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Effective number of accessed nodes in complex networks

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The measurement called accessibility has been proposed as a means to quantify the efficiency of the communication between nodes in complex networks. This article reports results regarding the properties of accessibility, including its relationship with the average minimal time to visit all nodes reachable after h steps along a random walk starting from a source, as well as the number of nodes that are visited after a finite period of time. We characterize the relationship between accessibility and the average number of walks required in order to visit all reachable nodes (the exploration time), conjecture that the maximum accessibility implies the minimal exploration time, and confirm the relationship between the accessibility values and the number of nodes visited after a basic time unit. The latter relationship is investigated with respect to three types of dynamics: traditional random walks, self-avoiding random walks, and preferential random walks.

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I. INTRODUCTION

A critical issue in the study of complex systems regards the interdependency between connectivity and dynamics [1–3]. For instance, given a specific network topology, it would be interesting to be able to predict how it would behave with respect to several types of dynamics. It has been shown, for example, that reaction-diffusion dynamics spreads more quickly in scale-free complex networks [4] than in uniformly random networks. Also, consensus dynamics tends to converge faster in small-world topologies [5]. A possible way to address this problem is to obtain meaningful measurements of the network topology and then try to correlate them with relevant properties of the dynamics. This analysis can be performed at local or global levels, which provide a complementary characterization of the studied relationship between structure and dynamics.

Particularly important types of dynamics include communications, flow, and diffusion [6–9]. Several real-world complex systems are underlaid by this type of dynamics, including accesses to WWW pages [10], disease spread [11], power distribution collapse [12], and underground and highway systems [13,14]. Frequently the activation of these systems starts at a specific node, or set of nodes, henceforth called sources, and unfolds into the remainder of the network in ways that are intrinsically dependent on the network topology [15]. More specifically, it would be desirable to quantify how effectively a given source can influence the overall network dynamics. By "effectively" is meant the time that is required for the activation to reach specific levels at a given set of nodes, or the total activation at such a set after a given period of time. These concepts are closely related to the so-called coupon-collector problem [16,17]: Given a number of coupons (i.e., nodes), each with a respective probability of occurrence, how many attempts will be required, on average, until all coupons are obtained? Alternatively, it is also important to identify how many nodes will be accessed after a given period of time. The current work addresses these problems through the concept of *accessibility* [18], which quantifies, for a given source node, the number of effectively accessible nodes at a given distance and with respect to a specific dynamics. In this sense, this measure complements the traditional hierarchical degree [19], providing valuable information about the network structure. Note that accessibility takes into account not only the number of nodes at a given distance, but also the transition probabilities between the source and these nodes.

The potential of accessibility to provide valuable insights about the structure and dynamics of complex networks has been confirmed with respect to many applications (Sec. II), including the definition and identification of the borders of complex networks [20]. However, some important aspects of this measurement remain to be formalized in a more comprehensive fashion. For instance, how is accessibility related to the minimum average time required for accessing all reachable nodes? Or in which sense does accessibility quantify the number of effectively accessed nodes? To answer these important questions in a satisfying way constitutes the main objective of the present article, as this paves the way not only to more complete interpretations of the obtained results but also to different types of applications and interpretations. In particular, we show that accessibility can be interpreted in conceptually meaningful way as being related to the number of nodes that can be visited along a given period of time.

This work starts by revising the several applications of accessibility already reported in the literature. Then we define and illustrate the accessibility concept, following by establishing the relationship with the coupon collector problem and showing that accessibility is related to the number of nodes effectively accessed after a period of time.

II. APPLICATIONS

Several different applications have been reported by using the accessibility concept. For instance, it has been shown [18] that, in geographical networks, nodes located close to the peripheral regions have lower values of accessibility. By extending this result to nongeographical networks, it has been possible to define the border of complex networks as the set of nodes with accessibility smaller than a given threshold value [20]. Moreover, recent investigations have showed that the

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position of nodes (inside or outside borders) drastically affects the activity of nodes [21,22]. Other applications unveiled correlations between accessibility and real-world properties of nodes. In particular, in Ref. [23] the authors investigated the network obtained from the theorems in Wikipedia (www.wikipedia.org). In such a network, each theorem is a node, and two nodes are connected whenever a hyperlink is found between the respective theorems. Considering the proof data, the results indicate that the older theorems have higher accessibility values, while newer theorems exhibit lower accessibility values. Consequently new theorems are located at the periphery of the network, defining the frontier of mathematical knowledge. Accessibility has also been used to investigate the effects of underground systems on the transportation properties of large cities. It was shown that overall transportation can be enhanced by incorporating the underground networks [24]. These results were obtained for the London and Paris transportation networks.

III. THE EFFECTIVE NUMBER OF ACCESSIBLE NODES

Given a source node i, suppose it is possible to reach $N_i(h)$ different nodes by performing walks with length h departing from i. Then we say that i has N_i reachable neighbors at distance h. Each neighbor is reached with a different probability, which is represented by the vector $\mathbf{p}_i^{(h)} = \{p_1^{(h)}, p_2^{(h)}, \ldots, p_{N_i(h)}^{(h)}\}$. Given this vector, accessibility of the node i, at scale h, is defined as

$$\kappa_i(h) = \exp\left(-\sum_j p_j^{(h)} \log p_j^{(h)}\right). \tag{1}$$

Accessibility values are in the range $[1, N_i(h)]$, the maximum being obtained for the homogeneous case, when all probabilities have the same value $1/N_i(h)$. This measurement, which is related to the heterogeneity of the vector $\mathbf{p}$, provides a generalization of the classical concept of hierarchical (or concentric) degree [19], as explained in Fig. 1. The hierarchical degree of a source node i, at distance h is defined as $k_i(h) = N_i(h)$; i.e., it is the number of nodes that are at distance h from node i. It is important to note that the value of $k_i(h)$ does not take into account a dynamical process or respective edge weights in the case of weighted networks. Accessibility generalizes the concept of hierarchical degree by considering that a specific dynamics is unfolding in the network. We show in this article that accessibility can be understood as kind of *effective hierarchical degree*.

In Fig. 1(a) we show the hierarchical levels around the source node i up to the distance h. The network topology, as well as a type of random walk adopted, will define the transition probabilities, i.e., the components of the vector $\mathbf{p}$. In Figs. 1(b) and 1(c) we represent these probabilities by using different widths for the edges. Observe that in both cases the source node is able to reach $N_i(h) = 3$ nodes. In the first case, all nodes have the same probability, while in the second case one of the nodes has higher probability than the others. It means that, in the first case, the source node accesses its neighbors in a more uniform manner, which yields an accessibility value equal to 3, as shown in Fig. 1(d). On the other hand, the interaction between the source and its neighbors in the second case is biased to a given node, which decreases the effective hierarchical degree to almost 1.9, as shown in Fig. 1(e).

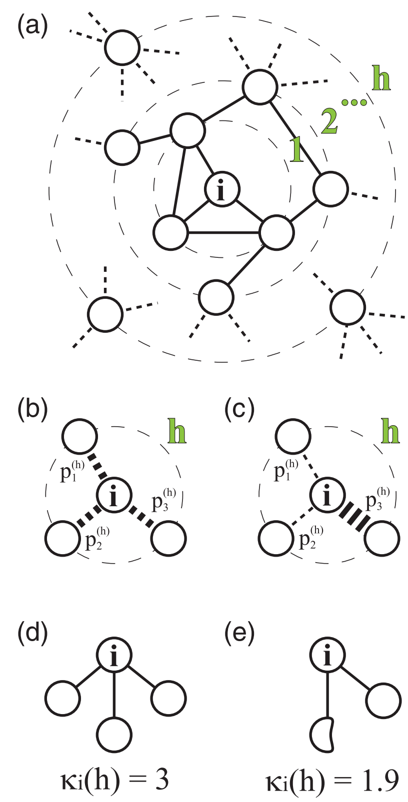

FIG. 1. (Color online) (a) Hierarchical (or concentric) organization around the source node (node i). (b) Homogeneous case, where all neighbors are reached with the same probability. (c) Heterogeneous case, where one node has higher probability to be reached. Accessibilities for the (d) homogeneous and (e) heterogeneous case.

It is important to note that the idea of measuring the heterogeneity among first-neighbors nodes in weighted networks was previously proposed in Refs. [25,26], with the so-called *disparity*. More recently, in Ref. [27] the authors showed a generalization of this measure, namely, the *Rényi disparity*, which is based on the *Rényi entropy*. In a particular case, the *Rényi disparity* uses the *Shannon entropy* in order to quantify the heterogeneity of weights attached to the edges of a node. This particular case has a similar equation to Eq. (1). However, in our case we consider not only the first neighbors, but all nodes that can be reached at distance h by a specific dynamic. In this sense our approach can be also applied to nonweighted networks, since we consider the transition probabilities instead of the edge weights.

Another way to think about the interaction between a source node and its neighbors is by considering the coupon collector problem. This problem [16,17] deals with the following question: On average, how many walks with length h departing from i are required in order to visit all neighboring nodes of i after h steps at least one time? We will call this quantity the *exploration time* of the node i and denote it by $\tau_i(h)$, since we can consider the displacement velocity through the network constant. Then the number of walks is proportional to the time needed to visit all $N_i(h)$ nodes. This problem can be mapped into a Poisson problem [28] with independent variables, which yields

$$\tau_i(h) = \int_0^\infty \left[1 - \prod_{j=1}^{N_i(h)} \left(1 - e^{-xp_j^{(h)}}\right)\right] dx. \tag{2}$$

A conjecture has been proposed [17,29] that τ_i reaches its minimum value for the homogeneous case, where all neighboring nodes are reached with the same probability, i.e., $p_j^{(h)} = 1/N_i(h)$ for any j. In this case, it is not difficult to show that Eq. (2) can be rewritten as

$$\tau_i^{\text{hom}}(h) = N_i(h) \sum_{m=1}^{N_i(h)} \frac{1}{m}. \tag{3}$$

Therefore, by using the conjecture cited above, we can say that accessibility is maximum whenever the exploration time is minimum. This characteristic is illustrated in Fig. 2(a), which shows a scatter plot between accessibility κ and the exploration time $\tau(h)$ for 10^5 randomly generated vectors $\mathbf{p}$ with length $N = 6$ (each component is chosen from the uniform distribution and then normalization is imposed). A set of important curves is also shown in the scatter plot, which provides a more comprehensive characterization of the probability configuration. They correspond to the specific cases where exactly $n\,(\leqslant \lfloor N/2 \rfloor)$ probabilities of $\mathbf{p}$ have a value ϵ, while all the other $(N-n)$ probabilities are also identical among each other (so that the sum of all these probabilities becomes equal to one). Therefore, the straight line is related to the case where $n = 1$, so that the other $N-1$ probabilities have the same value. Also, this line corresponds to the bounding value of accessibility given $\tau(h)$, meaning that all the possible configurations of $\mathbf{p}$ are enclosed by this curve. The dashed line corresponds to the configurations where $n = 2$. Similarly, the dotted line corresponds to the situations where $n = 3$; in this case, half of each of the probabilities are equal between themselves. One can use the parametrization ϵ in Eqs. (1) and (2) in order to obtain a general equation (indexed C) characterizing these curves:

$$\tau_C(\epsilon) = \frac{1}{\epsilon}\sum_{m=1}^{n}\frac{1}{m} + \frac{1}{p}\sum_{m=1}^{N-n}\frac{1}{m} - \sum_{m=1}^{d}\sum_{m'=1}^{N-n}(-1)^{m+m'}\binom{n}{m}\binom{N-n}{m'}\frac{1}{m\epsilon + m'p} \tag{4}$$

and

$$\kappa_C(\epsilon) = \frac{1}{p}\left(\frac{p}{\epsilon}\right)^{\epsilon n}, \tag{5}$$

where $p = (1 - n\epsilon)/(N-n)$. Importantly, observe that ϵ lies in the interval $[0,1/n]$. When $\epsilon < 1/N$, the upper part of the curves is obtained. In this case we have $\kappa_C \to N - n$ and $\tau_C \to \infty$ for $\epsilon \to 0$. For $\epsilon > 1/N$, we have the bottom part of the curves, for which $\kappa_C \to n$ and $\tau_C \to \infty$, when $\epsilon \to 1/n$. When $\epsilon = 1/N$, we reach the homogeneous case, where accessibility is maximum and the exploration time is minimum. The arrows in the figure indicate the direction in which ϵ grows. Note also that, for the specific case of $n = 3$, there is no difference between the lower and upper part of the curve, since exactly half of the probabilities have a value equal to ϵ. Figure 2(b) complements the results providing the conditional density of the generated vectors for three different values of κ.

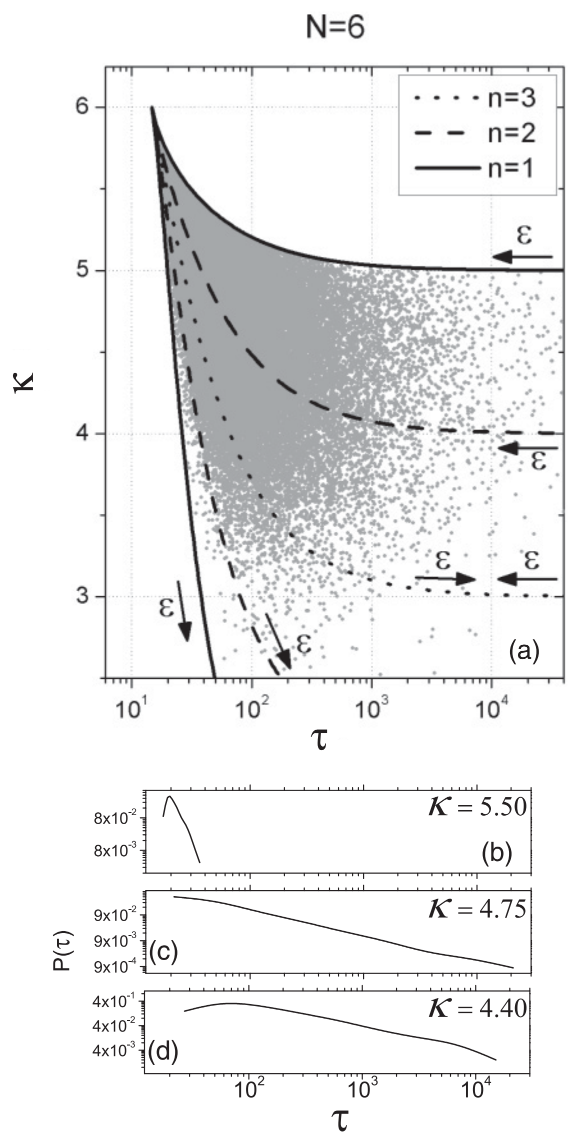

FIG. 2. (a) Scatter plot between accessibility and the exploration time for 10^5 random vectors with length $N = 6$. The lines correspond to the cases where the probabilities are divided in two groups having the same values among themselves as described by Eqs. (4) and (5) with parametrization ϵ. (b–d) Conditional distribution of τ for three different values of κ.

A. Probabilities in uniformly random networks

Now we investigate the coupon collector problem in uniformly random networks. More specifically, we used 5000 strongly connected realizations of the Erdős-Rényi model with 200 nodes and average degree 4, and then derived the transition probabilities from these respective networks. We performed random walks originating from each of the nodes in the networks so as to obtain the respective transition probabilities (the set of $\mathbf{p}$) by using the powers of the transition matrix [30].

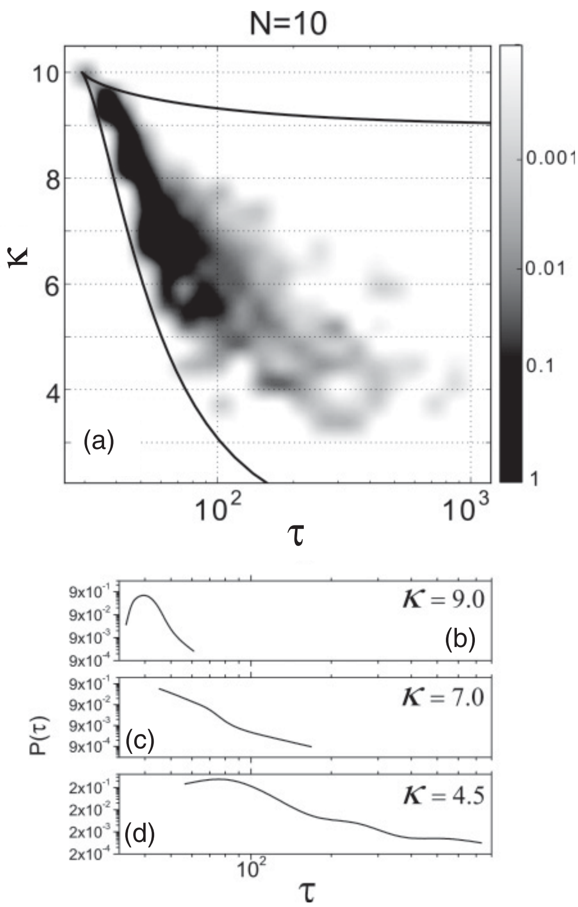

FIG. 3. (a) Distribution of the probability configurations in the $\kappa \times \tau$ space obtained for 5000 Erdős-Rényi networks. We considered the cases where 10 nodes are accessible after h steps ($2 \leqslant h \leqslant 15$) along random walks originating from each of the nodes of the networks. (b–d) Conditional distribution of τ for three different values of κ.

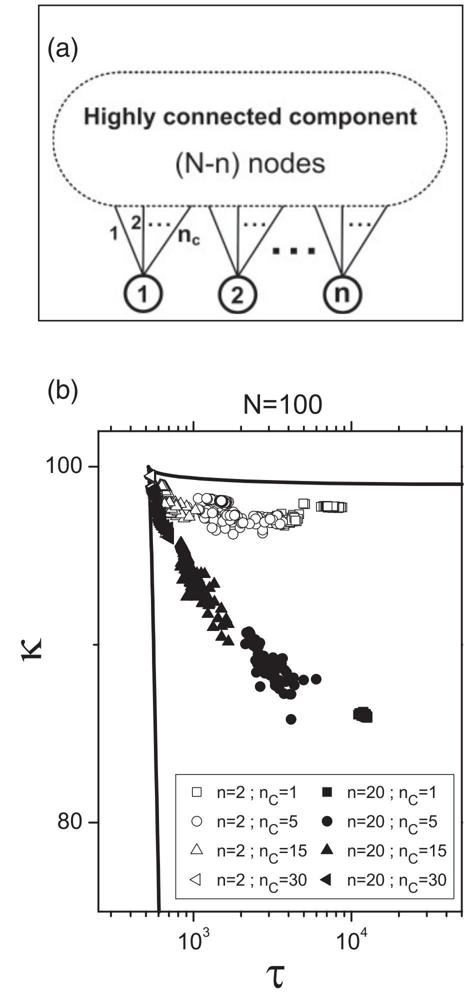

FIG. 4. (a) Example of a possible configuration where the upper region of the $\kappa \times \tau$ space is occupied. (b) Results obtained for random walks in the considered network for $n = 2$ (empty symbols) and $n = 20$ (filled symbols) and different number of connections n_c.

Figure 3(a) presents the distribution of the cases in the $\kappa \times \tau$ space. This result takes into account all cases where the number of accessible nodes $N_i(h)$ is equal to 10 for values of h in the interval [2,15]. The gray levels correspond to the density of cases, although the conditional distribution of τ is also provided for some values of κ [Fig. 3(b)]. Remarkably, the density is highly skewed toward the lower bound of the ϵ curve, and virtually no cases are obtained for the upper half of the probability region. This means that it is extremely unlikely to obtain probability configurations having the majority of nodes with higher probability, as illustrated in Fig. 2.

However, it is possible to obtain configurations that occupy the upper boundary region in the $\kappa \times \tau$ space, where the minority of the probabilities have smaller values. Figure 4(a) presents a particular situation exemplifying this case considering an artificial network with N nodes consisted of two groups: (1) a highly connected ER component with $(N - n)$ nodes and average degree $\langle k \rangle_c$; and (2) n loosely connected nodes with n_c ($n_c \ll \langle k \rangle_c$) links to the previous subgraph. This topological division implies that the nodes in the ER component will be much more accessed than the others when considering random walks in this network, irrespective of the starting node and the length h. Thus, in the case of $n \ll N$, the probability vectors $\mathbf{p}$ will have the majority of their components with higher values, thus occupying the upper region of the $\kappa \times \tau$ space. This property is verified for the simulations presented in Fig. 4(b) through random walks departing from each node for values of h (varied from 2 to 15) where all nodes are reachable. We considered a single realization of the network with $N = 100$ and ER component with $\langle k \rangle_c$ equal to 50. It was assumed $n = 2$ (empty symbols) and $n = 20$ (filled symbols) with n_c links, varying from 1 to 30, as indicated in the figure. Observe that, as the value of n_c decreases, the n nodes become less accessible, and the points move away from the origin (the homogeneous case), as expected. Although we assumed that the single nodes are directly connected to the ER component, this example can be immediately extended considering the presence of tails of nodes with different sizes. While this network can be artificially created, obtaining similar results for the occupation of the $\kappa \times \tau$ space, it has been shown [31] that tails are unlikely to occur in great variety of real networks, even for tails with short size. Results for real networks will be shown in the next section.

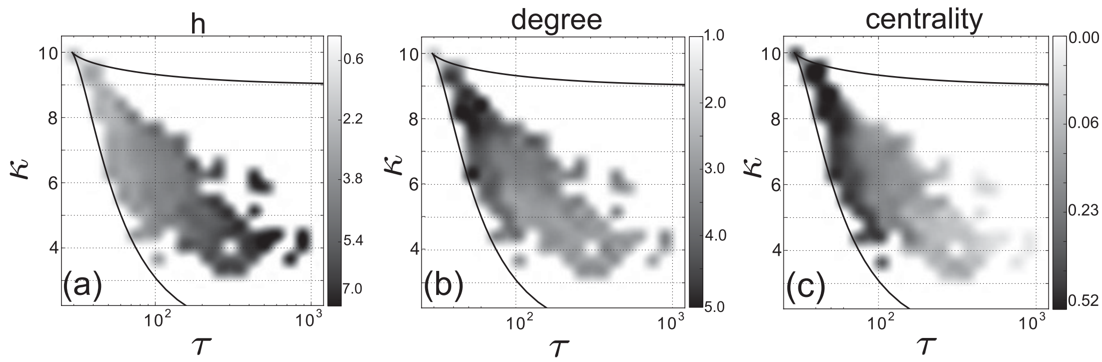

FIG. 5. Local average measurements for the probability configurations obtained for 5000 Erdős-Rényi networks. (a) Number of steps necessary to reach 10 nodes after departing from the source node, (b) the degree of the source node, and (c) the eigenvector centrality of the source node.

Figure 5 complements the characterization of the $\kappa \times \tau$ space. It shows (a) the local average number of steps necessary to reach 10 nodes after departing from the source node, (b) the degree of the source node, and (c) its eigenvector centrality obtained for the probability configurations. It is clear from Fig. 5(a) that random walks with larger number of steps (i.e., h) tend to have smaller accessibility and longer exploration time. On the other hand, random walks starting from nodes with larger degree [Fig. 5(b)] tend to have larger accessibility and shorter exploration times, though in a less definite fashion than that observed in Fig. 5(a). Furthermore, Fig. 5(c) shows a remarkable centrality pattern: It tends to increase with κ while decreasing with τ, apparently following the level set curves in Fig. 2. It should be observed that these results are specific for the uniformly random ER networks, in the sense that different trends may be obtained for other theoretical network models.

B. Probabilities from real-world networks

We also considered probabilities obtained from real-world networks, namely, circuits [32], power grids [33], German highways [34], protein interactions [35], e-mails [36], and coauthorships in network science [37]. The probability configurations obtained from these networks are shown in Fig. 6. Again, most of the cases tend to appear near the lower boundary in the $\kappa \times \tau$ space, which is characterized by low exploration time and varying accessibility. This is particularly interesting, as it suggests a universal asymmetry in both real and random uniform networks in which probability vectors containing a large number of entries with high values are unlikely. Therefore, the exploration time tends to be minimized at the expense of varying accessibilities.

Now we proceed to a related problem in which we are interested to know how many nodes, on the average, are visited during the time interval t, while performing a specific type of random walk. This quantity will be denoted by $\eta_i(t,h)$, providing information about how the network topology around the source node affects the interaction with its neighbors. After a long time, we expect that the source node will be able to visit all $N_i(h)$ neighbors, i.e., $\lim_{t\to\infty} \eta_i(t,h)) \to N_i(h)$, independent of the vector $\mathbf{p}_i^{(h)}$. Therefore, we can consider that the value of $\eta_i(t,h)$ provides an estimate of the average number of visited nodes during a finite time. This is confirmed in Fig. 7 for the US airlines network [38] and two random counterparts: *Erdős-Rényi model* and *Configuration model* [39,40]. In order to obtain the transition probabilities, we considered three different types of random walks: (1) traditional random walk (TRW), (2) preferential random walk (PRW), and (3) self-avoid random walk (SARW). They were estimated for $h = \ell$, where $\ell = 3$ is the network diameter. The TRW and PRW dynamics were calculated by using powers of the transition matrix, while the SARW was estimated through agent-based simulation. This calculation was repeated 10^6 times for each source node. It should be noted that for the PRW dynamics the probability of transition from a node i to a node j is proportional to the degree of j, i.e., $p_{i\to j} = k_j / \sum_{m\in\Gamma_i} k_m$, where Γ_i denotes the neighbors of i. In the case of the SARW dynamics, if an agent cannot proceed further, it remains at the final node contributing to the probabilities for the next steps [18]. Then the transition probabilities were used in Eq. (1) to evaluate accessibility of node i. The values of $\eta_i(t,h)$ were obtained for each node i

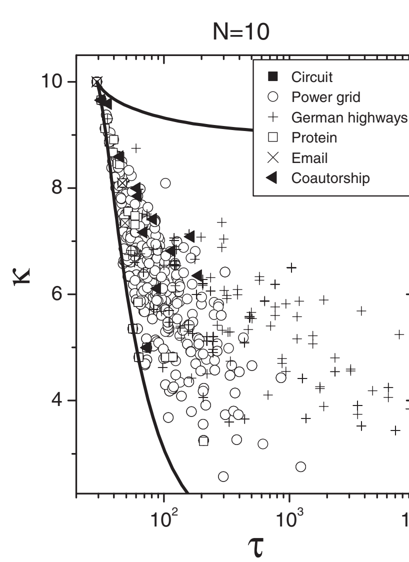

FIG. 6. Scatter plot between accessibility and the exploration time for six real networks considering the cases where $N = 10$ nodes are reached after $h = \{2,3,\ldots,20\}$.

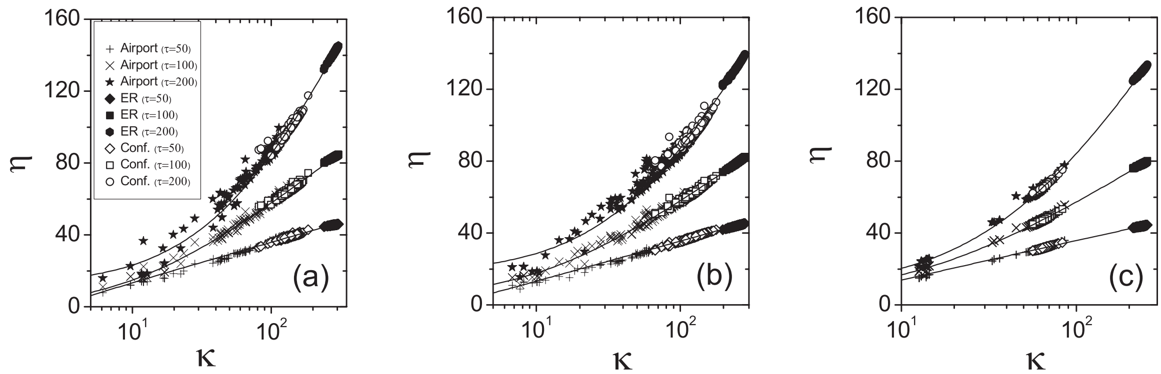

FIG. 7. Fraction of reached nodes $\eta_i(t,h)$ as a function of the node accessibility κ_i for (a) classical random walk, (b) self-avoid random walk, and (c) preferential random walk. The results are illustrated for the US airlines network and two random counterparts: *Erdős-Rényi model* and the respective *Configuration model* [39,40]. All of them have the same number of nodes ($N = 332$) and the same average degree ($\langle k \rangle = 12.81$).

as follows: First, we draw t neighbors of node i at distance $h = \ell$ according to the obtained probabilities $p_j(\ell)$. Then we count how many different nodes were drawn. The average over several realizations gives us an estimate of $\eta_i(t,h)$. The behavior of κ versus η is showed in Figs. 7(a)–7(c) for TRW, SARW, and PRW respectively. The results have been found to be well fitted by the functional form:

$$\eta = a + b\log(\kappa + c). \tag{6}$$

Observe that, irrespective of the adopted dynamics, the values obtained for the ER network are higher and more concentrated than those verified for other networks. This is a directed consequence of the low topological variability observed in these model. Indeed, $\kappa(h)$ and $\eta(t,h)$ are limited to the number of reachable nodes at distance h and become greater with the homogeneity of probability profile. Thus, the presence of hubs or degree heterogeneity tends to lower both measurements, as can be observed for the airport network as well as its configuration model. Comparing the dynamics, no significant difference was observed between TRW and SARW. However, for preferential random walk, the walks are biased to pass through nodes with high connectivity, implying the results to change significantly. Therefore, the effective number of reached nodes is smaller, enhancing the differences found for the airport network and the configuration model.

IV. CONCLUSIONS

The accessibility concept was introduced recently [18] as a means to quantify the potential of a node to interact with other nodes in a complex network. Given the many promising results obtained so far, it became important to better understand the accessibility concept, especially regarding optimization aspects. The present work focused on the investigation of accessibility regarding the coupon collector problem as well as its relationship with the average number of nodes visited along a random walk during a given time interval.

A number of remarkable results were obtained about the relationship between accessibility and the exploration time. First, we have that the minimal exploration time is obtained for maximum accessibility. No relationship between these two properties has been observed otherwise; i.e., when we consider all possible probability configurations. However, in the cases of uniformly random and real-world networks, a stronger correlation is verified, with the cases tending to lie near the lower boundary in the $\kappa \times \tau$ space. As a matter of fact, there is a very low probability of having cases occupying the upper half portion of this space. Although this could suggest some intrinsic impossibility of having such cases, we showed at least one type of topology leading to a configuration lying over the upper boundary. This configuration involves the coexistence of two rather distinct groups of degrees, one with very high values (the connected kernel) and another very low (the surrounding attached nodes). Indeed, real and model networks are almost invariably characterized by a much smoother degree distribution, implying their respective nodes occupy the lower left portion of the accessibility × exploration time diagram. In other words, most existing networks are intrinsically optimized. This remarkable result shows that in all considered networks the transition probability configurations tend to be characterized by small exploration time at the expense of varying accessibilities.

Regarding the relationship between accessibility and the average number of nodes visited along a random walk during a given time interval, we showed that the concept of accessibility can be understood as a generalization of the classical degree, in the sense that accessibility quantifies the effective number of nodes that can be reached from the source node after a given number of steps. In order to confirm this statement, we also showed a strong relationship between accessibility and the inverse coupon collector problem, which deals with the number of visited nodes in a finite time interval.

Future work could take into a account activations originating from multiple nodes as well as how other dynamical properties can be predicted from the accessibility values. It would be particularly interesting to identify more general theoretical models and real networks capable of covering the $\kappa \times \tau$ more uniformly.

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