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Superfluid ^4He: Brief Notes on Collective Energy Excitations and Specific Heat.

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Superfluid ^{4}He: Brief Notes on Collective Energy Excitations and Specific Heat.

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Abstract. These brief notes about some amazing properties of the helium superfluid have been written to graduate and postgraduate students of physics. We have estimated some collective energy excitations (*linear vortices, energy spectrum of the quasiparticles* and *solitons*) and calculated the specific heat of the superfluid helium assuming that the transition superfluid liquid → liquid is an order-disorder transition.

1) Introduction.

Few weeks ago we found an interesting book written by Walecka[1] named "Introduction to Modern Physics". Published at 2008 it is an up to date book written to graduate and postgraduate students of physics. It analyzes problems like, for instance, quantum electrodynamics, relativistic quantum mechanics, quarks, general relativity, cosmology, quantum fluids and quantum fields. He analyzes in Chapter 11 the quantum fluids (^{4}He superfluid and superconducting metals) that are macroscopic many-body systems whose behavior reflects the underlying quantum mechanics. Reading this Chapter we remembered our studies on hydrodynamics of the ^{4}He superfluid at 1974. As a result of these studies we have written a textbook on fluid dynamics[2] and also proposed a naïve phenomenological model[3] to explain the specific heat of the superfluid helium using an order-disorder transition approach. Now, inspired by Walecka´s book[1] we decided to write this didactical article briefly analyzing some aspects of the energy collective excitations like *linear vortices, energy spectrum of the quasiparticles* and *solitons*. Again is shown our calculations on the specific heat of the liquid helium. It will be explained only the basic aspects of collective excitations and it will be mentioned only a few references where one can find detailed experimental results and theoretical approaches. In Section 2 are presented some remarkable properties[4,5] of superfluid liquid helium. In Section 3, as helium atoms are bosons that interact weakly, we explain how to treat statistically the liquid helium as an "*degenerate Bose-Einstein gas*".[4,6] In Section 4 it will be estimated the energy collective excitations of the "*Bose condensate*" using the Hartree-Fock approach and the Gross-Pitaevskii equation. In Section 5 the specific heat of the liquid helium is calculated assuming that the transition *superfluid liquid → liquid* is an order-disorder transition.

2) Some Remarkable Properties of the Liquid Helium.

In London´s book[4] and in many other papers[5] one can learn about the amazing properties of the liquid helium which show that it is a substance entirely different from the normal liquids. Liquid helium is a "superfluid". In Fig. 1 is seen the phase diagram[4] of helium in the P-T plane. There are two kinds of liquids: "helium I" (He I) and "helium II" (He II). These two phases are separated by the λ-line that for $P \to 0$ has the end point, named "λ-point", at the temperature $T_\lambda = 2.16$ K. Note that there is no triple point between the solid, liquid and gaseous states. Instead of one there are actually two triple points: at the ends of the λ-line which separates the two liquid phases, He I and He II.

The He II that exists for $P < 20$ atm and for temperatures $T < T_\lambda$ has "super" properties like "superfluidity", "linear and ring quantum vortices", "thermal superconductivity", "fountain effect" and "supersurface film" (see references 4 and 5).

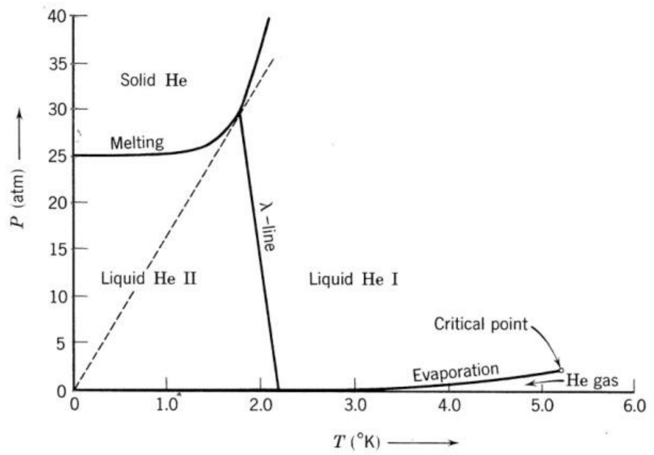

Figure 1.Phase diagram showing the four states of the helium in the P-T plane.

In Figs.1 and 2 are presented two amazing properties of this superfluid. In Fig. 2 is shown the viscosity η of the He II e He I. For $T > T_\lambda$ we see that the He I has a viscosity $\eta > 2$ cp, similar to the water viscosity[2] $\eta(\text{water}) \sim 1$ cp. The He II viscosity for $T \leq T_\lambda$ the decreases abruptly, tending to zero for $T \to 0$ when He II shows the superfluidity behavior.

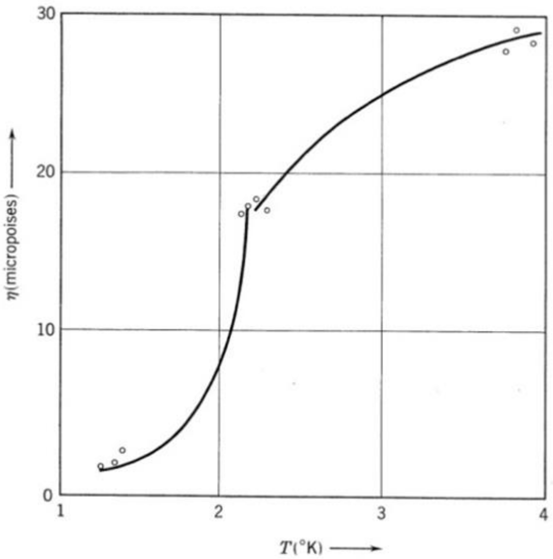

Figure 2.Viscosity η,[4] measured in micropoise, of the liquid helium as function of T(K).

Figure 3 gives[4] the specific heat C(cal/g.K) of the liquid helium under its own vapor pressure as a function of the temperature T(K). The curve seen in Fig.3 shows a shape of the letter λ. The specific heat shows a singularity, named "λ-singularity"[4] of at the "λ-point" (see Section 5).

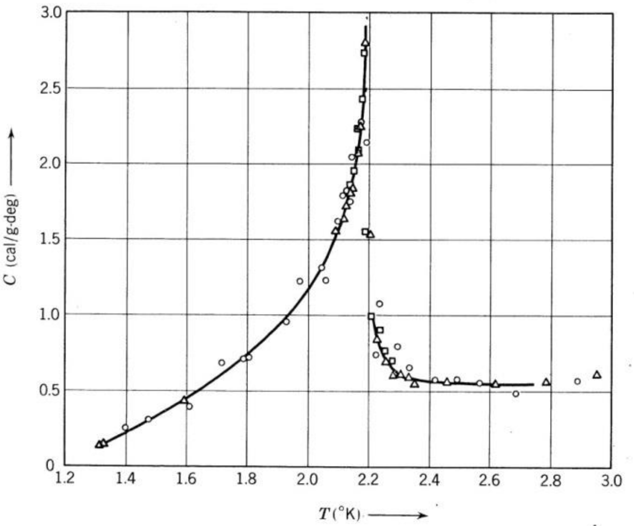

Figure 3. Specific heat of the liquid helium under its own vapor pressure.[4]

3) Degenerate Bose-Einstein Gas.

According to the Bose-Einstein statistics,[4,6,7] for a degenerate ideal gas with a very large number of particles N the average number n_i of particles in the energy level ϵ_i of statistical weight g_i is given by :

$$n_i = g_i/[\exp(\epsilon_i/k_BT+\alpha) -1] \qquad (3.1),$$

where k_B is the Boltzmann constant, $\alpha = - \mu/k_BT$ and μ is the chemical potential that for the Bose-Einstein gas must be always negative or zero.[4,6,7] In Fermi statistics μ can be positive or negative. It is important to note that the gas occupies a very large volume V and that the interaction potential between the particles is negligible. However, there is a non-local quantum interaction between the particles expressed by the bosonic quantum state symmetry of the system. As will be seen in Section 4 the energy of this global interaction is described by the chemical potential[4,6,7] $\mu = (\partial E/\partial N)_{S,V}$, where E is the total energy and S is the entropy of the gas. In what follows it will be assumed the bosons have spin zero (S=0).

Using (3.1) the total number of particles would be given by

$$N = \Sigma_i n_i \qquad (3.2).$$

Assuming that the energies of the individual particles in the fundamental state is $\epsilon_0 = 0$ it can be shown that for T = 0 K the number n_o of bosons with *spin zero* found in the fundamental state is given by, with $g_o = 1$,

$$n_o = 1/[\exp(\alpha) -1] \qquad (3.3).$$

Particles with mass m contained in a very large volume V that are not in the fundamental state have kinetic energy $\epsilon = p^2/2m$. For these particles the weight g is given by a smooth function

$$g(\epsilon) = (2\pi V/h^3)(2m)^{3/2}\epsilon^{1/2} \qquad (3.4).$$

In this way, the number $dn(\epsilon)$ of these particles with energy between ϵ and $\epsilon+d\epsilon$ is given by

$$dn(\epsilon)= (2\pi V/h^3)(2m)^{3/2}\, d\epsilon\, \sqrt{\epsilon}/[\exp(\epsilon/k_BT + \alpha) -1] \qquad (3.5).$$

Note that since $g(\epsilon) = g(\epsilon=0) = 0$ (3.5) does not describe n_o given by (3.3). That is, (3.5) can used only to estimate the contribution of states with $\epsilon > 0$. Consequently, the total number N of bosons is given by,

$$N = n_o + (2\pi V/h^3)(2m)^{3/2} \int_o^\infty d\epsilon\sqrt{\epsilon}/[\exp(\epsilon/k_BT + \alpha) -1]$$

$$= n_o + V(2\pi mk_BT/h^2)^{3/2} F_{3/2}(\alpha) \qquad (3.6),$$

where $F_{3/2}(\alpha)$ is the case $\sigma = 3/2$ of the functions $F_\sigma(\alpha)$ that are defined by[4]

$$F_\sigma(\alpha) = [1/\Gamma(\sigma)] \int_o^\infty dy\, y^{\sigma-1}/[\exp(y + \alpha) -1]$$

Defining a *critical* temperature T_c by

$$T_c = (h^2/2\pi mk_B)[N/VF_{3/2}(0)]^{2/3} = (h^2/2\pi mk_B)(N/2.612V)^{2/3} \qquad (3.7)$$

we verify that

$$N = 1/[\exp(\alpha) -1] + N(T/T_c)^{3/2} [F_{3/2}(\alpha)/F_{3/2}(0)] \qquad (3.8).$$

The α values are obtained solving (3.8).Taking into account that[4] for $\alpha < 1$ $F_{3/2}(\alpha) \approx 2.612 - 2\sqrt{\pi\alpha} + \ldots$ we get from (3.8), for $|T-T_c| << T_cN^{-1/3}$:

$$\alpha = (1/NC)^{2/3} [1 + (T/T_c -1)(N/C^2)^{1/3}] \qquad (3.9),$$

where $C = 2\sqrt{\pi}/2.612 = 1.36$. From (3.9) we see that, for $N \to\infty$ and $T < T_c$, $\alpha \approx N^{-2/3} \to 0$, that is, α can put α equal to zero.

According to detailed analysis performed by London[4] it can be shown that for $N \to\infty$ eq.(3.8) can be written as,

$$n_o = N[1 - (T/T_c)^{3/2}] \text{ (for } T < T_c) \quad \text{and} \quad n_o = 0 \text{ (for } T > T_c) \qquad (3.10)$$

Eqs. (3.10) show that for low temperatures, that is, $T < Tc$ the bosons will tend to condense into the single-particle ground state. At T = 0 K all them will be in the fundamental state forming a collective macroscopic quantum state named “Bose condensate”. As well known[4,6,7] a degenerate gas, Fermi-Dirac or Bose-Einstein gas, realizes a characteristic state of order when approaching O K temperature. A Fermi-Dirac gas does this by settling down in a kind of lattice order in momentum space and a Bose-Einstein gas by crowding the particles into the state of smallest momentum. A figure illustrating the bosonic condensation $n_o = n_o(T)$ is seen in ref.4 (Fig.21,pag.41). Finally, let us calculate the energy U of the bosonic system:

$$U = \Sigma_i n_i \epsilon_i = 2\pi V(2m/h^2)^{3/2} \int_o^\infty d\epsilon\, \epsilon^{3/2}/[\exp(\epsilon/k_BT + \alpha) -1]$$

$$= (3/2)Vk_BT(2\pi mk_BT/h^2)^{3/2} F_{5/2}(\alpha) \qquad (3.11),$$

remembering that the ground state (ϵ_o=0) does not contribute to the energy.

4) Collective excitations in Superfluid Helium.

Let us see how to calculate the collective excitations in the superfluid helium using the Hartree-Fock approximation[8] and the Gross-Pitaevskii equation.[9,10] We obtain only the *quantum linear vortices (*or simply*, linear vortices)* the *quasi-particles spectrum* and the *solitons*. Studies on "*ring vortices*" or "*smoke rings*" can be seen elsewhere.[5] It will be assumed that the system is not submitted to an external potential.

As is well known[1] the interaction between the helium atoms in the liquid is very weak.[1,4] Let us indicate by V(**x**) the interaction potential between two atoms and by *n*(**x**) the particle density at the point **x**. Using the Hartree approximation,[1,8] the helium atom at the point **x** is submitted to a potential $V_H(\mathbf{x})$ given by

$$V_H(\mathbf{x}) = \int d^3\mathbf{y}\ V(\mathbf{x}-\mathbf{y})\ n(\mathbf{y}) \qquad (4.1).$$

Since the interaction between the atoms is very weak we can assume that the He atom does not lose its individuality. So, the liquid is composed by N identical spin-zero bosons each one with mass m that in the fundamental state has energy $\mathcal{E}_o$ and is represented by the state function $\phi_o(\mathbf{x})$. Thus, liquid is represented by the bosonic symmetric state function[1,8]

$$\Phi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N) = \phi_o(\mathbf{x}_1)\ \phi_o(\mathbf{x}_2)\ldots\phi_o(\mathbf{x}_N) \qquad (4.2),$$

where $\phi_o(\mathbf{x})$ obeys the stationary Schrödinger´s equation, where Δ = laplacian operator,

$$\{ -(\hbar^2/2m)\Delta + V_H(\mathbf{x}) \}\ \phi_o(\mathbf{x}) = \mathcal{E}_o\ \phi_o(\mathbf{x}) \qquad (4.3).$$

At the condensate the particle density $n_o(\mathbf{x})$ is given by $n_o(\mathbf{x}) = |\phi_o(\mathbf{x})|^2$ and the energy E_o of the (fundamental) state is equal to $E_o = N\ \mathcal{E}_o$.

It is convenient[1] to define a new single-particle wave function that scales out the factor of N which represents the condensate,

$$\Psi_o(x) = \sqrt{N}\ \phi_o(\mathbf{x}) \qquad (4.4)$$

Thus, using (4.1), (4.3) and (4.4) we write following *Hartree equation* for the condensate

$$\{ -(\hbar^2/2m)\ \Delta + \int d^3\mathbf{y}\ V(\mathbf{x}-\mathbf{y})\ |\Psi_o(\mathbf{y})|^2\}\Psi_o(\mathbf{x}) = \mathcal{E}_o\Psi_o(\mathbf{x}) \qquad (4.5),$$

which is a non-linear, integro-differential Schrödinger equation that can be solved by iteration in the general case, or analytically in some particular cases. In this way the particle density $n_o(x)$ and the particle current $\mathbf{j}(\mathbf{x})$ of the condensate are given, respectively by

$$n_o(x) = |\Psi_o(x)|^2$$

and (4.6)

$$\mathbf{j}(\mathbf{x}) = n_o(\mathbf{x})\, \mathbf{v}(\mathbf{x}) = (\hbar/2mi)\{\Psi_o^* \mathrm{grad}(\Psi_o) - \mathrm{grad}(\Psi_o)^* \Psi_o\}.$$

Parameterizing the wave function in terms of the modulus and phase as

$$\Psi_o(x) = F(\mathbf{x}) \exp[i\varphi(\mathbf{x})] \qquad (4.7),$$

where $F(\mathbf{x})$ and $i\varphi(\mathbf{x})$ are real functions. We verify using (4.6) and (4.7) that the condensate density $n_o(\mathbf{x})$ and velocity $\mathbf{v}_o(\mathbf{x})$ are given by

$$n_o(\mathbf{x}) = |F(\mathbf{x})|^2 = |\Psi_o(\mathbf{x})|^2$$

and (4.8).

$$\mathbf{v}_o(\mathbf{x}) = (\hbar/m)\, \mathrm{grad}[\varphi(\mathbf{x})]$$

The second equation is very interesting. As is known from fluid mechanics[2] if the velocity field comes from the gradient of a given function, in our case the phase $\varphi(\mathbf{x})$, the velocity field is *irrotational*, that is,

$$\mathrm{rot}\, \mathbf{v}_o(\mathbf{x}) = 0 \qquad (4.9)$$

which means that the particles flow is *irrotational*.

4.a)Quantum Vortex: Quantized Circulation.

Putting the liquid He II into rotation it is observed[1,5,11,12] that *linear vortices*[2] are created in the bulk fluid. By *linear vortex* we mean a fluid rotation with a hole in the center as occurs in tornados and in flow of water in wash basin.[2] In this case the circulation of the fluid around a circle ○ involving the rectilinear vortex[2] is not null, that is,

$$\int_\circ \mathbf{v}\cdot d\boldsymbol{\ell} \neq 0 \qquad (4.10).$$

In this way, using (4.8) we get

$$\int_\circ \mathbf{v}_o\cdot d\boldsymbol{\ell} = (\hbar/m)\int_\circ \mathrm{grad}[\varphi(\mathbf{x})]\cdot d\boldsymbol{\ell} = (\hbar/m)\int_\circ d\varphi$$

$$= (\hbar/m)\{\varphi(2\pi) - \varphi(0)\} \neq 0 \qquad (4.11).$$

Since the bosonic wave function is assumed to be a single-valued function throughout the fluid the difference of phase $\varphi(2\pi) - \varphi(0)$ must be an integral number of 2π. So, (4.10) can be written as

$$\int_{\circ} \mathbf{v}\cdot d\boldsymbol{\ell} = (\hbar/m)(2\pi n) = (h/m)n, \quad \{n = 0,1,2,\ldots\} \qquad (4.12),$$

showing that the circulation around a vortex must be quantized in units of h/m. For ^{4}He the unit of circulation has the value

$$h/m_{He} = 0.997\ 10^{-3} cm^2/s \qquad (4.12).$$

Its remarkable[1] that properties of the macroscopic fluid flow of the condensed Bose system have been obtained from single-particle wave function.

4.b)Gross-Pitaevskii Equation.

First, let us assume that the interaction $V(\mathbf{x} - \mathbf{y})$ between two Bose particles is given delta function ($g > 0$ for repulsive and $g < 0$ for attractive interaction):

$$V(\mathbf{x} - \mathbf{y}) = g\ \delta(\mathbf{x}-\mathbf{y}) \qquad (4.13)$$

and let us generalize the Hartree equation (4.5) substituting the single-particle energy $\mathcal{E}_o$ by the chemical potential μ[1,6,7]

$$\mu = (\partial E/\partial N)_{S,V} = \mathcal{E}_o \qquad (4.14),$$

which is an experimental observable. The use of the chemical potential allow us to take into account the effect of the interactions that at $T \neq 0$ take some particles out of the Bose condensate distributing them over the single-particle states with energies higher than $\mathcal{E}_o$. The number of particles n_o in the condensate is not a conserved quantity, changing according to (3.10). The use of μ allows one to take this into account. Assuming that the system is in the condensate state (4.5) we obtain, becomes using (4.13) and (4.14), the local non-linear differential equation named *Gross-Pitaevskii equation.*[9,10]

$$\{ -(\hbar^2/2m)\Delta + g\ |\Psi_o(\mathbf{x})|^2\}\ \Psi_o(\mathbf{x}) = \mu\ \Psi_o(\mathbf{x}) \qquad (4.15),$$

noting that for the condensate $\Psi_o(\mathbf{x,t}) = \Psi_o(\mathbf{x}) \exp(-i\mu t/\hbar)$.

Considering that the solution of (4.14) is a linear vortex we look for a function $\Psi_o(\mathbf{x})$ with a cylindrical symmetry (using r instead of ρ to indicate the distance of a point from the symmetry axis z)

$$\Psi_o(r,\theta) = \sqrt{n_o}\, f(r) \exp(i\theta) \qquad (4.16),$$

where n_o is the condensate density and f(r) is real. Thus, from (4.8) and (4.16) we obtain the tangential velocity of the fluid around the symmetry axis z of the linear vortex (see Figs.11.4 and 11.5 of ref.1)

$$\mathbf{v}(r,\theta) = (\hbar/m)\,\mathrm{grad}(\theta) = (\hbar/mr)\mathbf{e}_\theta \qquad (4.17),$$

where $\mathbf{e}_\theta$ is the unit vector tangent to the circular trajectory of radius r. Note that the velocity v(r) falls with 1/r.

Since for the circle we have $d\boldsymbol{\ell} = r\, d\theta\, \mathbf{e}_\theta$ the circulation (4.10) about the origin is given by

$$\oint \mathbf{v}\cdot d\boldsymbol{\ell} = \int_o^{2\pi} (\hbar/mr) r d\theta = 2\pi\hbar/m = h/m \qquad (4.18).$$

where h/m is taken as *one unit of circulation*.

Taking (4.16) and the laplacian in polar cylindrical coordinates the Gross-Pitaevskii equation (4.15) becomes

$$(\hbar^2/2m)\{(1/r)d/dr(rd/dr) - 1/r^2\}f(r) + \mu f(r) - n_o g f(r)^3 = 0 \qquad (4.19).$$

Note that far away from the center of the vortex, that is, for $r \to \infty$ we must have the boundary condition $|\Psi_o(r,\theta)|^2 \to n_o$ which implies that

$$\lim_{r\to\infty} f(r) = 1 \qquad (4.20).$$

It follows from (4.19) and (4.20) that $\mu = g n_o$ which is the energy required to insert a boson into the condensate. Thus, defining $\xi = (\hbar^2/2mgn_o)^{1/2}$ and $\zeta = r/\xi$ the equation (4.19) becomes written as

$$d^2f/d\zeta^2 + (1/\zeta)\, df/d\zeta - (1/\zeta^2)f + f - f^3 = 0 \qquad (4.21)$$

which now obeys the boundary condition $\lim_{\zeta\to\infty} f(\zeta) = 1$. The solution of (4.21) gives the *Gross-Pitaevskii vortex* which is a non-uniform system.

As $\zeta \to \infty$ a power series solution in $1/\zeta^2$ gives

$$f(\zeta) = 1 - 1/2\zeta^2 + \ldots \qquad (4.22).$$

As $\zeta \to 0$ the third term (the angular momentum barrier) of (4.21) dominates and it is easily verified that the solution of (4.21) takes the following form

$$f(\zeta) = C\zeta \qquad (4.23),$$

where C is a constant. Note that, according to (4.23), since f(0) = 0, that the liquid is excluded from the vortex core, as advertised. The size of the vortex can be estimated putting

$$r_{core} \sim \xi = (\hbar^2/2mgn_o)^{1/2} \qquad (4.24).$$

According to Fetter and Walecka[13] the speed of sound in the weakly interacting Bose gas is given by $c_{sound} = (n_o g/m)^{1/2}$. In this way the core dimension r_{core} of the vortex (4.24) can be written by

$$r_{core} \sim (\hbar/2m)^{1/2}/c_{sound} \qquad (4.25).$$

Putting $m = m_{He}$ and assuming that that the ordinary velocity of sound[4] for helium at lowest temperatures is $c_{sound} \approx 237$m/s we see that the roughly estimated value (4.25) is $r_{core} \sim 0.5$ Å, in fair agreement with experimental results $r_{core} \sim 1$ Å.[9,10]

Numerical integration of (4.21) can be carried out for all values of ζ using the Runge-Kutta algorithm in Mathcad11. The result of the calculations are shown in Fig.4.[1]

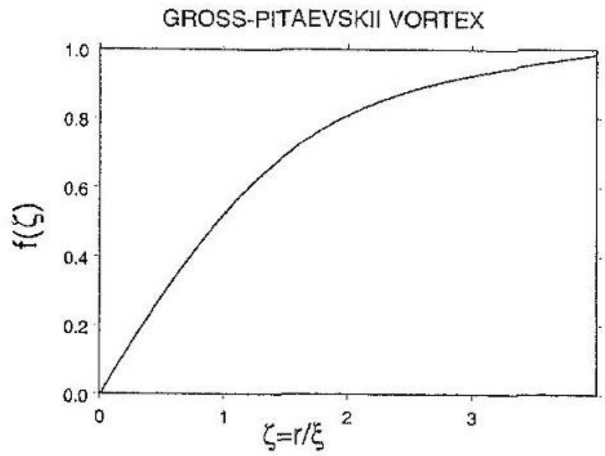

Fig.4. Numerical values of f(ζ) x ζ for a unit vortex circulation.

The vortex energy E_v, per unit of length, is given by[13] $E_v \approx (N\pi\hbar^2/m)\ln(1.46R/\xi)$ where R is a cutoff at large distances that may be interpreted as the radius of the rotating container.

"The Gross-Pitaevskii equation is also applicable to cold, isolated, laser-trapped Bose systems, whose experimental study provides one of the more fascinating aspects of modern physics.[1,13,14]"

Finally, the GP equation is derived with more sophistication in the book of Fetter and Walecka.[13]

4.b)Quasiparticles Excitations.

Besides the vortices there are another collective excitations in He II that have a dispersion relation that will be indicated by $\varepsilon_k(k)$. These are experimentally accessible through specific heat measurements,[1,4] or more directly, through neutron scattering.[1,4,15] The quanta of these excitations, with momentum $k = 2\pi/\lambda$, were called "quasiparticles" by Landau.[6] In Fig.5 is shown the measured low-temperature (T =1.12 K) quasiparticle spectrum in He II obtained by neutron scattering.[15] At long wavelengths (k < 1), dashed linear region, we have phonons that are the quanta of the sound waves in the fluid with $\varepsilon_k = \hbar k c_{sound}$ At higher k ($k \geq 1Å^{-1}$) we have "rotons"[6] which are excitations described by the dispersion relation

$$\varepsilon_k = \Delta + \hbar^2(k - k_o)^2/2m_r \qquad (4.26),$$

where $\Delta = 8.6\ k_B$, $k_o = 1.91Å^{-1}$ and m_r = "roton mass" = $0.16\ m_{He}$.

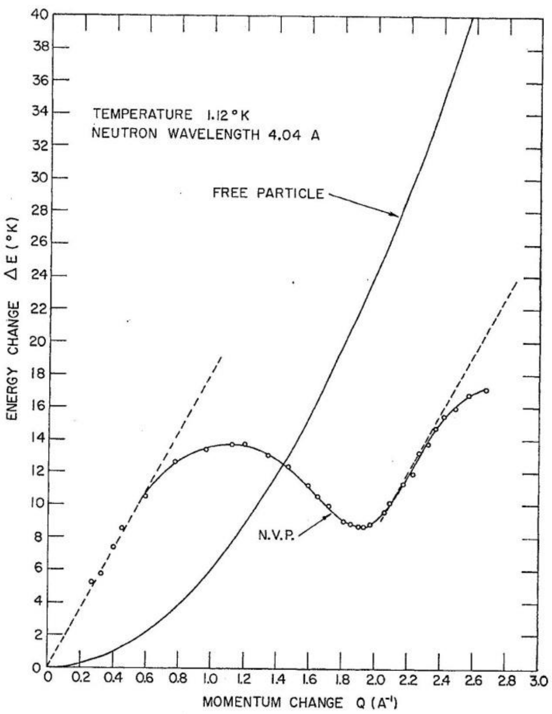

Fig.5. Experimental low-temperature quasiparticle spectrum $\varepsilon_k(k)/k_B = \Delta E(K)$ measured in K degrees by Henshaw and Woods[15] as function of $k(Å^{-1}) = Q(Å^{-1})$.

Taking into account the roton spectrum Landau[6] gave a simple argument based on conservation of energy and momentum to understand the superfluidity. He assumed that an object with a large mass M is moving with velocity **v** through the He II and that due to a collision process it creates an excitation[1] in the condensate as shown in Fig.6.

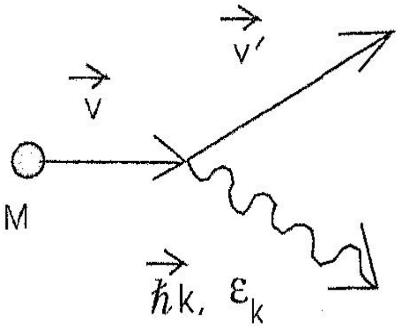

Fig.6. Creation of an excitation[1] with momentum $\hbar\mathbf{k}$ and energy ε_k in the Bose condensate by a heavy object with mass M moving through it.

Due to energy and momentum conservation in the collision we have

$$(1/2)M\mathbf{v}^2 = (1/2)M\mathbf{v}'^2 + \varepsilon_k$$
$$M\mathbf{v} = M\mathbf{v}' + \hbar\mathbf{k} \qquad (4.27).$$

Substituting the second equation in first gives, neglecting the term $\hbar^2\mathbf{k}^2/M^2$,

$$(1/2)M\mathbf{v}^2 \approx (1/2)M\{\mathbf{v}^2 - 2\hbar\mathbf{k}\cdot\mathbf{v}/M\} + \varepsilon_k \qquad (4.28).$$

From (4.28) we get

$$\varepsilon_k = \hbar\mathbf{k}\cdot\mathbf{v} \qquad (4.29).$$

This implies that if $\varepsilon_k > \hbar kv$ eq.(4.29) cannot be satisfied. This leads to a *critical velocity*

$$v_{critical} = (\varepsilon_k/\hbar k)_{min} \qquad (4.30),$$

which is known as Landau´s criterion of superfluidity. That is, if an object is moved in the condensate at a velocity inferior to v_{crit} it will not be energetically favorable to produce excitations and it will move without dissipation, which is a characteristic of a superfluid. That is, if the velocity of M is less than $v_{critical}$ it cannot create excitations in the fluid and hence there will be no viscosity effect on the moving object. From (4.17) and (4.30) is clear that[13] the minimum value for the $v_{critical}$ is at the minimum of the roton curve

$$v_{critical} = (\Delta/\hbar k_o) \approx 60 \text{ m/s} \qquad (4.31).$$

Such roton-limited critical velocities have been observed with ions in He II under pressure.[13] Similarly, the absence of viscosity for He II moving in tubes and channels can also be explained for flows with velocity smaller than a critical velocity. However, the estimated roton-limited critical velocity (4.31) is too large to explain the observed breakdown of superfluid flows in these conditions.[13]

It is important to note that the time independent Eq.(4.15) has a homogeneous solution $\Psi_o(\mathbf{x}) = \sqrt{n_o}$, where n_o if $\mu = g n_o$. Let us consider the case where atoms are trapped in a cubic box with a very large size L assuming periodic boundary conditions. So, let us try to explain the quasiparticle spectrum $\varepsilon_k(\mathbf{k})$ in the condensate using the Bogoliubov-de Gennes approximation.[16] To do this we first write, instead of (4.15), a time dependent equation

$$\{ -(\hbar^2/2m)\Delta + g\,|\varphi(\mathbf{x},t)|^2\}\varphi(\mathbf{x},t) = i h\partial\varphi(\mathbf{x},t)/\partial t \qquad (4.32),$$

where $\varphi(\mathbf{x},t) = \varphi_o(\mathbf{x},t) + \delta\varphi(\mathbf{x},t)$ where $\varphi_o(\mathbf{x},t) = \sqrt{n_o}\,\exp(-i\mu t/\hbar)$ and $\delta\varphi(\mathbf{x},t)$ is a small perturbation. Inserting this $\varphi(\mathbf{x},t)$ and its complex conjugate $\varphi^*(\mathbf{x},t)$ in (4.32) we have, in a first order approximation:

$$\begin{aligned} &-(\hbar^2/2m)\Delta\,\delta\varphi + g(2n_o\,|\varphi_o|^2\,\delta\varphi + \varphi^2\,\delta\varphi^*) = i h\partial(\delta\varphi)/\partial t \\ &-(\hbar^2/2m)\Delta\,\delta\varphi^* + g(2\,n_o|\varphi_o|^2\,\delta\varphi^* + \varphi^2\,\delta\varphi) = i h\partial(\delta\varphi^*)/\partial t \end{aligned} \qquad (4.33)$$

Putting $\delta\varphi = \exp(-i\mu t/\hbar)\{u(\mathbf{x})\exp(i\omega t) - v^*(\mathbf{x})\exp(i\omega t)\}$ into (4.23) results

$$\begin{aligned} &\{-(\hbar^2/2m) + 2n_o\,g - \mu - \hbar\omega)\}\,u - g n_o\,v = 0 \\ &\{-(\hbar^2/2m) + 2n_o g - \mu + \hbar\omega)\}\,v - g n_o u = 0 \end{aligned} \qquad (4.34)$$

Considering in addition that $u = A\exp(i\mathbf{k}\bullet\mathbf{r})$ and $v = B\exp(i\mathbf{k}\bullet\mathbf{r})$ are plane waves with momentum **k** one can see that the solution of the homogeneous system (4.34) leads to the energy spectrum

$$\varepsilon_k = \hbar\omega = \{ (\hbar^2\mathbf{k}^2/2m)\{ \hbar^2\mathbf{k}^2/2m \pm 2|g|n_o]\}^{1/2} \qquad (4.35),$$

where the signal + is for repulsive interaction (g > 0) and the signal – for attractive interaction (g > 0).

The dispersion relation (4.35), for small **k** predicts the phonon,

$$\varepsilon_k = c\hbar k,$$

where $c = \sqrt{n_o}|g|/m$ is the speed of sound in the condensate and for large **k** it gives the energy of free particles

$$\varepsilon_k = \hbar^2\mathbf{k}^2/2m.$$

One can easily verify that (4.35) does not predict the rotons for intermediate values of **k**. However, we see that for g > 0 the minimum value of $\varepsilon_k/\hbar k$ obeys the condition $(\varepsilon_k/\hbar k)_{min} > c$ showing, according to Landau´s criterion, that condensate is a superfluid. Note that for g < 0 we verify that there appear inconsistencies in the predicted energy spectrum $\varepsilon_k(k)$ given by (4.35) for very small k values.

4.c)Solitons.

In a recent paper[17] we have shown how to obtain solitons for general non-linear quantum mechanical equations similar to the Gross-Pitaevskii equation. To describe the solitons in the BE condensate it necessary to adopt another approach that can be seen, for instance, in reference 16.We will show here only a simple description of the BE condensate solitons.[1] So, according to (4.15) for the condensate state in a stationary state we have :

$$\{(\hbar^2/2m)\Delta + \mu\}\Psi_o(\mathbf{x}) - g\,|\Psi_o(\mathbf{x})|^2\,\Psi_o(\mathbf{x}) = 0 \qquad (4.36).$$

Putting $\Psi_o(\mathbf{x}) = F(\mathbf{x})\exp[i\varphi(\mathbf{x})]$ in (4.36), the real and imaginary parts can be written as

$$\begin{aligned} &\mathrm{div}\{F^2\,(\hbar/m)\mathrm{grad}(\varphi)\} = \mathrm{div}(\mathbf{j}_o(\mathbf{x})) = 0 \\ &\mu/m = F^2 g/m - (\hbar^2/2m^2F)\,\Delta F + (\hbar\,\mathrm{grad}(\varphi\,)/m\sqrt{2})^2 \qquad (4.37) \\ &\qquad = F^2 g/m - (\hbar^2/2m^2F)\,\Delta F + \mathbf{v}_o^{\,2}/2 \end{aligned}$$

where $\mathbf{v}_o(\mathbf{x})=(\hbar/m)\,\mathrm{grad}[\varphi(\mathbf{x})]$ is the condensate velocity. The first equation of (4.37) is recognized as the continuity equation for the condensate and the second one as a quantum analog of Bernouilli´s equation for steady flow.[2]

Let us consider a condensate confined to a semi-infinite domain (x > 0) and that $\varphi(\mathbf{x})$ = constant = 0 (“static approximation”). In one-dimensional geometry $\Psi_o(\mathbf{x})$ will be written as

$$\Psi_o(\mathbf{x}) = \sqrt{n_o}\,F(x) \qquad (4.38).$$

In this way the last equation of (4.37) becomes

$$\xi^2(d^2F/dx^2) \pm (\mu/|g|n_o)F - F^3 = 0 \qquad (4.39),$$

where the characteristic length $\xi = (\hbar^2/2mn_o g)^{1/2}$ and the signal + is for repulsive interaction (g > 0) and – when the interaction is attractive (g < 0).

A) *Repulsive interaction (g>0). Dark Soliton.*

Taking the boundary conditions of (4.39) as $\Psi_o(\mathbf{x}) = F(x) = 0$ at x = 0 and $F \rightarrow 1$ as $x \rightarrow \infty$. The last condition implies that $\mu = |g|n_o$. Consequently, (4.39) becomes

$$\xi^2(d^2F_d/dx^2) + F_d - F_b^3 = 0 \qquad (4.40).$$

We verify that a first integral of (4.40) is given by[1]

$$\xi^2(dF_d/dx)^2 = (1-F_d^2)^2/2 \qquad (4.41),]$$

that is easily integrated to yield[1], putting $k = 1/(\xi\sqrt{2})$:

$$F_d(x) = \tanh(kx) \qquad (4.42)$$

Consequently, for *dark solitons* we obtain

$$\Psi_o^{(d)}(\mathbf{x}) = \sqrt{n_o}\,\tanh(kx) \qquad (4.43).$$

A rough description of a freely propagating *dark soliton* along the x-axes is given by the wave function:[16]

$$\Psi_o^{(d)}(\mathbf{x,t}) = A\sqrt{n_o}\,\tanh[k(x-x_o-vt)]\,\exp[i\gamma(x,t)] \qquad (4.44),$$

where A is the amplitude and v is the velocity of propagation of the soliton.

B) *Attractive interaction (g < 0). Bright Soliton.*

Similarly, the wavefunction[18] of a freely propagating *bright soliton* along the x-axes (that resembles a classical particle) can be written as

$$\Psi_o^{(b)}(\mathbf{x,t}) = A\sqrt{n_o}(\beta/2)^{1/2}\mathrm{sech}[\beta(x-x_o-vt)]\,\exp[i\theta(x,t)/\hbar] \qquad (4.45),$$

where $\beta = \sqrt{(2m|\mu|/\hbar^2)}$, $\theta(x,t) = mvx - Et$ and $E = mv^2/2 + \mu$.

As pointed out by U.Al Khawaja et al.[19] properties of dark solitons have been extensively studied theoretically. They have also been created experimentally in elongated Bose-condensates. Much less is known about bright solitons, which have only recently been created in with Bose-condensates of ^7L_i atoms.

5) Specific Heat at the λ-Point.

Many years ago, between 1940 and 1955 various attempts have been made[4] to modify the energy spectrum of the ideal Bose-Einstein gas to fit, for instance, the experimental data of the specific heat of the liquid helium. The conclusion drawn from these attempts is that something drastic as an *energy gap* is required.[4] Note that this gap is not necessarily the (4.16) roton gap. In a precedent paper[3] we have calculate the specific heat of the liquid helium taking into account also an energy gap (that will be later interpreted) and assuming, in addition, that the energy spectrum depends on the temperature of system and that the superfluid liquid → liquid phase transition is an order-disorder transition. Let us reproduce here, slightly modified, our earlier calculations.[3]

For low temperatures ($T < 0.6$ K) all thermal energy is associated with longitudinal phonons excitations. In these conditions the de Broglie wavelength ℓ of the excitations is bigger than the mean intermolecular separation a. As the temperatures rises, local atomic motions become relatively more significant than the collective excitations so that $\ell \leq a$. In this way, we assumed that these new energy levels E_n ($n = 0,1,2,\ldots$) of the atoms are $E_o = 0$ and $E_n = \Delta + \varepsilon_n$ for $n =1,2,\ldots$with $\varepsilon_1 = 0$. In our approach the energy gap Δ, which is a constant adjustable parameter, is the minimum energy that a particle can assume in local motions (for these energy values $\ell \leq a$). As will be seen in what follows we have found $\Delta/k_BT_\lambda = 2.6$, where $T_\lambda = 2.19$ K is the λ-point temperature.

Due to the weak interactions between the helium atoms we must expect that the energy spectrum ε_n is quite similar to the free particle spectrum. Without trying to incorporate into a consistent scheme both phonon and the individual excitations that are practically individual atomic motion, we take an additive superposition of these two contributions. The phonon energy contribution can be seen, for instance, in London´s book.[4]

Let us calculate the contribution of the "local" atomic motions. If N is total the number of helium atoms we have , using the Bose-Einstein statistics and assuming that the energy ε_n spectrum is quasi-continuum, that is, $\varepsilon_{n+1} - \varepsilon_n << k_BT$ we have, according to Section 3:

$$N = n_o + N_{exc} = n_o + k_BT \int_o^\infty \rho(\varepsilon,T)\, d(\varepsilon/k_BT)/[\exp(\epsilon/k_BT + \alpha') -1] \qquad (5.1),$$

where $n_o =1/[\exp(\alpha) -1]$ is the number of excitations in ground state, $\rho(\varepsilon,T)$ is the density of states in the energy interval $d\varepsilon$ and $\alpha' = \alpha + \Delta/k_BT$. In our preceding paper[3] we put $\rho(\varepsilon,T) = 1/\psi(\varepsilon,T)$.

As the temperature increases the energy spectrum tends to the spectrum of free particles so that $\rho(\varepsilon,T) \to g(\epsilon) = (2\pi V/h^3)(2m)^{3/2}\epsilon^{1/2}$, according to (3.4). On the other side, if the particles, instead of free, were vibrating harmonically with a fundamental frequency ν, with energy $\varepsilon_n =$

$(n+1/2)h\nu$ around an equilibrium center we would have[8] $\rho(\varepsilon,T) \to (h\nu)^{-1}$, that is, $\rho = 1/\psi$ would be independent of ε. So, it seems reasonable to expect that $\rho = 1/\psi \sim \varepsilon^{\delta}$ where δ is closer to ½ than to 0. As will be seen in what follows, our predictions for $T < T_\lambda$ are practically independent of the δ value that are in the interval $0 \leq \delta \leq ½$. It is significant only for $T \geq T_\lambda$. So, for $T < T_\lambda$, to simplify the calculations we put $\delta = ½$ writing $\rho(\varepsilon,T)$ as

$$\rho(\varepsilon,T) = 2g(\varepsilon)/\psi(T)\surd\pi \qquad (5.2).$$

In these conditions the number of excited particles N_{exc}, using (5.1) and (5.2), is given by[4]

$$N_{exc}= (2/\surd\pi)\ [(k_BT)^{3/2}/\psi(T)\ ] \int_o^\infty dx\ x^{1/2}/[\exp(\alpha'+ x) -1]$$

$$= [(k_BT)^{3/2}/\psi(T)]\ F_{3/2}(\alpha + \Delta/k_BT) \qquad (5.3),$$

noting that for $T \geq T_\lambda$ the condition $N_{exc} = N$ must be satisfied.

The total energy U, using (3.11), is now given by

$$U = [2\ /\psi(T)\surd\pi] \int_o^\infty d\varepsilon\ (\varepsilon+\Delta)\varepsilon^{1/2}/[\exp(\alpha'+ \Delta/k_BT) -1]$$

$$= N_{exc}\ \{(3k_BT/2)\ F_{5/2}(\alpha + \Delta/k_BT)/F_{3/2}(\alpha + \Delta/k_BT) + \Delta\ \} \qquad (5.4).$$

Now our problem is to determine the function $\psi(T)$. At low temperatures Keesom and Taconis[20] making an x-ray analysis of liquid helium deduced that the helium atoms seems to form, during short time intervals, locally ordered structures. Note that the diffuseness of the x-ray pattern excludes a crystal structure which is not expected to exist in a fluid anyhow.[4] As the temperature increases the existence of these locally ordered structures tend to disappear. Inspired by Fröhlich[21] we will assume that λ-point shape of the specific heat curve is due an order-disorder transition *superfluid liquid → liquid*. According to the order-disorder phase transition formalism[22] the order parameter X obeys the equation $X = \tanh[(T_c/T)X]$, where $T_c = T_\lambda$ is the critical temperature. It may seem unrealistic to treat a liquid using a lattice model. This objection is quite valid in general but many properties of liquids are calculated approximately using the lattice model.[23] We expect that the energy spectrum ε_n is the free particle spectrum when the system is completely disordered, that is, when $X = 0$ at the temperature $T = T_\lambda$. So, according to (5.2), $\psi(T)$ must decrease when X decreases, that is, when $X \to 0$. In this way, let us assume that

$$\psi(T) = \eta\ (1 + \xi X^{\theta}) \qquad (5.5),$$

where η is determined using the condition $N_{exc}(T_\lambda) = N$ in (5.3) and θ and ξ are adjustable parameters.

5.a) Specific Heat $C_v^{(-)}$ for $T < T_\lambda$.

Taking into account that for these temperatures, according to (3.9), $\alpha = 0$ and that the experimental C_v values will require that $\Delta/k_BT \sim 3$ the functions $F_\sigma(\alpha + \Delta/k_BT)$ can be written as[4] $F_\sigma(\alpha + \Delta/k_BT) = F_\sigma(\Delta/k_BT_\lambda) \approx \exp(-\Delta/k_BT)$. With these approximations, using (5.3)-(5.5), we calculate the specific heat per unit of mass $C_v^{(-)} = (\partial U/\partial T)$:

$$C_v^{(-)} = (k_B/m)\,(T/T_\lambda)^{3/2}(1 + \xi X^\theta)^{-1}\,[15/4+3\chi(T_\lambda/T)+(\chi T_\lambda/T)^2]\,\exp[\chi(1-T_\lambda/T)]$$
$$+ (k_B/m)\,(T/T_\lambda)^{3/2}\,\theta\xi X^\theta\,(1 + \xi X^\theta)^{-2}\,[3/2+\chi(T_\lambda/T)]\,\exp[\chi(1-T_\lambda/T)]$$
$$/[\cosh^2(XT_\lambda/T) - T_\lambda/T] \qquad (5.1),$$

where $\chi = (\Delta/k_BT_\lambda)$.

5.b) Specific Heat $C_v^{(+)}$ for $T > T_\lambda$.

For $T > T_\lambda$ the specific heat per unit of mass $C_v^{(+)}$ is given by

$$C_v^{(+)} = (3/2)k_B/m \qquad (5.2),$$

which is the specific heat of an ideal gas.

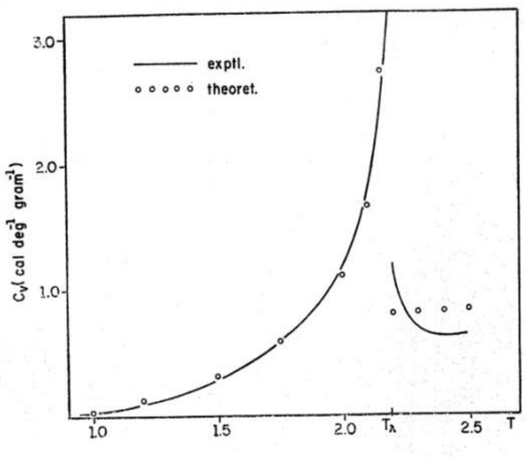

Figure 7. Experimental results for $C_v^{(-)}$ and $C_v^{(+)}$ of Keesom and Clusius[24] and Keesom and Keesom[25] compared with our theoretical predictions obtained using (5.1) and (5.2).

In Fig.7 are shown the experimental results for $C_v^{(-)}$ and $C_v^{(+)}$ of Keesom and Clusius[24] and Keesom and Keesom[25] (see Fig.3) compared with our theoretical predictions obtained using (5.1) and (5.2). We have also taken into account the phonons contributions to the specific heat using Eqs.(5) seen in pag.94 of ref.4 which are negligible compared with those given by (5.1). The best agreement with the experimental results was found putting $\theta = 0.22$, $\chi = \Delta/k_B T_\lambda = 2.60$ and $\xi = 8.00$. At the λ-point our predictions for $C_v^{(-)}$ diverges as $(T_\lambda - T)^{-0.89}$ and experimentally it diverges as $\log(T_\lambda - T)$.

Taking into account that the adjusted parameter $\chi = \Delta/k_B T_\lambda = 2.60$ is a reasonable value compared with "roton" value $\Delta/k_B T_\lambda = 8.6/T_\lambda \sim 4$ and that there is a good agreement between theory and experiment for $T \leq T_\lambda$ we see that our order-disorder model is able to give a fair description of the transition *superfluid liquid → liquid*. Thus, according to the Italian poet:

"Se non è vero, è bene trovato".

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