{\theta}_{10})\quad\text{and}\tag{19}$$

$$\text{Score:}\quad S=\boldsymbol{U}_1(\widehat{\boldsymbol{\theta}}_0)^\top\boldsymbol{F}^{11}(\widehat{\boldsymbol{\theta}}_0)\boldsymbol{U}_1(\widehat{\boldsymbol{\theta}}_0).\tag{20}$$

When the null hypothesis is as H_{01} in (18), $\boldsymbol{\theta}_1=(\boldsymbol{\alpha}^\top,\boldsymbol{\beta}^\top)^\top$ and $(\mu,\phi)^\top$ plays the role of a nuisance parameter. In this case, $r_R=2r$. The maximum likelihood estimates of μ and ϕ are computed cycling through (8), (9), and (11) with $\boldsymbol{\alpha}=\boldsymbol{\alpha}_0$ and $\boldsymbol{\beta}=\boldsymbol{\beta}_0$.

The score and the likelihood ratio statistics require the maximum likelihood estimator of $\boldsymbol{\theta}$ under H_0. On the other side, it may be useful to have an alternative testing procedure that does not demand the computation of $\widehat{\boldsymbol{\theta}}_0$. Let $\widetilde{\boldsymbol{\theta}}_2$ be a consistent estimator of $\boldsymbol{\theta}_2$. The $C(\alpha)$ test statistic, given by

$$C(\alpha)=\boldsymbol{U}(\widetilde{\boldsymbol{\theta}})^\top\boldsymbol{F}(\widetilde{\boldsymbol{\theta}})^{-1}\boldsymbol{U}(\widetilde{\boldsymbol{\theta}})-\boldsymbol{U}_2(\widetilde{\boldsymbol{\theta}})^\top\boldsymbol{F}_{22}(\widetilde{\boldsymbol{\theta}})^{-1}\boldsymbol{U}_2(\widetilde{\boldsymbol{\theta}}),\tag{21}$$

where $\widetilde{\boldsymbol{\theta}}=(\boldsymbol{\theta}_{10}^\top,\widetilde{\boldsymbol{\theta}}_2^\top)^\top$, is asymptotically equivalent to the score test, under H_0 (Gourieroux and Monfort 1995; Bera and Bilias 2001). In our case, $\widetilde{\mu}=n^{-1}\sum_{i=1}^n X_i$ and $\widetilde{\phi}=(n-1)^{-1}\sum_{i=1}^n(X_i-\bar{X})^2-n^{-1}\sum_{i=1}^n\kappa_i$ are consistent estimators of μ and ϕ, respectively. Therefore, the test of hypotheses concerning $\boldsymbol{\theta}_1=(\boldsymbol{\alpha}^\top,\boldsymbol{\beta}^\top)^\top$ can be carried out using the $C(\alpha)$ test statistic.

Furthermore, with respect to the hypothesis H_0 in (17), let $\boldsymbol{h}\in\mathbb{R}^{2r+2}$ and the local alternative

$$H_1:\boldsymbol{\theta}=\boldsymbol{\theta}_0+\boldsymbol{F}^{1/2}(\boldsymbol{\theta}_0)^{-1}\boldsymbol{h},\tag{22}$$

where $\boldsymbol{\theta}_0=(\boldsymbol{\theta}_{10}^\top,\boldsymbol{\theta}_2^\top)^\top$. According to Fahrmeir (1988), LR, W, and $S\xrightarrow{d}\chi^2(r_R,\delta^2)$, as $n\to\infty$, under H_1, where r_R is the dimension of $\boldsymbol{\theta}_1$ and $\delta^2=(\boldsymbol{\theta}_1-\boldsymbol{\theta}_1^0)^\top\boldsymbol{F}^{11}(\boldsymbol{\theta})(\boldsymbol{\theta}_1-\boldsymbol{\theta}_1^0)$ is the non-centrality parameter.

4 A simulation study

In order to state some results in Sect. 2 and 3 we rest upon asymptotic theory. In view of this, we planned Monte Carlo simulations to assess some properties of the proposed methodologies.

4.1 Estimation

We begin dealing with point and interval estimation in two setups covering different degrees of heteroskedasticity. First, x_{0i}, $i=1,\ldots,n$, are independently drawn from a uniform distribution on the interval $(\xi_{0.25},\xi_{0.975})$, where ξ_q denotes the q-quantile of the $N(\mu,\phi)$ distribution. In Setup 1 we have $x_i\sim N(20,16)$, $k_i=(0.1x_{0i}+0.05)^2$, $\lambda_{1i}=\{0.15(\alpha_1+\beta_1x_{0i})+0.01\}^2$, and $\lambda_{2i}=\{0.25(\alpha_2+\beta_2x_{0i})+0.01\}^2$, whereas in Setup 2 the conditions are $x_i\sim N(30,20)$, $k_i=(0.05x_{0i}+0.05)^2$, $\lambda_{1i}=\{0.2(\alpha_1+\beta_1x_{0i})+0.01\}^2$, and $\lambda_{2i}=\{0.4(\alpha_2+\beta_2x_{0i})+0.01\}^2$. The error variances are kept fixed and the observations $\boldsymbol{Z}_i$ are sampled from (3), $i=1,\ldots,n$. Sample sizes are

30, 60, and 100. Computations were performed by using specific purpose Ox code (Doornik 2007).

Table 1 displays the true values of the regression coefficients and the results. For the estimators of the slopes, within the scope of our study the biases are negligible, even when the sample size is as small as $n = 30$. There is some bias in the estimates of the intercepts when $n = 30$, but the bias decreases when the sample size increases, as expected. Moreover, the root mean squared error of the estimates, the standard deviation of the estimates, and the average of the asymptotic standard errors entries are close together. The coverage probabilities of the 95% asymptotic confidence intervals differ from the nominal value by at most 0.6%. Overall, these results suggest a good performance of the point and interval estimators under the scenarios in our study.

4.2 Hypothesis testing

Now we turn our attention to the empirical level and the power of the test statistics (16), (19), (20), and (21) at two nominal significance levels. Favoring simplicity and motivated by our example in Sect. 5, we choose $r = 1$ in (18). The null hypothesis to be tested is

$$H_{01} : \alpha_1 = 0, \ \beta_1 = 1. \tag{23}$$

With respect to the method comparison problem motivated in Sect. 2, this hypothesis is frequently tested when evaluating the unbiasedness of the 'new' method (see, for example, Riu and Rius (1996); Cheng and Riu (2006)).

The simulations setting is as follows. Five values of intercept (α_1 = -4.0, -2.0, 0.0, 2.0, and 4.0) and slope (β_1 = 0.7, 0.85, 1.0, 1.15, and 1.3) create a range of conditions in a neighborhood of H_{01} in (23). The parameters of the distribution of the true x in (2) are $\mu = 160$ and $\phi = 2560$. Measurement error variance κ_i is set to be equal to $(0.15\, x_{i0} + 0.05)^2$, with x_{i0} as in Sect. 4.1, and $\lambda_{i1}^{1/2}$ follows a gamma distribution whit shape = 28 and rate = 1.4, $i = 1, \ldots, n$. Sample sizes are 30, 50, and 100. For each triplet (n, α_1, β_1), we generate the error variances as above; then, keeping these values fixed, observations $\boldsymbol{Z}_i$ are sampled from (3), $i = 1, \ldots, n$. Rejection rates of (23) are calculated from 5000 samples. Simulated scenarios resemble the data set of our example in Sect. 5. Graphics were drawn in the R system (R Core Team 2013).

Figure 1 shows examples of simulated samples with $n = 30$ and different alternatives (dashed lines) with respect to the null hypothesis in (23) to be tested (solid lines). The crosses at each point represent the standard deviations of the measurement errors.

In the simulations, and in Sect. 5, the EM algorithm iterates until

$$\max_{j=1,\ldots,2r+2} \left| \left(\widehat{\theta}_j^{(m+1)} - \widehat{\theta}_j^{(m)} \right) / \, \widehat{\theta}_j^{(m)} \right| \ < \ 10^{-5}. \tag{24}$$

Table 2 summarize the results. Rejection rates are close to the nominal significance levels when $\alpha_1 = 0.0$ and $\beta_1 = 1.0$, whichever the sample size, an evidence for there being agreement between empirical and theoretical distributions under the null hypothesis.

Table 1 Results from 10000 replications (Par: parameter to be estimated, True: true value of the parameter, Est: average of the estimates, RMSE: root mean squared error of the estimates, SD: standard deviation of the estimates, SE: average of the asymptotic standard errors, and CP: coverage probability of the 95% confidence interval)

n	Setup 1							Setup 2						
	Par	True	Est	RMSE	SD	SE	CP	Par	True	Est	RMSE	SD	SE	CP
30	α_1	2.0	1.77	4.98	4.97	4.94	0.954	α_1	−1.0	−1.58	17.14	17.12	16.86	0.951
	α_2	−1.0	−1.35	7.84	7.83	7.69	0.955	α_2	1.0	0.94	9.01	9.01	8.94	0.952
	β_1	1.3	1.31	0.25	0.25	0.24	0.953	β_1	2.0	2.02	0.57	0.57	0.56	0.952
	β_2	1.5	1.52	0.39	0.39	0.38	0.956	β_2	0.5	0.50	0.30	0.30	0.30	0.950
60	α_1	2.0	1.87	3.47	3.47	3.45	0.951	α_1	−1.0	−1.13	11.23	11.23	11.25	0.955
	α_2	−1.0	−1.12	5.40	5.40	5.38	0.953	α_2	1.0	0.95	6.08	6.08	6.00	0.948
	β_1	1.3	1.31	0.17	0.17	0.17	0.952	β_1	2.0	2.00	0.37	0.37	0.37	0.955
	β_2	1.5	1.51	0.26	0.27	0.26	0.953	β_2	0.5	0.50	0.20	0.20	0.20	0.948
100	α_1	2.0	1.91	2.80	2.79	2.79	0.952	α_1	−1.0	−1.14	8.64	8.64	8.61	0.948
	α_2	−1.0	−1.17	4.44	4.43	4.39	0.949	α_2	1.0	0.92	4.64	4.63	4.58	0.949
	β_1	1.3	1.30	0.14	0.14	0.14	0.954	β_1	2.0	2.01	0.29	0.29	0.28	0.949
	β_2	1.5	1.51	0.22	0.22	0.22	0.950	β_2	0.5	0.50	0.15	0.15	0.15	0.948

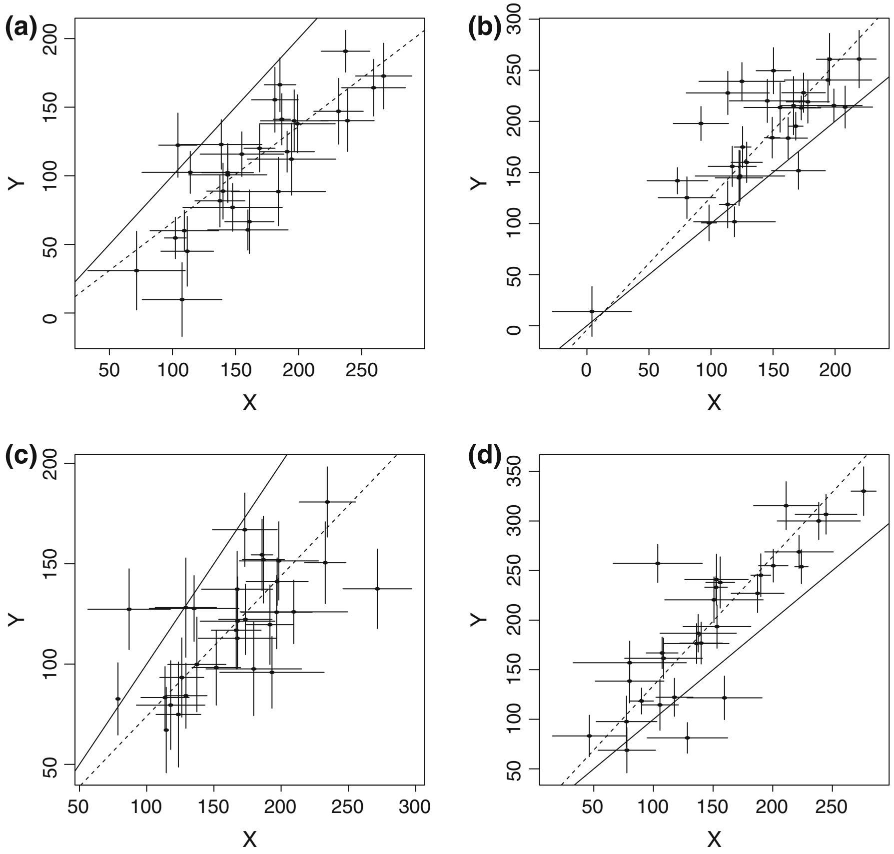

Fig. 1 Simulated samples of size 30 ∓ measurement error standard deviations ($\mu = 160$ and $\phi = 2560$). *Solid lines*: $\alpha_1 = 0.0, \beta_1 = 1.0$; *dashed lines*: **a** $\alpha_1 = -4.0, \beta_1 = 0.7$, **b** $\alpha_1 = -4.0, \beta_1 = 1.3$, **c** $\alpha_1 = 4.0, \beta_1 = 0.7$, and **d** $\alpha_1 = 4.0, \beta_1 = 1.3$

Table 2 Rejection rates of the hypothesis $H_{01} : \alpha_1 = 0, \beta_1 = 1$ from 5000 replications

Significance level	Test statistic	n		
		30	50	100
0.01	LR	0.0094	0.0100	0.0080
	Wald	0.0156	0.0120	0.0106
	Score	0.0096	0.0090	0.0080
	$C(\alpha)$	0.0144	0.0134	0.0116
0.05	LR	0.0472	0.0438	0.0522
	Wald	0.0532	0.0444	0.0548
	Score	0.0468	0.0440	0.0538
	$C(\alpha)$	0.0550	0.0490	0.0530

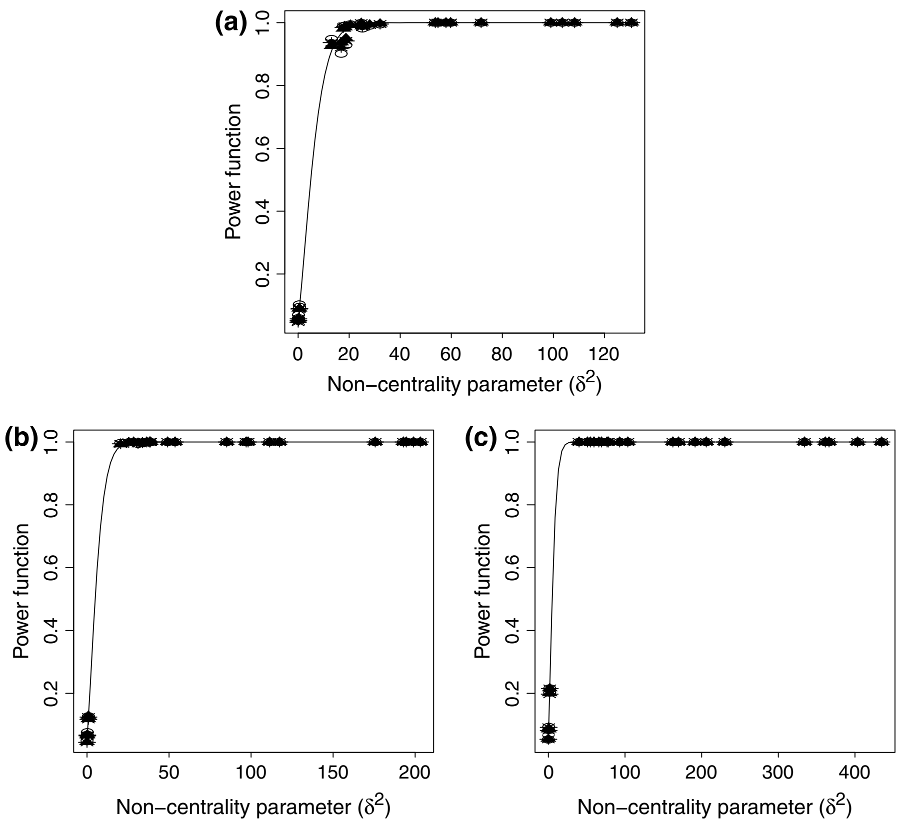

Fig. 2 Asymptotic power function (*solid line*) and rejection rates in the simulations with significance level 0.05: **a** $n = 30$, **b** $n = 50$, and **c** $n = 100$ (•: LR, ▲: Wald, $*$: score, and $\circ$: $C(\alpha)$)

Figure 2 displays the rejection rates of (23) and the theoretical probabilities from the non-central chi-square distribution. The choice of $\boldsymbol{h}$ in (22) ensures that the (α_1, β_1) pairs match the range of values described before. We emphasize that in the scenarios in Fig. 1b, c, the deviations from the null hypothesis are harder to detect than in Fig. 1a, d. For the proposed test statistics, the concordance between empirical and theoretical results is worth to point out. The pattern in Fig. 2 is also observed when the significance level is 0.01.

5 Application

Now we give an illustrative example of the methodology developed. Cheng and Riu (2006) present a data set related to the analysis of Ca^{2+} contents in water samples by means of two methods. Atomic absorption spectroscopy (AAS) is the reference method, whereas sequential injection analysis (SIA) is the new method. The data set with $n = 26$ pairs of observations and the standard deviations of the measurement errors is displayed in Fig. 2 in their paper. Cheng and Riu (2006) also provide details

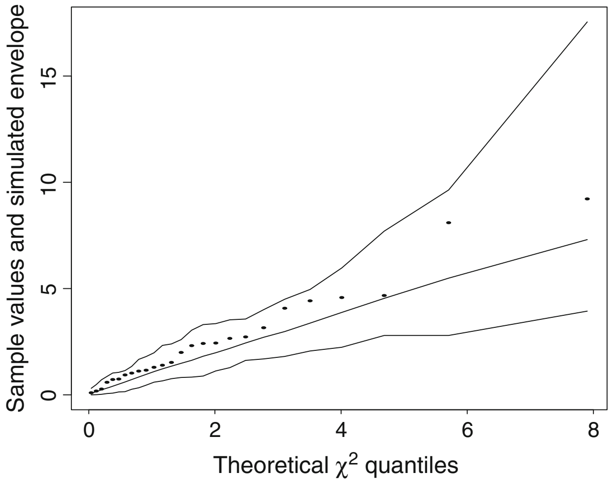

Fig. 3 QQ plot of the squared distances and simulated envelope

about the derivation of the variances of the measurement errors. The biases of the SIA method are assessed by testing the hypothesis in (23).

First of all, we are concerned with goodness of fit and it should be stressed that this issue is often neglected. For the assessment of the model adequacy, in order to implement the graphical tool suggested by Atkinson (1985), first we simulate J samples from (3) using the MLE $\widehat{\boldsymbol{\theta}}$. For the j-th simulated sample we compute the MLE of $\boldsymbol{\theta}$ and $\widehat{q}_{j1}, \ldots, \widehat{q}_{jn}$ from (4), which are ordered as $\widehat{q}_{j(1)} \leq \cdots \leq \widehat{q}_{j(n)}$. The pairs $(F^{-1}_{\chi^2_2}((i-1/2)/n), \widehat{q}_{(i)})$ are drawn in a graph, where $F^{-1}_{\chi^2_2}(\cdot)$ denotes the quantile function of the χ^2_2 distribution and $\widehat{q}_i$ is computed with the MLE $\widehat{\boldsymbol{\theta}}$. The limits of the envelope, given by $\min_{j=1}^{J} \widehat{q}_{j(i)}$ and $\max_{j=1}^{J} \widehat{q}_{j(i)}$, and the line connecting the points $(F^{-1}_{\chi^2_2}((i-1/2)/n), \sum_{j=1}^{J} \widehat{q}_{j(i)}/J)$, $i = 1, \ldots, n$, are also drawn in the graph. This plot forms the basis to guide us on assessing the adequacy of the model. The plot in Fig. 3 does not seem to indicate serious departures from the postulated model.

The MLE of the parameters (and estimated standard errors) are $\widehat{\alpha}_1 = 42.08$ (12.98), $\widehat{\beta}_1 = 0.7400$ (0.07746), $\widehat{\mu} = 159.8$ (10.12), and $\widehat{\phi} = 2560$ (736.8). The bootstrap standard errors based on 5000 samples are 15.77, 0.1117, 10.37, and 648.0, respectively, so that the asymptotic standard errors differ from the bootstrap ones by at most 31%. From (16), (19), (20), and (21), $LR = 10.98$, $W = 11.63$, $S = 11.76$, and $C(\alpha) = 11.71$. Taking the χ^2 distribution with two degrees of freedom as basis, H_{01} in (23) is rejected at a 1% level, since $p = 0.0041$, 0.0030, 0.0028, and 0.0029, respectively. Hence, using the tests proposed in Sect. 3 applied to this data set, the SIA method can not be declared unbiased with respect to the AAS method.

6 Concluding remarks

We have presented inferential tools for a model with a wealth of applications in Engineering and in the Sciences. The comparison of measurement methods having different costs, accuracies, operation requirements, or speeds of response has been reported as a key issue in many research areas. In particular, Barnett (1969), Theobald and Mallinson (1978), Shyr and Gleser (1986), Fuller (1987), and Vilca-Labra et al. (2011) deal with this problem using homoskedastic measurement error models.

We call attention to the fact that interval estimation and hypotheses testing are much less explored in the literature than point estimation methods. It should be emphasized that the $C(\alpha)$ statistic in Sect. 3 does not involve any iterative scheme and yields satisfactory performance in our simulation study (see also de Castro et al. (2008)). Moreover, a graphical device for checking the model is implemented.

Knowledge of the variances $(\kappa, \boldsymbol{\lambda})$ and normality of the distribution of the true values x in (2) are key assumptions. Replicated observations, if available, allow to propose a model in which the variances of the measurement errors are estimated. Also, more flexible distributions could be assumed for the true values, so that there is room for extensions to the model in this work.

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Appendix A: Derivatives related to the score vector

The derivatives in (12) are given by

$$\frac{\partial \log |\boldsymbol{V}_i|}{\partial \boldsymbol{\alpha}} = \boldsymbol{0}, \quad \frac{\partial \log |\boldsymbol{V}_i|}{\partial \boldsymbol{\beta}} = 2c_i \boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta}, \quad \frac{\partial \log |\boldsymbol{V}_i|}{\partial \mu} = 0,$$

$$\frac{\partial \log |\boldsymbol{V}_i|}{\partial \phi} = \phi^{-1}(1 - c_i\phi^{-1}), \quad \text{and} \quad \frac{\partial q_i}{\partial \boldsymbol{\gamma}} = \frac{\partial q_{i1}}{\partial \boldsymbol{\gamma}} - 2c_i q_{i2}\frac{\partial q_{i2}}{\partial \boldsymbol{\gamma}} - q_{i2}^2\frac{\partial c_i}{\partial \boldsymbol{\gamma}},$$

where

$$\frac{\partial q_{i1}}{\partial \boldsymbol{\alpha}} = -2\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)(\boldsymbol{Y}_i - \boldsymbol{\alpha} - \mu\boldsymbol{\beta}), \quad \frac{\partial q_{i1}}{\partial \boldsymbol{\beta}} = \mu\frac{\partial q_{i1}}{\partial \boldsymbol{\alpha}},$$

$$\frac{\partial q_{i1}}{\partial \mu} = -2\kappa_i^{-1}(X_i - \mu) - 2\boldsymbol{\beta}^\top \boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)(\boldsymbol{Y}_i - \boldsymbol{\alpha} - \mu\boldsymbol{\beta}), \quad \frac{\partial q_{i1}}{\partial \phi} = 0,$$

$$\frac{\partial q_{i2}}{\partial \boldsymbol{\alpha}} = -\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta}, \quad \frac{\partial q_{i2}}{\partial \boldsymbol{\beta}} = \boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)(\boldsymbol{Y}_i - \boldsymbol{\alpha} - 2\mu\boldsymbol{\beta}), \quad \frac{\partial q_{i2}}{\partial \mu} = -\boldsymbol{b}^\top \boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{b},$$

$$\frac{\partial q_{i2}}{\partial \phi} = 0, \quad \frac{\partial c_i}{\partial \boldsymbol{\alpha}} = \boldsymbol{0}, \quad \frac{\partial c_i}{\partial \boldsymbol{\beta}} = -2c_i^2\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta}, \quad \frac{\partial c_i}{\partial \mu} = 0, \quad \text{and} \quad \frac{\partial c_i}{\partial \phi} = c_i^2\phi^{-2},$$

$i = 1, \ldots, n$.

Appendix B: Observed and expected information matrices

The elements of $\boldsymbol{L}_i(\boldsymbol{\theta})$ in (13) have general expression

$$\boldsymbol{L}_{i\gamma\tau} = -\frac{1}{2}\frac{\partial^2 \log|\boldsymbol{V}_i|}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\tau}^\top} - \frac{1}{2}\frac{\partial^2 q_i}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\tau}^\top} = -\frac{1}{2}(\boldsymbol{d}_{i\gamma\tau} + \boldsymbol{q}_{i\gamma\tau}),$$

for $\boldsymbol{\gamma}, \boldsymbol{\tau} = \boldsymbol{\alpha}, \boldsymbol{\beta}, \mu, \phi$. Deriving once more the expressions in Sect. 2.1 leads to

$$\boldsymbol{d}_{i\alpha\alpha} = \boldsymbol{d}_{i\alpha\beta} = \boldsymbol{0},\ \boldsymbol{d}_{i\alpha\mu} = \boldsymbol{d}_{i\alpha\phi} = \boldsymbol{d}_{i\beta\mu} = \boldsymbol{0}, \quad d_{i\mu\mu} = d_{i\mu\phi} = 0,$$

$$\boldsymbol{d}_{i\beta\beta} = 2c_i\{-2c_i\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta}\boldsymbol{\beta}^\top + \boldsymbol{I}_r\}\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i),$$

$$\boldsymbol{d}_{i\beta\phi} = 2(1 + \phi\boldsymbol{b}^\top\boldsymbol{\Phi}_i^{-1}\boldsymbol{b})^{-2}\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta},\ d_{i\phi\phi} = -\phi^{-4}(\phi - c_i)^2, \quad \text{and}$$

$$\begin{aligned}\boldsymbol{q}_{i\gamma\tau} = {} & \frac{\partial^2 q_{i1}}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\tau}^\top} - 2c_i\left(\frac{\partial q_{i2}}{\partial\boldsymbol{\gamma}}\frac{\partial q_{i2}}{\partial\boldsymbol{\tau}^\top} + q_{i2}\frac{\partial^2 q_{i2}}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\tau}^\top}\right) \\ & - 2q_{i2}\left(\frac{\partial q_{i2}}{\partial\boldsymbol{\gamma}}\frac{\partial c_i}{\partial\boldsymbol{\tau}^\top} + \frac{\partial c_i}{\partial\boldsymbol{\gamma}}\frac{\partial q_{i2}}{\partial\boldsymbol{\tau}^\top}\right) - q_{i2}^2\frac{\partial^2 c_i}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\tau}^\top},\end{aligned}$$

where

$$\frac{\partial^2 q_{i1}}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}^\top} = 2\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i),\ \frac{\partial^2 q_{i1}}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\beta}^\top} = 2\mu\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i),\ \frac{\partial^2 q_{i1}}{\partial\boldsymbol{\alpha}\partial\mu} = 2\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta},$$

$$\frac{\partial^2 q_{i1}}{\partial\boldsymbol{\alpha}\partial\phi} = \boldsymbol{0},\ \frac{\partial^2 q_{i1}}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}^\top} = 2\mu^2\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i),\ \frac{\partial^2 q_{i1}}{\partial\boldsymbol{\beta}\partial\mu} = -2\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)(\boldsymbol{Y}_i - \boldsymbol{\alpha} - 2\mu\boldsymbol{\beta}),$$

$$\frac{\partial^2 q_{i1}}{\partial\boldsymbol{\beta}\partial\phi} = \boldsymbol{0},\ \frac{\partial^2 q_{i1}}{\partial\mu^2} = 2\boldsymbol{b}^\top\boldsymbol{\Phi}_i^{-1}\boldsymbol{b},\ \frac{\partial^2 q_{i1}}{\partial\mu\partial\phi} = \frac{\partial^2 q_{i1}}{\partial\phi^2} = 0,\ \frac{\partial^2 q_{i2}}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}^\top} = \boldsymbol{0},$$

$$\frac{\partial^2 q_{i2}}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\beta}^\top} = -\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i),\ \frac{\partial^2 q_{i2}}{\partial\boldsymbol{\alpha}\partial\mu} = \frac{\partial^2 q_{i2}}{\partial\boldsymbol{\alpha}\partial\phi} = \boldsymbol{0},\ \frac{\partial^2 q_{i2}}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}^\top} = -2\mu\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)$$

$$\frac{\partial^2 q_{i2}}{\partial\boldsymbol{\beta}\partial\mu} = -2\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta},\ \frac{\partial^2 q_{i2}}{\partial\boldsymbol{\beta}\partial\phi} = \boldsymbol{0},\ \frac{\partial^2 q_{i2}}{\partial\mu^2} = \frac{\partial^2 q_{i2}}{\partial\mu\partial\phi} = \frac{\partial^2 q_{i2}}{\partial\phi^2} = 0,$$

$$\frac{\partial^2 c_i}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}^\top} = \frac{\partial^2 c_i}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\beta}^\top} = \boldsymbol{0},\ \frac{\partial^2 c_i}{\partial\boldsymbol{\alpha}\partial\mu} = \frac{\partial^2 c_i}{\partial\boldsymbol{\alpha}\partial\phi} = \boldsymbol{0},$$

$$\frac{\partial^2 c_i}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}^\top} = -2c_i^2\{\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i) - 4c_i\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta}\boldsymbol{\beta}^\top\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\},$$

$$\frac{\partial^2 c_i}{\partial \boldsymbol{\beta} \partial \mu} = \mathbf{0}, \ \frac{\partial^2 c_i}{\partial \boldsymbol{\beta} \partial \phi} = -4c_i^3\phi^{-2}\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta}, \ \frac{\partial^2 c_i}{\partial \mu^2} = 0, \ \frac{\partial^2 c_i}{\partial \mu \partial \phi} = 0,$$
$$\text{and} \quad \frac{\partial^2 c_i}{\partial \phi^2} = 2c_i^2\phi^{-3}(c_i\phi^{-1} - 1), \ i = 1, \ldots, n.$$

After computing expectations of $\boldsymbol{L}_i(\boldsymbol{\theta})$ in (13), we conclude that the elements of $\boldsymbol{F}_i(\boldsymbol{\theta})$ in (14) are

$$\boldsymbol{F}_{i\alpha\alpha} = \boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i) - c_i\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta}\boldsymbol{\beta}^\top\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i), \ \boldsymbol{F}_{i\alpha\beta} = \mu\boldsymbol{F}_{i\alpha\alpha},$$

$$\boldsymbol{F}_{i\alpha\mu} = c_i\phi^{-1}\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta}, \ \boldsymbol{F}_{i\alpha\phi} = \mathbf{0},$$

$$\boldsymbol{F}_{i\beta\beta} = c_i(2c_i - \phi - \mu^2)\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta}\boldsymbol{\beta}^\top\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i) + (\phi + \mu^2 - c_i)\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i),$$

$$\boldsymbol{F}_{i\beta\mu} = \mu c_i\phi^{-1}\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta}, \ \boldsymbol{F}_{i\beta\phi} = \phi^{-2}c_i(\phi - c_i)\boldsymbol{D}^{-1}(\boldsymbol{\lambda}_i)\boldsymbol{\beta},$$

$$F_{i\mu\mu} = \phi^{-1}(1 - c_i\phi^{-1}), \ F_{i\mu\phi} = 0, \ \text{and } F_{i\phi\phi} = \frac{1}{2}\phi^{-2}(c_i\phi^{-1} - 1)^2, \ i = 1, \ldots, n.$$

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