2012

STABLE PIECEWISE POLYNOMIAL VECTOR FIELDS

ELECTRONIC JOURNAL OF DIFFERENTIAL EQUATIONS, SAN MARCOS, v. 40, n. 2, supl., Part 1-2, pp. 861-866, SEP 22, 2012

http://www.producao.usp.br/handle/BDPI/41494

STABLE PIECEWISE POLYNOMIAL VECTOR FIELDS

CLAUDIO PESSOA
Universidade Estadual Paulista, UNESP–IBILCE
Av. Cristovão Colombo, 2265
15.054–000, S. J. Rio Preto, SP, Brasil
pessoa@ibilce.unesp.br

JORGE SOTOMAYOR
Instituto de Matemática e Estatística, Universidade de São Paulo
Rua do Matão 1010, Cidade Universitária
05.508-090, São Paulo, SP, Brasil
sotp@ime.usp.br

Abstract. Consider in $\mathbb{R}^2$ the semi-planes $N = \{y > 0\}$ and $S = \{y < 0\}$ having as common boundary the straight line $D = \{y = 0\}$. In N and S are defined polynomial vector fields X and Y, respectively, leading to a discontinuous piecewise polynomial vector field $Z = (X, Y)$. This work pursues the stability and the transition analysis of solutions of Z between N and S, started by Filippov (1988) and Kozlova (1984) and reformulated by Sotomayor–Teixeira (1995) in terms of the regularization method. This method consists in analyzing a one parameter family of continuous vector fields Z_ϵ, defined by averaging X and Y. This family approaches Z when the parameter goes to zero. The results of Sotomayor–Teixeira and Sotomayor–Machado (2002) providing conditions on (X, Y) for the regularized vector fields to be structurally stable on planar compact connected regions are extended to discontinuous piecewise polynomial vector fields on $\mathbb{R}^2$. Pertinent genericity results for vector fields satisfying the above stability conditions are also extended to the present case. A procedure for the study of discontinuous piecewise vector fields at infinity through a compactification is proposed here.

1. INTRODUCTION

One of the most accomplished stability theories for dynamical systems is that of Andronov–Pontryagin [2] and Peixoto [11] for C^1 vector fields in the plane and on surfaces. Elements of this theory provide characterization and genericity results for structurally stable vector fields. Extensions of this theory to the class of discontinuous, piecewise smooth, vector fields have been provided by Filippov [5] and Kozlova

1991 *Mathematics Subject Classification.* Primary 34C35, 58F09; Secondary 34D30.

Key words and phrases. structural stability, piecewise vector fields, infinity, compactification.

The first author is supported by FAPESP-Brazil Project 2011/13152-8 and by Programa Primeiros Projetos-PROPe/UNESP. The second author is fellow of CNPq and participates in the Project Fapesp 2008/02841-4.

[9]. The need for such an extended theory goes back to Andronov et al. [1].

In [5], Filippov defined the rules (revisited below) for the transition of the orbits crossing the line D of discontinuity which separates two regions N and S on which the field, given respectively by X and Y, is smooth. He also prescribed when the orbit slides along D. This leads to an orbit structure that is not always a flow on the surface obtained gluing N and S along D. The work of Kozlova [9, 5] pursues the setting established by Filippov.

In [14], Sotomayor and Teixeira developed the *regularization method*, taking as domain the sphere S^2 and the equation as the discontinuity line D. This method consists in defining a one parameter family of continuous vector fields that, when the parameter goes to zero, approaches the discontinuous one. To this end, a transition function φ is used to average X and Y in order to get the family of continuous vector fields. Sotomayor and Teixeira provided conditions on $Z = (X, Y)$, which imply that the regularized vector fields are in the class of Andronov–Pontryagin [2] and Peixoto [11] for C^1 vector fields and consequently are structurally stable. Moreover, Sotomayor and Machado [10] applied the method outlined above to the case of a compact planar region M, with a smooth border ∂M and having as discontinuity line either a segment with extremes on ∂M or a closed curve disjoint of ∂M. The conditions given in [14] are extended to this case and their genericity, not discussed in [14], is established.

Other developments in this direction can be found in Garcia–Sotomayor [6], where piecewise linear vector fields are studied and in Buzzi–da Silva–Teixeira [4], where the method of singular perturbations is used to study certain discontinuous piecewise smooth vector fields. For interesting examples in applied subjects of discontinuous systems the reader is addressed to [1] and [3].

In this paper we deal with discontinuous piecewise vector fields Z defined by a pair (X, Y), where X and Y are polynomial vector fields in the plane.

A *polynomial vector field X in* $\mathbb{R}^2$ is a vector field of the form

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y},$$

where P and Q are polynomials in the variables x and y with real coefficients. We define the *degree* of the polynomial vector field X as $\max\{\deg P, \deg Q\}$. We can write $P(x, y) = \sum a_{ij}x^iy^j$ and $Q(x, y) = \sum b_{ij}x^iy^j$, $0 \leq i + j \leq m$. Hence X has degree $\leq m$. The $l = (m+1)(m+2)$ real numbers $\{a_{i,j}, b_{ij}\}$ are called the *coefficients of* X.

The space of these vector fields, endowed with the structure of affine $\mathbb{R}^l$-space where X is identified with the l-tuple $(a_{00}, a_{10}, \dots, a_{0m}, b_{00}, \dots, b_{0m})$ of its coefficients, is denoted by χ_m.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = y$. In what follows we use the following notation: $D = f^{-1}(0)$, $N = f^{-1}(0, \infty)$ and $S = f^{-1}(-\infty, 0)$.

Let Ω_m be the space of vector fields $Z = (X, Y)$ defined by:

$$Z(q) = \begin{cases} X(q) & \text{if } f(q) \geq 0, \\ Y(q) & \text{if } f(q) \leq 0, \end{cases}$$

where $X, Y \in \chi_m$ and $\deg X = \deg Y = m$. We write $Z = (X, Y)$, which will be allowed to be bi-valued at points of D. In general the degrees of X and Y can be different, but in the present study, to simplify the notation and some computations, we take them to be equal.

The Poincaré compactification of $X \in \chi_m$ is defined to be the unique analytic vector field $\mathcal{P}(X)$ tangent to the sphere $S^2 = \{x^2+y^2+z^2 = 1\}$ whose restriction to the northern hemisphere $S^2_+ = \{S^2 : z > 0\}$ is given by $z^{m-1}\wp^*(X)$, where $\wp$ is the central projection from $\mathbb{R}^2$ to S^2_+, defined by $\wp(u, v) = (u, v, 1)/\sqrt{u^2 + v^2 + 1}$. See [7] for a verification of the uniqueness and analyticity of $\mathcal{P}(X)$.

Through the Poincaré compactification, the discontinuous piecewise polynomial vector field $Z = (X, Y)$ induces a discontinuous piecewise analytic vector field tangent to S^2, with S^1 invariant, defined by $\mathcal{P}(Z) = (\mathcal{P}(X), \mathcal{P}(Y))$. Notice that, for $\mathcal{P}(Z)$ restricted to the northern hemisphere, the function f becomes $f(x, y, z) = y$ with $(x, y, z) \in S^2_+$, the set of discontinuity is given by $D = \{S^2 : y = 0\}$ and the semi-planes N and S become the semi-hemispheres $N = \{S^2 : z > 0 \text{ and } y > 0\}$ and $S = \{S^2 : z > 0 \text{ and } y < 0\}$, respectively. Thus, $\mathcal{P}(Z)$ can be used to study the global structure of the orbits of Z.

By a *transition function* we mean a C^∞ function $\varphi : \mathbb{R} \to \mathbb{R}$ such that: $\varphi(t) = 0$ if $t \leq -1$, $\varphi(t) = 1$ if $t \geq 1$ and $\varphi'(t) > 0$ if $t \in (-1, 1)$.

Definition 1. *The φ_ϵ-compactification of $Z = (X, Y) \in \Omega_m$ is the one parameter family of C^∞ vector fields $\mathcal{P}(Z)_\epsilon$ in S^2 given by*

$$\mathcal{P}(Z)_\epsilon(q) = (1 - \varphi_\epsilon(f(q)))\mathcal{P}(Y)(q) + \varphi_\epsilon(f(q))\mathcal{P}(X)(q),$$

where $\varphi_\epsilon(t) = \varphi(\frac{t}{\epsilon})$.

Denote by $\chi^r(S^2, S^1)$ the space of C^r vector fields on S^2, $r \geq 1$, such that S^1 is invariant by the flow of the vector fields.

Definition 2. *$X \in \chi^r(S^2, S^1)$ is said to be structurally stable if there is a neighborhood V of X and a map $h : V \to Hom(S^2, S^1)$ (homeomorphisms of S^2 which preserve S^1) such that $h_X = Id$ and h_Y maps orbits of $\mathcal{P}(X)$ onto orbits of $\mathcal{P}(Y)$, for every $Y \in V$.*

Definition 3. *We call $\Sigma^r(S^2, S^1)$ the subset of $\chi^r(S^2, S^1)$ of vector fields that have all their singularities hyperbolic, all their periodic orbits hyperbolic and do not have saddle connections in S^2 unless they are contained in S^1.*

We have that the elements of $\Sigma^r(S^2, S^1)$ are structurally stable in the sense of definition 2.

In the next sections of this paper we will extend to the case of discontinuous piecewise polynomial vector fields in $\mathbb{R}^2$ the study performed in [14, 10] for piecewise smooth vector fields. To this end we will give sufficient conditions on $Z = (X, Y) \in \Omega_m$ which determine the structural stability of its φ_ϵ-compactification $\mathcal{P}(Z)_\epsilon$ (Definition 1), for any transition function φ and small ϵ. More precisely in Section 3 will be defined a set G_m (Definition 19) of discontinuous piecewise polynomial vector fields that satisfy sufficient conditions, reminiscent to those which define $\Sigma^r(S^2, S^1)$, in order to have a structurally stable φ_ϵ-compactification. In Section 4, the genericity of G_m will be established. A preliminary analysis of relevant local aspects of discontinuous piecewise polynomial vector fields is developed in Section 2. There is studied the effect of φ_ϵ-compactification on singular points, closed orbits and polytrajectoris (Definition 10) in $\mathbb{R}^2$ and in S^1 and on saddle separatrices.

2. φ_ϵ-Compactification of Singular Points, Closed and Saddle Separatrix Poly-Trajectories

In this section, using the notations, definitions and results of [14, 10], we define the regular and singular points of Z (resp. $\mathcal{P}(Z)$), the closed *poly-trajectories* and then we study the effects of the φ_ϵ-compactification on vector fields around these points and poly-trajectories. The main goal here is to determine the conditions for the φ_ϵ-compactification to have only regular points, hyperbolic singularities and hyperbolic closed orbits.

2.1. **Regular and Singular Points.** Given any $Z = (X, Y) \in \Omega_m$, following Filippov terminology (as [5]), we distinguish the following arcs in D:

- Sewing Arc (SW): characterized by $(Xf)(Yf) > 0$ (see Figure 1 (a)).
- Escaping Arc (ES): given by the inequalities $Xf > 0$ and $Yf < 0$ (see Figure 1 (b)).
- Sliding Arc (SL): given by the inequalities $Xf < 0$ and $Yf > 0$ (see Figure 1 (c)).

As usual, here and in what follows, Xf will denote the derivative of the function f in the direction of the vector X, i.e., $Xf = \langle \nabla f, X \rangle$.

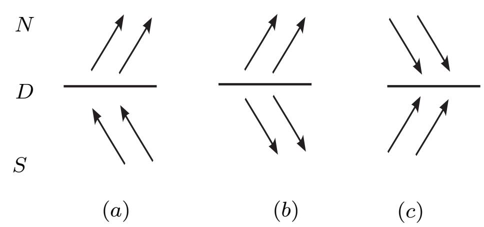

FIGURE 1. Arcs on D

On the arcs ES and SL we define the *Filippov vector field* F_Z associated to $Z = (X, Y)$, as follows: if $p \in SL$ or ES, then $F_Z(p)$ denotes the vector in the cone spanned by $X(p)$ and $Y(p)$ that is tangent to D, see Figure 2.

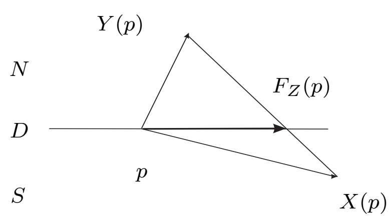

FIGURE 2. Filippov vector field

Definition 4. *A point $p \in D$ is called a D-regular point of Z if one of the following conditions holds:*

(1) $Xf(p).Yf(p) > 0$. *This means that $p \in SW$;*
(2) $Xf(p).Yf(p) < 0$ *but* $det[X, Y](p) \neq 0$. *This means that p belongs either to SL or ES and it is not a singular point of F_Z (see Figure 3).*

Now, we define the notion of hyperbolicity for the singular points of F_Z.

Definition 5. *A point $p \in D$ is called a singular point of F_Z if $Xf(p).Yf(p) < 0$ and $det[X, Y](p) = 0$. If we have $d(det[X, Y]|_D)(p) \neq 0$, then p is called a hyperbolic singular point of F_Z. Here $d(det[X, Y]|_D)(p)$ denote the derivative of $det[X, Y]|_D$ at point p.*

Let $p \in D$ be a hyperbolic singular point of F_Z. The point p is called a saddle if $p \in SL$ and $d(det[X, Y]|_D)(p) > 0$ or $p \in ES$ and $d(det[X, Y]|_D)(p) < 0$. The point p is called a node if $p \in SL$ and

$d(det[X,Y]|_D)(p) < 0$ or if $p \in ES$ and $d(det[X,Y]|_D)(p) > 0$ (see Figure 4).

In the next definition we extend the notion of hyperbolic singular point, located in D, for Z.

Definition 6. *A point $p \in D$ is an elementary D-singular point of $Z = (X,Y)$ if one of the following conditions is satisfied:*

1. *The point p is a fold point of $Z = (X,Y)$. This means that: either p is a fold point of X: $Yf(p) \neq 0, Xf(p) = 0$ and $X^2f(p) \neq 0$; or p is a fold point of Y: $Xf(p) \neq 0, Yf(p) = 0$ and $Y^2f(p) \neq 0$ (see Figure 5);*
2. *The point p is a hyperbolic singular point of F_Z.*

The definitions above can be reformulated in a similar way in the case of discontinuous piecewise analytic vector field $\mathcal{P}(Z)$ in S^2.

To determine the behavior of singular points and periodic orbits of $\mathcal{P}(Z)$ we will obtain an expression of $\mathcal{P}(Z)$ in polar coordinates. Take coordinates (θ,ρ), 2π-periodic in θ, defined by the covering map from $(-1,1) \times \mathbb{R}$ onto $S^2 \setminus \{(0,0,\pm 1\}$, given by $(\theta,\rho) \mapsto (x,y,z) = (1+\rho^2)^{-1/2}(\cos\theta, \sin\theta, \rho)$.

The expression for $z^{m-1}\wp^*(X)$, $X = (P,Q) \in \chi_m$, in these coordinates is

$$(1+\rho^2)^{(1-m)/2}\left[\left(\sum \rho^i A_{m-i}(\theta)\right)\frac{\partial}{\partial\theta} - \rho\left(\sum \rho^i R_{m-i}(\theta)\right)\frac{\partial}{\partial\rho}\right],$$

where $i = 0,1,\ldots,m$ and

$$A_k(\theta) = A_k(X,\theta) = Q_k(\cos\theta,\sin\theta)\cos\theta - P_k(\cos\theta,\sin\theta)\sin\theta,$$

$$R_k(\theta) = R_k(X,\theta) = P_k(\cos\theta,\sin\theta)\cos\theta + Q_k(\cos\theta,\sin\theta)\sin\theta,$$

with $P_k = \sum a_{ij}x^iy^j$, $Q_k = \sum b_{ij}x^iy^j$, $i+j = k$. Now, we perform a change in the time variable to remove the factor $(1+\rho^2)^{(1-m)/2}$ and to obtain a vector field defined in the whole plane (θ,ρ), i.e. we have the vector field

$$\left(\sum \rho^i A_{m-i}(\theta)\right)\frac{\partial}{\partial\theta} - \rho\left(\sum \rho^i R_{m-i}(\theta)\right)\frac{\partial}{\partial\rho}, \tag{1}$$

with $i = 0,1,\ldots,m$. Note that we also can obtain (1) directly from $X = (P,Q)$ introducing in the plane (x,y) the change of variables $x = \cos\theta/\rho$, $y = \sin\theta/\rho$. Moreover, the axis θ, i.e. $\{(\theta,\rho) : \rho = 0\}$, is invariant by (1) and corresponds to the points at infinity of $\mathbb{R}^2$. Therefore, to study the behavior of solutions of $\mathcal{P}(Z)$, $Z = (X,Y) \in$

Ω_m with $X = (P_1, Q_1)$ and $Y = (P_2, Q_2)$, is equivalent by (1) to study the discontinuous piecewise trigonometric vector field

$$(2) \quad \begin{cases} \left(\sum \rho^i A_{1,m-i}(\theta), -\rho \sum \rho^i R_{1,m-i}(\theta)\right), & \text{if } \theta \in [0,\pi], \rho \geq 0, \\ \left(\sum \rho^i A_{2,m-i}(\theta), -\rho \sum \rho^i R_{2,m-i}(\theta)\right), & \text{if } \theta \in [\pi,2\pi], \rho \geq 0, \end{cases}$$

with $i = 0, 1, \ldots, m$, where $A_{1,k}(\theta) = A_k(X,\theta)$, $A_{2,k}(\theta) = A_k(Y,\theta)$, $R_{1,k}(\theta) = R_k(X,\theta)$ and $R_{2,k}(\theta) = R_k(Y,\theta)$.

We remark that $S^1 \cap D = \{(\pm 1, 0, 0)\}$. Hence, if $p \in S^1 \cap D$ is not a singular point of $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ then, as S^1 is invariant by $\mathcal{P}(Z)$ and so by $\mathcal{P}(Z)_\epsilon$ (Definition 1) , it follows that p is a point of sewing arc SW or p is a singular point of the Filippov vector field $F_{\mathcal{P}(Z)}$.

Suppose that $(1, 0, 0)$ is a singular point of $F_{\mathcal{P}(Z)}$. This point corresponds to the point $(0, 0)$ in the chart (θ, ρ) and in this chart $D = \{(0,\rho) : \rho \geq 0\} \cup \{(\pi,\rho) : \rho \geq 0\} \cup \{(2\pi,\rho) : \rho \geq 0\}$. Therefore, by (2), it follows that $\det[\mathcal{P}(X), \mathcal{P}(Y)]|_{(0,\rho)} =$

$$-\rho \left[\sum \rho^i A_{1,m-i}(0) \sum \rho^i R_{2,m-i}(0) - \sum \rho^i R_{1,m-i}(0) \sum \rho^i A_{2,m-i}(0)\right],$$

and so $\dfrac{d}{d\rho}\left(\det[\mathcal{P}(X), \mathcal{P}(Y)]|_{(0,\rho)}\right)(0) =$

$$R_{1,m}(0)A_{2,m}(0) - A_{1,m}(0)R_{2,m}(0).$$

Hence, $(1, 0, 0)$ is a hyperbolic singular point of $F_{\mathcal{P}(Z)}$ if and only if

$$(3) \qquad P_{1,m}(1,0)Q_{2,m}(1,0) - Q_{1,m}(1,0)P_{2,m}(1,0) \neq 0,$$

where $P_{k,m}$ and $Q_{k,m}$ are the homogeneous parts of degree m of P_k and Q_k, respectively, $k = 1, 2$.

Now, in a similar way we have that if $(-1, 0, 0)$ is a singularity of $F_{\mathcal{P}(Z)}$ then it is hyperbolic if (3) holds.

The proofs of the propositions below are analogous to those proofs of the respective propositions (Proposition 6 page 230, Proposition 8 page 231 and Proposition 9 pag. 233) established in [10].

Proposition 7. *Let $p \in S^2_+ \cup S^1$ be a D-regular point of $\mathcal{P}(Z)$ with $Z = (X, Y) \in \Omega_m$. Then, given a transition function φ, there exists a neighborhood V of p and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, $\mathcal{P}(Z)_\epsilon$ has no singular points in V (see Figure 3).*

Proposition 8. *Given $Z = (X, Y) \in \Omega_m$, let p be a hyperbolic singular point of $F_{\mathcal{P}(Z)}$. Then, given a transition function φ, there is a neighborhood V of p in $S^2_+ \cup S^1$ and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, $\mathcal{P}(Z)_\epsilon$ has near p a unique singular point which is a hyperbolic saddle or a hyperbolic node (see Figure 4).*

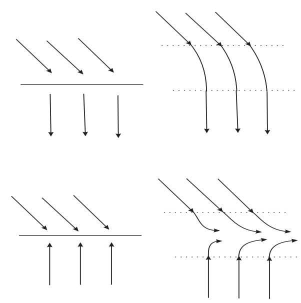

FIGURE 3. D-regular points and their φ_ϵ-compactification

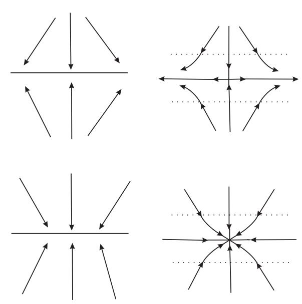

FIGURE 4. D-singular points and their regularizations

Proposition 9. *Let p be a fold point of $\mathcal{P}(Z)$ with $Z = (X, Y) \in \Omega_m$. Then, given a transition function φ, there is a neighborhood V of p and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0, \mathcal{P}(Z)_\epsilon$ has no singular points in V (see Figure 5).*

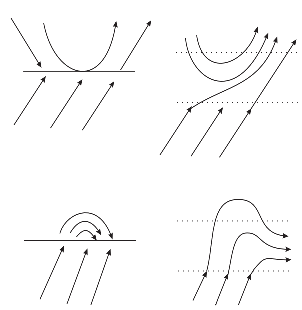

FIGURE 5. Fold points and their φ_ϵ-compactification

2.2. **Closed and Saddle Connections Poly-Trajectories.**

Definition 10. *A continuous curve γ consisting of regular trajectory arcs of X and/or of Y and/or of F_Z is called a poly-trajectory if:*

(1) *γ has arcs of at least two fields among X, Y and F_Z, or consists of a single arc of F_Z;*
(2) *the transition between arcs of X and Y happens on the sewing arc;*
(3) *the transition between arcs of X or Y and F_Z occurs at fold points or regular points of the sliding or the escaping arcs, preserving the sense of the arcs (see Figure 6).*

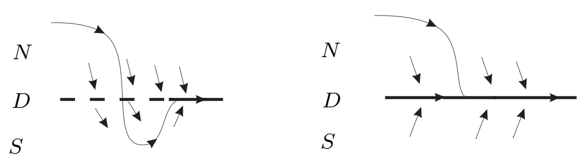

FIGURE 6. Poly-Trajectories

Now we define saddle connections on Z.

Definition 11.

a) *A separatrix of Z is a trajectory of X, Y or F_Z such that its α or ω-limit sets are saddle points of X, Y or F_Z.*
b) *A double separatrix of Z is a trajectory of X, Y or F_Z such that their α and ω-limit sets are saddles or a separatrix of X (resp. Y) that meets D at a saddle of F_Z.*
c) *A saddle connection of Z is a double separatrix or a poly-trajectory that contains a double separatrix or two separatrices (see Figure 7).*

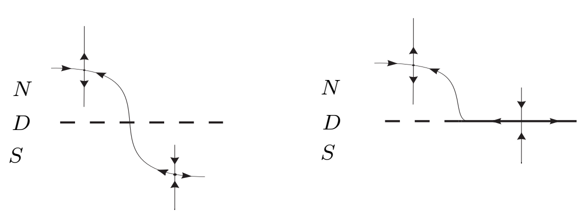

FIGURE 7. Saddle Connections

Now we define closed trajectories of Z that have points or arcs of D.

Definition 12. *Let γ be a closed poly-trajectory of $Z = (X, Y)$.*

(1) *γ is called a closed poly-trajectory of type 1 if γ meets D only at sewing points;*
(2) *γ is called a closed poly-trajectory of type 3 if it has at least one fold point and one sliding or escaping arc of Z (see Figure 8).*

Next definition extends the notion of hyperbolic orbits for closed poly-trajectory of Z.

Definition 13. *Let γ be a closed poly-trajectory of $Z = (X, Y) \in \Omega_m$. It is called elementary if one of the cases below holds:*

(1) *γ is of type 1 and has a first return map η with $\eta' \neq 1$;*
(2) *γ is of type 3 and all arcs of F_Z are sliding or all are escaping.*

The definitions above can be reformulated in similar way for the discontinuous piecewise analytic vector field $\mathcal{P}(Z)$ in S^2.

Now, we will study the stability of S^1 when it is a closed poly-trajectory of $\mathcal{P}(Z)$. Note that in this case S^1 is necessarily of type 1. Moreover m is odd, otherwise always there are singular points of $\mathcal{P}(Z)$ in S^1. We will need the following result that can be found in [2, 13].

Proposition 14. *Let X be a C^1 planar vector field. Given a point $p_0 \in \mathbb{R}^2$, denote by $\phi(t, p_0)$ the orbit of X such that $\phi(0, p_0) = p_0$ and by p_1 the point $\phi(T_0, p_0)$. Let Σ_0 and Σ_1 be transversal sections of X at the points p_0 and p_1, respectively. If $\sigma : I \to \mathbb{R}^2$ and $\hat{\sigma} : \hat{I} \to \mathbb{R}^2$ are the respective parameterizations of Σ_0 and Σ_1 with $\sigma(s_0) = p_0$ and $\hat{\sigma}(\hat{s}_0) = p_1$, then the derivative of the transition map $\Pi : \Sigma_0 \to \Sigma_1$ at the point p_0, defined by the flow of X, is given by*

$$\Pi'(p_0) = \frac{\det \begin{pmatrix} X(p_0) \\ \sigma'(s_0) \end{pmatrix}}{\det \begin{pmatrix} X(p_1) \\ \hat{\sigma}'(\hat{s}_0) \end{pmatrix}} e^{\displaystyle\int_0^{T_0} \operatorname{div} X(\phi(t, p_0))dt}.$$

Denote by $\tilde{Z} = (\tilde{X}, \tilde{Y})$ the discontinuous piecewise polynomial vector field which gives rise to system (2). In the plane (θ, ρ) the points $p_0 = (0, 0)$, $p_2 = (2\pi, 0)$, correspond to the point $(1, 0, 0)$ of $S^1 \cap D$, and $p_1 = (\pi, 0)$ corresponds to the other point $(-1, 0, 0)$. As S^1 is a closed poly-trajectory of type 1, we can take the following transversal sections $\Sigma_0 = \{(0, \rho) : 0 \leq \rho \leq \delta_0\}$, $\Sigma_1 = \{(\pi, \rho) : 0 \leq \rho \leq \delta_0\}$ and $\Sigma_2 = \{(2\pi, \rho) : 0 \leq \rho \leq \delta_0\}$ of (2) with δ_0 small enough. Hence, we define the following transition maps $\Pi_1 : \Sigma_0 \to \Sigma_1$, $\Pi_2 : \Sigma_1 \to \Sigma_2$ and obtain the Poincaré map Π of $\mathcal{P}(Z)$ associated to S^1 in the coordinates (θ, ρ), given by $\Pi = \Pi_2 \circ \Pi_1$. We have that $\Pi'(p_0) = \Pi_2'(\Pi_1(p_0))\Pi_1'(p_0) = \Pi_2'(p_1)\Pi_1'(p_0)$. Thus, by Proposition 14 and expression (2), it follows that

$$\Pi_1'(p_0) = -\frac{Q_{1,m}(1,0)}{Q_{1,m}(-1,0)} e^{\int_0^{T_1} \operatorname{div}\tilde{X}(\theta(t),0)dt} = e^{\int_0^{T_1} \left(-R_{1,m}(\theta(t)) + \frac{dA_{1,m}}{d\theta}(\theta(t))\right) dt},$$

with $\dot{\theta}(t) = A_{1,m}(\theta(t))$, $\theta(0) = 0$ and $\theta(T_1) = \pi$. Therefore,

$$\Pi_1'(p_0) = \frac{A_{1,m}(\pi)}{A_{1,m}(0)} e^{-\int_0^{\pi} \frac{R_{1,m}(\theta)}{A_{1,m}(\theta)} d\theta} = e^{-\int_0^{\pi} \frac{R_{1,m}(\theta)}{A_{1,m}(\theta)} d\theta}.$$

Analogously, we have

$$\Pi_2'(p_1) = e^{-\int_\pi^{2\pi} \frac{R_{2,m}(\theta)}{A_{2,m}(\theta)} d\theta} = e^{-\int_0^{\pi} \frac{R_{2,m}(\theta)}{A_{2,m}(\theta)} d\theta}.$$

Hence,

$$\Pi'(p_0) = e^{-\int_0^{\pi} \left(\frac{R_{1,m}(\theta)}{A_{1,m}(\theta)} + \frac{R_{2,m}(\theta)}{A_{2,m}(\theta)}\right) d\theta}.$$

Note that we have performed the computations above supposing that S^1 is oriented in the counterclockwise sense.

Now we can state the following proposition.

Proposition 15. *Suppose that $\mathcal{P}(Z)$, $Z \in \Omega_m$, with m odd, does not have singular points in S^1. Then S^1 is a closed poly-trajectory of type 1 and the derivative of the Poincaré map associated to a transversal section at the point $p_0 \in S^1 \cap D$ is given by*

$$\Pi'(p_0) = e^{\sigma\mu} = e^{\sigma \int_0^{\pi} \left(\frac{R_{1,m}(\theta)}{A_{1,m}(\theta)} + \frac{R_{2,m}(\theta)}{A_{2,m}(\theta)}\right) d\theta},$$

where $\sigma = -1$, if S^1 is oriented in the counterclockwise sense, and $\sigma = 1$, otherwise. Moreover, S^1 is an attractor if $\sigma\mu < 0$ and a repeller if $\sigma\mu > 0$.

We conclude that S^1 is an elementary closed poly-trajectory if and only if

$$\int_0^{\pi} \left(\frac{R_{1,m}(\theta)}{A_{1,m}(\theta)} + \frac{R_{2,m}(\theta)}{A_{2,m}(\theta)}\right) d\theta \neq 0. \tag{4}$$

The proof of the proposition below is analogous to the proof of Proposition 13, page 234, established in [10].

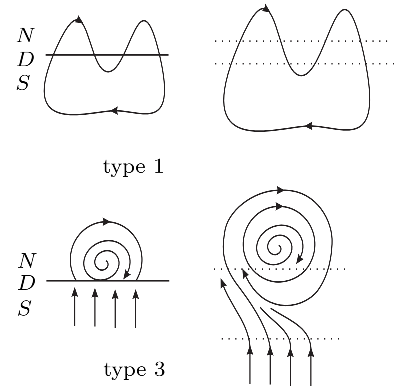

Figure 8. Closed poly-trajectories and their φ_ϵ-compactification

Proposition 16. *Let γ be an elementary closed poly-trajectory of $\mathcal{P}(Z)$ with $Z = (X, Y) \in \Omega_m$. Then, given a transition function φ, there is a neighborhood V of γ and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, $\mathcal{P}(Z)_\epsilon$ has only one periodic orbit in V, and this orbit is hyperbolic (see Figure 8).*

3. Piecewise Polynomial Vector Fields with Structurally Stable φ_ϵ-Compactification

In this section we define a set G_m of discontinuous piecewise polynomial vector fields whose elements, Z, have structurally stable φ_ϵ-compactification $\mathcal{P}(Z)_\epsilon$ (Definition 1), for any transition function φ and small ϵ.

The notion of structural stability in χ_m is defined in similar way as in $\chi^r(S^2, S^1)$ (see Definition 2). Denote by Σ_m the set of $X \in \chi_m$ that are structurally stable.

Definition 17. *We call $\mathcal{S}_m$ the set of all polynomial vector fields $X \in \chi_m$ for which $\mathcal{P}(X)$ satisfies the following conditions:*

1. *all its singular points are hyperbolic;*
2. *all its periodic orbits are hyperbolic;*
3. *it does not have saddle connections in S^2 unless they are contained in S^1.*

We have that $\mathcal{S}_m \subset \Sigma_m$ and it is an open and dense set of χ_m. However, it is an *unsolved problem* to prove (or disprove) that $\mathcal{S}_m = \Sigma_m$. See [12] for more details.

Remark 18. *By extension of the notation in definition 17, we will write in what follows $X|_N \in \mathcal{S}_m$ and $Y|_S \in \mathcal{S}_m$ to mean that conditions (1), (2) and (3) in this definition hold for $X|_N$ and $Y|_S$.*

Definition 19. *Write* $G_m = G_m(1) \cap G_m(2) \cap G_m(3)$, *where:*

(1) $G_m(1) = \{Z = (X,Y) \in \Omega_m : X|_N$ *and* $Y|_S \in \mathcal{S}_m$*; each D-singularity of* $\mathcal{P}(Z)$ *is elementary* $\}$.
(2) $G_m(2) = \{Z = (X,Y) \in \Omega_m : X|_N$ *and* $Y|_S \in \mathcal{S}_m$*; each closed poly-trajectory of* $\mathcal{P}(Z)$ *is elementary* $\}$.
(3) $G_m(3) = \{Z = (X,Y) \in \Omega_m : X|_N$ *and* $Y|_S \in \mathcal{S}_m$*;* $\mathcal{P}(Z)$ *does not have saddle connections in* S^2 *unless they are contained in* $S^1\}$.

Proposition 20. *Let* $Z = (X,Y) \in G_m(1)$*. Then, given a transition function* φ*, there is an* $\epsilon_0 > 0$ *such that for* $0 < \epsilon \leq \epsilon_0$*,* $\mathcal{P}(Z)_\epsilon$ *has only hyperbolic singularities in* S^2*.*

Proof. As $X|_N$ and $Y|_S \in \mathcal{S}_m$, it remains to prove that the singularities that appear due to the φ_ϵ-compactification process are hyperbolic. Indeed, let p be a point of D, then p can be a D-regular point, a hyperbolic singularity of $F_{\mathcal{P}(Z)}$ or a fold. For each case, there is a proposition that guarantees the existence a number $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0]$, $\mathcal{P}(Z)_\epsilon$ has no singularities near p (Propositions 7, 9) or has a unique hyperbolic singularity (Proposition 8). The union of these neighborhoods cover D, and, as D is compact, there is a sub covering made by a finite number of these neighborhoods. Then, we can chose ϵ_0 as the smallest ϵ_0 associated to these neighborhoods. □

Proposition 21. *Let* $Z = (X,Y) \in G_m(2) \cap G_m(3)$*. Then, given a transition function* φ*, there is an* $\epsilon_0 > 0$ *such that for* $0 < \epsilon \leq \epsilon_0$*,* $\mathcal{P}(Z)_\epsilon$ *has only hyperbolic periodic orbits in* S^2*.*

Proof. As $X|_N$ and $Y|_S \in \mathcal{S}_m$, all their periodic orbits are hyperbolic, so it remains to prove that the same occurs to the periodic orbits that appear by the φ_ϵ-compactification process. Let γ be an elementary closed poly-trajectory of $\mathcal{P}(Z)$. Then, by Proposition 16, there is an $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$, $\mathcal{P}(Z)_\epsilon$ has a hyperbolic closed orbit near γ. We can choose a unique positive ϵ_0 since the elementary poly-trajectories, are finite in number. As the singularities of X and Y are hyperbolic, there is no possibility of Hopf type bifurcation. So, the case of periodic orbits emerging from singularities by the φ_ϵ-compactification process is excluded. As $Z \in G_m(3)$, $\mathcal{P}(Z)$ does not have separatrix graphs in S^2 unless they are contained in S^1, so there is no possibility of appearance of a periodic orbit from such a graph. So, the periodic orbits emerging from the φ_ϵ-compactification of closed poly-trajectories are the only new periodic orbits of $\mathcal{P}(Z)_\epsilon$, for ϵ small. □

Proposition 22. *Let $Z = (X, Y) \in G_m(3) \cap G_m(2)$. Then, given a transition function φ, there is $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, $\mathcal{P}(Z)_\epsilon$ does not have saddle connections in S^2 unless they are contained in S^1.*

Proof. We claim that there is an $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$, $\mathcal{P}(Z)_\epsilon$ does not have saddle connections, except on S^1. Indeed, as $X|_N$ and $Y|_S \in \mathcal{S}_m$, and $\mathcal{P}(Z)$ does not have separatrix connections on S^2 unless they are contained in S^1, the only possibilities for $\mathcal{P}(Z)_\epsilon$ to have such separatrix connection on S^2, unless they are contained in S^1, are as follows:

(1) passing through points of the curve D;
(2) due to the presence of a semi-stable periodic orbit, which could disappear and allow a connection of two separatrices.

Possibility 2 is discarded, since $Z \in G_m(2)$. We must analyze possibility 1. Let δ be the minimum of the set $\{dist(e_i, e_j) : e_i$ is a separatrix of $\mathcal{P}(Z)$, and $i \neq j\}$. Of course, $\delta > 0$, since the number of separatrices is finite. Then, we diminish ϵ_0 so that the minimum distance of the separatrices for the regularized vector field can never be less than $\frac{\delta}{2}$. □

Recall that $\Sigma^r(S^2, S^1)$, $r \geq 1$, stands for structurally stable vector fields on S^2 inside $\chi^r(S^2, S^1)$ (see Definition 3).

Theorem 23. *If $Z = (X, Y) \in G_m$, then, given a transition function φ, there is $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, then $\mathcal{P}(Z)_\epsilon \in \Sigma^r(S^2, S^1)$, $r \geq 1$.*

Proof. It follows from Propositions 20, 21 and 22. □

4. Genericity

In this section we prove that the set G_m is open and that each discontinuous piecewise polynomial vector field Z of Ω_m can be approximated by fields of G_m, i.e. we prove the genericity of G_m.

Theorem 24. *The set G_m is open in Ω_m.*

Proof. Let $Z = (X, Y)$ be a vector field in G_m. It will be proved that there is $\delta > 0$ such that if $\widehat{Z} = (\widehat{X}, \widehat{Y}) \in \Omega_m$ and $|Z - \widehat{Z}| = \max \{|X - \widehat{X}|, |Y - \widehat{Y}|\} < \delta$, then $\widehat{Z} \in G_m$. For doing this, we have to prove that $\widehat{Z} \in G_m(i), i = 1, 2, 3$.

- We claim that there is a $\delta_1 > 0$ such that if $|Z - \widehat{Z}| < \delta_1$, then $\widehat{Z} \in G_m(1)$. Indeed, as $Z = (X, Y) \in G_m(1)$, we have $X|_N$ and $Y|_S \in \mathcal{S}_m$, and from the openness of $\mathcal{S}_m$, there is $\delta_1 > 0$ such that if $|Z - \widehat{Z}| < \delta_1$, then $\widehat{X}|_N$ and $\widehat{Y}|_S \in \mathcal{S}_m$. Now, it remains to prove that if p is an elementary D-singularity of

$\mathcal{P}(Z)$ and $\widehat{Z}$ is close to Z, then there is a point $\widehat{p}$ near p which is an elementary D-singularity of $\mathcal{P}(\widehat{Z})$.

Let p be a fold of Z. We can suppose that $Xf(p) = 0$, $X^2f(p) \neq 0$ and $Yf(p) \neq 0$. As $Xf(p) = 0$ and $X^2f(p) \neq 0$, the curve $\{Xf = 0\}$ crosses transversally the curve D at the point p, and, by continuity, the same occurs to the curve $\{\widehat{X}f = 0\}$, for $\widehat{Z}$ near Z. This means that there is $\widehat{p}$ near p such that $\widehat{X}f(\widehat{p}) = 0$ and $\widehat{X}^2f(\widehat{p}) \neq 0$. If δ_1 is small enough, we can assume that it is also true that $\widehat{Y}f(\widehat{p}) \neq 0$. So, $\widehat{p}$ is a fold of $\widehat{Z}$. Hence, as there are no folds of $\mathcal{P}(Z)$ in S^1, it follows that if p is a fold of $\mathcal{P}(Z)$ then $\widehat{p}$ near p is a fold of $\mathcal{P}(\widehat{Z})$.

Let p be a hyperbolic singularity of F_Z. We have that $Xf(p)Yf(p) < 0$, $det[X,Y](p) = 0$ and $d(det[X,Y])|_D(p) \neq 0$. Similarly to the fold case, the curve $\{det[X,Y]|_D(p) = 0\}$ crosses transversally the curve D at the point p, and the same is true for $\widehat{Z}$ near Z. So, there is a $\widehat{p}$ near p such that $\widehat{X}f(\widehat{p})\widehat{Y}f(\widehat{p}) < 0$, $det[\widehat{X},\widehat{Y}](\widehat{p}) = 0$ and $d(det[\widehat{X},\widehat{Y}])|_D(\widehat{p}) \neq 0$. This implies that $\widehat{p}$ is a hyperbolic singular point of $F_{\widehat{Z}}$. As δ_1 can be chosen so that none of the involved function change sign, and therefore $\widehat{p}$ is a singularity of the same kind as p. As the D-singularities are isolated, δ_1 can be chosen strictly positive. We have that $\widehat{Z}$ does not have other singularities. This is due to the openness of the conditions that exclude this type of singularities.

Now, if $p \in S^1 \cap D$ is a hyperbolic singularity of $F_{\mathcal{P}(Z)}$ then, as S^1 is invariant by $\widehat{Z}$, by the previous case, it follows that p is a hyperbolic singularity of $F_{\mathcal{P}(\widehat{Z})}$. Thus, $\widehat{Z} \in G_m(1)$.

- We claim that there is a $\delta_2 > 0$ such that if $|Z - \widehat{Z}| < \delta_2$, then $\widehat{Z} \in G_m(2)$.

  As $Z = (X,Y) \in G_m(2)$, we have $X|_N$ and $Y|_S \in \mathcal{S}_m$, and each closed poly-trajectory of Z is elementary.

  Let γ be an elementary closed poly-trajectory of type 1 of Z. Associated to γ there is a first return map η, differentiable and such that $\eta'(p) \neq 1$, for $p \in \gamma$. This means that p is a hyperbolic fixed point of the diffeomorphism η. So, there is a number $k > 0$ such that if μ is a diffeomorphism with $|\eta - \mu|_1 < k$, then μ has a hyperbolic fixed point p_μ near p. Then, it is enough to choose $\delta_2 > 0$ small as necessary for if $|Z - \widehat{Z}| < \delta_2$, the first return map $\widehat{\eta}$ associated to $\widehat{Z}$ satisfies $|\eta - \widehat{\eta}|_1 < k$. So, $\widehat{\eta}$ has a hyperbolic fixed point $\widehat{p}$ which corresponds to an elementary closed poly-trajectory of type 1 of $\widehat{Z}$. In the same way, if S^1 is a poly-trajectory of $\mathcal{P}(Z)$ and so it is of type 1, as S^1 is invariant by $\mathcal{P}(\widehat{Z})$, it follows that S^1 is also a poly-trajectory of $\mathcal{P}(\widehat{Z})$ and it is therefore of type 1.

Let γ be an elementary closed poly-trajectory of type 3 of Z. By the continuity of the functions involved, it can be shown that there is $\delta_2 > 0$ such that if $|Z - \widehat{Z}| < \delta_2$, $\widehat{Z}$ has an elementary closed poly-trajectory $\widehat{\gamma}$ of type 3 near γ.

As the number of poly-trajectories is finite, we can choose $\delta_2 > 0$ small enough so that $\widehat{Z}$ has only elementary poly-trajectories. So, we have proved that $\widehat{Z} \in G_m(2)$.

- We claim that there is a $\delta_3 > 0$ such that if $|Z - \widehat{Z}| < \delta_3$, then $\widehat{Z} \in G_m(3)$.

  Indeed, as $Z = (X, Y) \in G_m(3)$, we have $X|_N$ and $Y|_S \in \mathcal{S}_m$ and there is $\delta_3 > 0$ such that if $|Z - \widehat{Z}| < \delta_3$, then $\widehat{X}|_N$ and $\widehat{Y}|_S \in \mathcal{S}_m$. So, $\widehat{X}$ and $\widehat{Y}$ do not have separatrix connections in N and in S, respectively. It remains to analyze the appearance of a connection with at least one point in D. We know that $\mathcal{P}(Z)$ has only a finite number of separatrices and does not have a connection on S^2 unless they are contained in S^1. As $\mathcal{P}(\widehat{Z})$ has a unique separatrix corresponding to each separatrix of $\mathcal{P}(Z)$ (as follows from the uniqueness and continuous dependence of invariant manifolds of equilibrium of Vector Fields and fixed points of Diffeomorphisms, see [8]), it is easy to show that $\delta_3 > 0$ can be chosen so that $\mathcal{P}(\widehat{Z})$ does not have separatrix connections on S^2 unless they are contained in S^1. In this way, we have established that $\widehat{Z} \in G_m(3)$.

To finish the proof, we can take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then if $\widehat{Z} = (\widehat{X}, \widehat{Y}) \in \Omega_m$ and $|Z - \widehat{Z}| = \max\{|X - \widehat{X}|, |Y - \widehat{Y}|\} < \delta$, then $\widehat{Z} \in G_m$. As a consequence, the set G_m is open in Ω_m. □

Definition 25. *Assume that $Z = (X, Y) \in \Omega_m$. For each pair $(\sigma, v) \in \mathbb{R}^2 \times \mathbb{R}^2$, let $Z_{\sigma,v}$ be the field Z translated by $v = (v_1, v_2)$ and rotated by $\sigma = (\sigma_1, \sigma_2)$; this means that*

$$Z_{\sigma,v} = \mathcal{R}_\sigma(Z + v) = (\mathcal{R}_{\sigma_1}(X + v), \mathcal{R}_{\sigma_2}(Y + v)),$$

where

$$\mathcal{R}_{\sigma_1}(X + v) = \begin{pmatrix} \cos\sigma_1 & -\sin\sigma_1 \\ \sin\sigma_1 & \cos\sigma_1 \end{pmatrix} \begin{pmatrix} P_1 + v_1 \\ Q_1 + v_2 \end{pmatrix}.$$

Theorem 26. *The set G_m is dense in Ω_m.*

Proof. Let $Z = (X, Y)$ be a vector field of Ω_m with $X = (P_1, Q_1)$ and $Y = (P_2, Q_2)$. If $\mathcal{P}(Z)$ has singularities in S^1 we can suppose that the singularities of $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ in S^1 are all hyperbolic and these vector fields do not have singular points in $(\pm 1, 0, 0)$. Otherwise, from the continuous case (see [7] and [12]), we can approximate X and Y by other two vector fields with such properties. Now if some of the points $(\pm 1, 0, 0) \in S^1 \cap D$ are not hyperbolic singularities of $F_{\mathcal{P}(Z)}$, by (3), we can make these points hyperbolic by adding to X or Y a perturbation of type

$$\epsilon x^m \frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} \quad \text{or} \quad 0\frac{\partial}{\partial x} + \epsilon x^m \frac{\partial}{\partial y}.$$

Suppose that S^1 is a closed poly-trajectory of type 1 of $\mathcal{P}(Z)$ which is not elementary, i.e. by (4)

$$\int_0^\pi \left(\frac{R_{1,m}(\theta)}{A_{1,m}(\theta)} + \frac{R_{2,m}(\theta)}{A_{2,m}(\theta)} \right) d\theta = 0.$$

Then adding to X the perturbation

$$\epsilon(x^2+y^2)^k x \frac{\partial}{\partial x} + \epsilon(x^2+y^2)^k y \frac{\partial}{\partial y},$$

with $m = 2k+1$, it follows that the above equality becomes

$$\int_0^\pi \left(\frac{R_{1,m}(\theta)}{A_{1,m}(\theta)} + \frac{R_{2,m}(\theta)}{A_{2,m}(\theta)} + \frac{\epsilon}{A_{1,m}(\theta)} \right) d\theta = \int_0^\pi \frac{\epsilon}{A_{1,m}(\theta)} d\theta \neq 0.$$

This implies that S^1 can be made elementary.

Notice that if $X \in \chi_m$ then to $\tilde{X} = \mathcal{R}_{\sigma_1}(X + v)$, $(\sigma_1, v) \in \mathbb{R} \times \mathbb{R}^2$,

$$\begin{aligned} \tilde{A}_m(\theta) &= \cos\sigma_1 A_m(\theta) + \sin\sigma_1 R_m(\theta), \\ \tilde{R}_m(\theta) &= \cos\sigma_1 R_m(\theta) - \sin\sigma_1 A_m(\theta). \end{aligned}$$

Note also that we can write the condition (3) as

$$R_{1,m}(0)A_{2,m}(0) - A_{1,m}(0)R_{2,m}(0) \neq 0.$$

Hence, as the singularities of $\mathcal{P}(Z)$ in $S^1 \setminus \{(\pm 1, 0, 0)\}$ correspond by (1) the points $(\theta, 0)$ such that $A_{k,m}(\theta) = 0$ and they are hyperbolic if $A'_{k,m}(\theta)R_{k,m} \neq 0$, $k = 1, 2$, it follows that if (σ, v) is small enough then S^1 is still either an elementary closed poly-trajectory of $\mathcal{P}(Z_{\sigma,v})$ or all singularities of $\mathcal{P}(Z_{\sigma,v})$ in S^1 are hyperbolic. Now, by the continuous case (see [13]) and from the proof of Theorem 25 of [10], we have that the set of $(\sigma, v) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that $Z_{\sigma,v}$ has at least one non hyperbolic singularity, one non elementary D-singular point, one non hyperbolic closed orbit, one non elementary poly-trajectory or one connection of saddle separatrizes of $\mathcal{P}(Z)$ in $S^2 \setminus S^1$, has null Lebesgue measure in $\mathbb{R}^4$.

This finishes the proof of the theorem. □

References

[1] A. Andronov, V. Chaikin et al., *Theory of Oscillators*, Addison Wesley (1966).

[2] A. Andronov, I. Gordon, E. Leontovich and G. Maier, *Theory of Bifurcations of Dynamical Systems on a Plane, Israel Program for Scientific Translations*, John Wiley, New York (1973).

[3] M. di Bernardo, C.J. Budd, A. R. Champneys, P. Kowalczyk, *Piecewise Smooth Dynamical Systems Theory and Applications*, Springer-Verlag, New York (2008).

[4] C. Buzzi, P. da Silva and M. Teixeira, *A singular approach to discontinuous vector fields on the plane*, J. Differential Equations **231**, **2** (2006), 633–655.

[5] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer (1988).

[6] R. Garcia and J. Sotomayor, *Structural stability of piecewise-linear vector fields*, J. Differential Equations **192**, **2**, (2003), 553–565.

[7] E. González Velasco, *Generic Properties of Polynomial Vector Fields at Infinity*, Trans. Amer. Math. Soc. **143** (1969), 201–222.

[8] M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant manifolds. Lecture Notes in Mathematics*, **Vol. 583**, Springer-Verlag, Berlin-New York (1977).

[9] V.S. Kozlova, *Roughness of a Discontinuous System*, Vestnik Moskovskogo Universiteta, Matematika **5** (1984), 16–20.

[10] A.L.F. Machado and J. Sotomayor, *Structurally Stable Discontinuous Vector Fields in the Plane*, Qual. Th. Dynamical Systems **3** (2002), 227-250.

[11] M.M. Peixoto, *Structural Stability on Two-dimensional Manifolds*, Topology **1** (1962).

[12] J. Sotomayor, *Stable Planar Polynomial Vector Fields*, Rev. Mat. Iberoamericana **1**, **2** (1985), 15-23.

[13] J. Sotomayor, *Curvas Definidas por Equações Diferenciais no Plano*, Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro (1981).

[14] J. Sotomayor and M.A. Teixeira, *Regularization of Discontinuous Vector Fields*, International Conference on Differential Equation, Lisboa (1995), 207-223.