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On Equilibrium Distribution of a Reversible Growth Model

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Abstract We study a probabilistic model of interacting spins indexed by elements of a finite subset of the d-dimensional integer lattice, $d \geq 1$. Conditions of time reversibility are examined. It is shown that the model equilibrium distribution converges to a limit distribution as the indexing set expands to the whole lattice. The occupied site percolation problem is solved for the limit distribution. Two models with similar dynamics are also discussed.

Keywords Markov chain · Gibbs measure · Reversibility · Percolation

1 Introduction

We study a probabilistic model of interacting spins taking values in $\{0, 1, \dots, N\}$, where $N \geq 1$, and indexed by elements of a finite subset Λ of the d-dimensional integer lattice, $d \geq 1$. Our interest in the model has been originally motivated by modelling the dynamics of fracture. A spin can be interpreted as the number of microscopic cracks at a particular location. We assume that a crack is generated at a rate that depends *on the number* of existing cracks in a neighbourhood and disappears ("healing effect") at a constant rate. We show that under this assumption the corresponding Markov chain is time reversible if and only if its transition rates are uniquely parameterised by two parameters in a certain way. One of these

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parameters is responsible for the interaction between spins. Depending on the value of this parameter the interaction between spins can be either attractive, or repulsive or the spins evolve independently.

Loosely speaking a fracture is a (local) separation of a material into pieces and can be thought of as a result of propagation of microscopic cracks. The material breaks down when a macroscopic fracture is generated. This can be naturally formalised as a percolation problem for the model equilibrium measure.

It is plausible that existing cracks speed up the generation of new cracks around. Therefore the model with attractive interaction is of main interest. We show that the equilibrium measure of the model in the attractive case converges to a limit distribution as the set Λ expands to the whole lattice. We also solve the occupied site percolation problem for the limit distribution. These results are announced in [11].

It turns out that the equilibrium distribution in the case $N = 1$ is equivalent to a particular case of the ferromagnetic Ising model. Therefore the existence of the limit measure and its percolation properties in this case are implied by the well known results for the Ising model (see Remark 1 in Sect. 2). To the best of our knowledge the model in the case $N \geq 2$ has never been studied before though it can be related to Gibbs models in [10]. The equilibrium distribution of the model with attractive interaction possesses certain monotonicity properties which are similar to the well known monotonicity properties of the ferromagnetic Ising model. These monotonicity properties greatly facilitate proofs of both the convergence and the percolation results. In particular, these properties allow us to follow the known scheme of proving similar results for the ferromagnetic Ising model. This scheme is described in detail in survey [3]. We refer to this survey throughout, where further references to the original sources can be found.

We also consider two other models with similar Markovian dynamics. The first one is described by a denumerable Markov chain obtained by allowing the spins to take any non-negative integer values. The second one is a spatial birth-and-death process taking values in a set of finite point configurations of the Euclidean space $\mathbb{R}^d$. It turns out that both the denumerable Markov chain and the spatial birth-and-death process can be time reversible if and only if their transition rates are parameterised by two parameters in a similar manner as for the original Markov chain. Both the denumerable Markov chain and the spatial birth-and-death process are explosive in the case of attractive interaction and are ergodic in the case of repulsive interaction. It is interesting to notice that the equilibrium distribution of the ergodic spatial birth-and-death process is the Strauss point process [14]. The latter is one of the most well known point processes in spatial statistics. Moreover, the reversibility criterion (in both lattice and continuous cases) can be regarded as a reformulation of the well known characterisation result for the Strauss point process.

The paper is organised as follows. We formulate both the model with bounded spins and the main results in Sect. 2. The monotonicity properties are formulated in Sect. 3. The model with unbounded spins and the continuous space version of the original model are discussed in Sect. 4. All proofs are given in Sect. 5.

2 The Model and Results

Let $\mathbb{Z}$ be a set of all integers. For any finite set $\Lambda \subset \mathbb{Z}^d$ define $\Omega_{N,\Lambda} = \{0, \ldots, N\}^\Lambda$, where $N \geq 1$ is an integer. We denote by ξ_x, $x \in \Lambda$, components of a configuration $\xi \in \Omega_{N,\Lambda}$. A component ξ_x of $\xi \in \Omega_{N,\Lambda}$ is called a spin and can be interpreted as the number of particles located at x. Denote by $\mathbf{1}_B$ an indicator of any event B. For every $x \in \Lambda$ let $e^{(x)}$ be a configuration whose components are defined as follows

$$e^{(x)}_y = \mathbf{1}_{\{y=x\}}, \quad y \in \Lambda. \tag{1}$$

For $x \in \mathbb{Z}^d$ denote by $\|x\|$ its Euclidean norm in $\mathbb{R}^d$. We write $x \sim y$ for $x, y \in \mathbb{Z}^d$ if $\|x - y\| = 1$. Given $x \in \Lambda$ and $\xi \in \Omega_{N,\Lambda}$ define

$$n(x, \xi) = \xi_x + \sum_{y \in \Lambda : y \sim x} \xi_y. \tag{2}$$

Consider a continuous time Markov chain $\xi(t) = \{\xi_x(t),\ x \in \Lambda\} \in \Omega_{N,\Lambda}$ such that given a state $\xi(t) = \xi$

- the spin $\xi_x < N$ increases by 1 (a particle is created at x) at the rate $c_{n(x,\xi)}$, where c_k, $k = 0, 1, \ldots, (2d + 1)N - 1$, is a finite set of positive numbers,
- and the non-empty spin $\xi_x > 0$ decreases by 1 (a single particle dies at x) at a rate of 1.

This Markov chain describes a model of interacting spins which can be related to some known interacting particle systems and growth models. In particular, if $N = 1$, $c_k = \lambda k$, $k = 0, 1, \ldots, 2d$, where $\lambda > 0$, then the Markov chain resembles the contact process [6]. Further, if $N = 1$, the death rate were set to be zero and the birth rates c_k were chosen appropriately, then one would get finite volume versions of the following growth models with local interaction:

1. $c_k \equiv 1$, $k = 0, 1, \ldots, 2d$: Eden model [1];
2. $c_k = k$, $k = 0, 1, \ldots, 2d$: Richardson model [9];
3. arbitrary c_k, $k = 0, 1, \ldots, 2d$: contact interaction processes [13], monomer filling with cooperative effects [2].

Note also that all these models are special cases of the model with nearest-neighbour interaction introduced in [4].

It is easy to see that the Markov chain defined above is irreducible, and, hence, is ergodic. The following characterisation result takes place.

Lemma 1 (Reversibility conditions) *The Markov chain $\xi(t)$ is time reversible if and only if*

$$c_k = \alpha \gamma^k, \quad k = 0, 1, \ldots, (2d + 1)N - 1, \tag{3}$$

where $\gamma > 0$ and $\alpha > 0$. The corresponding invariant probability distribution in the time reversible case is a probability measure defined as follows

$$\mu_{\alpha,\gamma,N,\Lambda}(\xi) = \frac{\alpha^{h(\xi)} \gamma^{s(\xi)}}{\sum_{\zeta \in \Omega_{N,\Lambda}} \alpha^{h(\zeta)} \gamma^{s(\zeta)}}, \quad \xi \in \Omega_{N,\Lambda}, \tag{4}$$

where

$$s(\xi) = \sum_{x \in \Lambda} \xi_x(\xi_x - 1)/2 + \sum_{x,y \in \Lambda : x \sim y} \xi_x \xi_y, \tag{5}$$

$$h(\xi) = \sum_{x \in \Lambda} \xi_x. \tag{6}$$

Consider the probability measure $\mu_{\alpha,\gamma,N,\Lambda}$ on $\Omega_{N,\Lambda}$ defined by Eq. (4). For simplicity of notation and without loss of generality we assume in the sequel that $\alpha = 1$ and denote

$\mu_{\gamma,N,\Lambda} = \mu_{1,\gamma,N,\Lambda}$ throughout. Also, without loss of generality we assume in the sequel that Λ is a set of all integer points of $[-L, L]^d$, where $L \geq 1$ is an integer.

Let $\Omega_N = \{0, 1, \ldots, N\}^{\mathbb{Z}^d}$ be a set of infinite configurations equipped with the standard σ-algebra $\mathscr{F}_N$ generated by cylinder sets.

Theorem 1 *For any $\gamma \geq 1$ there exists a limit measure*

$$\mu_{\gamma,N} = \lim_{\Lambda \uparrow \mathbb{Z}^d} \mu_{\gamma,N,\Lambda},$$

on $(\Omega_N, \mathscr{F}_N)$, where convergence is understood in a sense of the weak convergence of finite-dimensional distributions.

To formulate the percolation result we need some definitions.

Definition 1 Given a configuration $\xi \in \Omega_N$

1. a site $x \in \mathbb{Z}^d$ is called occupied, if $\xi_x > 0$, and empty otherwise;
2. a set of occupied sites $U = \{x, y, \ldots\}$ is called an occupied cluster, if for any $x', x'' \in U$, there exists a finite subset $\{y_1, \ldots, y_n\} \subseteq U$, such that $x' = y_1$, $y_n = x''$ and $\|y_i - y_{i+1}\| = 1$, $i = 1, \ldots, n-1$.

Given $N \geq 1$ define an event

$$A_N = \{\text{there exists an infinite occupied cluster}\} \in \mathscr{F}_N. \tag{7}$$

Theorem 2 (Percolation properties)

(1) *If $d \geq 2$ and $N \geq 3$, then $\mu_{\gamma,N}(A_N) = 1$ for any $\gamma \geq 1$.*
(2) *If $d \geq 2$ and $N = 1$ or 2, then there exists a critical value $\gamma_c \geq 1$ such that*

$$\mu_{\gamma,N}(A_N) = \begin{cases} 0, & \text{if } \gamma < \gamma_c, \\ 1, & \text{if } \gamma > \gamma_c. \end{cases}$$

Remark 1 It should be noted that both statements of Theorem 1 and Theorem 2 in the case $N = 1$ are implied by the well known results for the Ising model. Indeed, the following linear transformation of spins

$$\sigma_x = 2\xi_x - 1 \in \{-1, 1\}$$

maps measure $\mu_{\gamma,1,\Lambda}$ to a measure on $\{-1, 1\}^\Lambda$. If $\gamma > 1$, then this induced probability measure corresponds to a finite volume measure (with empty boundary conditions) of the ferromagnetic Ising model with the inverse temperature $\beta = \log(\gamma)/4$ and the external field $r = 4d$. It is known (Proposition 4.14, [3]) that for the ferromagnetic Ising model with the inverse temperature $\beta > 0$ and the external field r there exists a unique Gibbs measure for any β provided that $r \neq 0$. This implies existence of a measure $\mu_{\gamma,1}$ for any $\gamma > 1$. Furthermore, for any $\beta > 0$ there exists r_c such that a unique infinite cluster formed by $+1$ spins exists almost surely provided that $r > r_c$ (Theorem 5.10, [3]). Moreover, if β is sufficiently large, then $r_c = 0$ (Theorem 8.2, [3]). This implies the percolation result for our model in the case $N = 1$.

3 Monotonicity Properties

Following Sect. 4.2 in [3] we recall the important notion of stochastic domination and monotonicity for probability measures adapted for the case of measures on $\Omega_{N,\Lambda}$ and Ω_N.

Note that both $\Omega_{N,\Lambda}$ and Ω_N are equipped with a natural partial order defined as follows. We write $\xi \leq \xi'$ for any $\xi, \xi' \in \Omega_{N,\Lambda}$ (Ω_N), if $\xi_x \leq \xi'_x$ for every $x \in \Lambda$. Also denote by $\mathscr{F}_{N,\Lambda}$ a collection of all subsets of $\Omega_{N,\Lambda}$.

Definition 2 An event $A \in \mathscr{F}_{N,\Lambda}$ ($\mathscr{F}_N$) is said to be increasing if

$$\mathbf{1}_{\{\xi \in A\}} \leq \mathbf{1}_{\{\xi' \in A\}},$$

whenever $\xi, \xi' \in \Omega_{N,\Lambda}$ (Ω_N) are such that $\xi \leq \xi'$.

Definition 3 Let μ and μ' be two probability measures on $\Omega_{N,\Lambda}$ (Ω_N). We say that μ is stochastically dominated by μ', writing $\mu \leq \mu'$, if for every increasing event $A \in \mathscr{F}_{N,\Lambda}$ ($\mathscr{F}_N$) we have that $\mu(A) \leq \mu'(A)$.

Definition 4 A probability measure μ on $\Omega_{N,\Lambda}$ is called *monotone* if

$$\mu(\xi_x \geq k | \xi = \zeta \text{ off } x) \leq \mu(\xi_x \geq k | \xi = \eta \text{ off } x)$$

for any $x \in \Lambda$, and $k \in \{0, 1, \ldots, N\}$ and where $\zeta, \eta \in \{0, 1, \ldots, N\}^{\Lambda \setminus \{x\}}$ are such that $\zeta \leq \eta$, $\mu(\xi = \zeta \text{ off } x) > 0$ and $\mu(\xi = \eta \text{ off } x) > 0$.

Our main technical result is the following lemma.

Lemma 2 (Monotonicity properties)

(1) *For any $\gamma \geq 1$ the probability measure $\mu_{\gamma,N,\Lambda}$ is monotone.*
(2) *For any γ_1 and γ_2 such that $0 < \gamma_1 \leq \gamma_2$ the limit measure $\mu_{\gamma_1,N}$ is stochastically dominated by the limit measure $\mu_{\gamma_2,N}$.*

Define a probability measure $\mu^{(0)}_{\gamma,N}$ on Ω_N by the following set of finite dimensional distributions

$$\mu^{(0)}_{\gamma,N}(\xi_{x_1} = k_1, \ldots, \xi_{x_n} = k_n) = \prod_{i=1}^{n} \frac{\gamma^{k_i(k_i-1)/2}}{\sum_{k=0}^{N} \gamma^{k(k-1)/2}}, \tag{8}$$

where $k_i \in \{0, 1, \ldots, N\}$, $x_i \in \mathbb{Z}^d$, $i = 1, \ldots, n$, and $n \geq 1$.

Corollary 1 *For any $\gamma > 1$ the measure $\mu^{(0)}_{\gamma,N}$ is stochastically dominated by $\mu_{\gamma,N}$.*

Both the notion of an irreducible measure (Sect. 4.2, [3]) and the Holley theorem (Theorem 4.8, [3]) will be used more than once in the proofs. For convenience we formulate them here in terms of probability measures on $\Omega_{N,\Lambda}$.

Definition 5 A probability measure μ on $\Omega_{N,\Lambda}$ is called irreducible if the set of configurations $\{\eta \in \Omega_{N,\Lambda} : \mu(\eta) > 0\}$ is connected in the sense that any element of $\Omega_{N,\Lambda}$ with positive μ-probability can be reached from any other element via successive coordinate changes without passing through elements with zero μ-probability.

Theorem 3 (Holley) *Let μ and μ' be probability measures on $\Omega_{N,\Lambda}$. Assume that μ' is irreducible and assigns positive probability to configuration $\xi \equiv N$ (the maximal element of $\Omega_{N,\Lambda}$ with respect to the partial order). Suppose further that*

$$\mu(\xi_x \geq k | \xi = \zeta \text{ off } x) \leq \mu'(\xi_x \geq k | \xi = \eta \text{ off } x) \tag{9}$$

whenever $x \in \Lambda$, $k \in \{1, \dots, N\}$, and $\zeta, \eta \in \Omega_{N,\Lambda \setminus \{x\}}$ are such that $\zeta \leq \eta$ component-wise, $\mu(\xi = \zeta \text{ off } x) > 0$ and $\mu'(\xi = \eta \text{ off } x) > 0$. Then μ is stochastically dominated by μ'.

4 The Models with Similar Dynamics

We are going to consider in this section two other Markov processes whose dynamics are similar to the original one.

We begin with a countable Markov chain obtained by setting formally $N = \infty$. Let $\mathbb{Z}_+$ be a set of all non-negative integers. For any finite $\Lambda \subset \mathbb{Z}^d$ consider a continuous time Markov chain $\xi(t) \in \mathbb{Z}_+^\Lambda$ whose generator G is defined as follows

$$\begin{aligned} Gf(\xi) = \sum_{x \in \Lambda} \big[f\big(\xi + e^{(x)}\big) - f(\xi) \big] c\big(n(x, \xi)\big) \\ + \sum_{x \in \Lambda} \big[f\big(\xi - e^{(x)}\big) - f(\xi) \big] \mathbf{1}_{\{\xi_x > 0\}}, \end{aligned} \tag{10}$$

where $c : \mathbb{Z}_+ \to (0, \infty)$ is a positive function and the quantity $n(x, \xi)$ is defined as before by Eq. (2). Similar to Lemma 1 it can be shown that the Markov chain specified by the generator (10) is time reversible if and only if $c(k) = \alpha \gamma^k$, $k = 0, 1, \dots$, where $\gamma > 0$ and $\alpha > 0$. The following classification takes place.

Theorem 4 *Suppose that $c(k) = \alpha \gamma^k$, $k = 0, 1, \dots$, where $\gamma > 0$ and $\alpha > 0$.*

(1) *If $0 < \gamma < 1$, then $\xi(t)$ is ergodic with the following stationary probability measure*

$$\nu_{\alpha,\gamma,\Lambda}(\xi) = \frac{\alpha^{h(\xi)} \gamma^{s(\xi)}}{\sum_{\zeta \in \mathbb{Z}_+^\Lambda} \alpha^{h(\zeta)} \gamma^{s(\zeta)}}, \quad \xi \in \mathbb{Z}_+^\Lambda, \tag{11}$$

where $s(\xi)$ and $h(\xi)$ are defined by Eqs. (5) and (6) respectively.

(2) *If $\gamma > 1$, then the Markov chain is explosive.*

(3) *If $\gamma = 1$, then the Markov chain is transient.*

Remark 2 It should be noted that if the death rate were set to zero, then the corresponding embedded Markov chain would be a d-dimensional version of the model for particle deposition in [12].

Let now $\Lambda \subset \mathbb{R}^d$ be a bounded Borel set with positive Lebesgue measure. Consider a continuous time, pure jump Markov process $X(t)$ whose states are finite point configurations $\{x_i \in \Lambda, i = 1, \dots, m\}$, $m \geq 1$, including the empty one denoted by $\emptyset$, and whose transition rates are specified as follows. Fix a positive number R and let $c(\cdot)$ be a positive function on $\mathbb{Z}_+$ as before. For any non-empty point configuration $X = \{x_i \in \Lambda, i = 1, \dots, m\}$ define

$$\tilde{n}(y, X) = \sum_{i=1}^{m} \mathbf{1}_{\{\|y - x_i\| \leq R\}}$$

and set $\tilde{n}(y,\emptyset)=0$. Given $X(t)=X$ a new point is added at location $y\in\Lambda$ with rate $c(\tilde{n}(y,X))$. If $X\neq\emptyset$, then an existing point $y\in X$ disappears at a constant unit rate. Again, similar to Lemma 1 it can be shown that the spatial birth-and-death process can be time reversible if and only if $c(k)=\alpha\gamma^k$, $k\geq 0$, where $\gamma>0$ and $\alpha>0$. Theorem 7.1 in [8] yields that the spatial birth-and-death process is explosive, if $\gamma>1$, and is well-defined and ergodic, if $0<\gamma\leq 1$. If $0<\gamma\leq 1$, then the stationary distribution is a probability measure specified by the following density with respect to the unit rate Poisson point process in Λ

$$f_\Lambda(X)=Z_\Lambda^{-1}\alpha^{|X|}\gamma^{\tilde{s}(X)},\tag{12}$$

where $|X|$ is the number of points in X, $\tilde{s}(X)$ is the number of R-closed pairs of points in X and Z_Λ is the normalising constant (making the function integrable to unity). This probability measure is known as the Strauss point process [14] which is one of the most known finite point processes in spatial statistics.

Theorem 1 in [5] yields that if g is a positive probability density with respect to the unit rate Poisson point process in Λ such that for any finite point configuration X and any $x\in\Lambda$ the following equation holds

$$g\big(X\cup\{x\}\big)=g(X)c\big(\tilde{n}(x,X)\big),\tag{13}$$

where $c:\mathbb{Z}_+\to(0,\infty)$ is a positive function, then necessarily $c(k)=\alpha\gamma^k$, for some $\alpha>0$ and $0<\gamma\leq 1$. Thus the probability density (12) is a unique positive density satisfying (13). It is easy to see that this statement is equivalent to saying that the spatial birth-and-death process described above is a unique time reversible spatial birth-and-death process specified by the unit death rate and by the birth rates $c(\tilde{n}(x,X))$, $x\in\Lambda$ (given a current state X), where $c(\cdot)$ is a positive function.

5 Proofs

5.1 Proof of Lemma 1

Let $x,y\in\Lambda$ be such that $\|x-y\|=1$. Consider $\xi\in\Omega_{N,\Lambda}$ satisfying the following conditions

$$\xi_x<N \quad\text{and}\quad n(x,\xi)=i<(2d+1)N-1,\tag{14}$$

$$\xi_y<N \quad\text{and}\quad n(y,\xi)=k<(2d+1)N-1.\tag{15}$$

Consider two *cycles* of successive states

$$\xi\to\xi+e^{(x)}\to\xi+e^{(x)}+e^{(y)}\to\xi+e^{(y)}\to\xi$$

and

$$\xi\to\xi+e^{(y)}\to\xi+e^{(y)}+e^{(x)}\to\xi+e^{(x)}\to\xi,$$

where $e^{(x)}$ and $e^{(y)}$ are configurations defined by Eq. (1) and where the addition of configurations is understood component-wise. By Kolmogorov's reversibility criterion the products of the corresponding successive transition rates must be equal, i.e.,

$$c_{n(x,\xi)}c_{n(y,\xi+e^{(x)})}=c_{n(y,\xi)}c_{n(x,\xi+e^{(y)})},\tag{16}$$

since the death rates are constants. Given conditions (14) and (15) Eq. (16) yields the following identity

$$\frac{c_{k+1}}{c_k}=\frac{c_{i+1}}{c_i}.$$

The preceding display implies formula (3) with $\gamma = c_1/c_0$ and $\alpha = c_0$, since $i, k \in \{0, 1, \dots, (2d+1)N - 1\}$ were arbitrary.

It is easy to see that for any $\xi \in \Omega_{N,\Lambda}$ such that $\xi_x < N$ we get that

$$\mu_{\alpha,\gamma,N,\Lambda}\big(\xi + e^{(x)}\big) = \mu_{\alpha,\gamma,N,\Lambda}(\xi)\alpha\gamma^{n(x,\xi)}, \tag{17}$$

because of the following identity

$$s\big(\xi + e^{(x)}\big) = s(\xi) + n(x, \xi). \tag{18}$$

Equation (17) is the detailed balance condition written for rates (3) and measure (4). This means time reversibility of the Markov chain with this invariant measure. The lemma is proved.

5.2 Proof of Lemma 2

We start with the following statement.

Proposition 1 *Let $(a_0, a_1, \dots, a_N)$ and $(b_0, b_1, \dots, b_N)$ be two finite sequences of positive numbers. Let P and Q be probability measures on the finite set $\{0, 1, \dots, N\}$ defined as follows*

$$\mathsf{P}\big(\{k\}\big) = \frac{a_k}{\sum_{i=0}^N a_i} \quad \textit{and} \quad \mathsf{Q}\big(\{k\}\big) = \frac{b_k}{\sum_{i=0}^N b_i}, \quad k = 0, 1, \dots, N.$$

If $a_i b_j \le a_j b_i$, $0 \le j < i \le N$, then P is stochastically dominated by Q.

Proof of Proposition 1 To prove this proposition we need to show that the following set of inequalities holds

$$\frac{\sum_{i=k}^N a_i}{\sum_{i=0}^N a_i} \le \frac{\sum_{i=k}^N b_i}{\sum_{i=0}^N b_i}, \quad k = 1, \dots, N. \tag{19}$$

It is easy to see that

$$\begin{aligned}
&(a_k + \cdots + a_N)(b_0 + b_1 + \cdots + b_N) - (b_k + \cdots + b_N)(a_0 + a_1 + \cdots + a_N) \\
&\quad = (a_k + \cdots + a_N)(b_0 + \cdots + b_{k-1}) - (b_k + \cdots + b_N)(a_0 + \cdots + a_{k-1}) \\
&\quad = \sum_{\substack{k \le i \le N; \\ 0 \le j \le k-1}} (a_i b_j - a_j b_i) \le 0,
\end{aligned}$$

for any $k = 1, \dots, N$, where the last inequality holds by assumption. Hence the inequalities (19) hold as required. The proposition is proved. □

Proof of Part (1) of Lemma 2 Let $\zeta, \eta \in \Omega_{N,\Lambda\setminus\{x\}}$ be such that $\zeta \le \eta$ and consider probability measures P and Q on the set $\{0, 1, \dots, N\}$ defined as follows

$$\begin{aligned}
\mathsf{P}\big(\{k\}\big) &= \mu_{\gamma,N,\Lambda}(\xi_x = k | \xi = \zeta \text{ off } x), \quad k = 0, 1, \dots, N, \\
\mathsf{Q}\big(\{k\}\big) &= \mu_{\gamma,N,\Lambda}(\xi_x = k | \xi = \eta \text{ off } x), \quad k = 0, 1, \dots, N.
\end{aligned}$$

It is easy to see that for any $x \in \Lambda$, $k = 0, 1, \dots, N$, and $w \in \Omega_{N,\Lambda\setminus\{x\}}$

$$\mu_{\gamma,N,\Lambda}(\xi_x = k | \xi = w \text{ off } x) = \frac{\gamma^{k(k-1)/2 + k\phi(x,w)}}{\sum_{i=0}^N \gamma^{i(i-1)/2 + i\phi(x,w)}}, \tag{20}$$

where

$$\phi(x,w)=\sum_{y\in\Lambda:y\sim x} w_y.$$

In terms of Proposition 1 measure P is defined by coefficients

$$a_k=\gamma^{k(k-1)/2+k\phi(x,\zeta)},\quad k=0,1,\dots,N,$$

and measure Q is defined by coefficients

$$b_k=\gamma^{k(k-1)/2+k\phi(x,\eta)},\quad k=0,1,\dots,N,$$

respectively. It is easy to see that for any $0\le j<i\le N$

$$\begin{aligned}a_ib_j-a_jb_i&=\gamma^{i(i-1)/2+j(j-1)/2}\big(\gamma^{i\phi(x,\zeta)+j\phi(x,\eta)}-\gamma^{j\phi(x,\zeta)+i\phi(x,\eta)}\big)\\&=\gamma^{i(i-1)/2+j(j-1)/2+i\phi(x,\zeta)+j\phi(x,\eta)}\big(1-\gamma^{(i-j)(\phi(x,\eta)-\phi(x,\zeta))}\big)\le 0,\end{aligned}$$

since the inequality $\zeta\le\eta$ implies that $\phi(x,\zeta)\le\phi(x,\eta)$. Applying Proposition 1 we get that measure P is dominated by measure Q. Part (1) of the lemma is proved. □

Proof of Part (2) It suffices to prove that the measure $\mu_{\gamma_1,N,\Lambda}$ is stochastically dominated by the measure $\mu_{\gamma_2,N,\Lambda}$ for any finite Λ since stochastic domination is preserved under the weak limit. It is easy to check that both measure $\mu_{\gamma_2,N,\Lambda}$ and measure $\mu_{\gamma_1,N,\Lambda}$ are irreducible in the sense of Definition 5 and these measures assign positive weights to the maximal element of $\Omega_{N,\Lambda}$, i.e., to configuration $\xi\equiv N$. To apply the Holley theorem it is left to show that the inequality (9) holds if we set $\mu=\mu_{\gamma_1,N,\Lambda}$ and $\mu'=\mu_{\gamma_2,N,\Lambda}$. The formula (20) yields that it suffices to show that a probability measure P on $\{0,1,\dots,N\}$ defined as follows

$$\mathsf{P}\big(\{k\}\big)=\frac{a_k}{\sum_{i=0}^N a_i},\quad k=0,1,\dots,N,$$

where $a_k=\gamma_1^{k(k-1)/2+ka}$, $k=0,1,\dots,N$, and $a\ge 0$, is stochastically dominated by a probability measure Q on $\{0,1,\dots,N\}$ defined as follows

$$\mathsf{Q}\big(\{k\}\big)=\frac{b_k}{\sum_{i=0}^N b_i},\quad k=0,1,\dots,N,$$

where $b_k=\gamma_2^{k(k-1)/2+kb}$, $k=0,1,\dots,N$, and $b\ge 0$. Indeed, it is easy to see that for any $0\le j<i\le N$

$$\begin{aligned}a_ib_j-a_jb_i&=\gamma_1^{i(i-1)/2+ia}\gamma_2^{j(j-1)/2+jb}-\gamma_1^{j(j-1)/2+ja}\gamma_2^{i(i-1)/2+ib}\\&=(\gamma_1\gamma_2)^{j(j-1)/2}\gamma_1^{ja}\gamma_2^{jb}\big(\gamma_1^{i(i-1)/2-j(j-1)/2+(i-j)a}-\gamma_2^{i(i-1)/2-j(j-1)/2+(i-j)b}\big)\\&\le 0,\end{aligned}$$

hence by Proposition 1 the measure P is dominated by the measure Q. Hence by the Holley theorem $\mu_{\gamma_2,N,\Lambda}$ stochastically dominates $\mu_{\gamma_1,N,\Lambda}$, if $\gamma_1\le\gamma_2$. Part (2) of the lemma is proved. □

Proof of Corollary 1 Let $\mu^{(0)}_{\gamma,N,\Lambda}$ be a restriction of the measure $\mu^{(0)}_{\gamma,N}$ on Λ. By applying Proposition 1 to measures P and Q specified by the coefficients $a_k=\gamma^{k(k-1)/2}$, $k=0,1,\dots,N$, and $b_k=\gamma^{k(k-1)/2+k\phi(x,\eta)}$, $k=0,1,\dots,N$, respectively, one gets that measure $\mu^{(0)}_{\gamma,N,\Lambda}$ is stochastically dominated by $\mu_{\gamma,N,\Lambda}$. It is left to notice that stochastic domination is preserved under the weak limit. □

5.3 Proof of Theorem 1

We already mentioned in Remark 1 that the statement of the theorem in the case $N = 1$ is implied by the well known results for the Ising model. The proof of the theorem in the case $N \geq 2$ is similar to the proof of Proposition 4.14 in [3] for the Ising model. We recall it here for completeness.

Consider two boxes $\Lambda = [-L, L]^d$ and $\Lambda' = [-L', L']^d$ and suppose that $L < L'$. Fix a finite $W \subset \Lambda$ and consider the event

$$A = \{\xi_x \geq k_x, \ x \in W\},$$

specified by some numbers $k_x \in \{0, 1, \dots, N\}$, $x \in W$. Event A belongs to both $\mathscr{F}_{N,\Lambda}$ and $\mathscr{F}_{N,\Lambda'}$. It is easy to see that if $\xi \leq \xi'$, then

$$\mathbf{1}_{\{\xi \in A\}} \leq \mathbf{1}_{\{\xi' \in A\}},$$

hence A is an increasing event. It is easy to show that the monotonicity property stated in Part (1) of Lemma 2 yields that

$$\mu_{\gamma,N,\Lambda}(A) \leq \mu_{\gamma,N,\Lambda'}(A),$$

therefore probability $\mu_{\gamma,N,\Lambda}(A)$ increases as $\Lambda \uparrow \mathbb{Z}^d$ and, hence, converges to a limit

$$\mu_{\gamma,N}(A) = \sup_{\Lambda} \mu_{\gamma,N,\Lambda}(A).$$

It can be shown in a standard way that the collection of limit probabilities such as in the preceding display uniquely defines the limit measure $\mu_{\gamma,N}(\cdot)$ on the measurable space $(\Omega_N, \mathscr{F}_N)$. The theorem is proved.

5.4 Proof of Theorem 2

Consider a map $T : \Omega_{N,\Lambda} \to \{0, 1\}^\Lambda$ defined as follows. For any $\xi \in \Omega_{N,\Lambda}$ let $T(\xi)$ be an element of $\{0, 1\}^\Lambda$ such that

$$\big(T(\xi)\big)_x = \mathbf{1}_{\{\xi_x > 0\}}, \quad x \in \Lambda.$$

For a measure μ on $\Omega_{N,\Lambda}$ this map induces a measure $\tilde{\mu}$ on $\{0, 1\}^\Lambda$ as follows

$$\tilde{\mu}(\eta) = \sum_{\xi \in T^{-1}(\eta)} \mu(\xi), \quad \eta \in \{0, 1\}^\Lambda. \tag{21}$$

Let $\tilde{\mu}_{\gamma,N,\Lambda}$ be the induced measure of the measure $\mu_{\gamma,N,\Lambda}$. It is easy to see that weak convergence of the sequence of measures $\mu_{\gamma,N,\Lambda}$ as $\Lambda \uparrow \mathbb{Z}^d$ implies weak convergence of the sequence of measures $\tilde{\mu}_{\gamma,N,\Lambda}$ as $\Lambda \uparrow \mathbb{Z}^d$. Let $\tilde{\mu}_{\gamma,N}$ be the corresponding limit measure on the set of infinite binary configurations $\{0, 1\}^{\mathbb{Z}^d}$. The finite-dimensional distributions of $\tilde{\mu}_{\gamma,N}$ are computable via the finite-dimensional distributions of the limit measure $\mu_{\gamma,N}$. For instance,

$$\tilde{\mu}_{\gamma,N}(\eta_x = 1, \ \eta_y = 0) = \mu_{\gamma,N}(\xi_x > 0, \ \xi_y = 0),$$

for any $x, y \in \mathbb{Z}^d$. It is easy to see that

$$\mu_{\gamma,N}(A_N) = 1 \quad \text{if and only if} \quad \tilde{\mu}_{\gamma,N}(A_1) = 1,$$

where events A_k are defined by Eq. (7).

For each $p \in (0,1)$ denote by $\psi_{p,d}$ the Bernoulli measure on $\{0,1\}^{\mathbb{Z}^d}$, i.e.

$$\psi_{p,d}(\xi_{x_1} = i_1, \dots, \xi_{x_n} = i_n) = p^{i_1+\dots+i_n}(1-p)^{n-i_1-\dots-i_n},$$

where $x_j \in \mathbb{Z}^d$, $i_j \in \{0,1\}$, $j = 1, \dots, n$ and $n \geq 1$.

Proof of Part (1) of Theorem 2 If $\gamma = 1$, then the probability measure $\mu_{1,N}$ is a probability distribution of a collection of independent uniformly distributed in $\{0, 1, \dots, N\}$ random variables labelled by elements of $\mathbb{Z}^d$. Therefore $\tilde{\mu}_{1,N} = \psi_{p_N,d}$ and

$$\mu_{1,N}(A_N) = \tilde{\mu}_{1,N}(A_1) = \psi_{p_N,d}(A_1),$$

where $\psi_{p_N,d}$ is the Bernoulli measure with parameter $p_N = N/(N+1)$. It is known that $\psi_{p,d}(A_1) = 1$ provided that $p > p_c(d)$, where $p_c(d)$ is the so-called critical probability in dimension d. It is also known (bound (19) in [3]) that $p_c(2) < 0.680$. This implies that $p_c(d) \leq 0.68$ for any dimension $d \geq 3$. Thus, if $N \geq 3$, then

$$p_N = \frac{N}{N+1} \geq 0.75 > p_c(d),$$

and, hence, $\psi_{p_N,d}(A_1) = \mu_{1,N}(A_N) = 1$. By Lemma 2 the measure $\mu_{1,N}$ is stochastically dominated by $\mu_{\gamma,N}$ for any $\gamma > 1$. Therefore, if $N \geq 3$, then

$$\mu_{\gamma,N}(A_N) = \mu_{1,N}(A_N) = \psi_{p_N,d}(A_1) = 1$$

for any $\gamma \geq 1$ as required. □

Proof of Part (2) of Theorem 2 Recall that the case $N = 1$ (Ising model) has been already discussed (Remark 1, Sect. 2) therefore let $N = 2$ in the rest of the proof. If we knew that $N/(N+1) = 2/3 > p_c(d)$, then it would be that $\gamma_c = 1$. Otherwise existence of the critical value γ_c can be shown as follows. If $2/3 < p_c(d)$, then

$$\mu_{1,2}(A_2) = \nu_{1,2}(A_1) = \psi_{2/3,d}(A_1) = 0.$$

It is well known that the probability of existence of an infinite cluster can take only two values, i.e. 0 and 1. Notice also that if $\gamma_1 \leq \gamma_2$, then

$$\mu_{\gamma_1,2}(A_2) \leq \mu_{\gamma_2,2}(A_2)$$

since $\mu_{\gamma_1,2}$ is stochastically dominated by $\mu_{\gamma_2,2}$ and A_2 is obviously an increasing event. Therefore it suffices to show that $\mu_{\gamma,2}(A_2) = 1$ for sufficiently large γ in order to show existence of the critical γ_c. Recall the measure $\mu^{(0)}_{\gamma,N,\Lambda}$ defined by Eq. (8). By Corollary 1 $\mu^{(0)}_{\gamma,2}$ is stochastically dominated by $\mu_{\gamma,2}$ for any $\gamma > 1$. Therefore $\mu^{(0)}_{\gamma,2}(A_2) \leq \mu_{\gamma,2}(A_2)$. It is left to notice that $\mu^{(0)}_{\gamma,2}(A_2) = \psi_{p(\gamma),d}(A_1)$, where $\psi_{p(\gamma),d}$ is the Bernoulli measure $\psi_{p(\gamma),d}$ specified by parameter

$$p(\gamma) = \frac{1+\gamma}{2+\gamma},$$

and $\psi_{p(\gamma),d}(A_1) = 1$, if γ is sufficiently large to ensure that $p_c(d) < p(\gamma)$. The theorem is proved. □

5.5 Proof of Theorem 4

Without loss of generality we assume that $\alpha = 1$ throughout the proof.

Proof of Part (1) of Theorem 4 By Theorem 1.4 in [7] to prove ergodicity it suffices to construct a function $f : \mathbb{Z}_+^\Lambda \to [0, \infty)$ such that for any $a > 0$ the set $\{\xi \in \mathbb{Z}_+^\Lambda : f(\xi) \le a\}$ is finite, $Gf(\xi) < \infty$ for all $\xi \in \mathbb{Z}_+^\Lambda$, where G is the generator of the Markov chain (defined by Eq. (10)), and such that the following inequality

$$Gf(\xi) \le -\varepsilon, \tag{22}$$

holds for all $\xi \in \{\xi \in \mathbb{Z}_+^\Lambda : f(\xi) > a\}$ for some $a > 0$ and $\varepsilon > 0$. We are going to show that the following function

$$f(\xi) = \sum_{x\in\Lambda} \xi_x^2$$

satisfies these conditions.

Fix $\varepsilon > 0$ and show that there exists $a = a(\varepsilon) > 0$ such that the inequality (22) holds provided that $\sum_{x\in\Lambda} \xi_x^2 > a$. Direct computation gives that for any $\xi \in \mathbb{Z}_+^\Lambda$

$$\begin{aligned} Gf(\xi) &= \sum_{x\in\Lambda} \big(\big[(\xi_x+1)^2 - \xi_x^2\big]\gamma^{n(x,\xi)} + \big[(\xi_x-1)^2 - \xi_x^2\big]\mathbf{1}_{\{\xi_x>0\}}\big) \\ &= \sum_{x\in\Lambda} 2\xi_x\big(\gamma^{n(x,\xi)} - \mathbf{1}_{\{\xi_x>0\}}\big) + \sum_{x\in\Lambda}\big(\gamma^{n(x,\xi)} + \mathbf{1}_{\{\xi_x>0\}}\big). \end{aligned}$$

The second sum in the preceding display can be bounded above by $2|\Lambda|$, where $|\Lambda|$ is the cardinality of Λ. It is also easy to see that for any $\xi_x \ge 0$

$$2\xi_x\big(\gamma^{n(x,\xi)} - \mathbf{1}_{\{\xi_x>0\}}\big) \le -2\xi_x(1-\gamma).$$

Therefore, if $\sum_{x\in\Lambda} \xi_x^2 > a > 0$, then $\sum_{x\in\Lambda} \xi_x > \sqrt{a}$ and, hence,

$$Gf(\xi) \le -2(1-\gamma)\sum_{x\in\Lambda} \xi_x + 2|\Lambda| \le -2(1-\gamma)\sqrt{a} + 2|\Lambda| \le -\varepsilon,$$

provided that $\sqrt{a} \ge (2|\Lambda| + \varepsilon)/(2(1-\gamma))$. Thus ergodicity of Markov chain $\xi(t)$ is proved.

It is easy to see that if $0 < \gamma < 1$, then

$$\sum_{\xi\in\mathbb{Z}_+^\Lambda} \gamma^{s(\xi)} \le \sum_{\xi\in\mathbb{Z}_+^\Lambda} \gamma^{\frac{1}{2}\sum_{x\in\Lambda}\xi_x(\xi_x-1)} = \left(\sum_{k=0}^{\infty} \gamma^{k(k-1)/2}\right)^{|\Lambda|} < \infty.$$

Thus the measure $\nu_{1,\gamma,\Lambda}$ is well defined. The detailed balance condition

$$\nu_{1,\gamma,\Lambda}\big(\xi + e^{(x)}\big) = \nu_{1,\gamma,\Lambda}(\xi)\gamma^{n(x,\xi)}$$

is implied by Eq. (18) and yields time reversibility of the Markov chain with the measure (11). □

Proof of Part (2) of Theorem 4 We prove that the Markov chain is explosive in the case $\gamma > 1$ by applying Theorem 1.5 in [7]. By the first part of this theorem a continuous time denumerable Markov chain with state space X and the generator Γ is explosive if there exists a non-negative function f in the generator domain and a positive ε such that

$$\Gamma f(y) \le -\varepsilon,$$

for *any* state $y \in X$. We are going to show that this condition is satisfied for the Markov chain under consideration with the following function

$$f(\xi) = \frac{1}{1 + \max_{x\in\Lambda} \gamma^{\xi_x}}.$$

(i) If $\xi \equiv 0$, then

$$Gf(\xi) = \left(\frac{1}{1+\gamma} - \frac{1}{2}\right)|\Lambda| = \frac{1-\gamma}{2(1+\gamma)}|\Lambda| \le \frac{1-\gamma}{2(1+\gamma)} = -\varepsilon_1 < 0.$$

(ii) Suppose $\xi \in \mathbb{Z}_+^\Lambda$ is such that there exists a unique maximal component equal to $m \ge 1$, i.e. there exists $x \in \Lambda$ such that $\xi_x > \xi_y$ for any $y \ne x$ and $\xi_x = m$. It is easy to see that in this case

$$\begin{aligned}
Gf(\xi) &= \left(\frac{1}{1+\gamma^{m+1}} - \frac{1}{1+\gamma^m}\right)\gamma^{n(x,\xi)} + \left(\frac{1}{1+\gamma^{m-1}} - \frac{1}{1+\gamma^m}\right)\\
&= \frac{\gamma^m(1-\gamma)}{(1+\gamma^{m+1})(1+\gamma^m)}\gamma^{n(x,\xi)} - \frac{\gamma^{m-1}(1-\gamma)}{(1+\gamma^m)(1+\gamma^{m-1})}\\
&\le \frac{\gamma^{2m}(1-\gamma)}{(1+\gamma^{m+1})(1+\gamma^m)} - \frac{\gamma^{m-1}(1-\gamma)}{(1+\gamma^m)(1+\gamma^{m-1})}\\
&= \frac{1-\gamma}{\gamma}\frac{\gamma^m}{(1+\gamma^m)}\left(\frac{\gamma^{m+1}}{1+\gamma^{m+1}} - \frac{1}{1+\gamma^{m-1}}\right),
\end{aligned}$$

where we used bound $n(x,\xi) \ge m$ to obtain the inequality. It is easy to check that

$$\delta = \min_{m\ge 1}\left(\frac{\gamma^{m+1}}{1+\gamma^{m+1}} - \frac{1}{1+\gamma^{m-1}}\right) > 0.$$

Therefore $Gf(\xi) \le -\varepsilon_2$, where $\varepsilon_2 = -\delta(1-\gamma)/\gamma > 0$.

(iii) If $\xi \in \mathbb{Z}_+^\Lambda$ is such that there exist two or more maximal components which are equal to $m \ge 1$, then

$$\begin{aligned}
Gf(\xi) &= \left(\frac{1}{1+\gamma^{m+1}} - \frac{1}{1+\gamma^m}\right)\sum_{x:\xi_x=m}\gamma^{n(x,\xi)}\\
&\le \frac{\gamma^m(1-\gamma)}{(1+\gamma^m)(1+\gamma^{m+1})}\gamma^m\\
&\le \frac{(1-\gamma)}{\gamma} = -\varepsilon_3 < 0.
\end{aligned}$$

It is easy to see that the three described types of configurations exhaust all possibilities.

Thus we have just shown that $Gf(\xi) \le -\varepsilon$, where $\varepsilon = \min(\varepsilon_1,\varepsilon_2,\varepsilon_3) > 0$, for any $\xi \in \mathbb{Z}_+^\Lambda$. Therefore Theorem 1.5 in [7] applies and the Markov chain is explosive. □

Proof of Part (3) If $\gamma = 1$, then the Markov chain is formed by $|\Lambda| \ge 3$ copies of independent simple random walks on $\mathbb{Z}_+$ with reflection at the origin and its transience is implied by transience of the simple random walk in dimension $|\Lambda|$. □

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