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# Nonlinear (magnetic) correction to the field of a static charge in an external field

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We find the first nonlinear correction to the field produced by a static charge at rest in a background constant magnetic field. It is quadratic in the charge and purely magnetic. The third-rank polarization tensor—the nonlinear response function—is written within the local approximation of the effective action in an otherwise model- and approximation-independent way within any *P*-invariant nonlinear electrodynamics, QED included.

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## I. INTRODUCTION

In Maxwell electrodynamics, the superposition principle is true, which reads that electromagnetic fields do not directly interact between themselves and may be linearly combined independently. This is not the case in nonlinear electrodynamics, wherein only small electromagnetic fields are mutually independent.

A popular example of a nonlinear electrodynamics in the vacuum is provided by the Born-Infeld model [1], and also by a noncommutative $U_\star(1)$ gauge theory, considered in this respect in Refs. [2,3]. Many issues of nonlinear electrodynamics are thoroughly elaborated in Ref. [4]. Another, practically most important, example is quantum electrodynamics (QED). The reason why it is nonlinear is that an electromagnetic field, say a photon, may create virtual electron-positron pairs that interact with this field itself and/or with any other "external" field. This makes a mechanism that lets electromagnetic fields sense each other.

The well-known nonlinear effect of QED, present already in the vacuum without any external field, is light-by-light scattering. When taken off the photon mass shell, the corresponding probability amplitude becomes as a matter of fact responsible for the leading nonlinear (cubic) correction to the electric Coulomb field [5][1] that can be conveniently written as

$$\mathcal{E}_{\rm nl} = \mathcal{E}\left(1 - \frac{2\alpha}{45\pi}\left(\frac{e\mathcal{E}}{m^2}\right)^2\right). \tag{1}$$

Here $\mathcal{E} = (q/4\pi r^2)$ is the standard Coulomb field[2] in Heaviside units produced by the point charge $q$ at the distance $r$, while $e$ and $m$ are the electron charge and mass, and $\alpha = (e^2/4\pi) = 1/137$ is the fine-structure constant. It is generally known, and also seen in this equation, that in QED the nonlinearity is determined by the ratio of the electromagnetic field to Schwinger's characteristic value $(m^2/e) = 4.4 \times 10^{13}$ in CGSE units, which makes $1.3 \times 10^{16}$ V/cm when one measures an electric field, and $4.4 \times 10^{13}$ G if a magnetic field is concerned. Electromagnetic fields should be comparable in strength to these values in order for the interaction between them to become essential. The nonlinear correction in Eq. (1) becomes valuable when one is interested in approaching a sufficiently small-sized charge sufficiently close—say, to approach the nucleus of a not-too-heavy atom within a few femtometers. On the other hand, electric fields, large in the Schwinger scale, up to $10^{18}$ to $10^{19}$ V/cm, occur [6] at the surfaces of strange quark stars [7], depending on whether the matter is in the superconducting state [8]. For such fields the vacuum is unstable, and the Schwinger effect of spontaneous electron-positron pairs by the vacuum becomes already efficient, which requires a special treatment (see Ref. [9]). We do not consider the corresponding complications in the present paper, however.

In this paper, we are dealing with another nonlinear phenomenon different from the one given by Eq. (1), also associated with strong electric fields, namely the production of a magnetic field by it: this magnetoelectric effect becomes possible if an external magnetic field is present.

The linear correction to the Coulomb field of a charge due to the vacuum polarization in a magnetic field was studied earlier, [10–12] with the finding that the hydrogen ground energy level saturates [10,12] as the magnetic field grows, and that a string is formed [10]. Some hints were thereby produced for considering [13] interquark potential in QCD. The nonlinear (purely magnetic) correction to the field of a charge in a magnetic field, which is to be considered now for the first time, is based on the known fact that there exists in this case not only photon-by-photon scattering, but also one photon splitting into two (and two photons merging into one). This splitting is enhanced by

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[1]The authors are indebted to M. I. Vysotsky, who attracted their attention to this result.

[2]The linear response to the applied charge $q$ due to the vacuum polarization known as the Uehling-Silber correction to the Coulomb potential [5] may be also included in $\mathcal{E}$.

 

the strength of the external magnetic field as compared to the vacuum case above. It was elaborated in theory [14] and is thought of as being efficient in a pulsar magnetosphere with magnetic fields above $10^{12}$ G [15], essentially contributing to the electron-positron plasma production and radiation pattern of pulsars. Again, the same as above, when taken outside the photon mass shell, the three-photon probability amplitudes become responsible for a nonlinear induction of time-independent current (and, hence, of the stationary magnetic field) by static charges or, equivalently, by the static electric fields they create. The magnetic field produced by a static charge in an external magnetic field is even (quadratic in the lowest order of nonlinearity) with respect to its magnitude and linearly disappears with the external field—in agreement with the generalized Furry theorem of Ref. [3] that states that the numbers of electric and magnetic legs in every diagram should each be even. It also agrees with this theorem, that there are no corrections to the static electric field in the lowest (second-power) nonlinear order. Previously, magnetoelectric effect was considered in Refs. [2,3] for classical noncommutative electrodynamics, and within QED as a linear response to a static charge by the vacuum filled with external electric and magnetic fields [16].

In Sec. II, for the most general case of a constant and homogeneous external electromagnetic field, we outline the derivation of nonlinear Maxwell equations, keeping only the first and second powers of the electromagnetic field living above that external field background, and define the notion of a current nonlinearly induced by a static electric field (or by a static charge). The nonlinear field equations are served by the second- and third-rank polarization tensors. In Sec. III, we restrict the external background to the magneticlike field; i.e., the one that is purely magnetic in a class of special Lorentz frames. Then the involved polarization tensors are given in the small four-momentum limit, also called the infrared or local approximation, in terms of the derivatives of the effective Lagrange density over the background field invariants, bearing in mind that in the local approximation this density does not depend upon space-time derivatives of the background field strength. In Sec. IV, we are working in a special frame, where the background field is purely magnetic and the static charge is at rest. We calculate the nonlinearly induced current and its magnetic field as expressed through the static electric field produced by the charge. The limiting cases of very large and very small background magnetic fields are discussed within QED, referring to the one-loop Euler-Heisenberg effective Lagrangian. In Sec. VI, the results are resumed, and numerical estimates of the domains of their applicability are given. Detailed calculations of the second- and third-rank derivatives of the effective action used in the work are presented in the accompanying paper [17] within the necessary local approximation.

## II. NONLINEAR ELECTROMAGNETIC FIELD EQUATIONS OVER A CONSTANT FIELD BACKGROUND

In QED and in any other $U(1)$-gauge-invariant nonlinear electrodynamics, the field equations, when written up to terms quadratic in the small electromagnetic field potential $a^\nu(x)$, have the form

$$[\eta_{\rho\nu}\Box - \partial^\rho\partial^\nu]a^\nu(x) + \int d^4x'\Pi_{\rho\nu}(x,x')a^\nu(x') + \frac{1}{2}\int d^4x'd^4x''\Pi_{\rho\nu\sigma}(x,x'x'')a^\nu(x')a^\sigma(x'') = j_\rho(x), \quad (2)$$

where $j_\rho(x)$ is a (small) source of the field, greek indices span the four-dimensional Minkowski space taking the values 1, 2, 3, 0, the metric tensor is $\eta_{\rho\nu} = \mathrm{diag}(1,1,1,-1)$, and $\Box = \nabla^2 - \partial_0^2$. The second- and third-rank polarization tensors—$\Pi_{\rho\nu}$ and $\Pi_{\rho\nu\sigma}$ here—are, in the presence of an external field potential $A^\beta(x) = \mathcal{A}^\beta_{\mathrm{ext}}(x)$, defined as

$$\Pi_{\mu\tau}(x,x') = \frac{\delta^2\Gamma}{\delta A^\mu(x)\delta A^\tau(x')}\bigg|_{A=\mathcal{A}_{\mathrm{ext}}}, \quad (3)$$

$$\Pi_{\mu\tau\sigma}(x,x',x'') = \frac{\delta^3\Gamma}{\delta A^\mu(x)\delta A^\tau(x')\delta A^\sigma(x'')}\bigg|_{A=\mathcal{A}_{\mathrm{ext}}} \quad (4)$$

in terms of the effective action

$$\Gamma = \int \mathfrak{L}(z)\mathrm{d}^4z, \quad (5)$$

the generating functional of all-rank polarization tensors—the vertex functions—known in QED as the Legendre transform of the generating functional of the Green functions [18]. The parameter of the power expansion, to which Eq. (2) provides the two lowest terms, depends on a field scale of a definite dynamical theory. We shall discuss this issue in Sec. IV below for QED.

We did not write the zero-power term $(a^\nu(x))^0$, an external macroscopic current, in Eq. (2), because we assumed that the external field had been subjected to the sourceless field equation

$$\frac{\delta S}{\delta A^\beta(y)}\bigg|_{A=\mathcal{A}_{\mathrm{ext}}} = 0, \quad (6)$$

where

$$S = \int L(z)\mathrm{d}^4z, \qquad L(z) = -\mathfrak{F}(z) + \mathfrak{L}(z) \quad (7)$$

are the total action and the total Lagrangian, respectively. Here $-\mathfrak{F}(z) = (1/4)F_{\mu\nu}F^{\nu\mu}$ is the (free) Maxwell Lagrangian, and $F_{\alpha\beta}(z) = \partial^\alpha A_\beta(z) - \partial^\beta A_\alpha(z)$ is the field strength tensor. In what follows, we shall only deal with the external fields $\mathcal{F}_{\alpha\beta} = \partial^\alpha\mathcal{A}^{\mathrm{ext}}_\beta - \partial^\beta\mathcal{A}^{\mathrm{ext}}_\alpha$, which are independent of the four-coordinate $z_\mu$, and with the case where the effective Lagrangian $\mathfrak{L}(z)$ may depend on $z_\mu$ only through the field tensor $F_{\alpha\beta}(z)$ and its space-time derivatives, and not explicitly. The latter property is fulfilled in QED and will be also assumed for other theories subject to our consideration. Under this assumption, the constant field does satisfy the exact sourceless nonlinear field equation [Eq. (6)]. To see this, we fulfill the variational derivative

$$\frac{\delta S}{\delta A_\beta(x)}\bigg|_{A=\mathcal{A}_{\rm ext}} = 2\sum_n \int \frac{\delta S}{\delta F^{(n)}_{\alpha\beta}(z)}\bigg|_{F=\mathcal{F}} \frac{\partial}{\partial z^\alpha}\delta^{4(n)}(x-z)\mathrm{d}^4 z,$$

where $(n)$ marks the derivative with respect to any space-time component. Once the variational derivative $\frac{\delta S}{\delta F^{(n)}_{\alpha\beta}(z)}$, when restricted onto the coordinate-independent fields $F_{\mu\nu}(z) = \mathcal{F}_{\mu\nu}$, cannot depend on $z$, the integration by parts turns this integral to zero.

The above presentation explains why Eq. (2) is the field equation for small electromagnetic perturbations $a^\beta(x) = A^\beta(x) - \mathcal{A}^\beta_{\rm ext}(x)$ over the external field of a constant field strength, caused by a small external current $j_\rho(x)$ and taken to the lowest-power nonlinearity.

Polarization tensors of every rank, $\Pi_{\mu\tau\dots\sigma}(x, x', \dots x'')$, satisfy the continuity relations with respect to every argument and every index (the transversality property)

$$\frac{\partial}{\partial x'_\tau}\Pi_{\mu\dots\tau\dots\sigma}(x, \dots x', \dots x'') = 0 \qquad (8)$$

necessary to provide invariance of every term in the expansion of $\Gamma$ in powers of the field $a^\nu$ under the gauge transformation of it. Note that this is the primary property of $\Gamma$ as a functional given on field strengths and their space-time derivatives only.

In our case of the external field with space- and time-independent strength, the translational invariance holds true, which makes the all-rank polarization tensors depend on their coordinate differences.

With the definition of the photon propagator $D_{\mu\nu}(x, x')$,

$$D^{-1}_{\mu\nu}(x-x') = [\eta_{\mu\nu}\square - \partial^\mu\partial^\nu]\delta^{(4)}(x'-x) + \Pi_{\mu\nu}(x-x'), \qquad (9)$$

the nonlinear field equations [Eq. (2)] take the form of (the set of) integral equations

$$a^\lambda(x) = \int d^4y D^{\lambda\rho}(x-y)j_\rho(y) + \int d^4y D^{\lambda\rho}(x-y)j^{\rm nl}_\rho(y), \qquad (10)$$

$$j^{\rm nl}_\mu(x) = -\frac{1}{2}\int d^4y d^4u \Pi_{\mu\tau\sigma}(x-u, y-u)a^\tau(y)a^\sigma(u), \qquad (11)$$

where we have introduced the notation $j^{\rm nl}_\mu(x)$ for what we shall be calling "nonlinearly induced current".

Before proceeding, the following explanation seems to be in order. Within the present approach, the electromagnetic field $a^\lambda(x)$ is not quantized; this is not needed unless we leave the electromagnetic sector. The nonlinear equations written in this section are classical and will be treated classically below in understanding that the effective action is known. In QED, the latter is the final product of quantum theory, obtained by continual integration over fermions [18]. The effective Lagrangian and all-rank polarization tensors involved are subject to approximate quantum calculations, and hence are functions containing the Planck constant, electron mass and charge. Available is the effective action in the local limit referred to in the next section, which is known as the Euler-Heisenberg action when it is calculated within the approximation of one electron-positron loop (see Ref. [5]), and as the Ritus action when it is calculated with two-loop accuracy [19]. The second-rank polarization tensor [Eq. (3)] was calculated in the one-loop approximation when the external background is formed by a constant and homogeneous electromagnetic field of the most general form (when both its invariants $\mathfrak{F}$ and $\mathfrak{G}$ are nonvanishing) in Ref. [20]. One-loop diagrams with three photon legs corresponding to the third-rank tensor [Eq. (4)] were calculated both on and off the photon mass shell for QED with external magneticlike ($\mathfrak{F} > 0$, $\mathfrak{G} = 0$) and crossed ($\mathfrak{F} = \mathfrak{G} = 0$) fields in Ref. [14], and for charge-asymmetric electron-positron plasma without an external field, using the temperature Green function techniques, in Ref. [21]. The calculations of Stoneham in Ref. [14] might become a basis for extending the results of the following sections beyond the local approximation used there, but they are overcomplicated and not well structured, so we leave this extension for future study. In the next sections, we stick to the general form of the effective Lagrangian and refer to its specific Euler-Heisenberg form only in the very last steps for getting numerical estimates.

## III. LOCAL LIMIT

From now on, we shall restrict ourselves only to slowly varying fields $a^\lambda(x)$ and, correspondingly, to consideration of the sources $j_\rho(y)$ that give rise to such fields via Eqs. (10) and (11). To this end, we may take the effective action in the local limit. This is equivalent to going to the infrared asymptotic limit in the second- and third-rank polarization operators; i.e., to keeping, respectively, only the second and third powers of the four-momentum $k_\mu$ in their Fourier transforms. Aiming at the local limit, we may admit that the effective Lagrangian $\mathfrak{L}$ depends only on (relativistic invariant combinations of) the (gauge-invariant) field strengths $F_{\rho\sigma}$. Moreover, as long as constant fields are concerned, all such combinations may be expressed as functions of the two field invariants $\mathfrak{F} = \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}$ and $\mathfrak{G} = \frac{1}{4}F^{\rho\sigma}\tilde{F}_{\rho\sigma}$, where the dual field tensor is defined as $\tilde{F}_{\rho\sigma} = \frac{1}{2}\epsilon_{\rho\sigma\lambda\kappa}F^{\lambda\kappa}$, with the completely antisymmetric unit tensor defined in such a way that $\epsilon_{1230} = 1$. Then the variational derivatives in Eqs. (3) and (4) can be calculated in terms of derivatives of $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ with respect to the field invariants reduced to the space- and time-independent external field. Henceforth, we shall be interested in the special case where the external field is a constant, purely magnetic field in a certain class of reference frames called "special" below. Since in other Lorentz frames the electric field is also present, we refer to this case as magneticlike. The invariant conditions that specialize the magneticlike case are $\mathfrak{F} > 0$, $\mathfrak{G} = 0$. Once the invariant $\mathfrak{G}$ is a pseudoscalar, the Lagrangian of a P-invariant theory, QED included, may contain it only in an even power. Hence, all the odd derivatives of $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ with respect to it disappear after being reduced to the external magneticlike field:

$$\frac{\partial \mathfrak{L}(\mathfrak{F}, \mathfrak{G})}{\partial \mathfrak{G}}\bigg|_{F=\mathcal{F}, \mathfrak{G}=0} = \frac{\partial^2 \mathfrak{L}(\mathfrak{F}, \mathfrak{G})}{\partial \mathfrak{G}\partial \mathfrak{F}}\bigg|_{F=\mathcal{F}, \mathfrak{G}=0} = 0,$$
$$\frac{\partial^3 \mathfrak{L}(\mathfrak{F}, \mathfrak{G})}{\partial \mathfrak{G}\partial \mathfrak{F}^2}\bigg|_{F=\mathcal{F}, \mathfrak{G}=0} = \frac{\partial^3 \mathfrak{L}(\mathfrak{F}, \mathfrak{G})}{\partial \mathfrak{G}^3}\bigg|_{F=\mathcal{F}, \mathfrak{G}=0} = 0. \quad (12)$$

We calculate Eqs. (3) and (4) in the Appendix to Ref. [17] using the rule

$$\frac{\delta F_{\alpha\beta}(z)}{\delta A^\mu(x)} = \left(\eta_{\mu\beta}\frac{\partial}{\partial z^\alpha} - \eta_{\mu\alpha}\frac{\partial}{\partial z^\beta}\right)\delta^4(x-z) \quad (13)$$

(understood as integrated over $z$ with any function of $z$) by repeatedly applying the relation

$$\frac{\delta \Gamma}{\delta A^\mu(x)} = \int\left[\frac{\partial \mathfrak{L}(\mathfrak{F}(z), \mathfrak{G}(z))}{\partial \mathfrak{F}(z)} F_{\alpha\mu}(z) + \frac{\partial \mathfrak{L}(\mathfrak{F}(z), \mathfrak{G}(z))}{\partial \mathfrak{G}(z)} \tilde{F}_{\alpha\mu}(z)\right]\frac{\partial}{\partial z_\alpha}\delta^4(x-z)\mathrm{d}^4 z \quad (14)$$

and reducing the final results onto the external field. Then, taking Eq. (12) into account and using the notations

$$\mathfrak{L}_{\mathfrak{F}} = \frac{\mathrm{d}\mathfrak{L}(\mathfrak{F}, 0))}{\mathrm{d}\mathfrak{F}}\bigg|_{F=\mathcal{F}}, \quad \mathfrak{L}_{\mathfrak{F}\mathfrak{F}} = \frac{\mathrm{d}^2\mathfrak{L}(\mathfrak{F}, 0)}{\mathrm{d}\mathfrak{F}^2}\bigg|_{F=\mathcal{F}},$$
$$\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} = \frac{\partial^2\mathfrak{L}(\mathfrak{F}, \mathfrak{G})}{\partial \mathfrak{G}^2}\bigg|_{F=\mathcal{F}, \mathfrak{G}=0}, \quad \mathfrak{L}_{\mathfrak{F}\mathfrak{F}\mathfrak{F}} = \frac{\mathrm{d}^3\mathfrak{L}(\mathfrak{F}, 0)}{\mathrm{d}\mathfrak{F}^3}\bigg|_{F=\mathcal{F}},$$
$$\mathfrak{L}_{\mathfrak{F}\mathfrak{G}\mathfrak{G}} = \frac{\mathrm{d}}{\mathrm{d}\mathfrak{F}}\frac{\partial^2\mathfrak{L}(\mathfrak{F}, \mathfrak{G})}{\partial \mathfrak{G}^2}\bigg|_{F=\mathcal{F}, \mathfrak{G}=0},$$

all with $F_{\mu\nu} = \mathcal{F}_{\mu\nu}$ substituted (hence, from now on, $\mathfrak{F} = \frac{1}{4}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma} > 0$ and $\mathfrak{G} = \frac{1}{4}\mathcal{F}^{\rho\sigma}\tilde{\mathcal{F}}_{\rho\sigma} = 0$), we get for the second-rank tensor

$$\Pi^{\mathrm{IR}}_{\mu\tau}(x-y) = \mathfrak{L}_{\mathfrak{F}}\left(\frac{\partial^2}{\partial x^\tau \partial x^\mu} - \eta_{\mu\tau}\Box\right)\delta^4(x-y) - \{\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}\mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\tau} + \mathfrak{L}_{\mathfrak{G}\mathfrak{G}}\tilde{\mathcal{F}}_{\alpha\mu}\tilde{\mathcal{F}}_{\beta\tau}\} \times \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\delta^4(x-y), \quad (15)$$

and for the third-rank tensor in the infrared limit

$$\Pi^{\mathrm{IR}}_{\mu\tau\sigma}(x-y, x-u) = -\mathcal{O}_{\mu\tau\sigma\alpha\beta\gamma}\frac{\partial}{\partial x_\alpha}\left(\left(\frac{\partial}{\partial x_\beta}\delta^4(y-x)\right) \times \left(\frac{\partial}{\partial x_\gamma}\delta^4(x-u)\right)\right), \quad (16)$$

where

$$\begin{aligned}\mathcal{O}_{\mu\tau\sigma\alpha\beta\gamma} = {}& \mathfrak{L}_{\mathfrak{G}\mathfrak{G}}[\tilde{\mathcal{F}}_{\gamma\sigma}\epsilon_{\alpha\mu\beta\tau} + \tilde{\mathcal{F}}_{\alpha\mu}\epsilon_{\beta\tau\gamma\sigma} + \tilde{\mathcal{F}}_{\beta\tau}\epsilon_{\alpha\mu\gamma\sigma}] \\ & + \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}[(\eta_{\mu\tau}\eta_{\alpha\beta} - \eta_{\mu\beta}\eta_{\alpha\tau})\mathcal{F}_{\gamma\sigma} \\ & + \mathcal{F}_{\alpha\mu}(\eta_{\tau\sigma}\eta_{\gamma\beta} - \eta_{\beta\sigma}\eta_{\gamma\tau}) \\ & + \mathcal{F}_{\beta\tau}(\eta_{\mu\sigma}\eta_{\gamma\alpha} - \eta_{\alpha\sigma}\eta_{\gamma\mu})] \\ & + \mathfrak{L}_{\mathfrak{F}\mathfrak{G}\mathfrak{G}}[\mathcal{F}_{\alpha\mu}\tilde{\mathcal{F}}_{\beta\tau}\tilde{\mathcal{F}}_{\gamma\sigma} + \tilde{\mathcal{F}}_{\alpha\mu}\mathcal{F}_{\beta\tau}\tilde{\mathcal{F}}_{\gamma\sigma} \\ & + \tilde{\mathcal{F}}_{\alpha\mu}\tilde{\mathcal{F}}_{\beta\tau}\mathcal{F}_{\gamma\sigma}] + \mathfrak{L}_{\mathfrak{F}\mathfrak{F}\mathfrak{F}}\mathcal{F}_{\alpha\mu}\mathcal{F}_{\beta\tau}\mathcal{F}_{\gamma\sigma}. \end{aligned} \quad (17)$$

(The reader may consult the Appendix in Ref. [17] for detailed calculations.) This tensor turns to zero when there is no external field, $\mathcal{F} = 0$, in agreement with the Furry theorem. The two transversality conditions [Eq. (8)] for Eq. (15) are provided for, in that the matrix in the brackets is antisymmetric under each permutation $\mu \Leftrightarrow \alpha$ and $\tau \Leftrightarrow \beta$, while the first term in Eq. (15) is transverse explicitly. The three transversality conditions [Eq. (8)] for Eq. (16) are provided for, in that the matrix [Eq. (17)] is antisymmetric under each permutation $\mu \Leftrightarrow \alpha$, $\tau \Leftrightarrow \beta$, and $\sigma \Leftrightarrow \gamma$. Thanks to the two latter antisymmetries, by using Eqs. (16) and (17) in Eq. (11), we obtain for the nonlinearly induced current the expression

$$j^{\mathrm{nl}}_\mu(x) = \frac{1}{8}\mathcal{O}_{\mu\tau\sigma\alpha\beta\gamma}\frac{\partial}{\partial x_\alpha}(f^{\beta\tau}f^{\gamma\sigma}), \quad (18)$$

that includes only the field intensity tensors $f^{\beta\tau} = \frac{\partial}{\partial x_\beta}a^\tau(x) - \frac{\partial}{\partial x_\tau}a^\beta(x)$. Therefore, the nonlinearly induced current is gauge invariant: it depends only on field intensities and, besides, it is conserved, $\frac{\partial}{\partial x_\mu}j^{\mathrm{nl}}_\mu(x) = 0$, due to the first antisymmetry, $\mu \Leftrightarrow \alpha$.

We have to approach the nonlinear set in Eqs. (10) and (11) by looking for its solution in a power series in the field $a^\lambda(x)$. Within the first iteration, to which we shall as a matter of fact confine ourselves, we substitute the linear approximation to the solution of Eq. (10)

$$a^{\mathrm{lin}}_\nu(x) = \int d^4x' D_{\nu\rho}(x-x')j^\rho(x') \quad (19)$$

for $a(x)$ into Eq. (11). In other words, we should use the electromagnetic field $f_{\beta\tau} = f^{\mathrm{lin}}_{\beta\tau} = \frac{\partial}{\partial x^\beta}a^{\mathrm{lin}}_\tau(x) - \frac{\partial}{\partial x^\tau}a^{\mathrm{lin}}_\beta(x)$

linearly produced by the source $j_\mu(x)$ in the expression for the nonlinearly induced current [Eq. (18)].

## IV. MAGNETIC FIELD OF A STATIC CHARGE AT REST IN EXTERNAL MAGNETIC FIELD

We are in a position to start studying the nonlinear effect of the production of a magnetic field by a static charge at rest in a constant and homogeneous external magnetic field in a special frame. The linear effect of the external magnetic field on the electrostatic field of a charge was studied earlier (beyond the infrared approximation) in Refs. [10–12].

In this frame, the external magnetic field is defined as $B_i = (1/2)\epsilon_{ijk}\mathcal{F}_{jk} = \tilde{\mathcal{F}}_{i0}$, $B = |\mathbf{B}|$, while the external electric field disappears, $E_i = \mathcal{F}_{0i} = 0$. The roman indices span the 3D subspace in this reference frame; $\epsilon_{ijk}$ is the fully antisymmetric tensor, and $\epsilon_{123} = 1$.

Consider now a static charge given in that frame by the four-current $j_\mu(x) = j_\mu(\boldsymbol{x})\delta_{\mu 0}$. In the linear approximation [Eq. (19)], naturally only an electrostatic field is generated in that frame. Hence, the components with $\alpha = \beta = \gamma = 0$, $\tau$, $\sigma \neq 0$ do not contribute to Eq. (18), so we need only the components

$$\begin{aligned}\mathcal{O}_{i00jmn} &= -\mathcal{F}_{ji}[\delta_{mn}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} - \tilde{\mathcal{F}}_{n0}\tilde{\mathcal{F}}_{m0}\mathfrak{L}_{\mathfrak{F}\mathfrak{G}\mathfrak{G}}] \\ &\quad + [\tilde{\mathcal{F}}_{m0}\epsilon_{jin0} + \tilde{\mathcal{F}}_{n0}\epsilon_{jim0}]\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} \\ &= \epsilon_{ijk}B_k[\delta_{mn}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} - B_nB_m\mathfrak{L}_{\mathfrak{F}\mathfrak{G}\mathfrak{G}}] \\ &\quad + [B_m\epsilon_{jin} + B_n\epsilon_{jim}]\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} \end{aligned} \tag{20}$$

in Eq. (18) [and those obtained from Eq. (20) by permutations between the second and fifth, and between the third and sixth indices], while $\mathcal{O}_{000jmn} = 0$ according to Eq. (17). Therefore $j_0^{\rm nl}(\boldsymbol{x}) = 0$; i.e., there is no nonlinear (quadratic) correction to the static charge within the current quadratic approximation—the induced current of Eq. (18) is purely spatial:

$$\begin{aligned} j_i^{\rm nl}(\boldsymbol{x}) &= \frac{1}{2}\mathcal{O}_{i00jmn}\frac{\partial}{\partial x_j}(f_{m0}^{\rm lin}f_{n0}^{\rm lin}) \\ &= \frac{1}{2}(\boldsymbol{\nabla}\times\boldsymbol{B})_i[\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}\boldsymbol{\mathcal{E}}^2 - \mathfrak{L}_{\mathfrak{F}\mathfrak{G}\mathfrak{G}}(\boldsymbol{B}\boldsymbol{\mathcal{E}})^2] \\ &\quad - \mathfrak{L}_{\mathfrak{G}\mathfrak{G}}(\boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}})_i(\boldsymbol{B}\boldsymbol{\mathcal{E}}), \end{aligned} \tag{21}$$

where $\mathcal{E}_n = \mathcal{E}_n(\boldsymbol{x}) = f_{0n}^{\rm lin} = \frac{-\partial}{\partial x_n}a_0^{\rm lin}(\boldsymbol{x})$ is the time-independent electric field, linearly produced following Eq. (19), and the differential operator $\boldsymbol{\nabla}$ acts on everything to the right of it. The magnetic field strength $\boldsymbol{h}(\boldsymbol{x})$ generated by this current according to the Maxwell equation $\boldsymbol{\nabla}\times\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{j}^{\rm nl}(\boldsymbol{x})$ is

$$h_i(\boldsymbol{x}) = \mathfrak{h}_i(\boldsymbol{x}) + \nabla_i\Omega, \tag{22}$$

where

$$\begin{aligned}\mathfrak{h}_i(\boldsymbol{x}) &= \frac{B_i}{2}[\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}(\boldsymbol{\mathcal{E}}(\boldsymbol{x}))^2 - \mathfrak{L}_{\mathfrak{F}\mathfrak{G}\mathfrak{G}}(\boldsymbol{B}\boldsymbol{\mathcal{E}}(\boldsymbol{x}))^2] \\ &\quad - \mathcal{E}_i(\boldsymbol{x})\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}(\boldsymbol{B}\boldsymbol{\mathcal{E}}(\boldsymbol{x}))\end{aligned} \tag{23}$$

because $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\Omega \equiv 0$, and the scalar function $\Omega$ should be subjected to the Poisson equation

$$\boldsymbol{\nabla}^2\Omega = -\nabla_j\mathfrak{h}_j(\boldsymbol{x})$$

to make the magnetic field $\boldsymbol{h}(\boldsymbol{x})$ obey the other Maxwell equation, $\boldsymbol{\nabla}\boldsymbol{h}(\boldsymbol{x}) = 0$. Hence, the magnetic field is the transverse part of Eq. (23):

$$h_i(\boldsymbol{x}) = \left(\delta_{ij} - \frac{\nabla_i\nabla_j}{\boldsymbol{\nabla}^2}\right)\mathfrak{h}_j(\boldsymbol{x}) = \mathfrak{h}_i(\boldsymbol{x}) + \frac{\nabla_i\nabla_j}{4\pi}\int\frac{\mathfrak{h}_j(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}d^3y. \tag{24}$$

Note that the substitution of the field of a pointlike charge into Eq. (24) through Eq. (23) would cause the divergency of the integral in Eq. (24) near $\mathbf{y} = 0$: the present approach fails near the point charge, since it is not applicable to its strongly inhomogeneous field. Dealing with the point charge would require going beyond the infrared approximation followed in the present work. Nevertheless, Eq. (24) is sound as applied to extended charges.

Equation (24) would coincide with the magnetic induction $\boldsymbol{b}(\boldsymbol{x}) = \boldsymbol{\nabla}\times\boldsymbol{a}^{\rm nl}(\boldsymbol{x})$ if the linear vacuum magnetization effect could be neglected; i.e., if the nonlinear correction to the field in Eq. (10),

$$a_{\rm nl}^\lambda(x) = \int d^4yD_\rho^\lambda(x-y)j_\rho^{\rm nl}(y), \tag{25}$$

could be taken without the contribution of the linear response function [Eq. (16)] $\Pi_{\mu\nu}(x-x')$ in the photon propagator [Eq. (9)]. Taking this contribution into account results in more complicated integrals. The situation remains simple, however, when we may disregard the anisotropy of the linear magnetic response. The inverse magnetic permeability tensor inherent in the second-rank polarization tensor [Eq. (15)] is, in the special frame, the constant tensor [22]

$$\mu_{ij}^{-1} = (1-\mathfrak{L}_{\mathfrak{F}})\delta_{ij} - \mathfrak{L}_{\mathfrak{F}\mathfrak{F}}B_iB_j,$$

whose two[3] eigenvalues $\mu_\perp^{-1} = 1-\mathfrak{L}_{\mathfrak{F}}$ and $\mu_\|^{-1} = 1 - \mathfrak{L}_{\mathfrak{F}} - 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}$ are responsible for magnetizations linearly caused by certain conserved, constant, straight-linear currents flowing along the external magnetic field and across it, respectively (see the Appendix in Ref. [23]). In QED, the values $\mathfrak{L}_{\mathfrak{F}}$ and $2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}$ are of the order of the fine structure constant $\alpha = 1/137$ but depend on the field $B$. When $B$ is very large ($B \gg m^2/e$), these quantities, as found from the Euler-Heisenberg one-loop effective Lagrangian (see, e.g., Ref. [24]), behave as

[3] The constant background magneticlike field makes a uniaxial medium in any of the special frames [22].

$$\mathfrak{L}_{\mathfrak{F}} \approx \frac{\alpha}{3\pi}\ln\frac{eB}{m^2}, \qquad 2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} \approx \frac{\alpha}{3\pi}.$$

So, when $\frac{eB}{m^2} \gg 2.7$, the contribution of $2\mathfrak{F}\mathfrak{L}_{\mathfrak{F}\mathfrak{F}}$ may be neglected as compared to $\mathfrak{L}_{\mathfrak{F}}$, and the linear magnetization becomes isotropic, $\mu_\perp^{-1} = \mu_{||}^{-1}$. Therefore, in this limit, we finally have for the nonlinear magnetic induction

$$\boldsymbol{b}(\boldsymbol{x}) = (1 - \mathfrak{L}_{\mathfrak{F}})^{-1}\boldsymbol{h}(\boldsymbol{x}). \tag{26}$$

The electric field $\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla} a_0^{\rm lin}(\boldsymbol{x})$ to be substituted in Eqs. (21) and (23) is the one that is linearly produced via Eq. (19) by a static charge distribution within the same infrared approximation. To determine it, note that in Eq. (26) only the propagator component $D_{00}$ participates, and that in the Fourier representation $D_{00} = (\boldsymbol{k}^2 - \varkappa_2)^{-1}$, with $\varkappa_2$ being one (out of three) eigenvalues of the second-rank polarization tensor [Eq. (4)] taken in the static limit $k_0 = 0$ in the special reference frame. Once the polarization tensor is considered in its infrared limit [Eq. (15)], this quantity is [23,24] $\varkappa_2 = \boldsymbol{k}^2\mathfrak{L}_{\mathfrak{F}} - k_{||}^2 2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}$. Here $\boldsymbol{k}^2 = \boldsymbol{k}_\perp^2 + k_{||}^2$ and $\boldsymbol{k}_\perp$, $k_{||}$ are the momentum components of the small electromagnetic field across and along $\boldsymbol{B}$, respectively. With the use of this propagator, the calculation of Eq. (19) for the pointlike charge $j_0(\boldsymbol{x}) = q\delta^3(\boldsymbol{x})$ results in the anisotropic Coulomb law

$$a_0^{\rm lin}(\boldsymbol{x}) = \frac{q}{4\pi}\frac{1}{\sqrt{\epsilon_\perp}\sqrt{\epsilon_\perp x_{||}^2 + \epsilon_{||}\boldsymbol{x}_\perp^2}}, \tag{27}$$

where $\epsilon_\perp = 1 - \mathfrak{L}_{\mathfrak{F}}$ and $\epsilon_{||} = 1 - \mathfrak{L}_{\mathfrak{F}} + 2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}}$ are eigenvalues of the dielectric tensor [22] $\epsilon_{ij} = (1 - \mathfrak{L}_{\mathfrak{F}})\delta_{ij} + \mathfrak{L}_{\mathfrak{G}\mathfrak{G}}B_iB_j$ responsible for polarizations, linearly caused by homogeneously charged planes parallel and orthogonal to $\boldsymbol{B}$, respectively, and $x_\perp$ and $x_{||}$ are the coordinate components across and along $\boldsymbol{B}$. For a large magnetic field, one gets the linearly growing asymptote from the Euler-Heisenberg Lagrangian $2\mathfrak{F}\mathfrak{L}_{\mathfrak{G}\mathfrak{G}} \approx \frac{\alpha}{3\pi}\frac{eB}{m^2}$. This means that if $\frac{eB}{m^2} > \frac{3\pi}{\alpha}$, the dielectric component $\epsilon_{||}$ dominates over $\epsilon_\perp$; i.e., the electrization becomes highly anisotropic, in contrast to the magnetization. In this asymptotic region, Eq. (27) becomes (if we disregard the polarization in $\epsilon_\perp$ by setting $\epsilon_\perp = 1$) the large-distance behavior of the potential of a point charge in a strong magnetic field calculated in the linear approximation in Refs. [10,11] beyond the infrared approximation of the polarization tensor. Note that Eq. (27), as well as its high-field limit, is only valid far from the charge. In that domain, however, it also fits any charge, with the total value $q$, distributed over a finite region.

The formal substitution of the electric field $\boldsymbol{\mathcal{E}}$ corresponding to the potential [Eq. (27)] into Eqs. (23) and (24) would result in a strong singularity $x^{-4}$ at the origin for the magnetic field of a pointlike charge. Beyond the infrared approximation, Eq. (27) is replaced in QED by an anisotropic Debye regime [10,11], actual at the distances, much smaller than the Compton length of the electron; however, the singularity in the origin would remain the same. Neither course of action is justified for the pointlike charge. To cover that case, one should use the third-rank polarization tensor [Eq. (4)] beyond the infrared limit [Eqs. (16) and (17)]. The behavior at the origin is governed by the ultraviolet limit $k \to \infty$. In this limit, the momentum $k$ dominates over the external magnetic field; hence, the latter may be neglected. However, without that field, the third-rank polarization tensor disappears due to the Furry theorem. For this reason, we may expect that the resulting magnetic field of the point charge will be less, if at all, singular at the origin. Any similar consideration is not helpful in the case of $x^{-6}$ singularity of Eq. (1) if applied to a pointlike charge, since this equation depends on the fourth-rank polarization tensor, nonzero already in the absence of an external field. However, in the case of Eq. (1), the singularity can be absorbed into charge renormalization.

## V. SOME NUMERICAL ESTIMATES

To analyze the large magnetic field limits of the induced current [Eq. (21)] of the resulting magnetic field [Eq. (24)] and of its induction [Eq. (26)], one should also bear in mind the asymptotic behavior $\mathfrak{L}_{\mathfrak{F}\mathfrak{G}\mathfrak{G}} = -\frac{\alpha e}{3\pi m^2 B^3}$. Then it follows from the large external magnetic field asymptotic behavior $\frac{eB}{m^2} \gg \frac{3\pi}{\alpha}$ of the other derivatives of the Euler-Heisenberg Lagrangian involved in Eq. (23) that were listed above that in this limit,

$$\frac{e\mathfrak{h}_{||}}{m^2} \sim \frac{\alpha}{6\pi}\left[-\left(\frac{e\mathcal{E}_{||}}{m^2}\right)^2 + \left(\frac{e\mathcal{E}_\perp}{m^2}\right)^2\frac{m^2}{eB}\right],$$
$$\frac{e\mathfrak{h}_\perp}{m^2} \sim -\frac{\alpha}{3\pi}\frac{e\mathcal{E}_{||}}{m^2}\frac{e\mathcal{E}_\perp}{m^2}, \qquad \frac{eB}{m^2} \gg \frac{3\pi}{\alpha}.$$

The minus sign in the first line indicates that the induced magnetic field diminishes the external field in the large external field regime.

We may apply the results of Eqs. (23) and (24) to a small external magnetic field, $(eB/m^2) \ll 1$, as well. With the Euler-Heisenberg Lagrangian density, one has in this regime

$$\mathfrak{L}_{\mathfrak{F}\mathfrak{F}} = \frac{4\alpha}{45\pi}\left(\frac{e}{m^2}\right)^2, \qquad \mathfrak{L}_{\mathfrak{G}\mathfrak{G}} = \frac{7\alpha}{45\pi}\left(\frac{e}{m^2}\right)^2,$$
$$\mathfrak{L}_{\mathfrak{F}\mathfrak{G}\mathfrak{G}} = \frac{\alpha}{315\pi}\left(\frac{e}{m^2}\right)^4.$$

The third coefficient $\mathfrak{L}_{\mathfrak{F}\mathfrak{G}\mathfrak{G}}$ does not contribute in the leading order in $(eB/m^2) \ll 1$ to the estimates

$$\mathfrak{h}_{||} \sim B\frac{\alpha}{45\pi}\left[2\left(\frac{e\boldsymbol{\mathcal{E}}}{m^2}\right)^2 - 7\left(\frac{e\mathcal{E}_{||}}{m^2}\right)^2\right],$$
$$\mathfrak{h}_\perp \sim -B\frac{7\alpha}{45\pi}\left(\frac{e\mathcal{E}_\perp}{m^2}\right)\left(\frac{e\mathcal{E}_{||}}{m^2}\right), \qquad \frac{eB}{m^2} \ll 1. \tag{28}$$

In this approximation, we may set $\epsilon_\perp = \epsilon_{||} = \mu_\perp^{-1} = \mu_{||}^{-1} = 1$. Therefore, $\boldsymbol{h} = \boldsymbol{b}$, and for the electric field of a charge outside of it, one may use here the standard Coulomb law $a_0^{\rm C}(\boldsymbol{x}) = (q/4\pi)\ \boldsymbol{x}/|x|^2$, $\boldsymbol{\mathcal{E}}^{\rm C} = (q/4\pi)\ \boldsymbol{x}/|x|^3$ instead of Eq. (27).

It is interesting that for a small external magnetic field $B$, the deflection of the linearly induced electric field $\boldsymbol{\mathcal{E}}^{\rm lin}$, as obtained from Eq. (27) by calculating the gradient $\boldsymbol{\mathcal{E}}^{\rm lin} = (\partial/\partial\boldsymbol{x})a_0^{\rm lin}(\boldsymbol{x})$ from the Coulomb field $\boldsymbol{\mathcal{E}}^{\rm C}$, is approximately expressed in a way very much symmetrical under the exchange $\boldsymbol{\mathcal{E}}\Leftrightarrow\mathbf{B}$ with Eq. (28):

$$\mathcal{E}_{||}^{\rm lin} - \mathcal{E}_{||} \sim \mathcal{E}_{||}\frac{\alpha}{45\pi}\left(\frac{eB}{m^2}\right)^2.$$

Note that $\alpha/45\pi = 5\times 10^{-5}$. So, for the electric field value close to Schwinger's $1.3\times 10^{16}$ V/cm, the nonlinearly produced magnetic field makes up to $3\times 10^{-4}$ of the external magnetic field, which must be kept below Schwinger's $4.4\times 10^{13}$ G in this case.

## VI. CONCLUSION

In this paper, we have found an expression for the magnetic field $\boldsymbol{h}(\boldsymbol{x})$ produced by a static charge $q$ placed into an external magnetic field $\boldsymbol{B}$ [Eqs. (23) and (24)]. It is shown that, in QED, this nonlinear magnetoelectric effect, not considered before, occurs already in the simplest approximation, where the effective Lagrangian $\mathfrak{L}$ is taken in its local limit, and only the second power of the charge $q$ and/or its electric field $\boldsymbol{\mathcal{E}}(\boldsymbol{x})$ are kept. As for the background magnetic field $\boldsymbol{B}$, to reveal the effect, it suffices to take it into account in the linear approximation $\sim B$, although a magnetic field $B$ of arbitrary magnitude is included in our result as well. The final formulas depend on the first three derivatives of the effective action $\mathfrak{L}$ with respect to the external field invariants, which complies with the fact that, minimally, diagrams with three photon legs are responsible for the effect in the given approximation.

The results are model independent and relate not only to QED, but also to any nonlinear electrodynamics provided the standard postulates of $U(1)$-gauge, Lorentz, translation, C, P, and T invariances are respected. When applying them to QED, we take the Euler-Heisenberg Lagrangian for $\mathfrak{L}$ to estimate the regimes of weak and strong $\boldsymbol{B}$. In QED, all electromagnetic fields appear in ratios to the Schwinger characteristic value $m^2/e$ of $4.4\times 10^{13}$ in CGSE units. The nonlinear magnetoelectric effect we are reporting on is efficient if the electric field of a charge is comparable to, but still smaller than, $m^2/e$. Such fields take place near atomic nuclei and at the surfaces of strange quark stars. Besides this, strange quark stars can be strongly magnetized [25]. When the Schwinger value is exceeded by the electric field, the nonlinearity can no longer be treated via the power expansion of Eq. (2), and electron-positron pair creation from the vacuum must also be taken into account.

To reveal the effect that the present nonlinear contribution to the field produced by the charged atomic nucleus may have on the atomic spectrum in a magnetic field, a separate study is needed. At present, we restrict ourselves to the remark that first of all, the interaction between the electron orbital momentum and the magnetic field due to the nucleus's magnetic moment is subject to modification. This interaction is known to be responsible, in the absence of the external magnetic field, for the hyperfine splitting.

The contribution to the magnetic field at the edge of the proton coming from Eq. (28) is about $\frac{\alpha^3 B}{45\pi a^4 m^4} = \frac{2\alpha^2}{45\pi}\mu_{\rm P}\frac{eB}{m^2}(\frac{m_{\rm P}}{m})^2\frac{1}{m_{\rm P}a^4}$. Here $\mu_{\rm P} = \frac{e}{2m_{\rm P}}$ is the nuclear magneton, with $m_{\rm P}$ being the proton mass, while the proton's electric radius $a$ may be taken as $a = 0.87$ Fm $\sim 5/m_{\rm P}$. The ratio of this "nonlinear" correction to the magnetic field at the edge of the proton due to the proton magnetic moment $\sim 2.8\mu_{\rm P}/a^3$ makes, thus, $\sim 10^6\frac{\alpha^2}{3\cdot 45\pi}\frac{eB}{m^2}$. At the extreme laboratory values of the magnetic field of the order of, say, $B\sim 10^{-7}(m^2/e)\approx 4.4\times 10^6$ G, this ratio, $\sim 10^{-6}$, seems to be too small for producing measurable effects on atomic spectra, but it may be essential for other possible effects associated with the proton magnetic moment, which—without the magnetic field—is known [26] to an accuracy of $10^{-8}\mu_{\rm P}$. On the contrary, in magnetic fields of the magnetar scale, $B\sim(1\div 10^2)m^2/e$, the nonlinear correction makes $\sim(1\div 10^2)\frac{\alpha^2}{225\pi}(\frac{m_{\rm P}}{m})^2$ part of the magnetic field of the nucleon magneton $\mu_{\rm P}$; i.e., it may even exceed it. Therefore, one may expect an influence on the magnetized neutron star matter.

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[1] M. Born and L. Infeld, Proc. R. Soc. Lond. A **144**, 425 (1934).
[2] A. Stern, Phys. Rev. Lett. **100**, 061601 (2008).
[3] T. C. Adorno, D. M. Gitman, A. E. Shabad, and D. V. Vassilevich, Phys. Rev. D **84**, 085031 (2011); **84**, 065003 (2011); T. C. Adorno, D. M. Gitman, and A. E. Shabad, Phys. Rev. D **86**, 027702 (2012).
[4] J. Plebański, *Lectures on Nonlinear Electrodynamics* (Nordita, Copenhagen, 1970).
[5] V. B. Berestetsky, E. M. Lifshits, and L. P. Pitayevsky, *Quantum Electrodynamics* (Nauka, Moscow, 1989; Pergamon Press, Oxford, New York, 1982).
[6] R. P. Negreiros, F. Weber, M. Malheiro, and V. Usov, Phys. Rev. D **80**, 083006 (2009); R. Picanço Negreiros, I. N. Mishustin, S. Schramm, and F. Weber, Phys. Rev. D **82**, 103010 (2010); M. Malheiro, R. Picanço Negreiros, F. Weber, and V. Usov, J. Phys. Conf. Ser. **312**, 042018 (2011).
[7] D. D. Ivanenko and D. F. Kurdgelaidze, Astrofiz. **1**, 479 (1965); N. Itoh, Prog. Theor. Phys. **44**, 291 (1970); E. Witten, Phys. Rev. D **30**, 272 (1984).
[8] L. P. Linares, M. Malheiro, A. R. Taurines, and M. Fiolhais, Braz. J. Phys. **36**, 1391 (2006).
[9] E. S. Fradkin, D. M. Gitman, and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer, Berlin, 1991).
[10] A. E. Shabad and V. V. Usov, Phys. Rev. Lett. **98**, 180403 (2007); Phys. Rev. D **77**, 025001 (2008); in *Particle Physics on the Eve of LHC (Proceedings of the 13th Lomonosov Conference on Elementary Particle Physics, Moscow, 2007*, edited by A. I. Studenikin (World Scientific, Singapore, 2009), p. 392.
[11] N. Sadooghi and A. Sodeiri Jalili, Phys. Rev. D **76**, 065013 (2007).
[12] B. Machet and M. I. Vysotsky, Phys. Rev. D **83**, 025022 (2011); S. I. Godunov, B. Machet, and M. I. Vysotsky, Phys. Rev. D **85**, 044058 (2012).
[13] M. A. Andreichikov, B. O. Kerbikov, and Y. A. Simonov, arXiv:1210.0227.
[14] S. L. Adler, J. N. Bahcall, C. G. Callan, and M. N. Rosenbluth, Phys. Rev. Lett. **25**, 1061 (1970); S. L. Adler, Ann. Phys. (N.Y.) **67**, 599 (1971); V. O. Papanyan and V. I. Ritus, Sov. Phys. JETP **34**, 1195 (1972); **38**, 879 (1974); R. J. Stoneham, J. Phys. A **12**, 2187 (1979).
[15] A. K. Harding and D. Lai, Rep. Prog. Phys. **69**, 2631 (2006).
[16] A. E. Shabad and V. V. Usov, Phys. Rev. D **81**, 125008 (2010).
[17] D. M. Gitman and A. E. Shabad, arXiv:1209.6287.
[18] S. Weinberg, *The Quantum Theory of Fields* (University Press, Cambridge, 2001).
[19] V. I. Ritus, in *Problems of Quantum Electrodynamics of Intense Field, Proceedings of the P. N. Lebedev Physics Institute* (Nauka, Moscow, 1986), Vol. 168, p. 5.
[20] I. A. Batalin and A. E. Shabad, Zh. Eksp. Teor. Fiz. **60**, 894 (1971); [Sov. Phys. JETP **33**, 483 (1971)].
[21] E. Ferrer, V. de la Incera, and A. E. Shabad, Fortschr. Phys. **32**, 261 (1984).
[22] S. Villalba-Chávez and A. E. Shabad, Phys. Rev. D **86**, 105040 (2012).
[23] A. E. Shabad and V. V. Usov, arXiv:0991.0640.
[24] A. E. Shabad and V. V. Usov, Phys. Rev. D **83**, 105006 (2011).
[25] B. J. Ahmedov, B. B. Ahmedov, and A. A. Abdujabbarov, Astrophys. Space Sci. **338**, 157 (2012).
[26] J. Beringer *et al.* (Particle Data Group), Phys. Rev. D **86**, 010001 (2012).