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Estimation of a measure of local correlation for independent samples and time series data

Sumaia A. Latif and Pedro A. Morettin
University of São Paulo, São Paulo, Brazil

Abstract

Different from measures of global dependence, measures of local dependence evaluate the dependence along the support of the variables. The aim of this paper is to study a measure of local dependence proposed by Bairamov, Kotz and Kozubowski (2003) in the context of variables not indexed by time and also for stationary time series. We propose similar estimators for both cases. The consistency of the estimators are obtained, and their behavior are studied through simulations. Some empirical illustrations are provided.

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1 Introduction

In this section, some measures of local dependence found in the literature are introduced briefly.

Holland and Wang (1987) proposed a measure of local dependence for two continuous variables by extending an rc contingence table for two discrete variables to the case of two continuous variables by taking partitions in a thin rectangular grid. The function of local dependence proposed by Holland and Wang (1987) does not consider the marginal distributions and is constant under the bivariate normal case.

Another local measure, proposed by Bjerve and Doksum (1993), is the correlation curve between X and Y, that is a generalization of Pearson correlation coefficient for linear models to nonlinear models. However, it is only conditional on X and not on X and Y. An application of this measure to financial markets was studied by Bradley and Taqqu (2005) under the case of random sample. Latif and Morettin (2012a) investigated this correlative curve in the context of univariate and bivariate stationary time series.

Sibuya (1960) proposed a function of dependence between two continuous random variables X and Y, that is the ratio between the joint distribution function and the product of their marginal distribution functions. It was initiated from the study of the notion of extreme statistics for the bivariate case. For this function, Latif and Morettin (2009, 2011) studied nonparametric estimators under random sample as well as for time series.

Aiming to create a measure of local dependence with an explicit representation for copulas, Anjos and Kolev (2005) proposed a measure that could be interpreted as a standardized distance between the copula and product copula. In Latif and Morettin (2012b), we can find the nonparametric estimation of this function for both random vectors and stationary time series.

Also in Nelsen (2006), among others, we can find the concept of copula (and copula density) which was studied in the context of time series by Fermanian and Scaillet (2003) who used kernel estimators and also by Morettin et al. (2011) who used estimators through wavelets.

The aim of this work is to study the estimation of the measure of local dependence of Bairamov et al. (2003), that is also called the local measure of dependence of Bairamov and Kotz by Mari and Kotz (2001). From now on we will call it measure of local correlation. Both the context of random variables and univariate and bivariate stationary process are considered. To this end, we propose estimators based on the estimator of Nadaraya–Watson for conditional expectations. The consistency of the three estimators are proved, and simulations are presented to assess the estimators properties. Empirical illustrations are shown considering two indices of performance of large companies in Brazil in 2006, and for the time series the daily returns of Petrobras (Brazilian Oil Company) and also of CAC 40 (French stock market index) and FTSE (Financial Times Stock Exchange - UK stock index) were used.

2 Measure of local correlation

The measure of local dependence between two variables X and Y proposed by Bairamov et al. (2003) is given by the following expression:

$$H(x,y) = \frac{E[(X - E[X|Y=y])(Y - E[Y|X=x])]}{\sqrt{E[(X - E[X|Y=y])^2]}\sqrt{E[(Y - E[Y|X=x])^2]}}, \ \forall (x,y) \in S, \tag{2.1}$$

which refers to the known Pearson's correlation coefficient with the replacement of $E[X]$ by $E[X|Y=y]$ and the replacement of $E[Y]$ by $E[Y|X=x]$.

Here S denotes the support of (X, Y). This measure can also be written as

$$H(x, y) = \frac{\rho_{XY} + \varphi_X(y)\varphi_Y(x)}{\sqrt{1 + \varphi_X^2(y)}\sqrt{1 + \varphi_Y^2(x)}}, \; \forall (x, y) \in S, \tag{2.2}$$

where

$$\varphi_X(y) = \frac{E[X] - E[X|Y = y]}{\sqrt{Var[X]}}, \; \varphi_Y(x) = \frac{E[Y] - E[Y|X = x]}{\sqrt{Var[Y]}}$$

and ρ_{XY} is the usual Pearson's correlation coefficient.

In Figure 1, the plots (a) to (d) and the corresponding plots of contour curves (e) to (h), show the theoretical behavior of this measure for a bivariate random vector with standard normal distribution and correlation coefficients equal to $+0.80$, -0.80, $+0.20$ and -0.20, in this order.

Bairamov et al. (2003) observe that:

- $H(x, y)$ refers to the localized version of the Pearson's correlation coefficient ρ_{XY};
- this measure of local dependence characterizes the effect of X on Y and the effect of Y on X, conditional to (X, Y) equal to (x, y), allowing the

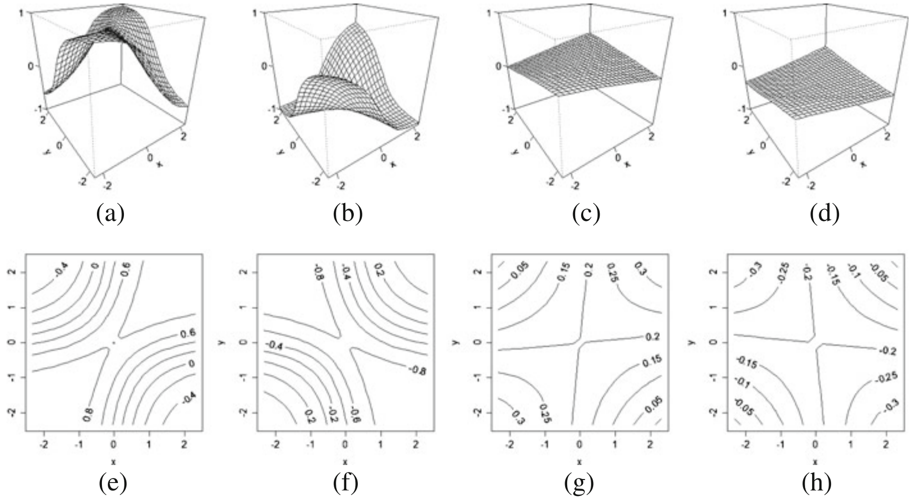


Figure 1: Plots and contour curves of the measure of local correlation (Equations (2.1) or (2.2)) for (X, Y) with standard normal distribution and correlation coefficient $\rho = +0.80$ in (a) and (e), $\rho = -0.80$ in (b) and (f), $\rho = +0.20$ in (c) and (g), and $\rho = -0.20$ in (d) and (h).

identification of the variable values with stronger or weaker association than the global one;

- $E[H(X,Y)]$ is nearly equal to Pearson's linear correlation coefficient. This approximation can be made through weighted integration of H with respect to the joint density f of (X,Y), and the result is always finite because $|H(x,y)| \leq 1$.

The form and symmetry of $H(x,y)$ for the symmetrical elliptical distributions can be found in Kotz and Nadarajah (2003), and for the distributions of extreme values in Nadarajah, Mitov and Kotz (2003).

Let (X,Y) be a continuous random vector with support S. Then $H(x,y)$ satisfies the following properties (see Bairamov et al., 2003):

(i) $|H(x,y)| \leq 1$, $\forall (x,y) \in S$;

(ii) if $H(x,y) = \pm 1$ for some $(x,y) \in S$, then $\rho_{XY} \neq 0$;

(iii) if $Y = aX + b$ almost surely, then $H(X,Y) = 1 \times \text{sign}(a)$;

(iv) if $\rho_{XY} = \pm 1$, then $H(X,Y) = \pm 1$ almost surely;

(v) if $U = a + bX$ and $V = c + dY$, with $bd \neq 0$, then $H_{UV}(u,v) = \text{sign}(bd) H_{XY}(x,y)$, where $u = a + bx$ and $v = c + dy$;

(vi) if X and Y are independent, then $H(x,y) = 0$, $\forall (x,y) \in S$;

(vii) if $H(x,y) = 0$, $\forall (x,y) \in S$, then $E[X] = E[X|Y = y]$ or $E[Y] = E[Y|X = x]$, $\forall (x,y) \in S$, and $\rho_{XY} = 0$;

(viii) the point (x^*, y^*) satisfying $\varphi_X(y^*) = \varphi_Y(x^*) = 0$ is a saddle point of H and, at this point, $H(x^*, y^*) = \rho_{XY}$, with $(x^*, y^*) \in S$;

(ix) $H(\mu_X, \mu_Y) = \rho_{XY}$ if (X,Y) has a normal distribution with vector mean equal to $(\mu_X, \mu_Y)'$.

For the case where (X,Y) has a Gaussian distribution with mean $\boldsymbol{\mu} = (\mu_X, \mu_Y)'$ and $\text{vec}(\boldsymbol{\Sigma}) = (\gamma_{XX}, \gamma_{YX}, \gamma_{XY}, \gamma_{YY})'$, where $\gamma_{XY} = \gamma_{YX}$, then $E[X|Y = y] = \mu_X + (\gamma_{XY}/\gamma_{YY})(y - \mu_Y)$ and $E[Y|X = x] = \mu_Y + (\gamma_{XY}/\gamma_{XX})(x - \mu_X)$, and for this case

$$H(x,y) = \frac{\rho_{XY}\left(1 + (\rho_{XY}/(\gamma_{XX}\gamma_{YY})^{1/2})(x - \mu_X)(y - \mu_Y)\right)}{\sqrt{1 + (\rho_{XY}^2/\gamma_{XX})(x - \mu_X)^2}\sqrt{1 + (\rho_{XY}^2/\gamma_{YY})(y - \mu_Y)^2}},$$

$\forall (x,y) \in S.$

To estimate the measure given by Equation (2.1) (or Equation (2.2)), Bairamov et al. (2003) suggest to use the moment estimators for $E[X]$, $E[Y]$, $Var[X]$, $Var[Y]$ and ρ_{XY}, and kernel estimators of conditional expectations.

Let $((X_1, Y_1), ..., (X_n, Y_n))$ be a random sample observed from (X, Y). Considering the estimator of Nadaraya-Watson (proposed independently by Nadaraya, 1964 and Watson, 1964) for conditional expectations, then an estimator of $H(x, y)$ can be given by:

$$\hat{H}(x,y) = \frac{R_{XY} + ((\bar{X} - \hat{m}_X(y))/S_X)((\bar{Y} - \hat{m}_Y(x))/S_Y)}{\sqrt{1 + ((\bar{X} - \hat{m}_X(y))/S_X)^2}\sqrt{1 + ((\bar{Y} - \hat{m}_Y(x))/S_Y)^2}}, \quad \forall (x,y) \in S, \tag{2.3}$$

where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \ , \ \ \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i,$$

$$S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \ , \ \ S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2,$$

$$R_{XY} = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{X_i - \bar{X}}{S_X}\right)\left(\frac{Y_i - \bar{Y}}{S_Y}\right),$$

$$\hat{m}_Y(x) = \frac{\sum_{i=1}^{n} Y_i K_1\left(\frac{x - X_i}{h_{1n}}\right)}{\sum_{i=1}^{n} K_1\left(\frac{x - X_i}{h_{1n}}\right)} \ , \ \ \hat{m}_X(y) = \frac{\sum_{i=1}^{n} X_i K_2\left(\frac{y - Y_i}{h_{2n}}\right)}{\sum_{i=1}^{n} K_2\left(\frac{y - Y_i}{h_{2n}}\right)},$$

with K_i being symmetric, bounded and real kernel functions such that $\int K_i(u)du = 1$ and $K_i(x; h_{in}) = 1/h_{in}K_i(x/h_{in})$, $i = 1, 2$, with $h_{in} > 0$ being functions of n such that $h_{in} \to 0$ as $n \to \infty$. Here $\hat{m}_Y(x)$ and $\hat{m}_X(y)$ are estimators of $m_Y(x) = E[Y|X = x]$ and $m_X(y) = E[X|Y = y]$, respectively.

THEOREM 2.1. *Let (X, Y) be a continuous random vector with support S whose components have finite second order moments, and $h_{in} \to 0$ such that $nh_{in} \to \infty$, $i = 1, 2$, then*

$$\hat{H}(x,y) \xrightarrow[n\to\infty]{P} H(x,y), \ \textit{for every } (x,y) \in S.$$

Proof. Because the second order moments of X and Y are finite, then we know that the estimators $\bar{X}$, $\bar{Y}$, S_X^2, S_Y^2, R_{XY} are consistent for their true parameters μ_X, μ_Y, σ_X^2, σ_Y^2, ρ_{XY}, respectively, as $n \to \infty$. According to Härdle (1991), if $h_{in} \to 0$ such that $nh_{in} \to \infty$, $i = 1, 2$, then $\hat{m}_Y(x)$ converges in probability to $m_Y(x)$, and $\hat{m}_X(y)$ to $m_X(y)$, as $n \to \infty$. Therefore, by the continuous mapping theorem, $\hat{H}(x, y)$ is a consistent estimator of $H(x, y)$.

3 Measure of local correlation for time series

Let $\{X_t,\ t \in \mathbb{Z}\}$ be a stochastic process with continuous values. Consider a random vector (X_{t_1}, X_{t_2}), $t_1, t_2 \in \mathbb{Z}$, $t_1 \neq t_2$, with support S. Then, using Equation (2.2), the measure of local correlation can be defined in the following way:

$$H(x_1, x_2; t_1, t_2) = \frac{\rho(t_1, t_2) + \varphi_{t_1,t_2}(x_2)\varphi_{t_2,t_1}(x_1)}{\sqrt{1 + \varphi^2_{t_1,t_2}(x_2)}\sqrt{1 + \varphi^2_{t_2,t_1}(x_1)}},\ \forall (x_1, x_2) \in S,\ \forall t_1, t_2 \in \mathbb{Z},$$

where

$$\varphi_{t_1,t_2}(x_2) = (E[X_{t_1}] - E[X_{t_1}|X_{t_2} = x_2])/\sqrt{Var[X_{t_1}]},$$

$$\varphi_{t_2,t_1}(x_1) = (E[X_{t_2}] - E[X_{t_2}|X_{t_1} = x_1])/\sqrt{Var[X_{t_2}]},$$

$$\rho(t_1, t_2) = E[(X_{t_1} - E[X_{t_1}])(X_{t_2} - E[X_{t_2}])]/\sqrt{Var[X_{t_1}]Var[X_{t_2}]}.$$

Now, assuming that the process $\{X_t,\ t \in \mathbb{Z}\}$ is strictly stationary, then, the finite dimensional distributions remain the same under translations of time. In particular, the univariate distributions are invariant under translations of time which imply that the mean and the variance are constants, and the bivariate distributions depend on the time lag τ and then ρ_τ, $E[X_{t+\tau}|X_t = x]$ and $E[X_t|X_{t+\tau} = x]$ depend of τ. Therefore, considering the random vector $(X_t, X_{t+\tau})$, $\forall t, \tau \in \mathbb{Z}$, $\tau \neq 0$, we have

$$H_\tau(x_1, x_2) = \frac{\rho_\tau + ((\mu - E[X_t|X_{t+\tau} = x_2])/\sigma)\,((\mu - E[X_{t+\tau}|X_t = x_1])/\sigma)}{\sqrt{1 + ((\mu - E[X_t|X_{t+\tau} = x_2])/\sigma)^2}\sqrt{1 + ((\mu - E[X_{t+\tau}|X_t = x_1])/\sigma)^2}}, \tag{3.1}$$

$$\forall (x_1, x_2) \in S,\ \forall t, \tau \in \mathbb{Z},\ \tau \neq 0.$$

Proposition 3.1. *Let $\{X_t,\ t \in \mathbb{Z}\}$ be a strictly stationary process with continuous values. Then, $H_\tau(x_1, x_2)$ satisfies the following properties:*

(i) $-1 \le H_\tau(x_1, x_2) \le +1,\ \forall (x_1, x_2) \in S,\ \forall \tau \in \mathbb{Z}^*$;

(ii) *if $H_\tau(x_1, x_2) = \pm 1$ for some $(x_1, x_2) \in S$, then $\rho_\tau \neq 0,\ \forall \tau \in \mathbb{Z}^*$;*

(iii) *the point (x_1^*, x_2^*) satisfying $\varphi(x_1^*; \tau) = \varphi(x_2^*; \tau) = 0$ is a saddle point of H_τ and, at this point, $H_\tau(x_1^*, x_2^*) = \rho_\tau$, with $(x_1^*, x_2^*) \in S,\ \forall \tau \in \mathbb{Z}^*$;*

(iv) $H_{-\tau}(x_1, x_2) = H_\tau(x_1, x_2),\ \forall (x_1, x_2) \in S,\ \forall \tau \in \mathbb{Z}^*$;

(v) *$H_\tau(\mu, \mu) = \rho_\tau,\ \forall \tau \in \mathbb{Z}^*$, if $\{X_t\}$ is a Gaussian process with mean μ.*

The proofs of these properties are immediate.

As an example, consider a strictly stationary, second order AR(1) process $\{X_t,\ t \in \mathbb{Z}\}$ of the form $X_t = \phi_0 + \phi_1 X_{t-1} + a_t,\ a_t \sim iid(0, \sigma_a^2)$. Then

$$H_\tau(x_1, x_2) = \frac{\phi_1^{|\tau|}\sigma_a^2(1-\phi_1)^2 + \phi_1^{2|\tau|}(1-\phi_1^2)\left((1-\phi_1)x_2 - \phi_0\right)\left((1-\phi_1)x_1 - \phi_0\right)}{\left[\prod_{i=1}^2 \left(\sigma_a^2(1-\phi_1)^2 + \phi_1^{2|\tau|}(1-\phi_1^2)\{(1-\phi_1)x_i - \phi_0\}^2\right)\right]^{1/2}},$$
$$|\tau| \ge 1,$$

which presents an exponential decay to zero from the saddle point, a behavior similar to the a.c.f. of an AR(1) process.

Now, if the strictly stationary second order process $\{X_t,\ t \in \mathbb{Z}\}$ follows a MA(1) model of the form $X_t = \theta_0 - \theta_1 a_{t-1} + a_t$, $a_t \sim (0, \sigma_a^2)$, we have that

$$H_\tau(x_1, x_2) = \begin{cases} \dfrac{-\theta_1\sigma_a^2 + (\theta_1 x_2 - \theta_0\theta_1)(\theta_1 x_1 - \theta_0\theta_1)}{\sqrt{\prod_{i=1}^2 \{\sigma_a^2(1+\theta_1^2) + (\theta_1 x_i - \theta_0\theta_1)^2\}}} & , |\tau| = 1 \\ 0 & , |\tau| \ge 2, \end{cases}$$

which is non null only on the first lag, as occurs with the a.c.f. of a MA(1) model.

When the process $\{X_t,\ t \in \mathbb{Z}\}$ follows an ARMA(1,1) model with zero mean of the form $X_t = \phi X_{t-1} - \theta a_{t-1} + a_t$, $a_t \sim iid(0, \sigma_a^2)$, it follows that

$$H_\tau(x_1, x_2) = \frac{\sigma_a^2\phi^{|\tau|-1}(1-\phi\theta)(\phi-\theta) + x_1 x_2 \phi^{2(|\tau|-1)}(\phi-\theta)^2(1-\phi^2)}{\sqrt{\prod_{i=1}^2 \left(\sigma_a^2(1+\theta^2-2\phi\theta) + \phi^{2(|\tau|-1)}(\phi-\theta)^2(1-\phi^2)x_i^2\right)}},$$
$$|\tau| \ge 1,$$

which shows a similar behavior to the a.c.f. of an ARMA(1,1) model.

In the context of time series, given observations $(X_1, ..., X_T)$ from a strictly stationary, second order process with continuous values, one estimator for $H_\tau(x_1, x_2)$ is

$$\hat{H}_\tau(x_1, x_2) = \frac{r_\tau + \left(\frac{\bar{X} - \hat{m}_1(x_2)}{\sqrt{c_0}}\right)\left(\frac{\bar{X} - \hat{m}_2(x_1)}{\sqrt{c_0}}\right)}{\sqrt{1 + \left(\frac{\bar{X} - \hat{m}_1(x_2)}{\sqrt{c_0}}\right)^2}\sqrt{1 + \left(\frac{\bar{X} - \hat{m}_2(x_1)}{\sqrt{c_0}}\right)^2}}, \ \forall (x_1, x_2) \in S, \ \forall t \in \mathbb{Z}, \tag{3.2}$$

where

$$\bar{X} = \frac{1}{T}\sum_{t=1}^{T} X_t,$$

$$c_\tau = \frac{1}{T}\sum_{t=1}^{T-\tau}(X_t - \bar{X})(X_{t+\tau} - \bar{X}), \quad \text{with} \ \ c_{-\tau} = c_\tau,$$

$$r_\tau = \frac{c_\tau}{c_0}, \quad \text{with} \ \ r_{-\tau} = r_\tau,$$

$$\hat{m}_1(x_2) = \frac{\sum_{t=1}^{T-\tau} X_t K_2\left(\frac{x_2 - X_{t+\tau}}{h_{2T}}\right)}{\sum_{t=1}^{T-\tau} K_2\left(\frac{x_2 - X_{t+\tau}}{h_{2T}}\right)}, \ \ \hat{m}_2(x_1) = \frac{\sum_{t=1}^{T-\tau} X_{t+\tau} K_1\left(\frac{x_1 - X_t}{h_{1T}}\right)}{\sum_{t=1}^{T-\tau} K_1\left(\frac{x_1 - X_t}{h_{1T}}\right)}$$

where $\tau = 1, 2, ..., T-1$, K_i is a kernel function with short tails, h_{iT} is a sequence of bandwidth converging to zero in an appropriate rate, $i = 1, 2$, and $\hat{m}_1(x_2)$ and $\hat{m}_2(x_1)$ are estimators of $E[X_t|X_{t+\tau} = x_2]$ and $E[X_{t+\tau}|X_t = x_1]$, respectively. We observe that the the first three estimators are those usually used in time series and the last is the estimator of Nadaraya-Watson whose version for time series can be found in Härdle, Lütkepohl and Chen (1997, page 55).

Consider the following regularity conditions:

(**C1**) $X_t = \mu + \sum_{j=-\infty}^{+\infty} \alpha_j e_{t-j}$, where $e_t \sim iid(0, \sigma^2)$, $E[e_t^4] = \eta\sigma^4$, $\sum_{j=-\infty}^{+\infty} |\alpha_j| < \infty$ and $\sum_{k=-\infty}^{\infty} |\gamma_k| < \infty$;

(**C2**) the smoothing parameter $h_{iT} > 0$ is such that $h_{iT} \to 0$ and $Th_{iT} \to \infty$, $i = 1, 2$, as $T \to \infty$;

(**C3**) the kernel K_i, $i = 1, 2$, is a bounded density function, symmetric (around zero) and such that

$$\lim_{x\to\infty} xK_i(x) = 0 \ \ \text{and} \ \ \int x^2 K_i(x)dx < +\infty;$$

(**C4**) K_i, $i = 1, 2$, is a Lipschitz continuous function of order γ on $\mathbb{R}$, that is, $|K_i(x_1) - K_i(x_2)| \leq c|x_1 - x_2|^\gamma$ with $x_1, x_2 \in \mathbb{R}$, $i = 1, 2$, and $\gamma > 0$;

(**C5**) X_t is an α-mixing process, with geometrical decay coefficients, that is, $\exists u \in]0; \infty[$ and $\exists v \in [0; 1[$ such that $\alpha(k) \leq uv^k$, $k \geq 1$;

(**C6**) r and f are functions twice continuously differentiable and with values on $\mathbb{R}$, such that

$$||f||_\infty = \inf\{a : P[f > a] = 0\} \leq b \;\text{ and }\; ||f^{(2)}||_\infty \leq b,$$

$$||r||_\infty = \inf\{a : P[r > a] = 0\} \leq b \;\text{ and }\; ||r^{(2)}||_\infty \leq b,$$

for some b, where $r(x_1) = \int x_2 f(x_1, x_2) dx_2$ is the numerator of the conditional expectation;

(**C7**) $E[\exp\{a|X_t|^s\}] < +\infty$, for some $a > 0$ and some $s > 0$, $\forall t \in \mathbb{Z}$;

(**C8**) $Th_T/(logT)^{2+1/s} \to +\infty$ when $T \to \infty$ and $s > 0$;

(**C9**) S is a compact set such that $\inf\limits_{x \in S} f(x) > 0$;

(**C10**) $h_T \simeq \left((logT)^{2-1/s}/T\right)^{1/5}$, $s > 0$.

THEOREM 3.1. *Let $\{X_t,\ t \in \mathbb{Z}\}$ be a strictly stationary, second order process, with continuous values and conditions (C1) to (C10) valid. Then*

$$\hat{H}_\tau(x_1, x_2) \xrightarrow[T\to\infty]{P} H_\tau(x_1, x_2),\ \textit{for every}\ (x_1, x_2) \in S,\ |\tau| \geq 1.$$

PROOF.
By (C1) and by Corollary 6.1.1.2 of Fuller (1996), we have that $\bar{X}$ is a consistent estimator for μ, and by (C1) and by Theorem 6.2.2 of Fuller (1996) c_τ is consistent for γ_τ. By Theorem 3.2 of Bosq (1998), with the conditions (C3) to (C10) valid, then $\hat{m}_i(x) \xrightarrow[T\to\infty]{a.s.} m_i(x)$, that is, $\hat{m}_i(x) \xrightarrow[T\to\infty]{P} m_i(x)$ for $h_{iT} \to 0$ and $Th_{iT} \to \infty$, both when $T \to \infty$, $i = 1, 2$, for every fixed $x \in S$. Therefore, by the continuous mapping theorem, we have that $\hat{H}_\tau(x_1, x_2) \xrightarrow[T\to\infty]{P} H_\tau(x_1, x_2)$.

Let $\{(X_t, Y_t),\ t \in \mathbb{Z}\}$ be a strictly stationary process with continuous values and support S. Take $\rho_{XY}(\tau)$ at $\tau = 0$, denoted by $\rho_{XY}(0)$, the contemporaneous or instantaneous correlation coefficient between X_t and Y_t. Then, the local measure of correlation can be written as

$$H_0(x,y) = \frac{\rho_{XY}(0) + ((\mu_X - E[X_t|Y_t = y])/\sigma_X)\,((\mu_Y - E[Y_t|X_t = x])/\sigma_Y)}{\sqrt{1 + ((\mu_X - E[X_t|Y_t = y])/\sigma_X)^2}\sqrt{1 + ((\mu_Y - E[Y_t|X_t = x])/\sigma_Y)^2}}, \tag{3.3}$$

$$\forall (x,y) \in S,\ \forall t, \tau \in \mathbb{Z}.$$

This measure satisfies properties similar to those of the local measure given by Equation (2.1) (see Section 2).

We propose the same type of estimator, that is, observed $((X_1, Y_1)$, ..., $(X_T, Y_T))$ from the process under study, then

$$\hat{H}_0(x,y) = \frac{r_{XY}(0) + \left((\bar{X} - \hat{m}_X(y))/\hat{\sigma}_X\right)\left((\bar{Y} - \hat{m}_Y(x))/\hat{\sigma}_Y\right)}{\sqrt{1 + \left((\bar{X} - \hat{m}_X(y))/\hat{\sigma}_X\right)^2}\sqrt{1 + \left((\bar{Y} - \hat{m}_Y(x))/\hat{\sigma}_Y\right)^2}},\ \forall (x,y) \in S, \tag{3.4}$$

where

$$\hat{\sigma}_X^2 = \frac{1}{T}\sum_{t=1}^{T}(X_t - \bar{X})^2,\ \ \hat{\sigma}_Y^2 = \frac{1}{T}\sum_{t=1}^{T}(Y_t - \bar{Y})^2,$$

$$r_{XY}(0) = \frac{c_{XY}(0)}{\hat{\sigma}_X \hat{\sigma}_Y} \quad \text{where} \quad c_{XY}(0) = \frac{1}{T}\sum_{t=1}^{T}(X_t - \bar{X})(Y_t - \bar{Y}),$$

$$\hat{m}_X(y) = \frac{\sum_{t=1}^{T} X_t K_2\left(\frac{y - Y_t}{h_{2T}}\right)}{\sum_{t=1}^{T} K_2\left(\frac{y - Y_t}{h_{2T}}\right)}, \quad \hat{m}_Y(x) = \frac{\sum_{t=1}^{T} Y_t K_1\left(\frac{x - X_t}{h_{1T}}\right)}{\sum_{t=1}^{T} K_1\left(\frac{x - X_t}{h_{1T}}\right)},$$

with K_i and h_{iT}, $i = 1, 2$, as before.

THEOREM 3.2. *Let $\{(X_t, Y_t),\ t \in \mathbb{Z}\}$ be a strictly stationary, second order process, with continuous values. With regularity conditions similar to (C1) to (C10) valid, we have*

$$\hat{H}_0(x,y) \xrightarrow[T\to\infty]{P} H_0(x,y),\ \textit{for every}\ (x,y) \in S.$$

4 Simulations

In simulations of this section, we use bivariate grid with 25×25 points corresponding to 98% of the central data, Gaussian kernel, optimal bandwidth according to Bosq (1998) (see the Section 3 for more details) and the R package.

(**1**) For the case of two random variables, we consider a bivariate random vector with Gaussian distribution with mean $\boldsymbol{\mu} = (3.05; 6.44)'$ and $\mathrm{vec}(\boldsymbol{\Sigma}) = (1.13; 1.49; 1.49; 3.99)'$, that is, X and Y have correlation equal to 0.70. The measure $H(x, y)$ was calculated using Equation (2.1) (or Equation (2.2)) and the estimator given by Equation (2.3) was simulated using 1,000 random samples of sizes 250, 500 and 1,000

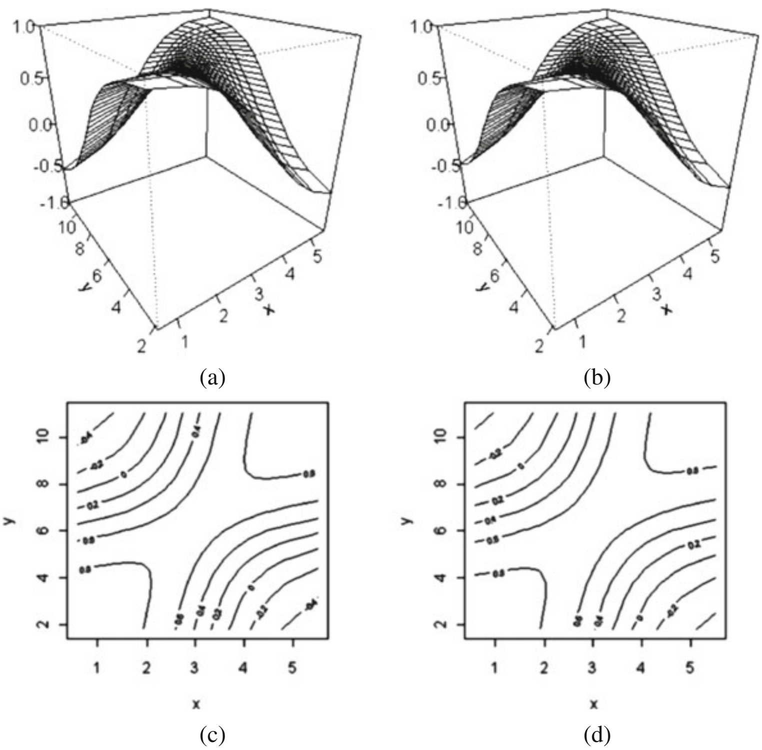


Figure 2: For a random vector (X, Y) with normal distribution, mean $\boldsymbol{\mu} = (3.05; 6.44)'$ and $\mathrm{vec}(\boldsymbol{\Sigma}) = (1.13; 1.49; 1.49; 3.99)'$ (correlation 0.70), we have (a) plot of H (Equation (2.1) or (2.2)), (c) the contour plot. For 1,000 random samples of size $n = 1,000$ observed from (X, Y), we have (b) plot of $\hat{H}$ (Equation (2.3)) and (d) contour plot.

observed from (X, Y). Then, some plots and statistics (bias and mean squared error) at specified points of the bivariate grid are obtained.

In Figure 2 we see the plots and the contour curves of H in (a) and (c) and of $\hat{H}$ in (b) and (d) based on 1,000 samples of size 1,000. We

Table 1: Actual value of H (Equation (2.1) or (2.2)), bias and mean squared error of $\hat{H}$ using Equation (2.3) at some points of the bivariate grid considering 1,000 series of size 1,000, observed from a normally distributed random vector (X, Y) with $\boldsymbol{\mu} = (3.05; 6.44)'$ and $\mathrm{vec}(\boldsymbol{\Sigma}) = (1.13; 1.49; 1.49; 3.99)'$ (correlation 0.70).

		0.01	**0.05**	**0.25**	**0.50**	**0.75**	**0.95**	**0.99**
Actual	**0.01**	0.918	0.883	0.690	0.366	-0.024	-0.401	-0.535
Bias		-0.012	-0.019	-0.021	0.013	0.057	0.071	0.063
MSE		0.000	0.000	0.001	0.002	0.005	0.007	0.006
Actual	**0.05**	0.883	0.870	0.734	0.460	0.103	-0.265	-0.401
Bias		-0.008	-0.014	-0.017	0.012	0.054	0.072	0.065
MSE		0.000	0.000	0.001	0.001	0.004	0.006	0.006
Actual	**0.25**	0.690	0.734	0.753	0.636	0.402	0.103	-0.024
Bias		0.004	0.001	-0.008	0.005	0.039	0.063	0.061
MSE		0.001	0.000	0.000	0.000	0.002	0.005	0.005
Actual	**0.50**	0.366	0.460	0.636	0.700	0.636	0.460	0.366
Bias		0.031	0.030	0.012	-0.001	0.011	0.031	0.033
MSE		0.002	0.002	0.001	0.000	0.001	0.002	0.002
Actual	**0.75**	-0.024	0.103	0.402	0.636	0.753	0.734	0.690
Bias		0.057	0.060	0.038	0.004	-0.007	0.002	0.006
MSE		0.005	0.005	0.002	0.000	0.000	0.000	0.001
Actual	**0.95**	-0.401	-0.265	0.103	0.460	0.734	0.870	0.883
Bias		0.065	0.073	0.057	0.013	-0.016	-0.014	-0.009
MSE		0.005	0.006	0.004	0.001	0.001	0.000	0.000
Actual	**0.99**	-0.535	-0.401	-0.024	0.366	0.690	0.883	0.918
Bias		0.064	0.074	0.062	0.016	-0.018	-0.019	-0.013
MSE		0.006	0.007	0.005	0.002	0.001	0.001	0.000

can see that the behavior of the measure and their estimator are very similar, both showing positive dependence. The actual values, biases and the mean squared errors for simulations with 1,000 samples of sizes 250, 500 and 1,000 were computed, and in Table 1 we present the case of $T = 1,000$. The biases and the mean squared errors generally decrease as n increases.

(**2**) To assess the behavior of the local correlation measure for a stationary process, we calculate the theoretical measure $H_\tau(x_1, x_2)$, $\tau = 1, 2, 3$, given by Equation (3.1) and simulate their estimator $\hat{H}_\tau(x_1, x_2)$ given by Equation (3.2) considering 1,000 experiments of Monte Carlo with series of sizes 250, 500 and 1,000 observed from a stationary Gaussian process with zero mean, unit variance and autoregressive structure, with $\phi_1 = 0.70$. Then, at some points of the bivariate grid, we calculate the biases and the mean squared errors.

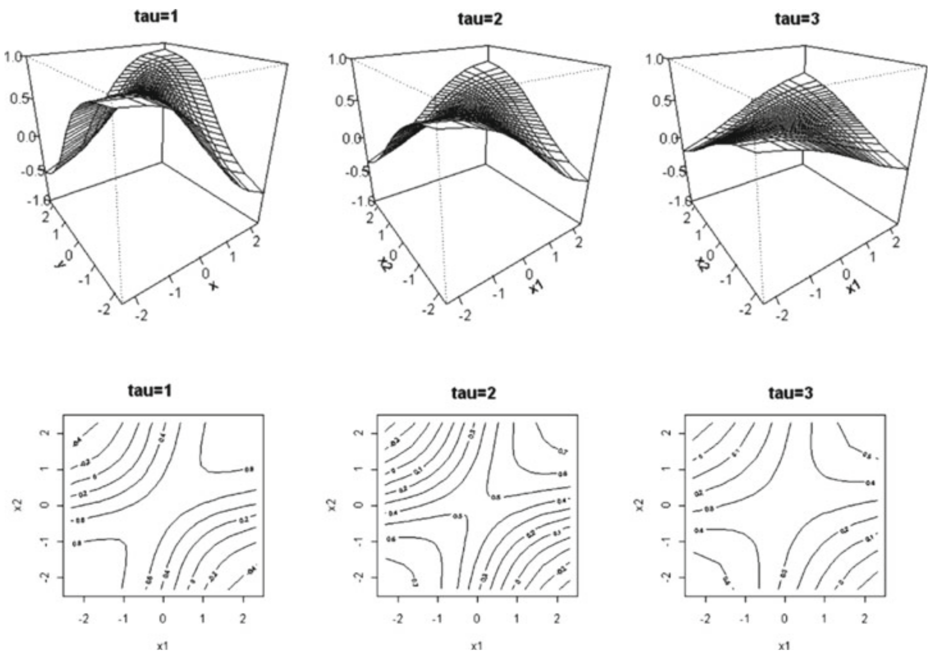


Figure 3: The first line shows H_τ, $\tau = 1, 2, 3$, (Equation (3.1)) and the second line shows the contour curves, for a Gaussian process with zero mean, unit variance and autoregressive autocorrelation, $\phi_1 = 0.7$.

In Figure 3, the plots and the contour curves of H_τ show positive dependence with decay from the saddle point along the lags. The corresponding plots of their estimator $\hat{H}_\tau$, are presented in Figure 4. We see that both theoretical and estimated measure have similar behavior. Table 2 shows the actual values, biases and the mean squared errors of the estimator for the case $T = 1,000$, showing a similar behavior to the case of random variables (Table 1).

(**3**) Finally we consider a bivariate stationary process, where we evaluate the estimator of the function of local correlation using 1,000 experiments of Monte Carlo with series of sizes 250, 500 and 1,000 observed from a VAR(1) model.

Let the stationary vector autoregressive model of order one given by

$$\mathbf{Z}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{Z}_{t-1} + \varepsilon_t,$$

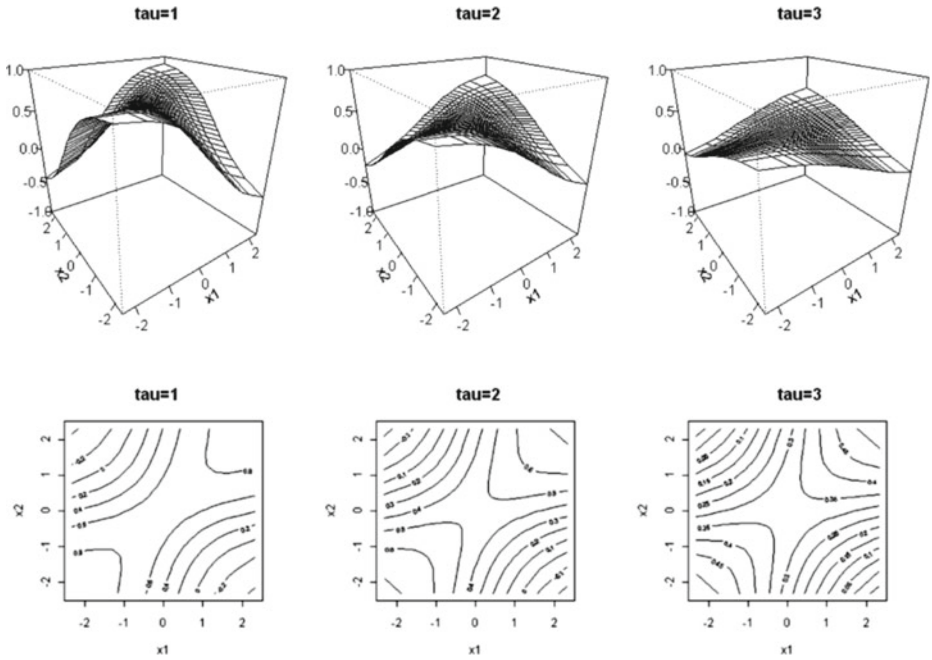


Figure 4: The first line shows $\hat{H}_\tau$, $\tau = 1, 2, 3$, (Equation (3.2)) and the second line shows the contour curves, for 1,000 series with $T = 1,000$ observed from a Gaussian process with zero mean, unit variance and autoregressive autocorrelation, $\phi_1 = 0.7$.

Table 2: Actual value of H_τ (Equation (3.1)), bias and mean squared error of $\hat{H}_\tau$ using Equation (3.2) at some points of the bivariate grid considering 1,000 series of size 1,000, observed from a Gaussian process with zero mean, unit variance and autoregressive autocorrelation with $\phi_1 = 0.7$.

		0.01	**0.05**	**0.25**	**0.50**	**0.75**	**0.95**	**0.99**
Actual	**0.01**	0.918	0.883	0.690	0.366	-0.024	-0.401	-0.535
Bias		-0.019	-0.021	-0.013	0.033	0.087	0.105	0.099
MSE		0.001	0.001	0.001	0.003	0.009	0.012	0.011
Actual	**0.05**	0.883	0.870	0.734	0.460	0.103	-0.265	-0.401
Bias		-0.021	-0.021	-0.013	0.029	0.084	0.109	0.106
MSE		0.001	0.001	0.001	0.002	0.008	0.013	0.012
Actual	**0.25**	0.690	0.734	0.753	0.636	0.402	0.103	-0.024
Bias		-0.013	-0.013	-0.013	0.010	0.055	0.088	0.092
MSE		0.001	0.001	0.001	0.001	0.004	0.009	0.010
Actual	**0.50**	0.366	0.460	0.636	0.700	0.636	0.460	0.366
Bias		0.032	0.028	0.010	-0.003	0.010	0.032	0.037
MSE		0.003	0.002	0.001	0.001	0.001	0.003	0.004
Actual	**0.75**	-0.024	0.103	0.402	0.636	0.753	0.734	0.690
Bias		0.087	0.084	0.055	0.010	-0.012	-0.011	-0.010
MSE		0.010	0.008	0.004	0.001	0.001	0.001	0.001
Actual	**0.95**	-0.401	-0.265	0.103	0.460	0.734	0.870	0.883
Bias		0.105	0.109	0.088	0.032	-0.011	-0.021	-0.021
MSE		0.012	0.013	0.009	0.002	0.001	0.001	0.001
Actual	**0.99**	-0.535	-0.401	-0.024	0.366	0.690	0.883	0.918
Bias		0.099	0.106	0.091	0.036	-0.010	-0.021	-0.019
MSE		0.011	0.012	0.010	0.004	0.001	0.001	0.001

where $\mathbf{Z}_t = (X_t, Y_t)$, $\mathbf{\Phi}_0 = (1,1)'$, $vec(\mathbf{\Phi}_1) = (0.25, 0.2, 0.2, 0.75)'$ and $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \mathbf{\Sigma})$ with $vec(\mathbf{\Sigma}) = (0.75, 0.5, 0.5, 1.25)'$. Then, the parameters of the stationary Gaussian distribution are $\boldsymbol{\mu} = (3.05, 6.44)'$ and $vec(\mathbf{\Gamma}(0)) = (1.13, 1.49, 1.49, 3.99)'$ (correlation 0.70), which are used to obtain the theoretical function H_0 (Equation (3.3)). The estimator $\hat{H}_0$ given by Equation (3.4) was simulated through 1,000 series of size 1,000 observed from this model. In Figure 5 we see the behavior of both theoretical (plots (a) and (c)) and estimated (plots (b) and (d)) functions, which are similar. The actual values, biases and mean squared errors of the estimator $\hat{H}_0$ for simulations with series of sizes 250, 500 and 1,000 show similar behavior as that of H, $\hat{H}$, H_τ and $\hat{H}_\tau$. See Table 3 for the case of $T = 1,000$.

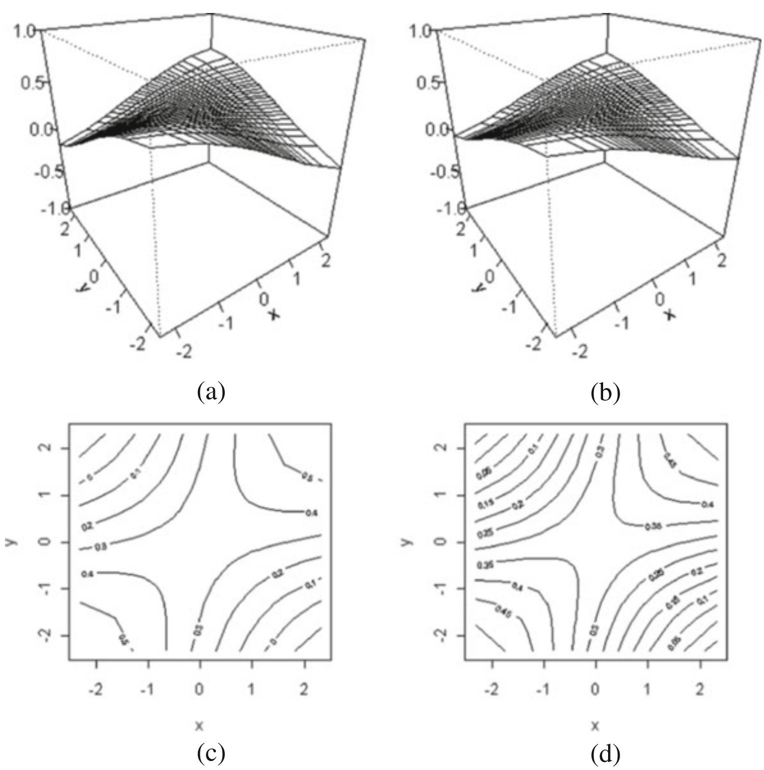


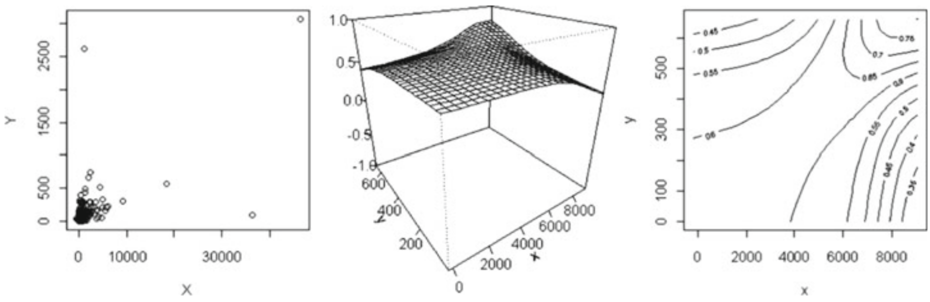
Figure 5: For a VAR(1) model with mean $(3.05; 6.44)'$, $vec(\mathbf{\Gamma}(0)) = (1.13; 1.49; 1.49; 3.99)'$ (correlation 0.70) and Gaussian innovations, we have the plot of H_0 (Equation (3.3)) in (a), the contour plot in (c), the plot of $\hat{H}_0$ (Equation (3.4)) in (b) and contour plot in (d), obtained from 1,000 series of size $T = 1,000$ observed from this model.

Table 3: Actual value of H_0 (Equation (3.3)), bias and mean squared error of $\hat{H}_0$ using Equation (3.4) at some points of the bivariate grid considering 1,000 series of size 1,000, observed from the VAR(1) model with $\boldsymbol{\mu} = (3.05, 6.44)'$, $vec(\boldsymbol{\Gamma}(0)) = (1.13, 1.49, 1.49, 3.99)'$ (correlation 0.70) and Gaussian innovations.

		0.01	**0.05**	**0.25**	**0.50**	**0.75**	**0.95**	**0.99**
Actual	**0.01**	0.920	0.885	0.692	0.370	-0.027	-0.404	-0.537
Bias		-0.014	-0.020	-0.021	0.013	0.059	0.073	0.067
MSE		0.000	0.001	0.002	0.003	0.006	0.007	0.007
Actual	**0.05**	0.855	0.873	0.737	0.462	0.101	-0.268	-0.404
Bias		-0.010	-0.015	-0.018	0.012	0.057	0.074	0.070
MSE		0.000	0.000	0.001	0.002	0.005	0.007	0.007
Actual	**0.25**	0.692	0.737	0.757	0.639	0.402	0.101	-0.026
Bias		0.004	0.000	-0.009	0.004	0.039	0.062	0.063
MSE		0.002	0.001	0.001	0.001	0.003	0.006	0.007
Actual	**0.50**	0.367	0.462	0.639	0.704	0.639	0.462	0.367
Bias		0.033	0.029	0.010	-0.001	0.011	0.030	0.034
MSE		0.005	0.003	0.001	0.001	0.001	0.003	0.005
Actual	**0.75**	-0.027	0.101	0.402	0.639	0.757	0.737	0.692
Bias		0.063	0.061	0.038	0.005	-0.008	0.001	0.005
MSE		0.008	0.006	0.003	0.001	0.001	0.001	0.002
Actual	**0.95**	-0.404	-0.268	0.101	0.462	0.737	0.873	0.885
Bias		0.068	0.071	0.055	0.011	-0.018	-0.014	-0.009
MSE		0.007	0.006	0.005	0.002	0.001	0.000	0.000
Actual	**0.99**	-0.537	-0.404	-0.026	0.367	0.692	0.885	0.920
Bias		0.067	0.073	0.060	0.015	-0.021	-0.020	-0.013
MSE		0.007	0.007	0.006	0.003	0.002	0.001	0.000

5 Empirical illustration

In what follows, we use bandwidth equal to the standard deviation of the data (as suggested by Bjerve and Doksum, 1993) and 99.7% of the central data.



(a) (b) (c)

Figure 6: For the net equity and salary (both in millions of dollars) of companies in 2006, we have the scatterplot in (a), the plot of $\hat{H}$ (given by Equation (2.3)) in (b) and the contour plot in (c), respectively.

(**1**) The first application consider the net equity (in millions of dollars) and salary (also in millions of dollars) for 687 companies chosen among the 1,018 largest companies in Brazil, except banks and insurers, in 2006 (see http://app.exame.abril.com.br/servicos/melhoresemaiores/). For these two variables, we obtain that the Spearman's correlation coefficient is 0.542, indicating positive association, and also the measure

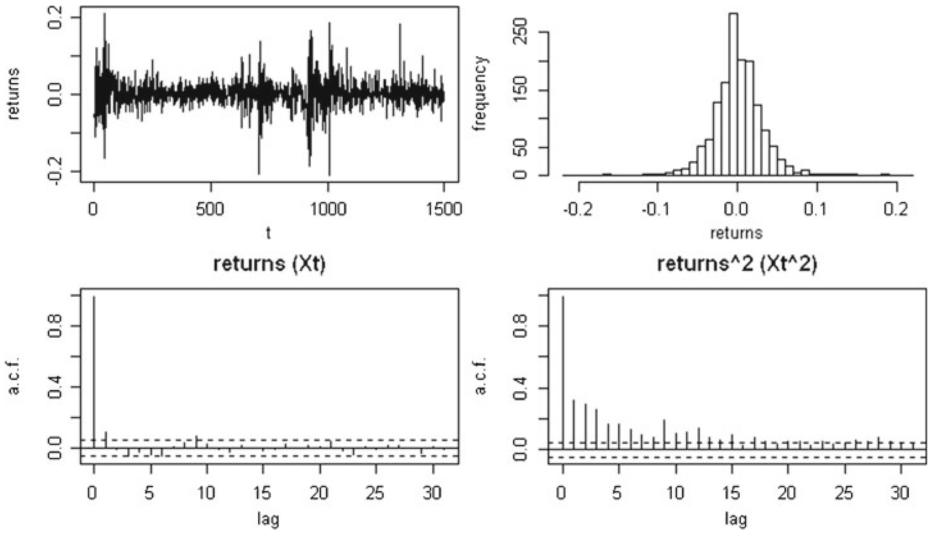


Figure 7: Plot of daily returns of Petrobras (X_t) (from 3/01/95 to 27/12/00), histogram, a.c.f. of the serie and squared serie.

of local correlation. Figure 6 shows the scatterplot in (a), the plot of $\hat{H}$ in (b) and the contour curves in (c), which show asymmetric positive dependence, and moreover larger values of one variable tend to be more correlated with larger values of the other one.

(2) Next, we analyze the daily log returns X_t of Petrobras (Brazilian oil company) from 3 January 1995 to 27 December 2000, which correspond to 1,498 observations. In Figure 7 we see the graphics of the series, its histogram and also the a.c.f. of the series and of the squared series

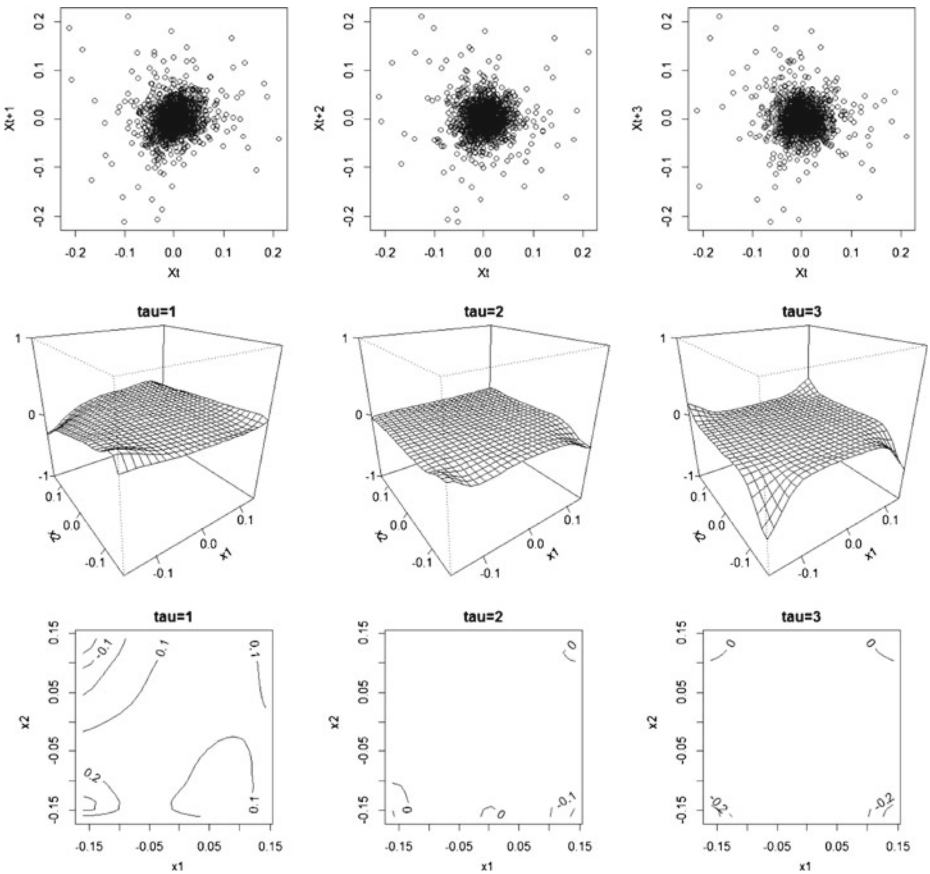


Figure 8: For the daily returns of Petrobras (X_t) (from 3/01/95 to 27/12/00), we have the scatterplots, plots of $\hat{H}_\tau$ (Equation (3.2)) and contour curves, lags 1 to 3.

showing serial correlation and dependence. Considering $(X_t, X_{t+\tau})$, $\tau = 1, 2, 3$, we see in Figure 8 the scatterplots, the plots and contour curves of the $\hat{H}_\tau$, showing positive dependence at lag one.

(**3**) Finally, to illustrate the case of two series, we consider the data of daily log returns of CAC 40 (Cotation Assiste en Continu) and FTSE (Financial Times Stock Exchange - UK stock index) from 03 January 1994 to 8 August 2000 with 1,722 observations. The returns of the CAC 40 (X_t) and FTSE (Y_t) have contemporaneous correlation coefficient

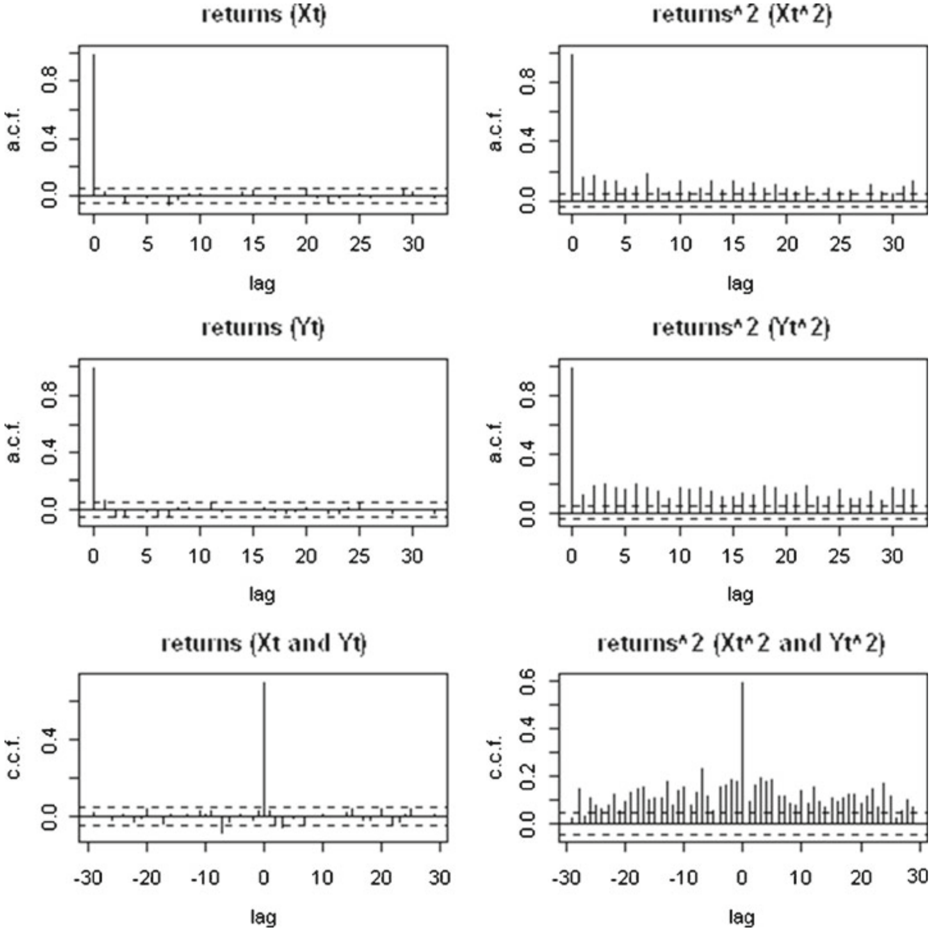
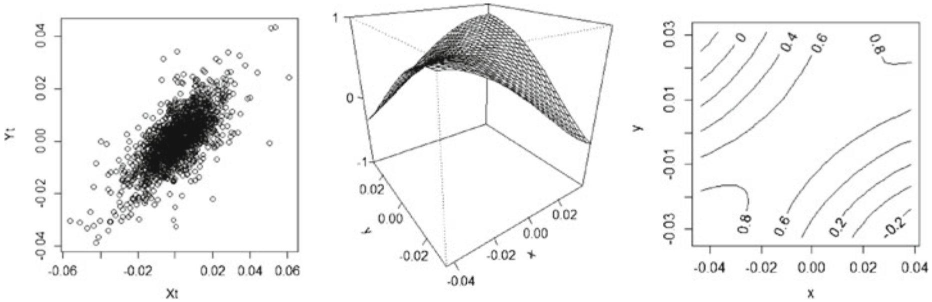


Figure 9: Autocorrelation functions of returns and squared returns of CAC 40 (X_t) and FTSE (Y_t) (from 3/01/94 to 08/08/00), and their cross-correlation function.



(a) (b) (c)

Figure 10: For the returns of CAC 40 (X_t) and FTSE (Y_t) (from 3/01/94 to 08/08/00) we have the scatterplot in (a) and the plots of $\hat{H}_0$ (Equation (3.4)) in (b) and (c).

0.71, Spearman's rho 0.67 and Kendall's tau 0.49. The a.c.f. and c.c.f. of the returns and squared returns are shown in Figure 9. In Figure 10, for the returns we have the scatterplot in (a) and the plots of $\hat{H}_0$ in (b) and (c). All these graphics show symmetric positive dependence between the series.

6 Concluding remarks

For the measure of local correlation of Bairamov et al. (2003), we proposed a smoothed kernel estimator for the conditional expectation. Then, consistency of this estimator was obtained. Using simulations of 1,000 samples, the estimator was shown to be similar to the theoretical measure, and with the increasing of the sample's sizes then the bias and the mean squared error decreased. The empirical illustration of this estimator was made using the net equity (in million of dollars) and salary (in million of dollars) of some companies in Brazil in 2006.

Next we considered the measure of local correlation for univariate and bivariate stationary time series. Estimators were proposed for these situations, again using kernel estimators, and their consistency are obtained. For the univariate process, a stationary Gaussian model was used to make the simulations, and for the bivariate process, time series are observed from a VAR(1) model. In both cases, the behavior of the estimator were similar to the theoretical ones. Empirical illustrations of these estimators were made using the daily returns of the Petrobras for the univariate process and the daily returns of the CAC 40 and FTSE for the bivariate process.

We considered only discrete time processes in the paper. It would be interesting to extend the results to continuous time.

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Sumaia A. Latif
School of Arts, Science and Humanities, University of São Paulo
Rua Arlindo Béttio, 1000
Ermelino Matarazzo, CEP: 03828-000
São Paulo, Brazil
E-mail: salatif@usp.br

Pedro A. Morettin
Department of Statistics
Institute of Mathematics and Statistics, University of São Paulo
Rua do Matão, 1010, Butantã
CEP: 05508-090, São Paulo, Brazil
E-mail: pam@ime.usp.br

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