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ORIGINAL PAPER

Singularities of equidistants and global centre symmetry sets of Lagrangian submanifolds

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Abstract We study the global centre symmetry set (GCS) of a smooth closed submanifold $M^m \subset \mathbb{R}^n$, $n \leq 2m$. The GCS includes both the centre symmetry set defined by Janeczko (Geometria Dedicata 60:9–16, 1996) and the Wigner caustic defined by Berry (Philos Trans R Soc Lond A 287:237–271, 1977). The definition of $GCS(M)$ uses the concept of an affine λ-equidistant of M, $E_\lambda(M)$, $\lambda \in \mathbb{R}$. When $M = L$ is a Lagrangian submanifold in the affine symplectic space $(\mathbb{R}^{2m}, \omega = \sum_{i=1}^m dp^i \wedge dq^i)$, we present generating families for singularities of $E_\lambda(L)$ and prove that the caustic of any simple stable Lagrangian singularity in a $4m$-dimensional Lagrangian fibre bundle is realizable as the germ of an affine equidistant of some $L \subset \mathbb{R}^{2m}$. We characterize the criminant part of $GCS(L)$ in terms of bitangent hyperplanes to L. Then, after presenting the appropriate equivalence relation to be used in this Lagrangian case, we classify the affine-Lagrangian stable singularities of $GCS(L)$. In particular we show that, already for a smooth closed convex curve $L \subset \mathbb{R}^2$, many singularities of $GCS(L)$ which are affine stable are not affine-Lagrangian stable.

Keywords Centre symmetry set · Symplectic geometry · Lagrangian singularities

Mathematics Subject Classification (1991) 57R45 · 58K40 · 53D12 · 58K25 · 58K50

1 Introduction

The centre of symmetry of an ellipse in $\mathbb{R}^2$ can be defined as the set (in this case consisting of a single element) of midpoints of intervals connecting pairs of points on the curve with

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parallel tangent vectors. For a generic smooth convex closed curve, this set is not a single point, but forms a curve with an odd number of cusps, in the interior of the smooth original curve, which has been known as the *Wigner caustic* of the smooth curve since the work of Berry in the 70's. Thus, the Wigner caustic is an affine-invariant generalization of the centre of symmetry of an ellipse and this definition of centre of symmetry extends to higher dimensional smooth closed submanifolds of $\mathbb{R}^n$.

On the other hand, the centre of symmetry of an ellipse in $\mathbb{R}^2$ can also be described as the envelope of all straight lines connecting pairs of points on the curve with parallel tangent vectors. For a generic smooth convex closed curve, this set is not a single point, but forms a curve with an odd number of cusps, in the interior of the smooth original curve, which has been known as the *centre symmetry set* of the smooth curve since the work of Janeczko in the 90's. Again, this is an affine-invariant generalization of the centre of a circle, which extends to higher dimensional smooth closed hypersurfaces of $\mathbb{R}^n$ [16].

The Wigner caustic and the centre symmetry set of a generic smooth convex closed curve are not the same singular curve. Instead, the Wigner caustic is interior to the centre symmetry set and the cusp points of the inner curve touches the outer one in its smooth part. A larger centre symmetry set, containing the two previous ones, can be defined in an affine-invariant way, for an arbitrary smooth closed m-dimensional submanifold M of $\mathbb{R}^n$, for $n/2 \leq m < n$. We call this new set the *global centre symmetry* set of M and denote it by $GCS(M)$.

Our definition is a slight modification of a definition introduced by Giblin and Zakalyukin [10–12] to study singularities of centre symmetry sets of hypersurfaces. A key notion in this definition is that of an affine λ-equidistant of the smooth submanifold M, denoted $E_\lambda(M)$, of which the Wigner caustic is the case $\lambda = 1/2$. The singularities of $E_\lambda(M)$ are then fundamental to characterize $GCS(M)$ and its own singularities.

In this paper, we study singularities of $E_\lambda(L)$ and $GCS(L)$, when L is a smooth closed *Lagrangian* submanifold of $(\mathbb{R}^{2m}, \omega)$, where ω is the canonical symplectic form. The paper is organized as follows.

In Sect. 2 we present the definitions of an affine λ-equidistant of M and of the global centre symmetry set of M, for a general smooth submanifold $M^m \subset \mathbb{R}^n$, $n \leq 2m$. In Sect. 3, for $M = L$ Lagrangian in $\mathbb{R}^{2m}$, we obtain the generating families for the affine equidistants $E_\lambda(L)$, cf. Theorem 3.8, relating their general classification to the well known classification by Lagrangian equivalence (chapters 18, 19, 21 in [2]). This is used in Sect. 4 to study singularities of affine equidistants. Theorem 4.1 states that the caustic of any simple stable Lagrangian singularity in a $4m$-dimensional Lagrangian fibre bundle is realizable as the germ of an affine equidistant $E_\lambda(L)$ of some $L \subset \mathbb{R}^{2m}$.

In Sect. 5 we obtain a geometric characterization for the criminant of $GCS(L)$ in terms of bitangent hyperplanes to the Lagrangian submanifold $L^m \subset \mathbb{R}^{2m}$, cf Theorem 5.5. This result is similar to results presented for a hypersurface $M^m \subset \mathbb{R}^{m+1}$ in [10–12].

In Sect. 6 we introduce the equivalence relation (also as an equivalence of generating families) that is used to classify the singularities of $GCS(L)$, cf. Definitions 6.1, 6.3 and 6.7. Then, we show that the only affine-Lagrangian stable singularities of $GCS(L)$ are singularities of the criminant, the smooth part of the Wigner caustic, or tangent union of both, cf. Theorems 6.12 through 6.16 and Lemma 6.13.

Section 7 is devoted to the *GCS* of curves in the affine symplectic plane. First, in Theorem 7.1 we collect results on the *GCS* of convex curves in non-symplectic plane, [3,9–13,16], and we obtain in Theorem 7.2 a new inequality on the number of cusps of the centre symmetry set and the Wigner caustic. Pictures illustrate these results.

Then, we obtain in Theorem 7.7 and Corollary 7.8 all the affine-Lagrangian stable singularities of the *GCS* of curves in symplectic plane. Comparison of Theorem 7.1 and Corollary

7.8 shows that most of the singularities of the *GCS* which are affine-stable when no symplectic structure is considered, are not affine-Lagrangian stable.

In other words, although any smooth curve on $\mathbb{R}^2$ is Lagrangian, the singularities of their *GCS* are sensitive to the presence of a symplectic form to be accounted for, that is, there is a breakdown of their stability. Thus, we end the paper with some discussion of this result, which is similar to some results in [4–7] showing a breakdown of the simplicity of some singularities due to a symplectic form.

2 Definition of the global centre symmetry set

Let M be a smooth closed m-dimensional submanifold of the affine space $\mathbb{R}^n$, with $n \leq 2m$. Let a, b be points of M. Let τ_{a-b} be the translation by the vector $(a - b)$, i.e., $\tau_{a-b} : \mathbb{R}^n \ni x \mapsto x + (a - b) \in \mathbb{R}^n$.

Definition 2.1 A pair $a, b \in M$ $(a \neq b)$ is a *weakly parallel* pair if

$$T_a M + \tau_{a-b}(T_b M) \neq T_a \mathbb{R}^n.$$

A weakly parallel pair $a, b \in M$ is called *k-parallel* if

$$\dim(T_a M \cap \tau_{b-a}(T_b M)) = k.$$

If $k = m$ the pair $a, b \in M$ is called *strongly parallel*, or just *parallel*. We also refer to k as the *degree of parallelism* of the pair (a, b).

Definition 2.2 A *chord* passing through a pair a, b, is the line

$$l(a, b) = \{x \in \mathbb{R}^n | x = \lambda a + (1 - \lambda)b, \lambda \in \mathbb{R}\}.$$

Definition 2.3 For a given λ, an *affine* λ*-equidistant* of M, $E_\lambda(M)$, is the set of all $x \in \mathbb{R}^n$ such that $x = \lambda a + (1 - \lambda)b$, for all weakly parallel pairs $a, b \in M$. $E_\lambda(M)$ is also called a (affine) *momentary equidistant* of M. Whenever M is understood, we write E_λ for $E_\lambda(M)$.

Note that, for any λ, $E_\lambda(M) = E_{1-\lambda}(M)$ and in particular $E_0(M) = E_1(M) = M$. Thus, the case $\lambda = 1/2$ is special:

Definition 2.4 $E_{1/2}(M)$ is called the *Wigner caustic* of M [3,17].

The *extended affine space* is the space $\mathbb{R}^{n+1}_e = \mathbb{R} \times \mathbb{R}^n$ with coordinate $\lambda \in \mathbb{R}$ (called *affine time*) on the first factor and projection on the second factor denoted by $\pi : \mathbb{R}^{n+1}_e \ni (\lambda, x) \mapsto x \in \mathbb{R}^n$.

Definition 2.5 The *affine extended wave front* of $M, \mathbb{E}(M)$, is the union of all affine equidistants each embedded into its own slice of the extended affine space: $\mathbb{E}(M) = \bigcup_{\lambda \in \mathbb{R}} \{\lambda\} \times E_\lambda(M) \subset \mathbb{R}^{n+1}_e$.

Note that, when M is a circle in the plane, $\mathbb{E}(M)$ is the (double) cone, which is a smooth manifold with nonsingular projection π everywhere, but at its singular point, which projects to the centre of the circle. From this, we generalize the notion of centre of symmetry. Thus, let π_r be the restriction of π to the affine extended wave front of M: $\pi_r = \pi|_{\mathbb{E}(M)}$. A point $x \in \mathbb{E}(M)$ is a *critical* point of π_r if the germ of π_r at x fails to be the germ of a regular projection of a smooth submanifold. We now introduce the main definition of this paper:

Definition 2.6 The *global centre symmetry* set of M, $GCS(M)$, is the image under π of the locus of critical points of π_r.

Remark 2.7 The set $GCS(M)$ is the bifurcation set of the family of affine equidistants (the family of chords of weakly parallel pairs) of M.

In general, $GCS(M)$ consists of two components: the *caustic* $\Sigma(M)$ being the projection of the singular locus of $\mathbb{E}(M)$ and the *criminant* $\Delta(M)$ being the (closure of) the image under π_r of the set of regular points of $\mathbb{E}(M)$ which are critical points of the projection π restricted to the regular part of $\mathbb{E}(M)$. Thus $\Delta(M)$ is the envelope of the family of regular parts of momentary equidistants, while $\Sigma(M)$ contains all the singular points of momentary equidistants.

The above definition (with its following remarks) is a slight modification of the definition that has already been introduced by Giblin and Zakalyukin [10]. However, in our present definition the whole manifold M is considered, as opposed to pairs of germs, as in [10], and weak parallelism is also taken into account. Considering the whole manifold in the definition leads to the following simple but important result:

Theorem 2.8 *The set $GCS(M)$ contains the Wigner caustic of M.*

Proof Let x be a regular point of $E_{\frac{1}{2}}(M)$. Then $x = \frac{1}{2}(a+b)$ for a weakly parallel pair $a, b \in M$. It means that x is a intersection point of the chords $l(a,b)$ and $l(b,a)$. Then $\mathbb{E}(M)$ contains the sets

$$\{(\lambda, \lambda a + (1-\lambda)b) | \lambda \in \mathbb{R}\},\ \{(\lambda, (1-\lambda)a + \lambda b) | \lambda \in \mathbb{R}\}.$$

If $(\frac{1}{2}, x)$ is a regular point of $\mathbb{E}(M)$ then the above sets are included in the tangent space to $\mathbb{E}(M)$ at $(\frac{1}{2}, x)$. Therefore the fiber $\{(\lambda, x) | \lambda \in \mathbb{R}\}$ is included in the tangent space of $\mathbb{E}(M)$. Thus if $(\frac{1}{2}, x)$ is a regular point of $\mathbb{E}(M)$ then x is in the criminant $\Delta(M)$. If $(\frac{1}{2}, x)$ is not a regular point of $\mathbb{E}(M)$ then x is in the caustic $\Sigma(M)$. □

If $M \subset \mathbb{R}^2$ is a smooth curve, then $E_{1/2}(M)$ is the bifurcation set for the number of chords connecting two points in M and having a given midpoint x, for any $x \in E_{1/2}(M)$ [3]. Similarly, if $\mathcal{R}_x : \mathbb{R}^2 \to \mathbb{R}^2$ denotes reflection through $x \in \mathbb{R}^2$, then $x \in E_{1/2}(M)$ when M and $\mathcal{R}_x(M)$ are not transversal [14,17]. Finally, let $A(x, \kappa)$ be the area of the planar region bounded by M and a chord, considered as a function of a point x on the chord and a variable κ locating one of the endpoints of the chord on the curve. Then, $A(x, \kappa)$ is a generating family for $E_{1/2}(M)$ [3,13]. Below we generalize this notion to every λ-equidistant of any Lagrangian submanifold.

3 Generating families

Consider the product affine space $\mathbb{R}^n \times \mathbb{R}^n$ with coordinates (x_+, x_-), the tangent bundle to $\mathbb{R}^n$, $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with coordinate system $(x, \dot{x})$, and standard projection $pr : T\mathbb{R}^n \to \mathbb{R}^n,\ (x, \dot{x}) \mapsto x$.

Definition 3.1 $\forall \lambda \in \mathbb{R} \setminus \{0, 1\}$, a λ-*chord transformation*

$$\Phi_\lambda : \mathbb{R}^n \times \mathbb{R}^n \to T\mathbb{R}^n,\ (x^+, x^-) \mapsto (x, \dot{x})$$

is a *linear* diffeomorphism defined by the λ-*point equation*:

$$x = \lambda x^{+} + (1-\lambda)x^{-}, \tag{3.1}$$

for the λ-point x, and a *chord equation*:

$$\dot{x} = \lambda x^{+} - (1-\lambda)x^{-}. \tag{3.2}$$

Now, let M be a smooth closed m-dimensional submanifold of the affine space $\mathbb{R}^n$ ($2m \geq n$) and consider the product $M \times M \subset \mathbb{R}^n \times \mathbb{R}^n$. Let $\mathcal{M}_\lambda$ denote the image of $M \times M$ by a λ-chord transformation,

$$\mathcal{M}_\lambda = \Phi_\lambda(M \times M).$$

Theorem 3.2 *The set of critical values of the standard projection $pr : T\mathbb{R}^n \to \mathbb{R}^n$ restricted to $\mathcal{M}_\lambda$ is $E_\lambda(M)$.*

Proof If a is a critical value of $pr|_{\mathcal{M}_\lambda}$, then

$$k = \dim T_{(a,\dot{a})}\mathcal{M}_\lambda \cap T_{(a,\dot{a})}pr^{-1}(a) \geq 2m - n.$$

Let $v_1, \ldots, v_k$ be a basis of $T_{(a,\dot{a})}\mathcal{M}_\lambda \cap T_{(a,\dot{a})}pr^{-1}(a)$ of the form $v_j = \sum_{i=1}^{n} \alpha_{ji} \frac{\partial}{\partial \dot{x}_i}|_{(a,\dot{a})}$ for $j = 1, \ldots, k$. We have $(\Phi_\lambda^{-1})_*(v_j) = \frac{1}{2\lambda}v_j^{+} - \frac{1}{2(1-\lambda)}v_j^{-}$, where

$$v_j^{+} = \sum_{i=1}^{n} \alpha_{ji} \frac{\partial}{\partial x_i^{+}}|_{a^+} \in T_{a^+}M, \quad v_j^{-} = \sum_{i=1}^{n} \alpha_{ji} \frac{\partial}{\partial x_i^{-}}|_{a^-} \in T_{a^-}M.$$

It implies that $v_j^{+} \in T_{a^+}M \cap \tau_{(a^+-a^-)}T_{a^-}M$ for $j = 1, \ldots, k$. Thus $T_{a^+}M + \tau_{(a^+-a^-)}T_{a^-}M \neq T_{a^+}\mathbb{R}^n$ and consequently a^+, a^- is a k-parallel pair. Hence $\lambda a^+ + (1-\lambda)a^- = a \in E_\lambda$.

Now, assume $a \in E_\lambda$. Then $a = \lambda a^+ + (1-\lambda)a^-$ for a weakly k-parallel pair a^+, a^- for $k > 2m - n$. Thus there exist linearly independent vectors $v_j^{+} = \sum_{i=1}^{n} \alpha_{ji} \frac{\partial}{\partial x_i^{+}}|_{a^+} \in T_{a^+}M \cap \tau_{(a^+-a^-)}T_{a^-}M$ for $j = 1, \ldots, k$. Consider linearly independent vectors $v_j = (\Phi_\lambda)_*((1-\lambda)v_j^{+} - \lambda\tau_{(a^--a^+)}v_j^{+})$ for $j = 1, \ldots, k$. Then, v_j belongs to $T_{(a,\dot{a})}\mathcal{M}_\lambda$ and $pr_*(v_j) = 0$ for $j = 1, \ldots, k$. Thus a is a critical value of $pr|_{\mathcal{M}_\lambda}$. □

Let $(\mathbb{R}^{2m}, \omega)$ be the affine symplectic space with canonical coordinates p_i, q_i, so that $\omega = \sum_{i=1}^{m} dp_i \wedge dq_i$, and let L be a smooth closed Lagrangian submanifold of $(\mathbb{R}^{2m}, \omega)$. For a fixed $\lambda \in \mathbb{R} \setminus \{0, 1\}$, consider the product affine space $\mathbb{R}^{2m} \times \mathbb{R}^{2m}$ with the λ-weighted symplectic form

$$\delta_\lambda\omega = 2\lambda^2\pi_1^*\omega - 2(1-\lambda)^2\pi_2^*\omega, \tag{3.3}$$

where π_i is the projection of $\mathbb{R}^{2m} \times \mathbb{R}^{2m}$ on ith factor for $i = 1, 2$.

Now, let Φ_λ be the λ-chord transformation (3.1) (3.2). Then,

$$\left(\Phi_\lambda^{-1}\right)^* (\delta_\lambda\omega) \;=\; \dot{\omega}. \tag{3.4}$$

where $\dot{\omega}$ is the canonical symplectic form on the tangent bundle to $(\mathbb{R}^{2m}, \omega)$, defined by $\dot{\omega}(x, \dot{x}) = d\{\dot{x} \lrcorner \omega\}(x)$ or, in Darboux coordinates,

$$\dot{\omega} = \sum_{i=1}^{m} d\dot{p}_i \wedge dq_i + dp_i \wedge d\dot{q}_i. \tag{3.5}$$

The fibers of $T\mathbb{R}^{2m}$ are Lagrangian for $\dot{\omega}$, so that $pr : T\mathbb{R}^{2m} \to \mathbb{R}^{2m}$ defines a *Lagrangian fiber bundle* with respect to $\dot{\omega}$, that is, a fiber bundle whose fibers are Lagrangian in the total symplectic space.

Denote the restriction of the projection pr of $(T\mathbb{R}^{2m}, \dot{\omega})$ to the Lagrangian submanifold

$$\mathcal{L}_\lambda = \Phi_\lambda(L \times L)$$

by $pr|_{\mathcal{L}_\lambda}$. According to chapter 18 in [2], $pr|_{\mathcal{L}_\lambda}$ is a Lagrangian map. The set of critical values of a Lagrangian map is called a *caustic*. Theorem 3.2 implies

Proposition 3.3 *The caustic of the Lagrangian map* $pr|_{\mathcal{L}_\lambda}$ *is* $E_\lambda(L)$.

Definition 3.4 $E_\lambda(L)$ and $E_\lambda(\widetilde{L})$ are *Lagrangian equivalent* if the Lagrangian maps $pr|_{\mathcal{L}_\lambda}$ and $pr|_{\widetilde{\mathcal{L}}_\lambda}$ are Lagrangian equivalent (see chapter 18 in [2]).

It follows from above definitions:

Proposition 3.5 *The classification of affine equidistants* $E_\lambda(L)$ *by Lagrangian equivalence is affine symplectic invariant, i.e., invariant under the standard action of the affine symplectic group on* $(\mathbb{R}^{2m}, \omega)$.

From the above, we also use the term *affine-Lagrangian equivalence* for Lagrangian equivalence (see chapter 18 in [2]) of $E_\lambda(L)$.

Remark 3.6 The definition of the λ-weighted symplectic form $\delta_\lambda\omega$ given by (3.3) is not arbitrary. When $\lambda = 1/2$, a Lagrangian submanifold $\Lambda \subset (\mathbb{R}^{2m} \times \mathbb{R}^{2m}, \delta_{1/2}\omega)$ defines a *canonical relation* in $(\mathbb{R}^{2m}, \omega)$ which can be locally described by a generating function of the midpoints $x_{1/2} = (x^+ + x^-)/2$, for $(x^+, x^-) \in \Lambda$, when $\mathcal{L}_{1/2} = \Phi_{1/2}(\Lambda)$ locally projects regularly to the zero section of $(T\mathbb{R}^{2m}, \dot{\omega})$, cf. [8,18]. Thus, a Lagrangian submanifold $\Lambda \subset (\mathbb{R}^{2m} \times \mathbb{R}^{2m}, \delta_\lambda\omega)$ defines a λ-*weighted canonical relation* in $(\mathbb{R}^{2m}, \omega)$ which can be locally described by a generating function of the λ-points $x_\lambda = \lambda x^+ + (1-\lambda)x^-$, when $\mathcal{L}_\lambda = \Phi_\lambda(\Lambda)$ locally projects regularly to the zero section of $(T\mathbb{R}^{2m}, \dot{\omega})$. Such generating functions give rise to the generating families, as described below, used to study singularities of the Lagrangian map $pr|_{\mathcal{L}_\lambda}$.

Let L^+ and L^- denote germs of L at points a^+ and a^-.

Proposition 3.7 *If the pair* a^+, a^- *is* k*-parallel, then there exist canonical coordinates* (p, q) *in* $\mathbb{R}^{2m}$ *and function germs* S^+ *and* S^- *such that*

$$L^+ : \ p_i = \frac{\partial S^+}{\partial q_i}(q_1, \ldots, q_m), \ \ i = 1, \ldots, m$$

$$L^- : \begin{cases} p_j = \frac{\partial S^-}{\partial q_j}(q_1, \ldots, q_k, p_{k+1}, \ldots, p_m), & j = 1, \ldots, k, \\ q_l = -\frac{\partial S^-}{\partial p_l}(q_1, \ldots, q_k, p_{k+1}, \ldots, p_m), & l = k+1, \ldots, m \end{cases} \tag{3.6}$$

and $d^2S^+(q^+_{a,1}, \ldots, q^+_{a,m}) = 0$ *and* $d^2S^-(p^-_{a,1}, \ldots, p^-_{a,k}, q^-_{a,k+1}, \ldots, p^-_{a,m}) = 0$, *where* $a^+ = (p^+_a, q^+_a)$ *and* $a^- = (p^-_a, q^-_a)$.

Proof We can find a linear symplectic change of coordinates such that $T_{a^+}L^+ = \{p = p^+_a\}$, where $a^+ = (p^+_a, q^+_a)$, and $T_{a^-}L^- = \{p_1 = p^-_{a,1}, \ldots, p_k = p^-_{a,k}, q_{k+1} = q^-_{a,k+1}, \ldots, q_m = q^-_{a,m}\}$, where $a^- = (p^-_a, q^-_a)$. Since L is a smooth Lagrangian submanifold, it follows from standard considerations that it can be described locally by differentials of generating functions of the forms stated above in neighborhoods of a^+ and a^-, in which case we have that $d^2S^+|a^+ = d^2S^-|a^- = 0$. □

Let $q=(q_1,\ldots,q_m)$, $p=(p_1,\ldots,p_m)$, $\dot{q}=(\dot{q}_1,\ldots,\dot{q}_m)$, $\dot{p}=(\dot{p}_1,\ldots,\dot{p}_m)$.

Also, let $\beta=(\beta_1,\ldots,\beta_m)$ and, for any $k<m$, let $[k]=\{1,\ldots,k\}$, so that $\beta_{[k]}=(\beta_1,\ldots,\beta_k)$, and $\alpha_{[m]\setminus[k]}=(\alpha_{k+1},\ldots,\alpha_m)$.

Let $L^+\times L^-$ denote the germ of $L\times L$ at the point $(a^+,a^-)\in L\times L$ so that $\mathcal{L}_\lambda=\Phi_\lambda(L^+\times L^-)$ is the germ at $(a,\dot{a})$, where $a=\lambda a^+ +(1-\lambda)a^-$, $\dot{a}=\lambda a^+ -(1-\lambda)a^-$, of a smooth Lagrangian submanifold of $(T\mathbb{R}^{2m},\dot{\omega})$.

Theorem 3.8 *If the pair a^+,a^- is k-parallel and germs L^+ and L^- are given by (3.6) then the germ of the generating family*

$$F_\lambda(p,q,\alpha_{[m]\setminus[k]},\beta)=2\lambda^2S^+\left(\frac{q+\beta}{2\lambda}\right)-2(1-\lambda)^2S^-\left(\frac{q_{[k]}-\beta_{[k]},\ p_{[m]\setminus[k]}-\alpha_{[m]\setminus[k]}}{2(1-\lambda)}\right)$$
$$-\sum\nolimits_{i=1}^{k}p_i\beta_i+\frac{1}{2}\sum\nolimits_{j=k+1}^{m}q_j\alpha_j-p_j\beta_j-\alpha_j\beta_j-p_jq_j \qquad (3.7)$$

generates the germ of $\mathcal{L}_\lambda$ at $(a,\dot{a})$ as follows:

$$\mathcal{L}_\lambda=\left\{(\dot{p},\dot{q},p,q):\exists(\alpha,\beta)\ \dot{p}=\frac{\partial F_\lambda}{\partial q},\ \dot{q}=-\frac{\partial F_\lambda}{\partial p},\ \frac{\partial F_\lambda}{\partial\alpha}=\frac{\partial F_\lambda}{\partial\beta}=0\right\}.$$

Proof The proof is a straightforward calculation. □

Remark 3.9 It follows from (3.7) that *the degree of parallelism is the corank of the singularity*, i.e. the corank of the Hessian of $F_\lambda(p_a,q_a,\alpha_{[m]\setminus[k]},\beta)$ as a function in $(\alpha_{[m]\setminus[k]},\beta)\in\mathbb{R}^{2m-k}$.

Theorem 3.10 ([2]) *Germs of Lagrangian maps are Lagrangian equivalent iff the germs of their generating families are stably $\mathcal{R}^+$-equivalent.*

Corollary 3.11 *Germs $E_\lambda(L)$ and $E_\lambda(\tilde{L})$ are Lagrangian equivalent iff germs of generating families for $\mathcal{L}_\lambda$ and $\tilde{\mathcal{L}}_\lambda$ are stably $\mathcal{R}^+$-equivalent.*

4 Singularities of equidistants of Lagrangian submanifolds

We have the following results on singularities of affine equidistants of closed Lagrangian submanifolds, up to Lagrangian equivalence:

Theorem 4.1 *The caustic of any simple stable Lagrangian singularity (A-D-E singularities) in the $4m$-dimensional symplectic tangent bundle $(T\mathbb{R}^{2m},\dot{\omega})$ is realizable as $E_\lambda(L)$, for some smooth closed Lagrangian submanifold L in $(\mathbb{R}^{2m},\omega)$.*

The generic Lagrangian maps for manifolds of dimension smaller than 6 have only simple stable Lagrangian singularities (chapter 21 in [2]). Therefore we obtain the following corollary.

Corollary 4.2 *Any germ of generic caustics on $2m$-dimensional manifold for $m=1,2$ is realizable as $E_\lambda(L)$, for some smooth Lagrangian submanifold L in $(\mathbb{R}^{2m},\omega)$.*

Proof of Theorem 4.1 We use the method described in chapters 8 and 21 in [2]. For a fixed λ, let $x=(p,q)$ and $\kappa=(\alpha,\beta)$. From (3.7) we easily see that

$$\mathrm{rank}_{(a,\dot{a})}\left[\frac{\partial^2F_\lambda}{\partial\kappa^2},\ \frac{\partial^2F_\lambda}{\partial\kappa\partial x}\right]=2m-k,$$

hence is equal to the dimension of κ-space. Let the arguments of the function S^+ be denoted by $(q_1^+, \dots, q_m^+)$ and the arguments of the function S^- by $(q_1^-, \dots, q_k^-, p_{k+1}^-, \dots, p_m^-)$.

We find S^+ and S^- such that $F_\lambda(x, \kappa)$ is a $\mathcal{R}^+$-versal deformation of A-D-E singularities. Let

$$\begin{aligned} S^+(q^+) &= \sum_{i=1}^{m} p_{a,i}^+(q_i^+ - q_{a,i}^+) + S_3^+(q^+ - q_a^+) \\ S^-(q_{[k]}^-, p_{[m]\setminus[k]}^-) &= \sum_{i=1}^{k} p_{a,i}^-(q_i^- - q_{a,i}^-) - \sum_{i=k+1}^{m} q_{a,i}^-(p_i^- - p_{a,i}^-) \\ &\quad + S_3^-(q_{[k]}^- - q_{a,[k]}^-, p_{[m]\setminus[k]}^- - p_{a,[m]\setminus[k]}^-), \end{aligned}$$

where we used Proposition 3.7 and where $S_3^\pm \in \mathfrak{m}^3$ ($\mathfrak{m}$ is the maximal ideal of the ring of smooth function-germs on $\mathbb{R}^n$ at 0). We write the generating families in coordinates $\tilde{p} = p - p_a, \tilde{q} = q - q_a, s = \alpha - \dot{p}_a, t = \beta - \dot{q}_a$, where $a = (p_a, q_a), \dot{a} = (\dot{p}_a, \dot{q}_a)$. By Theorem 3.8 we obtain

$$\begin{aligned} F_\lambda(\tilde{p}, \tilde{q}, s, t) &= 2\lambda^2 S_3^+\left(\frac{\tilde{q} + t}{2\lambda}\right) - 2(1-\lambda)^2 S_3^-\left(\frac{\tilde{q}_{[k]} - t_{[k]},\ \tilde{p}_{[m]\setminus[k]} - s_{[m]\setminus[k]}}{2(1-\lambda)}\right) \\ &\quad - \sum\nolimits_{i=1}^{k} \tilde{p}_i t_i + \frac{1}{2}\sum\nolimits_{j=k+1}^{m} \tilde{q}_j s_j - \tilde{p}_j t_j - s_j t_j - \tilde{p}_j \tilde{q}_j \\ &\quad + \sum\nolimits_{l=1}^{m} \dot{p}_{a,l}\tilde{q}_l - \dot{q}_{a,l}\tilde{p}_l \end{aligned} \tag{4.1}$$

$$\begin{aligned} f_\lambda(s,t) = F_\lambda(0,0,s,t) &= 2\lambda^2 S_3^+\left(\frac{t}{2\lambda}\right) \\ &\quad -2(1-\lambda)^2 S_3^-\left(\frac{-t_{[k]}, -s_{[m]\setminus[k]}}{2(1-\lambda)}\right) - \frac{1}{2}\sum\nolimits_{j=k+1}^{m} s_j t_j \end{aligned} \tag{4.2}$$

The following singularities are realizable by generating function-germs:

$$A_{2l}:\quad S_3^+(\tilde{q}^+) = \lambda(\tilde{q}_1^+)^3 + (\tilde{q}_1^+)^{2l+1} + \sum_{i=2}^{l} \tilde{q}_i^+(\tilde{q}_1^+)^{2i-1},$$

$$S_3^-(\tilde{q}_1^-, \tilde{p}_2^-, \dots, \tilde{p}_m^-) = -(1-\lambda)(\tilde{q}_1^-)^3 + \sum_{i=2}^{l-1} \tilde{p}_i^-(\tilde{q}_1^-)^{2(l-i+1)}.$$

$$A_{2l+1}:\quad S_3^+(\tilde{q}^+) = \lambda(\tilde{q}_1^+)^3 + (\tilde{q}_1^+)^{2l+2} + \sum_{i=2}^{l} \tilde{q}_i^+(\tilde{q}_1^+)^{2i-1},$$

$$S_3^-(\tilde{q}_1^-, \tilde{p}_2^-, \dots, \tilde{p}_m^-) = -(1-\lambda)(\tilde{q}_1^-)^3 + \sum_{i=2}^{l} \tilde{p}_i^-(\tilde{q}_1^-)^{2(l-i+2)}.$$

$$D_{2l}:\quad S_3^+(\tilde{q}^+) = \lambda(\tilde{q}_1^+)^3 + \tilde{q}_2^+(\tilde{q}_1^+)^2 \pm (\tilde{q}_2^+)^{2l-1} + \lambda(\tilde{q}_2^+)^3 + \sum_{i=2}^{l-1} \tilde{q}_{i+1}^+(\tilde{q}_2^+)^{2i-1},$$

$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\setminus[2]}^-) = -(1-\lambda)(\tilde{q}_1^-)^3 - (1-\lambda)(\tilde{q}_2^-)^3 + \sum_{i=2}^{l-2} \tilde{p}_{i+1}^-(\tilde{q}_2^-)^{2(l-i)}.$$

$$D_{2l+1}: \quad S_3^+(\tilde{q}^+) = \lambda(\tilde{q}_1^+)^3 + \tilde{q}_2^+(\tilde{q}_1^+)^2 \pm (\tilde{q}_2^+)^{2l} + \lambda(\tilde{q}_2^+)^3 + \sum_{i=2}^{l-1} \tilde{q}_{i+1}^+(\tilde{q}_2^+)^{2i-1},$$

$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\backslash[2]}^-) = -(1-\lambda)(\tilde{q}_1^-)^3 - (1-\lambda)(\tilde{q}_2^-)^3 + \sum_{i=2}^{l-1} \tilde{p}_{i+1}^-(\tilde{q}_2^-)^{2(l-i+1)}.$$

$$E_6: \quad S_3^+(\tilde{q}^+) = (\tilde{q}_1^+)^3 \pm (\tilde{q}_2^+)^4 + \lambda\tilde{q}_1^+(\tilde{q}_2^+)^2 + \lambda(\tilde{q}_2^+)^3 + \tilde{q}_1^+(\tilde{q}_2^+)^2\tilde{q}_3^+,$$

$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\backslash[2]}^-) = -(1-\lambda)\tilde{q}_1^-(\tilde{q}_2^-)^2 - (1-\lambda)(\tilde{q}_2^-)^3.$$

$$E_7: \quad S_3^+(\tilde{q}^+) = (\tilde{q}_1^+)^3 + \tilde{q}_1^+(\tilde{q}_2^+)^2 + \lambda\tilde{q}_1^+(\tilde{q}_2^+)^2 + \lambda(\tilde{q}_2^+)^3 + (\tilde{q}_2^+)^3\tilde{q}_3^+,$$

$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\backslash[2]}^-) = -(1-\lambda)\tilde{q}_1^-(\tilde{q}_2^-)^2 - (1-\lambda)(\tilde{q}_2^-)^3 + (\tilde{q}_2^-)^4\tilde{p}_3^-.$$

$$E_8: \quad S_3^+(\tilde{q}^+) = (\tilde{q}_1^+)^3 + (\tilde{q}_2^+)^5 + \lambda\tilde{q}_1^+(\tilde{q}_2^+)^2 + \lambda(\tilde{q}_2^+)^3 + \tilde{q}_1^+(\tilde{q}_2^+)^2\tilde{q}_3^+ + \tilde{q}_1^+(\tilde{q}_2^+)^3\tilde{q}_4^+,$$

$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\backslash[2]}^-) = -(1-\lambda)\tilde{q}_1^-(\tilde{q}_2^-)^2 - (1-\lambda)(\tilde{q}_2^-)^3 + (\tilde{q}_2^-)^3\tilde{p}_3^-.$$

By long but straightforward calculations one can show that (4.1) is a $\mathcal{R}^+$-versal deformation of (4.2) for the above choices of $S_3^{\pm}$. □

5 The GCS of a Lagrangian submanifold: the criminant

We now begin the study of singularities of the global centre symmetry set of a smooth closed Lagrangian submanifold $L \subset (\mathbb{R}^{2m}, \omega)$. Recall that in general the set $GCS(L)$ consists of the caustic and the criminant (see Remark 2.7). As part of the $GCS(L)$ caustic, the Wigner caustic of L has been almost entirely classified in Sect. 4. In a subsequent paper [5], we study $E_{1/2}(L)$ in a neighborhood of L, considering pairs of points of the type $(a, a) \in L \times L$ as strongly parallel pairs. In terms of the generating families of Sect. 4, these are odd functions of the variables, so we consider classification *in the category of odd functions*. This implies a hidden $\mathbb{Z}_2$-symmetry for these singularities [5].

This section is devoted to the criminant $\Delta(L)$. In order to study the global centre symmetry set, the whole λ-family must be considered together. Due to the Lagrangian condition, we resort to a classification via generating families. We know that $E_\lambda(L)$ is the caustic of $\mathcal{L}_\lambda = \Phi_\lambda(L \times L)$. The generating family for $\mathcal{L}_\lambda$ is given by $F_\lambda(p, q, \alpha, \beta)$ of the form (3.7). Since $\mathbb{E}(L)$ is the union of $\{\lambda\} \times E_\lambda$, the germ of $\mathbb{E}(L)$ is described in the following way (for $\kappa = (\alpha, \beta)$):

Proposition 5.1 $\mathbb{E}(L) = \left\{(\lambda, p, q) : \exists\kappa \ \frac{\partial F_\lambda}{\partial\kappa} = 0, \ \det\left[\frac{\partial^2 F_\lambda}{\partial\kappa_i\partial\kappa_j}\right] = 0\right\}$.

Let us consider the fiber bundle

$$Pr : T^*\mathbb{R} \times T\mathbb{R}^{2m} \ni ((\lambda^*, \lambda), (\dot{p}, \dot{q}, p, q)) \mapsto (\lambda, (p, q)) \in \mathbb{R} \times \mathbb{R}^m. \qquad (5.1)$$

The above bundle with the canonical symplectic structure

$$d\lambda^* \wedge d\lambda + \dot{\omega}$$

is a Lagrangian fiber bundle. For F_λ given by (3.7) in Theorem 3.8, let

$$F(\lambda, p, q, \alpha, \beta) = F_\lambda(p, q, \alpha, \beta). \qquad (5.2)$$

Proposition 5.2 *The germ of $\mathbb{E}(L)$ is the caustic of the germ of a Lagrangian submanifold $\mathcal{L}$ of $(T^*\mathbb{R} \times T\mathbb{R}^{2m}, d\lambda^* \wedge d\lambda + \dot{\omega})$ generated by the family F given by (3.7)–(5.2) in the following way ($\kappa = (\alpha, \beta)$):*

$$\left\{((\lambda^*, \lambda), (\dot{p}, \dot{q}, p, q)) : \exists \kappa \;\; \lambda^* = \frac{\partial F}{\partial \lambda},\; \dot{p} = \frac{\partial F}{\partial q},\; \dot{q} = -\frac{\partial F}{\partial p},\; \frac{\partial F}{\partial \kappa} = 0\right\}. \tag{5.3}$$

We find the condition for the tangency of $\mathbb{E}(L)$ to the fibers of the projection $\pi : (\lambda, p, q) \mapsto (p, q)$.

Proposition 5.3 *If (λ, a) is a regular point of $\mathbb{E}(L)$, then there exists a 1-parallel pair a^+, a^- such that $a = \lambda a^+ + (1 - \lambda)a^-$.*

Proof If (λ, a) is a regular point of $\mathbb{E}(L)$ then the rank of the map

$$\kappa \mapsto \left(\frac{\partial F}{\partial \kappa}(\lambda_a, p_a, q_a, \kappa),\; \det\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa)\right]\right) \tag{5.4}$$

is maximal $2m - k$. It implies that corank $\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa_a)\right]$ is 1. By Remark 3.9 we obtain that a^+, a^- is a 1-parallel pair. □

Proposition 5.4 *Let $(\lambda_a, a) = (\lambda_a, p_a, q_a)$ be a regular point of $\mathbb{E}(L)$. The fiber of $\pi_r = \pi|_{\mathbb{E}(M)}$ is tangent to $\mathbb{E}(L)$ at (λ_a, a) if and only if*

$$rank\left[\frac{\partial^2 F}{\partial \lambda \partial \kappa_j}, \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}\right] = rank\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}\right] = 2m - 2 \tag{5.5}$$

at $(\lambda_a, p_a, q_a, \kappa_a)$ s.t. $\frac{\partial F}{\partial \kappa}(\lambda_a, p_a, q_a, \kappa_a) = \det\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa_a)\right] = 0.$

Proof By Proposition 5.3 if (λ_a, p_a, q_a) is a regular point of $\mathbb{E}(L)$, the map (5.4) has maximal rank $2m - 1$. Also, rank $\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa_a)\right]$ is $2m - 2$ which implies one of the columns of this matrix is linearly dependent on the others. Assume this is the first column. Thus, $\kappa \mapsto \left(\frac{\partial F}{\partial \kappa_{[2m-1]\setminus[1]}}(\lambda_a, p_a, q_a, \kappa),\; \det\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa)\right]\right)$ has maximal rank. By implicit function theorem there is a smooth map germ $\mathcal{K} : \mathbb{R}_e^{2m+1} \to \mathbb{R}^{2m-1}$ at (λ_a, a), s.t. $\kappa = \mathcal{K}(\lambda, p, q)$ iff $\frac{\partial F}{\partial \kappa_{[2m-1]\setminus[1]}}(\lambda, p, q, \kappa) = 0,\; \det\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda, p, q, \kappa)\right] = 0$. Then the germ of $\mathbb{E}(L)$ at(λ_a, a) is $\mathbb{E}(L) = \left\{(\lambda, p, q) : \frac{\partial F}{\partial \kappa_1}(\lambda, p, q, \mathcal{K}(\lambda, p, q)) = 0\right\}$. The fiber of π_r is tangent to $\mathbb{E}(L)$ at (λ_a, a) iff

$$\frac{\partial^2 F}{\partial \lambda \partial \kappa_1}(\lambda_a, p_a, q_a, \kappa_a) + \sum_{j=1}^{2m-1} \frac{\partial^2 F}{\partial \kappa_j \partial \kappa_1}(\lambda_a, p_a, q_a, \kappa_a)\frac{\partial \mathcal{K}_j}{\partial \lambda}(\lambda_a, p_a, q_a) = 0. \tag{5.6}$$

Differentiating $\frac{\partial F}{\partial \kappa_{[2m-1]\setminus[1]}}(\lambda, p, q, \mathcal{K}(\lambda, p, q)) = 0$ w.r.t. λ we obtain

$$\frac{\partial^2 F}{\partial \lambda \partial \kappa_i}(\lambda_a, p_a, q_a, \kappa_a) + \sum_{j=1}^{2m-1} \frac{\partial^2 F}{\partial \kappa_j \partial \kappa_i}(\lambda_a, p_a, q_a, \kappa_a)\frac{\partial \mathcal{K}_j}{\partial \lambda}(\lambda_a, p_a, q_a) = 0. \tag{5.7}$$

Thus (5.6)–(5.7) imply (5.5). But also (5.7) and (5.5) imply (5.6). □

Theorem 5.5 *The point $a = \lambda a^+ + (1 - \lambda)a^-$ belongs to the criminant $\Delta(L)$ of $GCS(L)$ iff there is a bitangent hyperplane to L at a^+ and a^-.*

Proof If $(\lambda, a) \in \mathbb{E}(L)$ is regular, by Propositions 5.3-5.4, a^+, a^- are 1-parallel and $a = (p, q) \in \Delta(L)$ iff (λ, a) satisfies (5.5). Thus $\left[\frac{\partial^2 F}{\partial\kappa_i \partial\kappa_j}\right] =$

$$\frac{1}{2}\begin{bmatrix} \frac{\partial^2 S^+}{(\partial q_1^+)^2} - \frac{\partial^2 S^-}{(\partial q_1^-)^2} & \frac{\partial^2 S^+}{\partial q_1^+ \partial q_2^+} & \cdots & \frac{\partial^2 S^+}{\partial q_1^+ \partial q_m^+} & -\frac{\partial^2 S^-}{\partial q_1^- \partial p_2^-} & \cdots & -\frac{\partial^2 S^-}{\partial q_1^- \partial p_m^-} \\ \frac{\partial^2 S^+}{\partial q_1^+ \partial q_2^+} & \frac{\partial^2 S^+}{(\partial q_2^+)^2} & \cdots & \frac{\partial^2 S^+}{\partial q_2^+ \partial q_m^+} & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 S^+}{\partial q_1^+ \partial q_m^+} & \frac{\partial^2 S^+}{\partial q_2^+ \partial q_m^+} & \cdots & \frac{\partial^2 S^+}{(\partial q_m^+)^2} & 0 & \cdots & -1 \\ -\frac{\partial^2 S^-}{\partial q_1^- \partial p_2^-} & -1 & \cdots & 0 & -\frac{\partial^2 S^-}{(\partial p_2^-)^2} & \cdots & \frac{\partial^2 S^-}{\partial p_2^- \partial p_m^-} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial^2 S^-}{\partial q_1^- \partial p_m^-} & 0 & \cdots & -1 & \frac{\partial^2 S^-}{\partial p_2^- \partial p_m^-} & \cdots & -\frac{\partial^2 S^-}{(\partial p_m^-)^2} \end{bmatrix}$$

and $\frac{\partial^2 F}{\partial\lambda\partial\beta_1} = p_1^+ - p_1^- - \sum_{j=1}^n q_j^+ \frac{\partial^2 S^+}{\partial q_1^+ \partial q_j^+} + q_1^- \frac{\partial^2 S^-}{(\partial q_1^-)^2} + \sum_{j=2}^n p_j^- \frac{\partial^2 S^-}{\partial q_1^- \partial p_j^-}$, $\frac{\partial^2 F}{\partial\lambda\partial\beta_i} = p_i^+ - \sum_{i=1}^n q_j^+ \frac{\partial^2 S^+}{\partial q_i^+ \partial q_j^+}$, $\frac{\partial^2 F}{\partial\lambda\partial\alpha_i} = q_i^- + q_1^- \frac{\partial^2 S^+}{\partial p_i^- \partial q_1^-} + \sum_{j=2}^n p_j^- \frac{\partial^2 S^+}{\partial p_i^- \partial p_j^-}$, for $i = 2, \ldots, m$,, with $q^+ = \frac{q+\beta}{2\lambda}$, $p^+ = \frac{\partial S^+}{\partial q^+}$ and $q_1^- = \frac{q_1 - \beta_1}{2(1-\lambda)}$, $p_{[m]\backslash[2]}^- = \frac{p_{[m]\backslash[2]} - \alpha_{[m]\backslash[2]}}{2(1-\lambda)}$, $p_1^- = \frac{\partial S^-}{\partial q_1^-}$, $q_{[m]\backslash[2]}^- = -\frac{\partial S^-}{\partial p_{[m]\backslash[2]}^-}$. Then, (5.5) is equivalent to

$$(a^+ - a^-) \in T_{a^+}L^+ + T_{a^-}L^-, \tag{5.8}$$

since $T_{a^+}L^+$ is spanned by $\sum_{j=1}^m \frac{\partial^2 S^+}{\partial q_i^+ \partial q_j^+} \frac{\partial}{\partial p_j} + \frac{\partial}{\partial q_i}$ for $i = 1, \ldots, m$ and $T_{a^-}L^-$ is spanned by $\frac{\partial^2 S^-}{(\partial q_1^-)^2} \frac{\partial}{\partial p_1} - \sum_{j=2}^m \frac{\partial^2 S^-}{\partial q_1^- \partial p_j^-} \frac{\partial}{\partial q_j} + \frac{\partial}{\partial q_1}$ and $\frac{\partial^2 S^-}{\partial p_i^- \partial q_1^-} \frac{\partial}{\partial p_1} - \sum_{j=2}^m \frac{\partial^2 S^-}{\partial p_i^- \partial p_j^-} \frac{\partial}{\partial q_j} + \frac{\partial}{\partial p_i}$ for $i = 2, \ldots, m$. If a^+, a^- is 1-parallel, (5.8) means there is a bitangent hyperplane to L^+ at a^+ and to L^- at a^-. By continuity, a point in the closure of the set of points satisfying (5.8) also satisfies this condition. □

Corollary 5.6 *If, for some* λ, $\lambda a^+ + (1-\lambda)a^- = a \in \Delta(L) \subset GCS(L)$*, then the whole chord* $l(a^+, a^-) \subset GCS(L)$*. Equivalently, if there is a bitangent hyperplane to L at* a^+ *and* a^-*, then* $l(a^+, a^-) \subset GCS(L)$*.*

Thus, we generalize the notion of convexity of a curve on the plane.

Definition 5.7 A smooth closed Lagrangian submanifold L of $(\mathbb{R}^{2m}, \omega)$ is *weakly convex* if there is no bitangent hyperplane to L.

Corollary 5.8 *If L is a weakly convex closed Lagrangian submanifold of* $(\mathbb{R}^{2m}, \omega)$ *then the criminant* $\Delta(L)$ *of* $GCS(L)$ *is empty.*

6 Affine-Lagrangian stable singularities of the GCS

We now define an equivalence relation to classify the singularities of $GCS(L)$. Due to the Lagrangian condition, we look for an equivalence of generating families. For the classification of $\mathbb{E}(\lambda)$ and $GCS(L)$, because λ is no longer fixed it has become an extra parameter that unfolds the generating families F. The naive approach is to consider the extended parameter space $\mathbb{R} \times \mathbb{R}^{2m} \ni (\lambda, x)$ for unfolding the generating families $f(\lambda, \kappa) = f_\lambda(\kappa)$ and classify their stable unfoldings in the usual way. However, such a classification of $GCS(L)$ would not take into account the projection $\pi : \mathbb{R} \times \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ in a proper way, because it does not distinguish the affine time $\lambda \in \mathbb{R}$ from $x \in \mathbb{R}^{2m}$.

Now, if $\mathcal{A} = (A, a)$ is an element of the affine symplectic group $iSp^{2m}_{\mathbb{R}} = Sp(2m, \mathbb{R}) \ltimes \mathbb{R}^{2m}$, with $A \in Sp(2m, \mathbb{R})$, $a \in \mathbb{R}^{2m}$, then

$$\mathcal{A} : (\mathbb{R}^{2m}, \omega) \supset L \to L' \subset (\mathbb{R}^{2m}, \omega),\ x \mapsto \mathcal{A}x = Ax + a. \tag{6.1}$$

From this, we define the natural action

$$\begin{gathered} id_{T^*\mathbb{R}} \times \mathcal{A} \times \mathcal{A} : T^*\mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to T^*\mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^{2m}, \\ (\lambda, \lambda^*, x^+, x^-) \mapsto (\lambda, \lambda^*, \mathcal{A}x^+, \mathcal{A}x^-), \end{gathered}$$

which, via the chord transformation Φ_λ, induces an action

$$\begin{aligned} iSp^{2m}_{\mathbb{R}} \ni id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi \ &: \ T^*\mathbb{R} \times T\mathbb{R}^{2m} \supset \mathcal{L} \ \to \ \mathcal{L}' \subset T^*\mathbb{R} \times T\mathbb{R}^{2m}, \\ id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi &: (\lambda, \lambda^*, x, \dot{x}) \mapsto (\lambda, \lambda^*, Ax + a, A\dot{x} + (2\lambda - 1)a), \end{aligned} \tag{6.2}$$

that commutes with projection $id_{T^*\mathbb{R}} \times pr : T^*\mathbb{R} \times T\mathbb{R}^{2m} \to T^*\mathbb{R} \times \mathbb{R}^{2m}$, that is, defining the obvious action $id_{\mathbb{R}} \times \mathcal{A}$ on $\mathbb{R} \times \mathbb{R}^{2m}$, we have

$$(id_{\mathbb{R}} \times \mathcal{A}) \circ (id_{T^*\mathbb{R}} \times pr) = (id_{T^*\mathbb{R}} \times pr) \circ (id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi). \tag{6.3}$$

Definition 6.1 Germs of Lagrangian submanifolds $\mathcal{L}$, $\widetilde{\mathcal{L}}$ of the fiber bundle $(T^*\mathbb{R} \times T\mathbb{R}^{2m}, d\lambda^* \wedge d\lambda + \dot{\omega})$ are *(1,2m)-Lagrangian equivalent* if there exists a symplectomorphism-germ Υ of $T^*\mathbb{R} \times T\mathbb{R}^{2m}$ such that $\Upsilon(\mathcal{L}) = \widetilde{\mathcal{L}}$ and the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{L} \hookrightarrow T^*\mathbb{R} \times T\mathbb{R}^{2m} & \overset{Pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^{2m} & \overset{\pi}{\rightarrow} & \mathbb{R}^{2m} \\ \downarrow \Upsilon & & \downarrow & & \downarrow \\ \widetilde{\mathcal{L}} \hookrightarrow T^*\mathbb{R} \times T\mathbb{R}^{2m} & \overset{Pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^{2m} & \overset{\pi}{\rightarrow} & \mathbb{R}^{2m} \end{array}$$

The first two vertical diffeomorphism-germs (from right to left) read:

$$x \mapsto X(x),\ (\lambda, x) \mapsto (\Lambda(\lambda, x), X(x)).$$

Moreover, germs $\mathcal{L}$, $\widetilde{\mathcal{L}}$ at $(\frac{1}{2}, a, \dot{a})$ are *(1,2m)-Lagrangian equivalent for* $\lambda = \frac{1}{2}$ if, in addition, for every $x \in \mathbb{R}^{2m}$

$$\Lambda\left(\frac{1}{2}, x\right) = \frac{1}{2}. \tag{6.4}$$

Remark 6.2 Condition (6.4) is introduced for the classification of the Wigner caustic $E_{1/2}(L)$ as a part of $GCS(L)$.

Definition 6.3 $GCS(L)$ and $GCS(\widetilde{L})$ are *(1,2m)-Lagrangian equivalent* if $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are (1,2m)-Lagrangian equivalent.

Remark 6.4 From (6.3), it is clear that classification of $GCS(L)$ by $(1, 2m)$-Lagrangian equivalence of $\mathcal{L}$ is *affine symplectic invariant*.

Remark 6.5 $(1, 2m)$-Lagrangian equivalence of germs of Lagrangian submanifolds of $(T^*\mathbb{R} \times T\mathbb{R}^{2m}, d\lambda^* \wedge d\lambda + \dot{\omega})$ is the equivalence of bifurcations of Lagrangian maps (chapter 22 in [2]), that is, diagrams of the form:

$$D(\mathcal{L}) : \mathcal{L} \hookrightarrow T^*\mathbb{R} \times T\mathbb{R}^{2m} \overset{Pr}{\longrightarrow} \mathbb{R} \times \mathbb{R}^{2m} \overset{\pi}{\to} \mathbb{R}^{2m}$$

Definition 6.6 $\mathcal{L}$ is *(1,2m)-Lagrangian stable* if the diagram of maps $D(\mathcal{L})$ is stable, i.e. every $\widetilde{\mathcal{L}}$ with nearby diagram $D(\widetilde{\mathcal{L}})$ is $(1, 2m)$-Lagrangian equivalent to $\mathcal{L}$. $GCS(L)$ is *(1,2m)-Lagrangian stable* if $\mathcal{L}$ is (1,2m)-Lagrangian stable.

In view of Remark 6.4, we also use the term *affine-Lagrangian stability* for $(1, 2m)$-Lagrangian stability.

Definition 6.7 The function-germs $F, \widetilde{F} : \mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^k \to \mathbb{R}$ are *(1,2m)-$\mathcal{R}^+$-equivalent* if there exists a diffeomorphism-germ

$$(\lambda, x, \kappa) \mapsto (\Lambda(\lambda, x), X(x), K(\lambda, x, \kappa))$$

and a smooth function-germ $g : \mathbb{R} \times \mathbb{R}^{2m} \to \mathbb{R}$ such that

$$\widetilde{F}(\lambda, x, \kappa) = F(\Lambda(\lambda, x), X(x), K(\lambda, x, \kappa)) + g(\lambda, x).$$

Germs F and $\widetilde{F}$ with the common (λ, x)-space $\mathbb{R} \times \mathbb{R}^{2m}$ of parameters, and with different spaces of arguments, $\kappa \in \mathbb{R}^k$, $\tilde{\kappa} \in \mathbb{R}^{\tilde{k}}$, are *stably (1,2m)-$\mathcal{R}^+$-equivalent* if there are nondegenerate quadratic forms Q in new arguments ξ and $\widetilde{Q}$ in new arguments $\tilde{\xi}$ such that $F + Q$ and $\widetilde{F} + \widetilde{Q}$ are $(1, 2m)$-$\mathcal{R}^+$-equivalent. The germ F at $(\frac{1}{2}, a, \kappa_a)$ and the germ $\widetilde{F}$ at $(\frac{1}{2}, a, \tilde{\kappa}_a)$ are (stably) (1,2m)-$\mathcal{R}^+$-equivalent for $\lambda = \frac{1}{2}$ if, in addition, for every $x \in \mathbb{R}^m$ condition (6.4) is satisfied.

Remark 6.8 $(1, 2m)$-$\mathcal{R}^+$-equivalence is a special case of Wassermann's $(1, 2m)$-equivalence [19]. For relations between the (r, s)-classification of families of functions [19], the classification of bifurcations of caustics [1,20] and the classification of bifurcations of Lagrangian maps, see chapter 22 in [2].

We have the following result, whose proof is a minor modification for $(1, 2m)$-Lagrangian equivalence of the proof of Theorem 3.10 in [2].

Proposition 6.9 *Germs of Lagrangian submanifolds* $\mathcal{L}$, $\widetilde{\mathcal{L}}$ *of* $(T^*\mathbb{R} \times T\mathbb{R}^{2m}, d\lambda^* \wedge d\lambda + \dot{\omega})$ *are* $(1, 2m)$*-Lagrangian equivalent iff the germs of generating families* F *and* $\widetilde{F}$ *are stably* $(1, 2m)$*-*$\mathcal{R}^+$*-equivalent.*

Definition 6.10 A function-germ F at z is *(1,2m)-$\mathcal{R}^+$-stable* if for any neighborhood $U \ni z$ in $\mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^k$ and representative function F' of the germ F on U, there is a neighborhood V of F' in $C^\infty(U, \mathbb{R})$ (with weak C^∞-topology) s.t. for any function $G' \in V$ there is a point $z' \in U$ such that the germ of G' at z' is $(1, 2m)$-$\mathcal{R}^+$-equivalent to F.

Remark 6.11 $\mathcal{L}$ and $GCS(L)$ are $(1, 2m)$-Lagrangian stable if and only if the germ of generating family F (of $\mathcal{L}$) is $(1, 2m)$-$\mathcal{R}^+$-stable.

The following theorems show that the only affine-Lagrangian stable singularities of GCS are singularities of the criminant, the smooth part of the Wigner caustic and their "tangent" union.

Theorem 6.12 *Let $\lambda_a \neq \frac{1}{2}$. If F is the germ at (λ_a, a, κ_a) of a $(1, 2m)$-$\mathcal{R}^+$-stable unfolding of $f \in \mathfrak{m}^2$ then F is stably $(1, 2m)$-$\mathcal{R}^+$-equivalent to the germ of the trivial unfolding or to one of the following germs at $(0, 0, 0)$ of unfoldings of $f(t) = t^3$*

$$A_2^{A_k^{\pm}} \ : \ F(\lambda, x, t) = t^3 + t\left(\sum_{i=1}^{k} x_i \lambda^{i-1} \pm \lambda^{k+1}\right), \tag{6.5}$$

for $k = 0, 1, 2, \ldots, 2m$ (the notation $A_2^{A_k^{\pm}}$ is taken from [15]*).*

Proof If f has A_1 singularity then it is obvious that F is stably $(1, 2m)$-$\mathcal{R}^+$-equivalent to the trivial unfolding. Now we assume that f has A_2 singularity. Since F is stable, then F is stably $(1, 2m)$-$\mathcal{R}^+$-equivalent to $F(\lambda, x, t) = t^3 + tg(\lambda, x)$, where g is a smooth function-germ vanishing at 0. If g is a versal unfolding of the function-germ $\lambda \mapsto g(\lambda, 0)$ with A_k singularity we can reduce F to the form (6.5) by a diffeomorphism-germ of the form $(\lambda, x, t) \mapsto (\Lambda(\lambda, x), X(x), t)$. □

The following lemma shows that these are the only $(1, 2m)$-$\mathcal{R}^+$-stable unfoldings.

Lemma 6.13 *Unfoldings of $A_3^{\pm}$ singularity are not $(1, 2m)$-$\mathcal{R}^+$-stable.*

Proof If f has A_3 singularity then F is stable $(1, 2m)$-$\mathcal{R}^+$-equivalent to $F(\lambda, x, t) = \pm t^4 + t^2 g_2(\lambda, x) + tg_1(\lambda, x)$, where g_1, g_2 are smooth function-germs vanishing at 0. Now we use the standard arguments of the singularity theory that stability implies infinitesimal stability. In the case of $(1, 2m)$-$\mathcal{R}^+$-equivalence, the infinitesimal stability implies

$$\mathcal{E}_2 = \mathcal{E}_2\left\langle \frac{\partial F}{\partial t}|_{\mathbb{R}^2}\right\rangle + \mathcal{E}_1\left\langle 1, \frac{\partial F}{\partial \lambda}|_{\mathbb{R}^2}\right\rangle + \mathbb{R}\left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^2}, \ldots, \frac{\partial F}{\partial x_{2m}}|_{\mathbb{R}^2}\right\rangle + \mathfrak{m}_2^{2m+4}, \tag{6.6}$$

where $\mathbb{R}^2$ denotes the t, λ-plane $\{x = 0\}$, $\mathcal{E}_2$ is the ring of smooth function-germs in λ and t, $\mathfrak{m}_2$ is the maximal ideal in $\mathcal{E}_2$ and $\mathcal{E}_1$ is the ring of smooth function-germs in λ. Now we use the method from [19]. Let $V = \mathcal{E}_2 \Big/ \left(\mathcal{E}_2\langle \frac{\partial F}{\partial t}|_{\mathbb{R}^2}\rangle + m_2^{2m+4}\right)$ and let $\pi : \mathcal{E}_2 \to V$. We have $\pi(t^3) = \pi(\mp 1/2tg_2|_{\mathbb{R}^2} \mp 1/4g_1|_{\mathbb{R}^2})$ in V. Thus elements $\pi(t^i\lambda^j)$ for $i = 0, 1, 2$ and $j < 2m + 4 - i$ form a basis of V over $\mathbb{R}$. Thus, $\dim_{\mathbb{R}} V = 6m + 9$. Moreover, $\frac{\partial F}{\partial \lambda}|_{\mathbb{R}^2} = t\left(t\frac{\partial g_2}{\partial \lambda}|_{\mathbb{R}^2} + \frac{\partial g_1}{\partial \lambda}|_{\mathbb{R}^2}\right)$. Then $\dim_{\mathbb{R}} \pi\left(\mathcal{E}_1\langle 1, \frac{\partial F}{\partial \lambda}|_{\mathbb{R}^2}\rangle\right) \leq 4m + 7$, $\dim_{\mathbb{R}} \pi\left(\mathbb{R}\left\langle\frac{\partial F}{\partial x_1}|_{\mathbb{R}^2}, \ldots, \frac{\partial F}{\partial x_{2m}}|_{\mathbb{R}^2}\right\rangle\right) \leq 2m$. So, (6.6) implies $\dim_{\mathbb{R}} V \leq 6m+7 < 6m+9$, which is impossible. Thus F is not $(1, 2m)$-$\mathcal{R}^+$-stable, A_3 has no $(1, 2m)$-$\mathcal{R}^+$-stable unfoldings. □

For $E_{1/2}(L) \subset GCS(L)$, we consider the germ of F at $(1/2, a, \kappa_a)$.

Theorem 6.14 *If F is the germ at $(\frac{1}{2}, a, \kappa_a)$ of a $(1, 2m)$-$\mathcal{R}^+$-stable unfolding of $f \in \mathfrak{m}^2$ then F is stably $(1, 2m)$-$\mathcal{R}^+$-equivalent ($\lambda = 1/2$) to the germ of the trivial unfolding or to one of the following germs at $(\frac{1}{2}, 0, 0)$ of unfoldings of $f(t) = t^3$*

$$A_2^{B_k^{\pm}} \ : \ F(\lambda, x, t) = t^3 + t\left(\sum_{i=0}^{k-1} x_{i+1}\left(\lambda - \frac{1}{2}\right)^i \pm \left(\lambda - \frac{1}{2}\right)^k\right), \tag{6.7}$$

for $k = 1, 2, \ldots, 2m$ (the notation $A_2^{B_k^{\pm}}$ is taken from [15]*).*

Proof If f has A_1 singularity then F is stably $(1, 2m)$-$\mathcal{R}^+$-equivalent to the trivial unfolding. If f has A_2 singularity, then (since F is stable) F is stably $(1, 2m)$-$\mathcal{R}^+$-equivalent to $F(\lambda, x, t) = t^3 + tg(\lambda, x)$, where g is a smooth function-germ vanishing at $(1/2, 0)$. If g is a versal unfolding of the function-germ $\lambda \mapsto g(\lambda, 0)$ with $B_k^{\pm}$ singularity on a manifold (λ-space) with the boundary ($\lambda = \frac{1}{2}$) (see [1]) then we can reduce F to the form (6.7) by a diffeomorphism-germ of the form $(\lambda, x, t) \mapsto (1/2 + (\lambda - 1/2)\Lambda(\lambda, x), X(x), t)$. □

Theorem 6.15 *If F (generating $\mathcal{L}$) has $A_2^{A_k^{\pm}}$ singularity, for $k = 0, 1, \dots, 2m$, then $\mathbb{E}(L)$ is a germ of a smooth hypersurface in $\mathbb{R} \times \mathbb{R}^{2m}$.*

If F has $A_2^{A_0}$ singularity at (λ_a, a, κ_a) then $\mathbb{E}(L)$ is transversal at (λ_a, a) to the fibers of projection π.

If F has $A_2^{A_k^{\pm}}$ singularity for $k \geq 1$ at (λ_a, a, κ_a) then $\mathbb{E}(L)$ is k-tangent at (λ_a, a) to the fibers of π, a belongs to the criminant $\Delta(L)$ of GSC(L) and the germ of $\Delta(L)$ at a is the caustic of $A_k^{\pm}$ singularity.

Proof By Proposition 5.1 and the normal form of F for $A_2^{A_k^{\pm}}$ singularity we obtain $\mathbb{E}(L) = \{(\lambda, x) \in \mathbb{R} \times \mathbb{R}^{2m} : \sum_{i=1}^{k} x_i \lambda^{i-1} \pm \lambda^{k+1} = 0\}$. It is easy to see that $\mathbb{E}(L)$ is the germ at $(0, 0)$ of a smooth hypersurface and $\mathbb{E}(L)$ is transversal at $(0, 0)$ to $\{\lambda = 0\}$ for $k = 0$ and $\mathbb{E}(L)$ is k-tangent to $\{\lambda = 0\}$ at $(0, 0)$ for $k = 1, 2, \dots, 2m$. The germ of $\Delta(L)$ at 0 is

$$\{x \in \mathbb{R}^{2m} : \exists \lambda \ \sum_{i=1}^{k} x_i \lambda^{i-1} \pm \lambda^{k+1} = 0, \ \sum_{i=2}^{k} (i-1) x_i \lambda^{i-2} \pm (k+1)\lambda^k = 0\}.$$

So $\Delta(L)$ is a caustic of $A_k^{\pm}$ singularity. □

Theorem 6.16 *If the germ at $(\frac{1}{2}, a, \kappa_a)$ of F has $A_2^{B_k^{\pm}}$ singularity ($k = 1, \dots, 2m$), then $\mathbb{E}(L)$ is a germ of smooth hypersurface in $\mathbb{R} \times \mathbb{R}^{2m}$.*

If F has $A_2^{B_1}$ singularity at $(\frac{1}{2}, a, \kappa_a)$, then $\mathbb{E}(L)$ is transversal at $(\frac{1}{2}, a)$ to the fibers of projection π. The germ of GCS(L) at a is the germ of a smooth hypersurface of $\mathbb{R}^{2m}$—the Wigner caustic $E_{1/2}(L)$.

If F has $A_2^{B_k^{\pm}}$ singularity for $k \geq 2$ at $(\frac{1}{2}, a, \kappa_a)$, then $\mathbb{E}(L)$ is k-tangent at $(1/2, a, t)$ to the fibers of π. The germ of GCS(L) at a consists of two tangent components: the germ of a smooth hypersurface—$E_{1/2}(L)$—and the germ of the caustic of $B_k^{\pm}$ singularity—$\Delta(L)$.

Proof By Proposition 5.1 and the normal form of F for $A_2^{B_k^{\pm}}$ singularity we get $\mathbb{E}(L) = \{(\lambda, x) \in \mathbb{R} \times \mathbb{R}^{2m} : \sum_{i=0}^{k-1} x_{i+1}(\lambda - 1/2)^i \pm (\lambda - 1/2)^k = 0\}$. $E_{1/2}(L) = \{x \in \mathbb{R}^{2m} : x_1 = 0\}$ is a germ of smooth hypersurface. Thus $\mathbb{E}(L)$ is the germ at $(1/2, 0)$ of a smooth hypersurface and $\mathbb{E}(L)$ is transversal at $(1/2, 0)$ to $\{\lambda = 1/2\}$ for $k = 1$. For $k = 2, \dots, 2m$, $\mathbb{E}(L)$ is k-tangent to $\{\lambda = 1/2\}$ at $(1/2, 0)$. The germ of $\Delta(L)$ at 0 is

$$\{x \in \mathbb{R}^{2m} : \exists \tau \ \sum_{i=0}^{k-1} x_{i+1} \tau^i \pm \tau^k = 0, \ \sum_{i=1}^{k-1} i x_{i+1} \tau^{i-1} \pm k\tau^{k-1} = 0\}.$$

So $\Delta(L)$ is a caustic of $B_k^{\pm}$ and $E_{1/2}(L)$ is tangent to $\Delta(L)$ at 0. □

Remark 6.17 Not all $(1, 2m)$-$\mathcal{R}^+$-stable singularities can be realizable as singularities of generating families F for $\mathcal{L}$ which are of the special form given in Theorem 3.8. In the next section, in Theorem 7.7, we prove that the $A_2^{A_2}$ singularity is not realizable for Lagrangian curves.

7 Classifications of the GCS of Lagrangian curves

We now classify the singularities of the global centre symmetry set of a Lagrangian curve $L \subset (\mathbb{R}^2, \omega)$. To set the stage, we first state the results for the *GCS* of a curve on the affine plane $\mathbb{R}^2$, when no symplectic structure is considered.

Theorem 7.1 ([3,10,11,16]) *Affine stable GCS of a smooth convex closed curve* $M \subset \mathbb{R}^2$ *(no symplectic structure) consists of:*

i) The CSS, a smooth curve with (possible) self intersections and cusp singularities, ii) the Wigner caustic, a smooth curve with (possible) self intersections and cusp singularities lying on the smooth part of the CSS, and iii) the medial axis, which are smooth half-lines starting at the cusp points of the CSS.

The results stated in Theorem 7.1, originally obtained by various methods, can also be proved using the affine-invariant method of chord equivalence, the analogous of $(1, 2m)$-Lagrangian equivalence when no symplectic structure is considered, cf Definition 7.10, below.

Theorem 7.2 *Let* M *be a generic smooth convex closed curve in* $\mathbb{R}^2$*. The number of cusps of the Wigner caustic of* M *is odd and not smaller than 3. The number of cusps of the CSS of* M *is odd and not smaller than 3. The number of cusps of the Wigner caustic of* M *is not greater than the number of cusps of the CSS of* M.

The statement on the number of cusps of Wigner caustics was first proved by Berry [3], and the statement on the number of cusps of CSS by Giblin and Holtom [9]. The last inequality of the theorem is new. It follows immediately from the characterization in [9] of cusps of $E_{1/2}(M)$ by the curvature ratio being 1 and cusps of CSS of M by the derivative of the curvature ratio being 0, using Rolle's theorem.

Figures of *GCS*(M) where the number of cusps of the CSS and of the Wigner caustic are equal to three and neither curve is self intersecting can be found in [9]. We picture a case when the number of cusps of the Wigner caustic is three and the CSS is self intersecting and the number of its cusps is five, and another case when both the Wigner caustic and the CSS are self intersecting and both have five cusps (Figs. 1, 2).

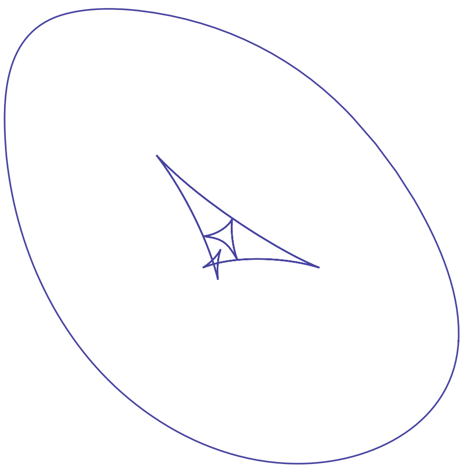
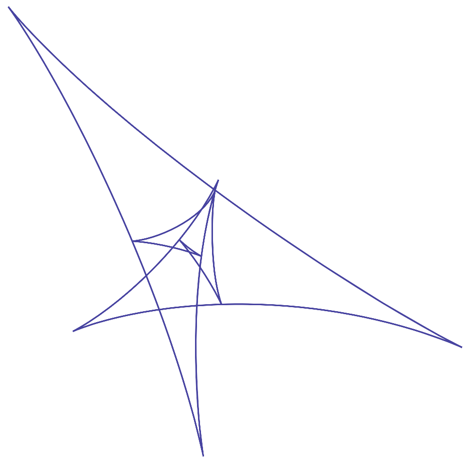


Fig. 1 GCS of an oval: CSS with five cusps and the Wigner caustic with three cusps (the medial axis are not shown here)

Fig. 2 GCS of an oval: CSS and the Wigner caustic with five cusps



7.1 Affine-Lagrangian classification of the *GCS* of Lagrangian curves

Let L be a smooth closed curve in $(\mathbb{R}^2, \omega = dp \wedge dq)$. Using the $(1, 2)$-Lagrangian equivalence introduced in Definitions 6.1 and 6.3, we classify the singularities of $GCS(L)$. In what follows, $a^+ = (p_a^+, q_a^+), a^- = (p_a^-, q_a^-)$ denote a parallel pair on L and $a_\lambda = \lambda a^+ + (1-\lambda)a^-, \dot{q}_\lambda = \lambda q_a^+ - (1-\lambda)q_a^-$. Let $S^\pm$ be germs of generating functions of L at $a^\pm$ satisfying the conditions in Proposition 3.7. The germ of generating family of $\mathcal{L}$ and the big wave front set are given by

$$F(\lambda, p, q, t) = 2\lambda^2 S^+\left(\tfrac{q+t}{2\lambda}\right) - 2(1-\lambda)^2 S^-\left(\tfrac{q-t}{2(1-\lambda)}\right) - pt.$$

$$\mathbb{E}(L) = \left\{(\lambda, p, q) \in \mathbb{R} \times \mathbb{R}^2 : \exists t\ \tfrac{\partial F}{\partial t}(\lambda, p, q, t) = \tfrac{\partial^2 F}{\partial t^2}(\lambda, p, q, t) = 0\right\}.$$

The following propositions present geometrical descriptions of positions of $\mathbb{E}(L)$ with respect to π in terms of functions F, S^+ and S^-.

Proposition 7.3 *The following conditions are equivalent*

(i) (λ, a_λ) *belongs to the regular part of* $\mathbb{E}(L)$,
(ii) $\exists t\ \frac{\partial^3 F}{\partial t^3}(\lambda, a_\lambda, t) \neq 0,\ \frac{\partial F}{\partial t}(\lambda, a_\lambda, t) = \frac{\partial^2 F}{\partial t^2}(\lambda, a_\lambda, t) = 0$,
(iii) $\frac{1}{\lambda}\frac{\partial^3 S^+}{\partial (q^+)^3}(q_a^+) + \frac{1}{1-\lambda}\frac{\partial^3 S^-}{\partial (q^-)^3}(q_a^-) \neq 0$,
(iv) $\frac{1}{\lambda}\kappa(a^+) + \frac{1}{1-\lambda}\kappa(a^-) \neq 0$, *where* $\kappa(x)$ *is the curvature of* L *at* x.

Proof Equivalence of (i) and (ii) follows from the definition of the regular part of $\mathbb{E}(L)$. Equivalence of (ii) and (iii) is obtained by direct calculations. (iv) is obvious since $\kappa(a^\pm) = \frac{\partial^3 S^\pm}{\partial (q^\pm)^3}(q_a^\pm)$. □

Proposition 7.4 *The following conditions are equivalent*

(v) *the regular part of* $\mathbb{E}(L)$ *is tangent to the fiber of* π *at* (λ, a_λ),
(vi) $\exists t$*:* (ii) *is satisfied and* $\frac{\partial^2 F}{\partial \lambda \partial t}(\lambda, a_\lambda, t) = 0$.
(vii) (iii) *is satisfied and* $p_a^+ = \frac{\partial S^+}{\partial q^+}(q_a^+) = \frac{\partial S^-}{\partial q^-}(q_a^-) = p_a^-$.
(viii) (iv) *is satisfied and* $l(a^+, a^-)$ *is bitangent to* a^+, a^- *to* L.

Proof All statements follow from Proposition 5.4 and Theorem 5.5. □

Proposition 7.5 *The following conditions are equivalent*

(ix) *the regular part of* $\mathbb{E}(L)$ *is* 1*-tangent to the fiber of* π *at* (λ, a_λ)*,*
(x) $\exists t$: (vi) *is satisfied and*

$$\left(\frac{\partial^3 F}{\partial\lambda\partial t^2}(\lambda, a_\lambda, t)\right)^2 - \frac{\partial^3 F}{\partial t^3}(\lambda, a_\lambda, t)\frac{\partial^3 F}{\partial\lambda^2\partial t}(\lambda, a_\lambda, t) \neq 0. \tag{7.1}$$

(xi) (vii) *is satisfied and* $\frac{\partial^3 S^+}{\partial(q^+)^3}(q_a^+)\frac{\partial^3 S^-}{\partial(q^-)^3}(q_a^-) \neq 0.$
(xii) (iv) *is satisfied and* $l(a^+, a^-)$ *is* 1*-tangent to* L *at* a^+ *and* a^-

Proof $(\lambda, a_\lambda) \in \mathbb{E}(L)$ is regular. By Proposition 7.3, $\frac{\partial^3 F}{\partial t^3}(\lambda, a_\lambda, t) \neq 0$. Thus, exists smooth function-germ T on $\mathbb{R}^3$ s.t. $\frac{\partial^2 F}{\partial t^2}(\lambda, p, q, t) = 0$ iff $t = T(\lambda, p, q)$. Then $\mathbb{E}(L) = \left\{(\lambda, p, q) : \frac{\partial F}{\partial t}(\lambda, p, q, T(\lambda, p, q)) = 0\right\}$. Then

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial F}{\partial t}(\lambda, p, q, T(\lambda, p, q))\right)\Big|_{(\lambda,a_\lambda)} = 0 \tag{7.2}$$

$$\frac{\partial^2}{\partial\lambda^2}\left(\frac{\partial F}{\partial t}(\lambda, p, q, T(\lambda, p, q))\right)\Big|_{(\lambda,a_\lambda)} \neq 0 \tag{7.3}$$

are equivalent to (ix). Using the formula

$$\frac{\partial T}{\partial\lambda}(\lambda, p, q) = -\left(\frac{\partial^2 F}{\partial t^3}(\lambda, p, q, T(\lambda, p, q)\right)^{-1}\frac{\partial^2 F}{\partial\lambda\partial t^2}(\lambda, p, q, T(\lambda, p, q)) \tag{7.4}$$

we see that (7.2)–(7.3) are equivalent to (x). Equivalence of (x) and (xi) is obtained by a direct calculation. The last equivalence is obvious. □

Proposition 7.6 *The following conditions are equivalent*

(xiii) *the regular part of* $\mathbb{E}(L)$ *is* 2*-tangent to the fiber of* π *at* (λ, a_λ)*,*
(xiv) $\exists t$*:* (vi) *is satisfied,* (7.1) *is not satisfied and*

$$\left\{\frac{\partial^4 F}{\partial\lambda^3\partial t}\left(\frac{\partial^3 F}{\partial t^3}\right)^3 - 3\frac{\partial^4 F}{\partial\lambda^2\partial t^2}\left(\frac{\partial^3 F}{\partial t^3}\right)^2\frac{\partial^3 F}{\partial\lambda\partial t^2}\right.$$
$$\left.+3\frac{\partial^4 F}{\partial\lambda\partial t^3}\frac{\partial^3 F}{\partial t^3}\left(\frac{\partial^3 F}{\partial\lambda\partial t^2}\right)^2 - \frac{\partial^4 F}{\partial t^4}\left(\frac{\partial^3 F}{\partial\lambda\partial t^2}\right)^3\right\}(\lambda, a_\lambda, t) \neq 0$$

(xv) (vii) *is satisfied and* $\left(\frac{\partial^3 S^+}{\partial(q^+)^3}(q_a^+) = 0 \wedge \frac{\partial^4 S^+}{\partial(q^+)^4}(q_a^+) \neq 0\right)$ *or* $\left(\frac{\partial^3 S^-}{\partial(q^-)^3}(q_a^-) = 0 \wedge \frac{\partial^4 S^-}{\partial(q^-)^4}(q_a^-) \neq 0\right)$
(xvi) (iv) *is satisfied and* $l(a^+, a^-)$ *is* 1*-tangent to* L *at one of points* a^+, a^- *and* 2*-tangent to* L *at the other.*

Proof (xiii) means that (7.2) is satisfied, (7.3) is not satisfied and $\frac{\partial^3}{\partial\lambda^3}\left(\frac{\partial F}{\partial t}(\lambda, p, q, T(\lambda, p, q))\right)\Big|_{(\lambda,a_\lambda)} \neq 0$. Using (7.4), we see that these conditions are equivalent to (xiv). By direct calculation we see that (xiv) $\iff$ (xv). Finally, (xvi) is the geometric description of (xv). □

Theorem 7.7 *Let* $\frac{1}{\lambda}\frac{\partial^3 S^+}{\partial (q^+)^3}(q_a^+) + \frac{1}{1-\lambda}\frac{\partial^3 S^-}{\partial (q^-)^3}(q_a^-) \neq 0$ *(for (1)–(2) below,* $\lambda = 1/2$*). Let* $l(a^+, a^-)$ *denote the chord passing through* (a^+, a^-)*.*

(1) If $l(a^+, a^-)$ *is not bitangent to L at* a^+, a^-*, then the germ of F at* $(1/2, a_{1/2}, \dot{q}_{1/2})$ *has* $A_2^{B_1}$ *singularity, and the germ of GCS at* $a_{1/2}$ *is a smooth curve (the smooth part of the Wigner caustic).*

(2) If $l(a^+, a^-)$ *is* 1*-tangent to L at* a^+ *and at* a^-*, then the germ of F at* $(1/2, a_{1/2}, \dot{q}_{1/2})$ *has* $A_2^{B_2}$ *singularity, and the germ of GCS at* $a_{1/2}$ *is a union of two* 1*-tangent smooth curves (the smooth part of the Wigner caustic and the smooth part of the criminant).*

(3) If $l(a^+, a^-)$ *is* 1*-tangent to L at* a^+ *and at* a^-*, then the germ of F at* $(\lambda, a_\lambda, \dot{q}_\lambda)$ *for* $\lambda \neq 1/2$ *has* $A_2^{A_1}$ *singularity and the germ of GCS at* a_λ *is a smooth curve (the smooth part of the criminant).*

(4) If $l(a^+, a^-)$ *is* 1*-tangent to L at one of the points* a^+, a^- *and* 2*-tangent at the other, then the germ of F at* $(\lambda, a_\lambda, \dot{q}_\lambda)$ *for* $\lambda \neq 1/2$ *is not* $(1,2)$*-*$\mathcal{R}^+$*-stable. In particular,* $A_2^{A_2}$ *is not realizable as stable singularity of the GCS of a Lagrangian curve.*

Proof By Proposition 7.3, if $\frac{1}{\lambda}\frac{\partial^3 S^+}{\partial (q^+)^3}(q_a^+) + \frac{1}{1-\lambda}\frac{\partial^3 S^-}{\partial (q^-)^3}(q_a^-) \neq 0$ then the germ of F is a unfolding of A_2 singularity. Therefore we can reduce F to the form $F'(\lambda, p, q, t) = t^3 + g(\lambda, p, q)t$, where g is a smooth function-germ vanishing at $(\lambda_a, 0)$ (for $\lambda_a = 0$ or $\lambda_a = 1/2$). By Proposition 7.4, if $l(a^+, a^-)$ is not bitangent to L at a^+, a^- then $\frac{\partial F'}{\partial t \partial \lambda}(1/2, 0, 0) \neq 0$ and this implies $\frac{\partial g}{\partial \lambda}(1/2, 0) \neq 0$. By Theorems 6.14 and 6.16 we obtain (1). If the chord $l(a^+, a^-)$ is tangent to L at a^+, a^- then by Proposition 7.4 we get that $p_a^+ = p_a^-$ and $\frac{\partial F'}{\partial t \partial \lambda}(\lambda_a, 0, 0) = 0$ and this implies $\frac{\partial g}{\partial \lambda}(\lambda_a, 0) = 0$. But $dg|_{(\lambda_a, 0)} \neq 0$ since $\frac{\partial F}{\partial t \partial p}(\lambda_a, a, \dot{q}_a) \neq 0$. By Proposition 7.5 if $l(a^+, a^-)$ is 1-tangent to L at a^+, a^- then $\left(\frac{\partial^3 F'}{\partial \lambda \partial t^2}(\lambda_a, 0, 0)\right)^2 - \frac{\partial^3 F'}{\partial t^3}(\lambda_a, 0, 0)\frac{\partial^3 F'}{\partial \lambda^2 \partial t}(\lambda_a, 0, 0) \neq 0$. But this implies $\frac{\partial^2 g}{\partial \lambda^2}(\lambda_a, 0,) \neq 0$. Thus if $\lambda_a = 1/2$ by Theorems 6.14 and 6.16 we obtain (2) and otherwise by Theorems 6.12 and 6.15 we obtain (2). Finally, assume that $l(a^+, a^-)$ is 1-tangent to L at a^+ and 2-tangent at a^-. By Proposition 7.6 we get $\frac{\partial^2 g}{\partial \lambda^2}(\lambda_a, 0,) = 0$ and $\left\{ \frac{\partial^4 F}{\partial \lambda^3 \partial t}\left(\frac{\partial^3 F}{\partial t^3}\right)^3 - 3\frac{\partial^4 F}{\partial \lambda^2 \partial t^2}\left(\frac{\partial^3 F}{\partial t^3}\right)^2 \frac{\partial^3 F}{\partial \lambda \partial t^2} + 3\frac{\partial^4 F}{\partial \lambda \partial t^3}\frac{\partial^3 F}{\partial t^3}\left(\frac{\partial^3 F}{\partial \lambda \partial t^2}\right)^2 - \frac{\partial^4 F}{\partial t^4}\left(\frac{\partial^3 F}{\partial \lambda \partial t^2}\right)^3 \right\}(\lambda_a, 0, 0) \neq 0$. Thus, $\frac{\partial^3 g}{\partial \lambda^3}(\lambda_a, 0,) \neq 0$. We know that $\frac{\partial g}{\partial p}(\lambda_a, 0,) \neq 0$ since $\frac{\partial^2 F}{\partial t \partial p}(\lambda_a, a, \dot{q}_a) \neq 0$. It is easy to see that $\frac{\partial^2 F}{\partial t \partial q}(\lambda_a, a, \dot{q}_a) = 0$. Thus F has $A_2^{A_2}$ singularity at $(\lambda_a, a, \dot{q}_a)$ iff $\frac{\partial^3 F}{\partial \lambda \partial q \partial t}(\lambda_a, a, \dot{q}_a)\frac{\partial^3 F}{\partial t^3}(\lambda_a, a, \dot{q}_a) - \frac{\partial^3 F}{\partial \lambda \partial t^2}(\lambda_a, a, \dot{q}_a)\frac{\partial^3 F}{\partial q \partial t^2}(\lambda_a, a, \dot{q}_a) \neq 0$. By direct calculation, this is equivalent to $\frac{(q_a^+ - q_a^-)}{\lambda_a(1-\lambda_a)}\frac{\partial^3 S^+}{\partial (q^+)^3}(q_a^+)\frac{\partial^3 S^-}{\partial (q^-)^3}(q_a^-) \neq 0$, which is not satisfied, since $l(a^+, a^-)$ is 2-tangent to L at a^-. □

Corollary 7.8 *Let L be a smooth closed convex curve in* $(\mathbb{R}^2, \omega)$*. The smooth part of* $E_{1/2}(L)$ *is* $(1,2)$*-Lagrangian stable, but the cusps of* $E_{1/2}(L)$*, seen as part of GCS*(L)*, are not* $(1,2)$*-Lagrangian stable; the medial axis and the whole CSS are not* $(1,2)$*-Lagrangian stable.*

Remark 7.9 For a convex curve $L \subset \mathbb{R}^2$, most singularities which are affine stable are not affine-Lagrangian stable (compare Theorem 7.1 and Corollary 7.8). Also, although the cusps of $E_{1/2}(L)$ are affine-Lagrangian stable when $E_{1/2}(L)$ is considered by itself, they are not affine-Lagrangian stable considering $E_{1/2}(L) \subset GCS(L)$, that is, the meeting of $E_{1/2}(L)$ and CSS is not affine-Lagrangian stable.

7.2 Discussion

Because of the large loss of stability for singularities of the GCS, when going from the affine to the affine-Lagrangian case, one wonders if it is possible to consider a coarsen classification of singularities of the GCS of Lagrangian submanifolds, which produces more stable singularities. In fact, the usual Lagrangian equivalence will do.

As mentioned at the beginning of Sect. 6, classification by usual Lagrangian equivalence amounts to considering the unfolding parameters $y = (\lambda, x) \in \mathbb{R} \times \mathbb{R}^{2m}$ on an equal footing. In this setting, Lagrangian equivalence of $\mathbb{E}(L)$ and $\mathbb{E}(\widetilde{L})$ is defined in terms of Lagrangian equivalence of $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ in the usual way, which means that their generating families must be stably $\mathcal{R}^+$-equivalent (Theorem 3.10), in other words, there is a symplectomorphism-germ Υ of $T^*\mathbb{R} \times T\mathbb{R}^{2m}$ such that $\Upsilon(\mathcal{L}) = \widetilde{\mathcal{L}}$ and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L} \hookrightarrow T^*\mathbb{R} \times T\mathbb{R}^{2m} & \overset{Pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^{2m} \\ \downarrow \Upsilon & & \downarrow \\ \widetilde{\mathcal{L}} \hookrightarrow T^*\mathbb{R} \times T\mathbb{R}^{2m} & \overset{Pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^{2m} \end{array}$$

where the right-vertical arrow is a diffeomorphism-germ of general form

$$\mathbb{R} \times \mathbb{R}^{2m} \ni (\lambda, x) \mapsto (\Lambda(\lambda, x), X(\lambda, x)) \in \mathbb{R} \times \mathbb{R}^{2m}.$$

Comparing with the classifying diagram in Definition 6.3 for $(1, 2m)$-Lagrangian equivalence, one expects that many singularities of $GCS(L)$ which are Lagrangian stable are not $(1, 2m)$-Lagrangian stable. In fact, for convex Lagrangian curves, it is easy to see that most of the singularities of Theorem 7.1 are Lagrangian stable in the above sense.

However, the fact that the last projection $\pi : \mathbb{R}^{1+2m} \to \mathbb{R}^{2m}$ is not taken into account is an obvious indication that usual Lagrangian equivalence is not the correct equivalence relation for classification of the singularities of $GCS(L)$, because this latter is the image under π of the locus of critical points of π restricted to $\mathbb{E}(L)$.

This becomes even clearer when we also analyze the non-symplectic case. In this case, consider the following *extended chord transformation*

$$\Gamma : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \times T\mathbb{R}^n, \ (\lambda, x^+, x^-) \mapsto (\lambda, \Gamma_\lambda(x^+, x^-)),$$

where $\Gamma_\lambda : \mathbb{R}^n \times \mathbb{R}^n \to T\mathbb{R}^n$ is a simpler λ-chord transformation,

$$\Gamma_\lambda(x^+, x^-) = (x, \dot{x}) = \left(\lambda x^+ + (1-\lambda)x^-, \frac{x^+ - x^-}{2}\right), \tag{7.5}$$

which differs from Φ_λ only in the kind of linear equation for $\dot{x}$ (compare (7.5) to (3.1) and (3.2)), this latter chosen in the symplectic case so that $(\Phi_\lambda^{-1})^*(\delta_\lambda \omega) = \dot{\omega}$ (no extra semi-basic form in the r.h.s.).

Now, let M and $\widetilde{M}$ be germs of m-dimensional smooth submanifolds of $\mathbb{R}^n$, $n \leq 2m$, and let $\mathbb{M}$ and $\widetilde{\mathbb{M}}$ be the chord transformed cylinders

$$\mathbb{M} = \Gamma(\mathbb{R} \times M \times M), \ \widetilde{\mathbb{M}} = \Gamma(\mathbb{R} \times \widetilde{M} \times \widetilde{M}).$$

Definition 7.10 Germs of $GCS(M)$ and $GCS(\widetilde{M})$ are *chord equivalent* if there is a diffeomorphism-germ Θ of $\mathbb{R} \times T\mathbb{R}^n$ s.t. $\widetilde{\mathbb{M}} = \Theta(\mathbb{M})$ and the following diagram commutes:

$$\begin{array}{ccccc} \mathbb{R} \times T\mathbb{R}^n & \overset{id_{\mathbb{R}} \times pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^n & \overset{\pi}{\longrightarrow} & \mathbb{R}^n \\ \downarrow \Theta & & \downarrow & & \downarrow \\ \mathbb{R} \times T\mathbb{R}^n & \overset{id_{\mathbb{R}} \times pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^n & \overset{\pi}{\longrightarrow} & \mathbb{R}^n \end{array}$$

where *vertical* arrows indicate diffeomorphism-germs, as follows:

$$\Theta : \mathbb{R} \times T\mathbb{R}^n \ni (\lambda, x, \dot{x}) \mapsto (\Lambda(\lambda, x), X(x), \dot{X}(\lambda, x, \dot{x})) \in \mathbb{R} \times T\mathbb{R}^n,$$
$$\mathbb{R} \times \mathbb{R}^n \ni (\lambda, x) \mapsto (\Lambda(\lambda, x), X(x)) \in \mathbb{R} \times \mathbb{R}^n,$$
$$\mathbb{R}^n \ni x \mapsto X(x) \in \mathbb{R}^n.$$

Definition 7.11 A singularity of $GCS(M)$ is *affine stable* if it is a stable singularity under its classification by the chord equivalence.

Using classification by the chord equivalence, one proves Theorem 7.1 for the *GCS* of convex curves by somewhat lengthy but straightforward computations. The classification of the singularities of $GCS(M)$ in the other known cases, for instance hyperplanes, can be similarly accomplished by chord equivalence, which gives the correct affine-invariant classification of the singularities of $GCS(M)$ for general m-dimensional submanifolds $M \subset \mathbb{R}^n, n \leq 2m$.

Comparison of the classifying diagram in Definition 7.10 for chord equivalence with the classifying diagram in Definition 6.3 for $(1, 2m)$-Lagrangian equivalence shows their obvious analogy.

On the other hand, the "obvious" analog of the classifying diagram for usual Lagrangian equivalence, when no symplectic form has to be accounted for, is

$$\begin{array}{ccc} \mathbb{R} \times T\mathbb{R}^n & \overset{id_{\mathbb{R}} \times pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^n \\ \downarrow \Theta & & \downarrow \\ \mathbb{R} \times T\mathbb{R}^n & \overset{id_{\mathbb{R}} \times pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^n \end{array}$$

where vertical arrows indicate diffeomorphism-germs of the form:

$$\Theta : \mathbb{R} \times T\mathbb{R}^n \ni (\lambda, x, \dot{x}) \mapsto (\Lambda(\lambda, x), X(\lambda, x), \dot{X}(\lambda, x, \dot{x})) \in \mathbb{R} \times T\mathbb{R}^n,$$
$$\mathbb{R} \times \mathbb{R}^n \ni (\lambda, x) \mapsto (\Lambda(\lambda, x), X(\lambda, x)) \in \mathbb{R} \times \mathbb{R}^n.$$

Of course, applying the above "obvious" and *wrong* equivalence relation to classify singularities of $GCS(M)$ for general submanifolds $M^m \subset \mathbb{R}^n, n \leq 2m$, produces many more stable singularities than when applying the *correct* classifying diagram of Definition 7.10.

Thus, choosing the correct classifying diagram in both the non-symplectic and the symplectic cases shows that most singularities of the *GCS* which are stable when no symplectic form has to be accounted for, cease to be stable when there is a symplectic form to be accounted for. In other words, there is breakdown of stability due to a symplectic form. Other similar cases, of breakdown of simplicity due to a symplectic form, can be found in [4,6] and especially in [7].

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