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Ward identities in Lifshitz-like field theories

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In this work, we develop a normal product algorithm suitable to the study of anisotropic field theories in flat space, apply it to construct the symmetries generators and describe how their possible anomalies may be found. In particular, we discuss the dilatation anomaly in a scalar model with critical exponent $z = 2$ in six spatial dimensions.

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I. INTRODUCTION

Aiming the ultraviolet improvement of the perturbative series without the introduction of ghost degrees of freedom, field theories with higher spatial derivatives have been proposed in the literature [1–7]. The characteristic feature of these proposals is provided by the different scaling properties of space and time, i.e., $x^i \rightarrow bx^i$ whereas $t \rightarrow b^z t$. The smallest value of the critical exponent z is one which corresponds to the usual Lorentz symmetric field theories; higher values of z furnishes models with better ultraviolet behavior at the expenses of breaking Lorentz invariance. Many investigations of theories with such anisotropy have been reported, namely, quantization of gravitational models [8–13], applications to cosmology [14–18], studies in Lorentz symmetry restoration and the renormalization group [19–24] and other aspects of field theories [25–36].

In a previous work [23] we studied properties of the renormalization of these anisotropic field theories and analyzed the renormalization group flows of relevant parameters in various models with special emphasis on the possibility of Lorentz symmetry restoration at low energies. We found that, the restoration requires that the interactions be infrared stable and some process of dimensional reduction must be furnished to treat the eventual divergences appearing whenever the higher derivative quadratic terms are eliminated. Besides the renormalization aspects, it is natural to investigate the underlying symmetries and their possible anomalies. In the usual relativistic setting, special interest is devoted to the conformal and chiral anomalies not only for their conceptual aspects but also for phenomenological applications (see [37] and references therein). For anisotropic space-time models these anomalies have been discussed in [38] for a scalar field coupled to a Hořava-Lifshitz background and in [20,39,40] for the chiral anomaly of Dirac fermions coupled to gauge and gravitational fields. Pursuing these studies, in the present work we generalize Zimmermann's normal product algorithm [41], which for Lorentz symmetric theories

has proved to be a very powerful tool [42]. We then apply the formalism to the analysis of the scale symmetry in a renormalizable φ^4 model with $z = 2$ in six spatial dimensions where it is renormalizable.

Our work is organized as follows. In Sec. II we recall and extend the basics of the normal product algorithm which shall be used in the sequel. Thus, in Sec. III we derive the conservation laws associated to space and time translations. Because of the breaking of Lorentz invariance inherent to theories with higher spatial derivatives, these two symmetries are considered separately. In Sec. IV we examine the dilatation current and explicitly analyze the anomaly in its conservation law. A summary and additional remarks are presented in the Conclusions. One Appendix to study Noether's theorem appropriated to higher derivative theories is included.

II. NORMAL PRODUCTS IN ANISOTROPIC FIELD THEORIES

Throughout this work we will consider the BPHZ renormalized anisotropic scalar field model, with $z = 2$ in $d = 6$ spatial dimensions, given by

$$\mathcal{L} = \frac{1}{2}(1+A)\partial_0\varphi\partial_0\varphi - \frac{(b^2+B)}{2}\partial_i\varphi\partial_i\varphi - \frac{(a^2+C)}{2}\Delta\varphi\Delta\varphi - \frac{(m^2+D)}{2}\varphi^2 - \frac{(\lambda+E)}{4!}\varphi^4, \quad (1)$$

where A , B , C , D and E are finite counterterms fixed according to appropriated renormalization conditions, b^2 , a^2 , m^2 and λ are the original parameters in the Lagrangian and Δ denotes the spatial Laplacian $\Delta \equiv \partial_i\partial_i$.

Before embarking into the analysis of the model (1), it is useful to recall and generalize the BPHZ normal product formalism [41,42]. Thus, let \mathcal{O} be some formal product of fields and their derivatives. Associated to the operator \mathcal{O} , there exists an infinite set of normal products N_δ , where δ , the degree of the normal product, must be an integer greater than or equal to the canonical dimension of the operator \mathcal{O} . More precisely,

$$\langle 0|TN_\delta[\mathcal{O}]X|0\rangle \equiv \text{Finite Part of } \langle 0|T\mathcal{O}X|0\rangle, \quad (2)$$

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where $X \equiv \prod_i \varphi_i(x_i)$ with φ_j designating the field of type “ j ”, either bosonic or fermionic. The prescription to get the finite part can be described as follows. We suppose that, without the normal product insertion, the Green’s function are already renormalized, according to the BPHZ scheme. However, in the presence of the formal product \mathcal{O} there are additional divergences that need to be subtracted. Indeed, the superficial degree of divergence for a generic proper graph G , with N_B and N_F bosonic and fermionic external lines, is in the anisotropic case

$$d(G) = d + z - \text{Dim}[\varphi]N_B - \text{Dim}[\psi]N_F - \sum_a (d + z - \text{Dim}[V_a]), \quad (3)$$

where the sum is over all vertices of G , the dimensions of the bosonic (φ) and fermionic (ψ) fields are

$$\text{Dim}[\varphi] = \frac{d - z}{2}, \quad \text{Dim}[\psi] = \frac{d}{2} \quad (4)$$

and $\text{Dim}[V_a] = D_a + \text{Dim}[\varphi]\nu_a^B + \text{Dim}[\psi]\nu_a^F$. ν_a^B and ν_a^F are the number of bosonic and fermionic lines joining at the vertex V_a and D_a counts the number of momenta factors in the vertex, z for a timelike and one for a spacelike component.

In a renormalizable model all vertices coming from the Lagrangian are assumed to have $\text{Dim}[V_a] = d + z$, whereas for the special vertex $V_{\mathcal{O}}$, associated to the operator \mathcal{O} , we make the replacement $\text{Dim}[\mathcal{O}] \rightarrow \delta$. This designation is convenient although it implies that super-renormalizable vertices are going to be oversubtracted. By taking into consideration these observations, the degree function $\delta(\gamma)$ for a generic proper graph γ is defined to be

$$\delta(\gamma) = \begin{cases} \delta - \text{Dim}[\varphi]N_B - \text{Dim}[\psi]N_F, & \text{if } V_{\mathcal{O}} \in \gamma \\ d + z - \text{Dim}[\varphi]N_B - \text{Dim}[\psi]N_F, & \text{if } V_{\mathcal{O}} \notin \gamma. \end{cases} \quad (5)$$

Finally, the finite part prescription consists in the application of the forest formula of the BPHZ scheme, such that for a proper primitively divergent diagram γ we must apply the Taylor operator of degree $\delta(\gamma)$,

$$t^{\delta(\gamma)} I_\gamma \equiv \sum_{s=0}^{[(\delta(\gamma))/z]} \frac{p_0^s}{s!} \frac{\partial^s}{\partial p_0^s} \sum_{n=0}^{\delta(\gamma)-sz} \frac{p_{i_1} \cdots p_{i_n}}{n!} \frac{\partial}{\partial p_{i_1}} \cdots \frac{\partial}{\partial p_{i_n}} I_\gamma, \quad (6)$$

where $[x]$ is the greatest integer less than or equal to x , p_0^s symbolically stands for the product of s timelike components of an independent set of external momenta; p_i denotes the i -th spacelike momentum (with the index of the component implicit) and all derivatives are computed at zero external momenta.

The normal products so defined satisfy a set of rules that enable us to derive Ward identities in a systematic way. To exemplify the general procedure let us consider the case of a partially conserved symmetry, i.e., a symmetry that

formally holds when the breaking parameters vanish (as the dilatation symmetry in (1) when b and m go to zero). In the normal product formalism, the conservation law associated with a partially conserved continuous symmetry is obtained by taking as the generator of the transformations the normal product of the corresponding Noether’s current. One then studies the object

$$\partial_\mu \langle 0 | TN[J^\mu](x) \prod \varphi(x_i) | 0 \rangle, \quad (7)$$

where the degree of the normal product, for each component of the current J^μ , should be the minimal leading to a well defined expression. As explained below, the derivative can be taken inside the normal product symbol by adequately increasing the degree of the normal product. The equations of motion are afterwards applied producing Dirac delta terms characteristics of the symmetry accompanied by other terms which break the symmetry. Generally, the breaking terms are oversubtracted and to investigate their persistence at the symmetry limit, Zimmermann identities, to be discussed shortly, are used to reduce the degrees of the normal products to the minimal ones what unveils the anomalies. To implement this algorithm the normal product formalism furnishes the following rules:

(1) Differentiation rule. It can be show that

$$\begin{aligned} \partial_0 \langle 0 | TN_\delta[\mathcal{O}]X | 0 \rangle &= \langle 0 | TN_{\delta+z}[\partial_0 \mathcal{O}]X | 0 \rangle \text{ and} \\ \partial_i \langle 0 | TN_\delta[\mathcal{O}]X | 0 \rangle &= \langle 0 | TN_{\delta+1}[\partial_i \mathcal{O}]X | 0 \rangle. \end{aligned} \quad (8)$$

In fact, these results follow easily from the observation that the anisotropic Taylor operator of degree δ satisfy $p^0 t_p^\delta f(p) = t_p^{\delta+z}(p^0 f(p))$ and $p^i t_p^\delta f(p) = t_p^{\delta+1}(p^i f(p))$.

(2) Equation of motion. It is the quantum version of the classical (Euler-Lagrange) equation of motion and for the model (1) it is given by

$$\begin{aligned} \langle 0 | TN_8[\varphi(\partial_0^2 - b^2 \Delta + a^2 \Delta^2 + m^2)\varphi](x)X | 0 \rangle &= -A \langle 0 | TN_8[\varphi \partial_0^2 \varphi](x)X | 0 \rangle \\ &+ B \langle 0 | TN_8[\varphi \Delta \varphi](x)X | 0 \rangle \\ &- C \langle 0 | TN_8[\varphi \Delta^2 \varphi](x)X | 0 \rangle \\ &- D \langle 0 | TN_8[\varphi^2](x)X | 0 \rangle \\ &- \frac{(\lambda + E)}{3!} \langle 0 | TN_8[\varphi^4](x)X | 0 \rangle \\ &- i \sum_{i=1}^N \delta(x - x_i) \langle 0 | TX | 0 \rangle. \end{aligned} \quad (9)$$

This expression may be derived by noting that in momentum space the operator applied on φ , in the left hand side of the above equation, is equal to $-i$ times the inverse of the free field propagator.

- (3) Zimmermann identities. These are identities relating normal products of different degrees associated to the same formal product. Using again the model (1), let us consider some examples that will be used later on. First, to the operator φ^2 , with dimension $\text{Dim}[\varphi^2] = 4$, we want to obtain the terms collectively denoted by \mathcal{R} in the relation

$$\langle 0|TN_8[\varphi^2]X|0\rangle = \langle 0|TN_4[\varphi^2]X|0\rangle + \mathcal{R}. \quad (10)$$

The degree functions for the N_4 and N_8 normal products are respectively: $\delta_1(\gamma) = 4 - 2N_\gamma$ and $\delta_2(\gamma) = 8 - 2N_\gamma$. Thus:

- (a) For graphs with $N_\gamma = 2$, $\delta_1(\gamma) = 0$ and $\delta_2(\gamma) = 4$. Notice first that the subtraction term without derivatives is present in both schemes so that it does not contribute to their difference. The subtractions with δ_2 produce the following additional terms: $\langle 0|TN_8[\varphi\Delta\varphi]X|0\rangle$ and $\langle 0|TN_8[\partial_i\varphi\partial_i\varphi]X|0\rangle$ with second order derivatives, and $\langle 0|TN_8[\varphi\partial_0^2\varphi]X|0\rangle$, $\langle 0|TN_8[\partial_0\varphi\partial_0\varphi]X|0\rangle$, $\langle 0|TN_8[\Delta\varphi\Delta\varphi]X|0\rangle$, $\langle 0|TN_8[\varphi\Delta^2\varphi]X|0\rangle$, $\langle 0|TN_8[\partial_i\partial_j\varphi\partial_i\partial_j\varphi]X|0\rangle$ and $\langle 0|TN_8[\Delta\partial_i\varphi\partial_i\varphi]X|0\rangle$ with fourth order derivatives. Note that the terms of the second order derivatives are oversubtracted and can be still reduced, i.e., may be written in terms of minimally subtracted normal products.
- (b) For graphs with $N_\gamma = 4$, $\delta_1(\gamma) = -4$ and $\delta_2(\gamma) = 0$. The difference between the two normal products is just the subtraction term without derivatives associated to the scheme with δ_2 . This counterterm is therefore $\langle 0|TN_8[\varphi^4]X|0\rangle$.

We could now collect all the contributions described above to explicitly write the terms denoted by \mathcal{R} in (10) but, instead, we will do that just for the integrated normal products, a procedure that avoids the dissemination of parameters and that will be useful later on. In this situation, we obtain

$$\begin{aligned} & \int dx^0 d^6x \langle 0|TN_8[\varphi^2](x)X|0\rangle \\ &= \int dx^0 d^6x \langle 0|T\{N_4[\varphi^2](x) \\ & \quad + pN_8[\varphi\Delta\varphi](x) + sN_8[\partial_0\varphi\partial_0\varphi](x) \\ & \quad + uN_8[\varphi\Delta^2\varphi](x) + vN_8[\varphi^4](x)\}X|0\rangle. \end{aligned} \quad (11)$$

The coefficients p , s , u , v are determined from normalization conditions, as we will show shortly. Incidentally, a simple dimensional analysis show us that: $\text{Dim}[p] = -2$ and $\text{Dim}[s] = \text{Dim}[u] = \text{Dim}[v] = -4$. This detail will be important in the study of the dilatation anomaly. After noticing that the term $\langle 0|TN_8[\varphi\Delta\varphi]X|0\rangle$ is oversubtracted we can follow a similar reasoning as before to verify that

$$\begin{aligned} & \int dx^0 d^6x \langle 0|TN_8[\varphi\Delta\varphi](x)X|0\rangle \\ &= \int dx^0 d^6x \langle 0|T\{N_6[\varphi\Delta\varphi](x) \\ & \quad + gN_8[\partial_0\varphi\partial_0\varphi](x) + kN_8[\varphi\Delta^2\varphi](x) \\ & \quad + lN_8[\varphi^4](x)\}X|0\rangle. \end{aligned} \quad (12)$$

From Eqs. (11) and (12) it follows now that

$$\begin{aligned} & \int dx^0 d^6x \langle 0|TN_8[\varphi^2](x)X|0\rangle \\ &= \int dx^0 d^6x \langle 0|T\{N_4[\varphi^2](x) + pN_6[\varphi\Delta\varphi](x) \\ & \quad + (s + pg)N_8[\partial_0\varphi\partial_0\varphi](x) \\ & \quad + (u + pk)N_8[\varphi\Delta^2\varphi](x) \\ & \quad + (v + pl)N_8[\varphi^4](x)\}X|0\rangle. \end{aligned} \quad (13)$$

- (4) Normalization conditions. As the normal products are defined by subtracting divergences through the application of Taylor operators, they satisfy simple normalization conditions as we will describe now. Thus, if D^M is a differential operator of degree $M \equiv zM_0 + M_1$ in the momenta,

$$D^M = \frac{\partial^{M_0+M_1}}{\partial p_1^0 \cdots \partial p_{M_0}^0 \partial p_{M_0+1}^{i_1} \cdots \partial p_{M_0+M_1}^{i_{M_1}}}, \quad (14)$$

then

$$\begin{aligned} & D^M \Gamma_{\mathcal{O},\delta}^{(N)}(p, p_1, \dots, p_N)|_{p=p_1=\dots=0} \\ &= \text{contribution of trivial diagram}, \end{aligned} \quad (15)$$

if $0 \leq M \leq \delta - \text{Dim}[\varphi]N_B - \text{Dim}[\psi]N_F$, i.e., the order of differential operator must be lower than the degree of the Taylor operator. $\Gamma_{\mathcal{O},\delta}^{(N)}(p, p_1, \dots, p_N)$ denotes the (amputated) vertex function of $N = N_B + N_F$ points with the insertion of the normal product $N_\delta[\mathcal{O}(x)]$:

$$\begin{aligned} & (2\pi)^{d+1} \delta(p + p_1 + \dots + p_N) \Gamma_{\mathcal{O},\delta}^{(N)}(p, p_1, \dots, p_N) \\ & \equiv \int dx^0 d^d x \prod_{i=1}^N dx_i^0 d^d x_i \\ & \quad \times e^{i(p_{x^0} + \sum p_k x_k)} \langle 0|TN_\delta[\mathcal{O}(x)]\varphi_1(x_1) \cdots \\ & \quad \times \varphi_N(x_N)|0\rangle^{\text{prop}}, \end{aligned} \quad (16)$$

where prop means proper diagrams, i.e., 1PI diagrams.

We may use now the property (15) to fix the parameters in the above Zimmermann identities. To determine g in Eq. (12), for example, we rewrite that equation for the vertex functions defined by (16) with $N = 2$, apply the

operator of fourth order $D^4 = \partial^2/\partial p_1^0 \partial p_1^0$ and then put the external momenta equal to zero. The only terms that survive are: the term accompanying g , i.e., $\Gamma_{\partial_0 \varphi \partial_0 \varphi, 8}^{(2)}$, which has only the trivial contribution; and the term involving the reduced normal product $\Gamma_{\varphi \Delta \varphi, 6}^{(2)}$. This yields

$$g = \frac{1}{4} \frac{\partial^2}{\partial p_1^0 \partial p_1^0} \int dx dx_1 dx_2 e^{ip_1 x_1 + ip_2 x_2} \times \langle 0 | TN_6[\varphi \Delta \varphi](x) \varphi(x_1) \varphi(x_2) | 0 \rangle^{\text{prop}} \Big|_{p_1=p_2=0}, \quad (17)$$

$$l = -\frac{1}{4!} \int dx \prod_{i=1}^4 dx_i e^{ipx+i \sum_k p_k x_k} \langle 0 | TN_6[\varphi \Delta \varphi](x) \varphi(x_1) \varphi(x_2) \varphi(x_3) \varphi(x_4) | 0 \rangle^{\text{prop}} \Big|_{p=p_1=\dots=p_4=0}. \quad (19)$$

From these expressions we may verify that all these coefficients start contributing in second order in the coupling constant. A similar analysis for Zimmermann identity (11) show us that also p , s , u , v all start at second order in the coupling constant. Thus, terms like pg appearing in $s + pg$ are of higher order when compared with s . This analysis of order, as well as the dimensions of parameters involved in the Zimmermann identities, will be important in the determination of the dilatation anomaly.

Now, we are ready to discuss the conservation laws at quantum level and then investigate the existence of anomalies. Let us start with space and time translations invariance.

III. ENERGY-MOMENTUM TENSOR

The classical energy-momentum tensor can be easily constructed from Noether's theorem for theories with higher derivatives (see the Appendix) applied to space-time translations $x^\mu \rightarrow x^\mu + \epsilon^\mu$. Because of the asymmetric form of the Lagrangian, we need to distinguish two cases:

- (a) Invariance under time translations. The components of the conserved current are

$$\begin{aligned} \Theta_{00} &= (1 + A) \partial_0 \varphi \partial_0 \varphi - \mathcal{L}, \\ \Theta_{i0} &= -(b^2 + B) \partial_i \varphi \partial_0 \varphi - (a^2 + C) \Delta \varphi \partial_i \partial_0 \varphi \\ &\quad + (a^2 + C) (\partial_i \Delta \varphi) \partial_0 \varphi, \end{aligned} \quad (20)$$

which satisfy $\partial_0 \Theta_{00} + \partial_i \Theta_{i0} = 0$ leading to the conservation of the energy $H = \int d^6 x \Theta_{00}$.

- (b) Invariance under spatial translations. In this case the components of the conserved current are

$$\Theta_{0j} = (1 + A) \partial_0 \varphi \partial_j \varphi, \quad (21)$$

$$\begin{aligned} \Theta_{ij} &= -(b^2 + B) \partial_i \varphi \partial_j \varphi - (a^2 + C) \Delta \varphi \partial_i \partial_j \varphi \\ &\quad + (a^2 + C) (\partial_i \Delta \varphi) \partial_j \varphi - \delta^{ij} \mathcal{L}, \end{aligned} \quad (22)$$

such that $\partial_0 \Theta_{0j} + \partial_i \Theta_{ij} = 0$ implies in the conservation of the j component of the momentum: $\int d^6 x \Theta_{0j}$.

where we defined $dx \equiv dx_0 d^6 x$. The others coefficients can be determined similarly and we get

$$k = -\frac{1}{384} \frac{\partial^4}{\partial p_1^i \partial p_1^i \partial p_1^j \partial p_1^j} \int dx dx_1 dx_2 e^{ipx + ip_1 x_1 + ip_2 x_2} \times \langle 0 | TN_6[\varphi \Delta \varphi](x) \varphi(x_1) \varphi(x_2) | 0 \rangle^{\text{prop}} \Big|_{p=p_1=p_2=0} \quad (18)$$

and

At the quantum level, to eliminate divergences, we should use normal products to define the energy and momentum operators. Thus, notice that these components have different dimensions: $\text{Dim}[\Theta_{00}] = 8$, $\text{Dim}[\Theta_{0i}] = 7$, $\text{Dim}[\Theta_{i0}] = 9$ and $\text{Dim}[\Theta_{ij}] = 8$. So, we shall use the following normal products $\langle 0 | TN_8[\Theta_{00}] X | 0 \rangle$, $\langle 0 | TN_7[\Theta_{0i}] X | 0 \rangle$, $\langle 0 | TN_9[\Theta_{i0}] X | 0 \rangle$ and $\langle 0 | TN_8[\Theta_{ij}] X | 0 \rangle$. According to the rules of normal products of the previous section, we get the analogous of the conservation equations at the quantum level:

$$\begin{aligned} \partial_0 \langle 0 | TN_8[\Theta_{00}] (x) X | 0 \rangle + \partial_i \langle 0 | TN_9[\Theta_{i0}] (x) X | 0 \rangle \\ = -i \sum_k \delta(x - x_k) \langle 0 | T \partial_0 \varphi(x) X_k | 0 \rangle \end{aligned} \quad (23)$$

and

$$\begin{aligned} \partial_0 \langle 0 | TN_7[\Theta_{0j}] (x) X | 0 \rangle + \partial_i \langle 0 | TN_8[\Theta_{ij}] (x) X | 0 \rangle \\ = -i \sum_k \delta(x - x_k) \langle 0 | T \partial_j \varphi(x) X_k | 0 \rangle, \end{aligned} \quad (24)$$

where X_k is equal to the X but with the field $\varphi(x_k)$ omitted. By integrating over the space-time, the left hand sides of (23) and (24) vanish and we get

$$\begin{aligned} \sum_k \langle 0 | T \partial_0 \varphi(x_k) X_k | 0 \rangle = 0 \quad \text{and} \\ \sum_k \langle 0 | T \partial_j \varphi(x_k) X_k | 0 \rangle = 0. \end{aligned} \quad (25)$$

These Ward identities reflect the invariance of the Green functions under space and time translations.

IV. DILATATION ANOMALY

A. Classical analysis

The crucial modification in anisotropic theories are just that of scale transformations, due to the anisotropic scaling between space and time,

$$x_i \rightarrow e^\alpha x_i \quad \text{and} \quad x_0 \rightarrow (e^\alpha)^z x_0. \quad (26)$$

Under this scaling, the scalar field transform as $\varphi(x) \rightarrow \varphi'(x') = e^{-\alpha \text{Dim}[\varphi]} \varphi(x)$, whose infinitesimal version is

$$\delta\varphi(x) \equiv \varphi'(x) - \varphi(x) = \epsilon[\text{Dim}[\varphi] + zx^0\partial_0 + x^i\partial_i]\varphi(x). \quad (27)$$

We are particularly interested in the study of the consequences of this anisotropic scaling for the dilatation anomaly. Let us consider the Lagrangian (1) without the finite counterterms that for our (classical) purpose are not necessary,

$$\mathcal{L} = \frac{1}{2}\partial_0\varphi\partial_0\varphi - \frac{b^2}{2}\partial_i\varphi\partial_i\varphi - \frac{a^2}{2}\Delta\varphi\Delta\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4. \quad (28)$$

The dimensional parameters b^2 and m^2 in this Lagrangian break the scale invariance as $\text{Dim}[m^2] = 4$ and $\text{Dim}[b^2] = 2$. On the other hand, when b^2 and $m^2 \rightarrow 0$, we have a classical conserved current. By splitting the Lagrangian in two parts, $\mathcal{L} = \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{br}}$, with

$$\mathcal{L}_{\text{inv}} \equiv \frac{1}{2}\partial_0\varphi\partial_0\varphi - \frac{a^2}{2}\Delta\varphi\Delta\varphi - \frac{\lambda}{4!}\varphi^4 \quad (29)$$

and

$$\mathcal{L}_{\text{br}} \equiv -\frac{b^2}{2}\partial_i\varphi\partial_i\varphi - \frac{m^2}{2}\varphi^2, \quad (30)$$

we observe that, under the transformation (26),

$$\delta\mathcal{L}_{\text{inv}} = \epsilon[2\partial_0(x^0\mathcal{L}_{\text{inv}}) + \partial_i(x^i\mathcal{L}_{\text{inv}})], \quad (31)$$

is a total derivative. However, the same does not happen with \mathcal{L}_{br} :

$$\delta\mathcal{L}_{\text{br}} = \epsilon[2\partial_0(x^0\mathcal{L}_{\text{br}}) + \partial_i(x^i\mathcal{L}_{\text{br}})] + \epsilon(b^2\partial_i\varphi\partial_i\varphi + 2m^2\varphi^2). \quad (32)$$

From these results, the components of the Noether's current can be constructed,

$$J_0 = 2\varphi\partial_0\varphi + 2x^0\Theta_{00} + x^i\Theta_{0i} \quad (33)$$

and

$$J^i = -3a^2\Delta\varphi\partial_i\varphi + 2a^2(\partial_i\Delta\varphi)\varphi - 2b^2\varphi\partial_i\varphi + 2x^0\Theta_{i0} + x^j\Theta_{ji}, \quad (34)$$

which satisfies

$$\partial_0 J^0 + \partial_i J^i = b^2\partial_i\varphi\partial_i\varphi + 2m^2\varphi^2, \quad (35)$$

such that when m^2 and $b^2 \rightarrow 0$, we have the conservation law. On the other hand, in the quantum case we need to consider the subtraction scheme to provide a precise meaning for the expressions involving operators at the same point, giving rise to the anomaly.

B. Quantum analysis

Now, let us apply the normal product method to the construction of the dilatation current. The dimension of components of the current are: $\text{Dim}[J_0] = 6$ and $\text{Dim}[J_i] = 7$. So, we need to consider the normal products $\langle 0|TN_6[J_0](x)X|0\rangle$ and $\langle 0|TN_7[J_i](x)X|0\rangle$. We are going to investigate the conservation law

$$\begin{aligned} & \partial_0\langle 0|TN_6[J_0]X|0\rangle + \partial_i\langle 0|TN_7[J_i]X|0\rangle \\ & \stackrel{?}{=} -i\sum_k\delta(x-x_k)\langle 0|T(\text{Dim}[\varphi] + 2x^0\partial_0 + x^i\partial_i)\varphi(x_k)X_k|0\rangle. \end{aligned} \quad (36)$$

Let us rewrite the formal components of the current including the finite counterterms

$$J_0 = d_a(1+A)\varphi\partial_0\varphi + 2x^0\Theta_{00} + x^i\Theta_{0i} \quad (37)$$

and

$$\begin{aligned} J^i &= -3(a^2+C)\Delta\varphi\partial_i\varphi + 2(a^2+C)(\partial_i\Delta\varphi)\varphi \\ & - 2(b^2+B)\varphi\partial_i\varphi + 2x^0\Theta_{i0} + x^j\Theta_{ji}, \end{aligned} \quad (38)$$

where d_a is the dimension of the scalar field. In lowest order of perturbation $d_a = 2$, but we know that in the quantum case it suffers radiative corrections, i.e., $d_a = 2 + \mathcal{O}(\lambda)$. In order to evaluate (36) it is convenient to use the identities

$$\langle 0|TN_8[x^i\partial_0\Theta_{0i}]X|0\rangle = x^i\partial_0\langle 0|TN_7[\Theta_{0i}]X|0\rangle \quad (39)$$

and

$$\langle 0|TN_8[x^0\partial_i\Theta_{i0}]X|0\rangle = x^0\partial_i\langle 0|TN_9[\Theta_{i0}]X|0\rangle. \quad (40)$$

Furthermore, employing the equations of the energy and momentum conservation (23) and (24), and the equation of motion (9), we get

$$\begin{aligned} \partial_0\langle 0|TN_6[J_0]X|0\rangle + \partial_i\langle 0|TN_7[J_i]X|0\rangle &= (d_a - 2)(1+A)\langle 0|TN_8[\partial_0\varphi\partial_0\varphi]X|0\rangle + (d_a - 2)(b^2 + B)\langle 0|TN_8[\varphi\Delta\varphi]X|0\rangle \\ & - (d_a - 2)(a^2 + C)\langle 0|TN_8[\varphi\Delta^2\varphi]X|0\rangle - (d_a - 4)(m^2 + D)\langle 0|TN_8[\varphi^2]X|0\rangle \\ & - (d_a - 2)\frac{(\lambda + E)}{3!}\langle 0|TN_8[\varphi^4]X|0\rangle + (b^2 + B)\langle 0|TN_8[\partial_i\varphi\partial_i\varphi]X|0\rangle \\ & - i\sum_k\delta(x-x_k)\langle 0|T(d_a + 2x^0\partial_0 + x^i\partial_i)\varphi(x_k)X_k|0\rangle. \end{aligned} \quad (41)$$

By integrating over the space and time, this expression reads

$$\begin{aligned}
 i \sum_k \langle 0 | T(d_a + 2x^0 \partial_0 + x^i \partial_i) \varphi(x_k) X_k | 0 \rangle &= \int dx^0 d^6 x \left\{ (d_a - 2)(1 + A) \langle 0 | TN_8[\partial_0 \varphi \partial_0 \varphi] X | 0 \rangle \right. \\
 &+ (d_a - 3)(b^2 + B) \langle 0 | TN_8[\varphi \Delta \varphi] X | 0 \rangle - (d_a - 2)(a^2 + C) \langle 0 | TN_8[\varphi \Delta^2 \varphi] X | 0 \rangle \\
 &\left. - (d_a - 4)(m^2 + D) \langle 0 | TN_8[\varphi^2] X | 0 \rangle - (d_a - 2) \frac{(\lambda + E)}{3!} \langle 0 | TN_8[\varphi^4] X | 0 \rangle \right\}.
 \end{aligned} \tag{42}$$

Now, we may use Zimmermann identities (12) and (13) to reduce the normal products involving oversubtractions $\langle 0 | TN_8[\varphi \Delta \varphi] X | 0 \rangle$ and $\langle 0 | TN_8[\varphi^2] X | 0 \rangle$. The final result is

$$\begin{aligned}
 i \sum_k \langle 0 | T(d_a + 2x^0 \partial_0 + x^i \partial_i) \varphi(x_k) X_k | 0 \rangle &= \int dx^0 d^6 x \left\{ -(d_a - 4)(m^2 + D) \langle 0 | TN_4[\varphi^2] X | 0 \rangle + [(d_a - 3)(b^2 + B) \right. \\
 &- (d_a - 4)(m^2 + D)p] \langle 0 | TN_6[\varphi \Delta \varphi] X | 0 \rangle + [(d_a - 2)(1 + A) + (d_a - 3)(b^2 + B)g \\
 &- (d_a - 4)(m^2 + D)\bar{s}] \langle 0 | TN_8[\partial_0 \varphi \partial_0 \varphi] X | 0 \rangle + [-(d_a - 2)(a^2 + C) \\
 &+ (d_a - 3)(b^2 + B)k - (d_a - 4)(m^2 + D)\bar{u}] \langle 0 | TN_8[\varphi \Delta^2 \varphi] X | 0 \rangle \\
 &\left. + \left[-(d_a - 2) \frac{(\lambda + E)}{3!} + (d_a - 3)(b^2 + B)l - (d_a - 4)(m^2 + D)\bar{v} \right] \langle 0 | TN_8[\varphi^4] X | 0 \rangle \right\}.
 \end{aligned} \tag{43}$$

where we have redefined: $\bar{s} \equiv s + pg$, $\bar{u} \equiv u + pk$ and $\bar{v} \equiv v + pl$. To have scale invariance symmetry of the Green functions, the coefficient of the three normal products N_8 must vanish independently, which leads to the conditions

$$(d_a - 2)(1 + A) + (d_a - 3)(b^2 + B)g - (d_a - 4)(m^2 + D)\bar{s} = 0, \tag{44}$$

$$-(d_a - 2)(a^2 + C) + (d_a - 3)(b^2 + B)k - (d_a - 4)(m^2 + D)\bar{u} = 0 \tag{45}$$

and

$$-(d_a - 2) \frac{(\lambda + E)}{3!} + (d_a - 3)(b^2 + B)l - (d_a - 4)(m^2 + D)\bar{v} = 0. \tag{46}$$

Let us discuss the consistency of these equations. First of all, as noticed before, the terms g , \bar{s} , k , \bar{u} , l and \bar{v} are of the second order in the coupling constant. So, by writing $d_a = 2 + d_a^{(1)} + d_a^{(2)} + \dots$, where the superscript denotes the order in the coupling constant, the Eq. (44) show us that $d_a^{(1)} = 0$. By writing explicitly the order of each term, we obtain the consistency condition at lowest non trivial order

$$d_a^{(2)} - b^2 g^{(2)} + 2m^2 \bar{s}^{(2)} = 0, \tag{47}$$

$$-a^2 d_a^{(2)} - b^2 k^{(2)} + 2m^2 \bar{u}^{(2)} = 0 \tag{48}$$

and

$$-b^2 l^{(2)} + 2m^2 \bar{v}^{(2)} = 0. \tag{49}$$

We will investigate this equations in the limit $b^2, m^2 \rightarrow 0$. Apparently, this equations are solved by letting $b^2, m^2 = 0$ and also $d_a^{(2)} = 0$. However, from the dimensional analysis of the parameters of the model, it follows that $g^{(2)} = \frac{\lambda^2}{b^2} \mathcal{F}(m^2/b^4, a^2)$, where \mathcal{F} is a dimensionless function of the parameters of the theory. Thus, the product $b^2 g^{(2)}$ does not vanish when $b^2, m^2 \rightarrow 0$, with $m^2/b^4 \rightarrow$ fixed. The

mass term behave analogously: $\bar{s}^{(2)}$ can be written as $\bar{s}^{(2)} = \frac{\lambda^2}{m^2} \mathcal{G}(m^2/b^4, a^2)$ and then the product $m^2 \bar{s}^{(2)}$ does not vanish in the limit $b^2, m^2 \rightarrow 0$, with $m^2/b^4 \rightarrow$ fixed. The same happens with the others terms in the Eqs. (48) and (49): in the specified limit none of the terms containing m^2 or b^2 vanishes. To see this explicitly, let us consider, for example, the condition (49) in the limit $b^2, m^2 \rightarrow 0$, but $m^2/b^4 \equiv c > 0$. The expression for $l^{(2)}$ is given by (19), whereas $\bar{v}^{(2)}$ has the same form that (19), but with the replacing of the normal product $N_6[\varphi \Delta \varphi]$ by $N_4[\varphi^2]$. So, the left hand side of Eq. (49) is

$$\frac{6i^3 \lambda^2}{4!} \int \frac{dk_0}{2\pi} \frac{d^6 k}{(2\pi)^6} \frac{(b^2 \mathbf{k}^2 + 2m^2)}{[k_0^2 - b^2 \mathbf{k}^2 - a^2 (\mathbf{k}^2)^2 - m^2 + i\epsilon]^3}. \tag{50}$$

By taking the limit above mentioned we get

$$\begin{aligned}
 &\frac{6i^3 \lambda^2}{4!} \int \frac{dk_0}{2\pi} \frac{d^6 k}{(2\pi)^6} \frac{(\mathbf{k}^2 + 2c)}{[k_0^2 - \mathbf{k}^2 - a^2 (\mathbf{k}^2)^2 - c + i\epsilon]^3} \\
 &= \frac{1}{4!} \frac{3}{512\pi^3} \frac{\lambda^2}{a^3},
 \end{aligned} \tag{51}$$

which is not zero for any finite a . This shows that Eq. (49) cannot be satisfied, and yields to the anomalous behavior of the quantum Ward identity.

V. CONCLUSIONS

In this work, we extended the method of the normal products to anisotropic theories, what enables us to proceed a systematic analysis of Ward identities. In particular, we investigated the symmetry under space and time translations in a φ^4 model and with $z = 2$ in six spatial dimensions, where the theory becomes renormalizable. Next, we studied the dilatation anomaly, caused by the necessity of the introduction of a scale in the renormalization procedure. When compared with the relativistic case ($z = 1$, $a = 0$, $d = 3$, with b dimensionless), in the anisotropic situation more dimensional parameters are involved, associated with the scale invariance breaking operators $b^2 \partial_i \varphi \partial_i \varphi$ and $m^2 \varphi^2$ in the Lagrangian and hence in the divergence of the dilatation current. In the context of the normal product formalism, these two operators appear oversubtracted, which means that they will produce additional contributions to the anomaly. So, we may intuit that the anomaly is different from the relativistic case, what was explicitly verified in (43) and in the consistency conditions subsequently listed.

As expected, the divergence of the dilatation current is related to the renormalization group beta function. In fact, a precise calculation show us that it is given essentially by (49), namely, $\beta(\lambda) = 4!(-b^2 l^{(2)} + 2m^2 \bar{v}^{(2)}) = (3/512\pi^3) \times (\lambda^2/a^3)$. This beta function was also obtained in [23]. These aspects of the dilatation anomaly are in contrast with the axial anomaly in gauge theories. As shown in [20,39,40] by using the functional integral method, the form of the axial anomaly in the anisotropic situation is exactly the same as in its relativistic counterpart. It has been argued that these differences are related to the ultra-violet character of the dilatation anomaly and the infrared nature of the axial anomaly [43].

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APPENDIX: NOETHER'S THEOREM FOR LAGRANGIANS WITH THIRD ORDER DERIVATIVES

For completeness, in this Appendix we present a derivation of Noether's theorem appropriated to the cases studied in the text. Let us suppose that the basic field changes continuously from $\varphi(x)$ to $\varphi(x) + \delta\varphi$ so that the Lagrangian density changes as

$$\begin{aligned} \delta \mathcal{L} = & \frac{\partial \mathcal{L}}{\partial \phi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \varphi} \delta \partial_\mu \partial_\nu \varphi \\ & + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \varphi} \delta \partial_\mu \partial_\nu \partial_\rho \varphi. \end{aligned} \quad (A1)$$

Now, using the Euler-Lagrange equation of motion,

$$\begin{aligned} -\frac{\partial \mathcal{L}}{\partial \varphi} + \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} - \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \varphi} + \partial_\mu \partial_\nu \partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \varphi} \\ = 0, \end{aligned} \quad (A2)$$

we get

$$\begin{aligned} \delta \mathcal{L} = & \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \varphi} \vec{\partial}_\nu \delta \varphi \right) \\ & + \left(\partial_\mu \partial_\nu \partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \varphi} \right) \delta \varphi \\ & + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \varphi} \delta \partial_\mu \partial_\nu \partial_\rho \varphi, \end{aligned} \quad (A3)$$

yielding

$$\begin{aligned} \delta \mathcal{L} = & \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \varphi} \vec{\partial}_\nu \delta \varphi \right) \\ & + \partial_\mu \left(\partial_\nu \partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \varphi} \delta \varphi \right) \\ & - \partial_\nu \left(\partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \varphi} \partial_\mu \delta \varphi \right) + \partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \varphi} \partial_\nu \partial_\mu \delta \varphi \\ & + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \varphi} \delta \partial_\mu \partial_\nu \partial_\rho \varphi, \end{aligned} \quad (A4)$$

where $A \vec{\partial}_\mu B \equiv A \partial_\mu B - (\partial_\mu A) B$. On the other hand, without using the equation of motion, if we could show that the change in the Lagrangian is a total derivative $\delta \mathcal{L} = \partial_\mu S^\mu$, we may therefore define a conserved current by

$$\begin{aligned} J^\mu = & \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta \varphi + \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \varphi} - \partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \varphi} \right) \vec{\partial}_\nu \delta \varphi \\ & + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \varphi} \partial_\nu \partial_\rho \delta \varphi - S^\mu. \end{aligned} \quad (A5)$$

This result may be generalized. If the Lagrangian contains derivatives up to fourth order we arrive at

$$\begin{aligned} J^\mu = & \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta \varphi + \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \varphi} - \partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \varphi} \right. \\ & \left. + \partial_\rho \partial_\sigma \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \varphi} \right) \vec{\partial}_\nu \delta \varphi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \varphi} \partial_\nu \partial_\rho \delta \varphi \\ & + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \varphi} \vec{\partial}_\nu (\partial_\rho \partial_\sigma \delta \varphi) - S^\mu. \end{aligned} \quad (A6)$$

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