2013

Sludge formation in the activated sludge process: mathematical analysis

Asma O. M Alharbi
University of Wollongong, aoma093@uowmail.edu.au

Mark I. Nelson
University of Wollongong, mnelson@uow.edu.au

Annette L. Worthy
University of Wollongong, annie@uow.edu.au

Harvinder S. Sidhu
University Of New South Wales, h.sidhu@adfa.edu.au

Publication Details

Research Online is the open access institutional repository for the University of Wollongong. For further information contact the UOW Library: research-pubs@uow.edu.au
Sludge formation in the activated sludge process: mathematical analysis

Abstract
One drawback associated with the activated sludge process is the production of 'sludge'. The expense for treating excess sludge can account for 50-60% of the running costs of a plant. Traditional methods for disposing of excess sludge, which include incineration, the use of landfill sites and dumping at sea are becoming increasingly regulated worldwide due to concerns about the presence of potentially toxic elements in it. Furthermore, a combination of the limited amount of land available for landfill, particularly in urban areas, with stringent legislation has seen the economic costs of using landfill sites increasing sharply. Thus there is a growing interest in methods that reduce the volume and mass of excess sludge produced as part of biological wastewater treatment processes. We investigate a simple model for the activated sludge process in which the influent contains a mixture of soluble and biodegradable particulate substrate. Within the bioreactor the biodegradable particulate substrate is hydrolyzed to form soluble substrate. The soluble organics are used for energy and growth by the biomass. We investigate how the amount of sludge formed depends upon both the residence time and how the use of a settling unit.

Keywords
mathematical, analysis, activated, process, formation, sludge

Disciplines
Engineering | Science and Technology Studies

Publication Details

This conference paper is available at Research Online: http://ro.uow.edu.au/eispapers/1648
Abstract—One drawback associated with the activated sludge process is the production of 'sludge'. The expense for treating excess sludge can account for 50-60% of the running costs of a plant. Traditional methods for disposing of excess sludge, which include incineration, the use of landfill sites and dumping at sea are becoming increasingly regulated worldwide due to concerns about the presence of potentially toxic elements in it. Furthermore, a combination of the limited amount of land available for landfill, particularly in urban areas, with stringent legislation has seen the economic costs of using landfill sites increasing sharply. Thus there is a growing interest in methods that reduce the volume and mass of excess sludge produced as part of biological wastewater treatment processes.

We investigate a simple model for the activated sludge process in which the influent contains a mixture of soluble and biodegradable particulate substrate. Within the bioreactor the biodegradable particulate substrate is hydrolyzed to form soluble substrate. The soluble organics are used for energy and growth by the biomass. We investigate how the amount of sludge formed depends upon both the residence time and how the use of a settling unit.

Keywords-activated sludge; modelling; recycling; settling unit; wastewater; water treatment

I. INTRODUCTION

Historically waste products such as agricultural and industrial wastewaters and sewage were placed on fields, where natural environmental processes broke them down. In 1913 Ardern and Lockett at Davyhulme (Manchester, UK) discovered the activated sludge process, which is now the most commonly process for treating sewage and industrial wastewater using air [1, page 4]. This process commonly uses two units, an aerated biological reactor, where the pollutants are degraded by bacteria, and a settling unit (or clarifier) where the activated sludge settles into the bottom of the unit. Activated sludge is recycled from the bottom of the clarifier to the biological reactor.

The first models for bioreactors were developed in the 1960s. These models assume that there is one limiting substrate and one limiting microorganism. Such models are easy to calibrate and often provide useful insights into process behavior; accordingly they remain widely used. Larger models have been developed, such as the IWA Activated Sludge models [2,3,4]. Here we analyse a common extension of the standard model which adds an insoluble substrate and allows a fraction of dead biomass to be recycled into the pool of insoluble substrate. This allows a more realistic depiction of the biochemical processes than the standard model and is also the simplest model that can be used to investigate sludge production in the activated sludge process. It has the advantage over the IWA models that it is amenable to mathematical analysis, rather than relying on numerical simulations. We find the steady-state solutions and determine their stability as a function of process parameters. These are used to calculate the steady-state chemical oxygen demand and the steady-state volatile suspended solids as a function of the residence time. We investigate the circumstances under which it is possible to operate the reactor at, or below, a target value for the volatile suspended solids.

II. MODEL DEVELOPMENT

In this section we develop the mathematical model. We first discuss the biochemistry of the model before proceeding to give the dimensional dimensionless model formulations.

III. BIOCHEMISTRY

An overview of the biochemical processes is shown in Figure 1.

---

**Keywords**- activated sludge; modelling; recycling; settling unit; wastewater; water treatment
In this figure \( (X_b) \) is the concentration of biomass, \( (X_i) \) is the concentration of non-biodegradable particulate material, \( (X_s) \) is the concentration of biodegradable particulate substrate, \( (S) \) is the concentration of soluble substrate, \( \mu(s) \) is the specific growth rate, \( (k_d) \) is the death coefficient, and \( (k_h) \) is the hydrolysis rate of insoluble organic compound \( (X_s) \). There are three biological processes occurring in the model:

1. Biodegradable particulate substrate \( (X_s) \) is hydrolysed to soluble substrate \( (S) \)

\[
eX_s \xrightarrow{k_h} (c\alpha_h)S, \tag{1}
\]

where \( \alpha_h \) is the yield factor for hydrolysis of insoluble organic compounds.

2. The soluble organic materials \( (S) \) are used as substrates for energy and growth by the biomass \( (X_b) \)

\[
aS \xrightarrow{\mu(s)} a\alpha_h X_b, \tag{2}
\]

where \( \alpha_g \) is the yield factor for growth of biomass.

3. The death of biomass adds to the pool of soluble substrate \( (S) \) in addition to producing an inert residue \( (X_i) \)

\[
bX_b \xrightarrow{k_d} f_bX_i + f_s\alpha_s bS, \tag{3}
\]

where \( f_i \) is the fraction of dead biomass converted to inert material, \( f_s \) is the fraction of dead biomass converted to soluble substrate and \( \alpha_s \) is the yield factor for conversion of dead biomass to soluble substrate.

It is assumed that after death the biomass is converted either into inert residue or soluble substrate. Thus the total of the fraction of inert residue and the fraction of soluble substrate must be one.

\[
f_i + f_s = 1.
\]

The sequence of biological reactions (2)–(3) converts substrate through biomass back to substrate.

This basic biochemical model was originally proposed by Chung and Neethling [5].

Mass conversation imposes the restriction that

\[
0 < f_s\alpha_g\alpha_s \leq 1. \tag{4}
\]

If \( S \) and \( X_i \) are measured in the same units the following additional inequalities must hold

\[
\alpha_g \leq 1,
\]

\[
f_s\alpha_s \leq 1.
\]

Chung and Neethling [5] calibrated their model against experimental steady-state data and used it to estimate the fraction of the total volatile suspended solids that was due to biomass. They were interested in how this fraction varied as a function of the sludge age.

Here we find analytical representations for the steady-state solutions in this model. (The reactor model considered by Chung and Neethling did not contain a settling unit.) We principally use our steady-state expressions to identify when the total volatile suspended solids can maintained at a target value.

Chemical oxygen demand (COD) is perhaps the mostly widely used parameter for characterizing the organic carbon content in wastewaters of municipal or domestic origin. Although COD does not differentiate between biodegradable organic and non biodegradable compounds, new experimental techniques now enable separate experimental identification of biodegradable and non biodegradable fractions [6, page 21]. In our model the COD of the wastewater is defined by

\[
COD = S + \alpha_h X_s,
\]

where \( \alpha_h \) is a conversion factor that is required because the units of \( S \) may differ from that of \( X_s \).

The volatile suspended solids (VSS) are the solids which are lost if dried sludge is ignited. They represent the solid organic carbon in the sludge. The VSS content is given by

\[
X_t = X_b + X_i + X_s. \tag{5}
\]

In this model we assume that a target value for the volatile suspended solids has been set,

\[
VSS = 12000 \text{ mg L}^{-1}. \tag{6}
\]

This value is slightly lower than the typical steady-state VSS in a conventional membrane bioreactor system (< 15,000 mg L\(^{-1}\)) [6].

We investigate how the residence time required to keep the volatile suspended solid below this target value is affected by the operation of the settling unit. In equations (5-12) the settling unit is characterized by the parameters \( C \) and \( R \).

IV. THE DIMENSIONAL MODEL

The model equations are

Soluble substrate

\[
\frac{dS}{dt} = FS_0 - S + V\alpha_h k_h X_s + VF\alpha_s k_d X_b - \frac{VX_b\mu(s)}{\alpha_g}. \tag{5}
\]

Biomass

\[
\frac{dX_b}{dt} = F(-X_b) + RF(C - 1)X_b + VX_b\mu(s) - Vk_d X_b. \tag{6}
\]

Non-biodegradable particulate material

\[
\frac{dX_i}{dt} = F(X_{i0} - X_i) + RF(C - 1)X_i + VFk_d X_b. \tag{7}
\]

Biodegradable particulate substrate.
In these equations the settling unit is equally effective at concentrating the biological component ($X_b$) and the particulates ($X_p$ and $X_t$). In the following we denote the units of soluble substrate by $[S]$ and the units of biomass, non-biodegradable particulate material and biodegradable particulate material by $[X]$. We have assumed that the settling unit is equally effective at concentrating the biological component ($X_b$) and the particulates ($X_p$ and $X_t$). In the following we denote the units of soluble substrate by $[S]$ and the units of biomass, non-biodegradable particulate material and biodegradable particulate material by $[X]$.

In these equations $C_i$ is the recycle concentration factor (-), the value of this factor depends on the settling unit design and operation, $F$ is the flow rate $(dm^3 \cdot hr^{-1})$, $K_x$ is the Monod constant $(g \cdot dm^{-3})$, $R$ is the recycle ratio based on volumetric flow rates(-), $S$ is the substrate concentration within the bioreactor ([S]), $S_0$ is the concentration of substrate flowing into the reactor ([S]), $V$ is the volume of the bioreactor (dm3), $X_i$ is the concentration of biomass ([X]), $X_i$ is the concentration of of non-biodegradable particulate material ([X]), $X_i$ is the concentration of biodegradable particulate substrate ([X]), $X_i, (i = s, p)$ is a concentration flowing into the reactor ([X]), $X_{i0}$ is total biomass ([X]), $t$ is time $(hr^{-1})$, $\mu_S$ is the specific growth rate model $(hr^{-1})$, $\mu_m$ is the maximum specific growth rate $(hr^{-1})$, and $\tau$ is the residence time $(hr)$. Note that $C \geq 1$, as otherwise recycle leads to a decrease in the performance of the bioreactor.

The following parameters were defined in the Biochemistry section: $f_i$, $f_p$, $K_d$, $k_h$, $\alpha_s$, $\alpha_p$, $\alpha_t$.

For a specific wastewater, a given biological community, and a particular set of environmental conditions, the parameters $K_d$, $k_h$, $\alpha_s$, $\alpha_p$, and $\mu_m$ are fixed. The parameters that can be varied are $S_0$, $X_{i0}$, $X_i, (i = 0, s, p)$, and $\tau$. In the standard model the components $X_t$ and $X_o$ are not included and it is assumed that $f_o = 0$.

The chemical oxygen demand in the influent is given by:

$$\text{COD}_{in} = S_0 + \alpha_{gh}X_{s0}.$$  

V. THE DIMENSIONLESS MODEL

By introducing dimensionless variables for concentrations of the substrate $[S'] = [S/K_x]$, microorganism and particulates we have assumed that the settling unit is equally effective at concentrating the biological component ($X_b$) and the particulates ($X_p$ and $X_t$). In the following we denote the units of soluble substrate by $[S]$ and the units of biomass, non-biodegradable particulate material and biodegradable particulate material by $[X]$. We have assumed that the settling unit is equally effective at concentrating the biological component ($X_b$) and the particulates ($X_p$ and $X_t$). In the following we denote the units of soluble substrate by $[S]$ and the units of biomass, non-biodegradable particulate material and biodegradable particulate material by $[X]$.

In these equations $C_i$ is the recycle concentration factor (-), the value of this factor depends on the settling unit design and operation, $F$ is the flow rate $(dm^3 \cdot hr^{-1})$, $K_x$ is the Monod constant $(g \cdot dm^{-3})$, $R$ is the recycle ratio based on volumetric flow rates(-), $S$ is the substrate concentration within the bioreactor ([S]), $S_0$ is the concentration of substrate flowing into the reactor ([S]), $V$ is the volume of the bioreactor (dm3), $X_i$ is the concentration of biomass ([X]), $X_i$ is the concentration of of non-biodegradable particulate material ([X]), $X_i$ is the concentration of biodegradable particulate substrate ([X]), $X_i, (i = s, p)$ is a concentration flowing into the reactor ([X]), $X_{i0}$ is total biomass ([X]), $t$ is time $(hr^{-1})$, $\mu_S$ is the specific growth rate model $(hr^{-1})$, $\mu_m$ is the maximum specific growth rate $(hr^{-1})$, and $\tau$ is the residence time $(hr)$. Note that $C \geq 1$, as otherwise recycle leads to a decrease in the performance of the bioreactor.

The following parameters were defined in the Biochemistry section: $f_i$, $f_p$, $K_d$, $k_h$, $\alpha_s$, $\alpha_p$, $\alpha_t$.

For a specific wastewater, a given biological community, and a particular set of environmental conditions, the parameters $K_d$, $k_h$, $\alpha_s$, $\alpha_p$, and $\mu_m$ are fixed. The parameters that can be varied are $S_0$, $X_{i0}$, $X_i, (i = 0, s, p)$, and $\tau$. In the standard model the components $X_t$ and $X_o$ are not included and it is assumed that $f_o = 0$.

The chemical oxygen demand in the influent is given by:

$$\text{COD}_{in} = S_0 + \alpha_{gh}X_{s0}.$$  

By introducing dimensionless variables for concentrations of the substrate $[S'] = [S/K_x]$, microorganism and particulates

\[ X_i' = X_i / (\alpha_g K_x) \quad (j = i, s) \] and time \([t' = \mu_m t]\) the dimensional model, equations (5)-(8), can be written in the dimensionless form

\[ \frac{dS'}{dt'} = \frac{1}{\tau} (S'_{0} - S') + \alpha_{gh} k_h X_b' S' + f_i \alpha_{gh} k_d X_b' - \frac{S' X_b'}{1 + S'}, \]  

\[ \frac{dX_{p}'}{dt'} = \frac{1}{\tau} (-X_{p}' + X_{p}') + \frac{k_d S'}{1 + S'} - k_d X_b', \]  

\[ \frac{dX_{t}'}{dt'} = \frac{1}{\tau} \left( X_{i0}' - X_{t}' \right) + \frac{R}{\tau} X_{t}' + f_i k_d X_b', \]  

\[ \frac{dX_{s}'}{dt'} = \frac{1}{\tau} \left( X_{i0}' - X_{s}' \right) + \frac{R}{\tau} X_{s}' - k_d X_b', \]  

The chemical oxygen demand COD

$$\text{COD}' = S' + \alpha_{gh} X_b'. $$

Total Volatile Suspended Solids VSS

$$X_i' = X_i' + X_t' + X_s'. $$

VI. RESULTS

In this section we present a steady-state analysis of the model. The stability analysis calculations are not provided, but the results are summarized. Finally, we find simplifying expressions for the steady-state solutions for large values of the residence time.

A. Steady-State Analysis

The steady-state solutions are found by putting the derivative equal to zero in equations (13-16) and then solving the resulting system of the equations.

The steady-state solutions are given by Washout branch

\[ (S', X_b', X_i', X_s') = (S_W', 0, X_{i0}, X_{s0}) \]  

\[ S_W' = S_0 + \alpha_{gh} k_d X_b' t' \frac{1}{1 + R t' + k_d t'}. \]
The washout branch corresponds to process failure due to removal of all the biomass.

No-washout branch

\[ (S^*, X^*_b, X^*_w, X^*_v) = \left( S^*_{WN}, X^*_{WN}, X^*_v, X^*_w \right) \]

\[ S^*_{WN} = \frac{1 - R^* + k_d^* t^*}{(1 - k_d^*) t^* - 1 + R^*} \]

\[ X^*_b,WN = \frac{(1 + S^*_{WN})(S^*_{WN} - S^*_{WN})}{(S^*_{WN} - (1 + S^*_{WN})\alpha_{g,h}k_d^*)t^*} \]

\[ X^*_v,WN = X^*_v \frac{f_d k_d^* X^*_w}{1 + R^*} + \frac{X^*_w}{1 + R^*} \]

\[ X^*_w,WN = X^*_w \frac{f_d}{1 + R^*} + \frac{k_d^* X^*_w}{1 + R^*} \]

\[ COD^*_WN = \frac{1 - R^* + k_d^* t^*}{(1 - k_d^*) t^* - 1 + R^*} + \frac{\alpha_{g,h}X^*_w}{1 + R^* + k_d^* t^*} \]

\[ VSS^*_WN = (1 + \frac{f_d k_d^*}{1 + R^*})X^*_w + \frac{X^*_w}{1 + R^*} + \frac{X^*_w}{1 + R^* + k_d^* t^*} \]

It can be shown that the no-washout branch is only physically meaningful when the decay rate is sufficiently small

\[ k_d^* < k_d^{cr,*} = \frac{S^*_a + \alpha_{g,h}X^*_w}{S^*_a + \alpha_{g,h}X^*_w + 1} \]

and the residence time is sufficiently large

\[ \tau^* > \tau^*_{cr} = (1 - R^*)\tau_{cr}^{*} \]

where

\[ \tau_{cr}^{*} = \frac{b^* + \sqrt{b^* - 4a^*}}{2a} \]

\[ a = \left( S^*_0 + \alpha_{g,h}X^*_w \right) - (1 + S^*_0 + \alpha_{g,h}X^*_w)k_d^{*}k_h^{*} \]

\[ b^* = (1 + S^*_0)(k_h^{*} + k_d^{*}) + k_d^{*}\alpha_{g,h}X^*_w - S^*_0 \]

\[ c^* = (1 + S^*_0) \]

For the parameter value in the appendix we have

\[ \tau_{cr}^{*} = 1.2018 \]

When the no-washout branch is not physically meaningful then the only stable steady state is the washout solution. Therefore, process failure must occur. This case can therefore be ignored.

B. Stability of the Steady-State solutions

When the decay rate is larger than the critical value \( k_d^{*} \geq k_d^{cr,*} \) then that the biomass must die out \( \lim_{t \to \infty} X_b = 0 \). When the decay rate is smaller than the critical value \( k_d^{*} < k_d^{cr,*} \) then there is a critical value of the residence time \( \tau_{cr}^{*} \). If the residence time is smaller than the critical value \( \tau^* < \tau_{cr}^{*} \) then the washout solution is globally stable. When \( \tau^* > \tau_{cr}^{*} \) the no-washout branch is stable.

The no-washout solution is physically meaningful and stable if \( k_d^* < k_d^{cr,*} \) and \( \tau^* > \tau_{cr}^{*} \).

These statements are not proved here, but can be shown using technique of linear stability analysis.

C. Asymptotic Results

Asymptotic solutions at large residence time for the chemical oxygen demand and the volatile suspended solids are given by

\[ COD^*_{WN} = \frac{k_d^*}{1 - k_d^*} + \frac{(1 - R^*)}{(1 - k_d^*)} + \frac{\alpha_{g,h}X^*_w}{1 - R^*} + \frac{1}{(1 - k_d^*)} + 0 \left( \frac{1}{\tau^{*}} \right) \]

\[ VSS^*_{WN} = \frac{X^*_w}{1 - R^*} + \frac{f_d k_d^*}{1 - R^*}a_0 + \left( a_0 + \frac{f_d k_d^*}{1 - R^*}a_1 + \frac{X^*_w}{k_d^*} \right) \frac{1}{(1 - k_d^*)} + 0 \left( \frac{1}{\tau^{*}} \right) \]

(21)

where

\[ a_0 = \frac{\text{COD}^*_{WN}(1 - k_d^*) - k_d^*}{k_d^*(1 - k_d^*)(1 - f_d\alpha_{g,h})} \]

\[ a_1 = \frac{(1 - R^*)}{D} (B - A - C) \]

\[ A = \frac{\alpha_{g,h}X^*_w}{k_d^*} + \frac{1}{(1 - k_d^*)} \]

\[ B = \frac{\left( S^*_0 + \alpha_{g,h}X^*_w \right)(1 - k_d^*) - k_d^*}{(1 - k_d^*)} \]

\[ C = \frac{\left( S^*_0 + \alpha_{g,h}X^*_w \right)(1 - k_d^*) - k_d^*}{k_d^*(1 - k_d^*)} \]

\[ D = k_d^* \left( 1 - f_d\alpha_{g,h} \right) \]

VII. DISCUSSION

In this section we apply the results of the previous section to investigate how the steady-state values of the chemical oxygen demand and the volatile suspended solids vary as a function of the residence time. For the latter we are particularly interested in whether it is possible to maintain the value below the target value. We show that this is only possible if the effective recycle ratio is below a critical value.

A. Steady-state chemical oxygen demand

The steady-state chemical oxygen demand along the washout branch is given by equation (19) and the steady-state chemical oxygen demand along the non-washout branch is given by equation (20).

Figure 2 shows the steady-state chemical oxygen demand as a function of the residence time.
When there is no recycle ($R^* = 0$) the chemical oxygen demand is constant along the washout branch ($\tau^* < \tau^*_w = 1.2018$).

$$\text{COD}_W = S_0 + \frac{(1 + k_g \tau^*)a_{b,0}X_{s,0}}{1 + R^* + k_h \tau^*} = S_0 + \alpha_{b,0}X_{s,0} = 10.179.$$  

Along the no-washout branch the chemical oxygen demand is a decreasing function. It has limiting value when the residence time approaches infinity ($\tau^* \to \infty$) given by

$$\lim_{\tau^* \to \infty} \text{COD}^*_W = \frac{k_d}{1-k_d}.$$  

The efficiency at which the chemical oxygen demand is removed can be defined by

$$E_{\text{COD}}^* = \frac{\text{COD}^*_\text{in} - \text{COD}^*_\text{out}}{\text{COD}^*_\text{in}}.$$  

From equation (21) the efficiency can not be increased over the limiting value $E_{\text{COD}}^*(\tau^* = \infty) = 0.99283$. This means that it is impossible to remove more that 99.28% of the chemical oxygen demand flowing into the reactor.

When $E_{\text{COD}}^* = 0.90$, the chemical oxygen demand in the effluent has been reduced to 10% of the value in the influent. The required value of the residence time is $\tau^* = 3.4123$ (15.5 days). On figure 2 the target value $E_{\text{COD}}^* = 0.9$ is indicated by the horizontal line.

In order to reduce the chemical oxygen demand in the effluent to 1% of the value in the influent ($E_{\text{COD}}^* = 0.99$) the required value of the residence time is $\tau^* = 97.006$ (440.9 days). Thus in practice an efficiency ($E_{\text{COD}}^* = 0.99$) can not be achieved.

When the effective recycle parameter takes its theoretical maximum ($R^* = 1$) then the residence time to achieve an efficiency $E_{\text{COD}}^* = 0.99$ is reduced to $\tau^* = 56.739$. This is still too large for practical purposes.

### B. Steady-State volatile suspended solids

Figure 3 shows a steady state diagram for the volatile suspended solids as a function of the residence time. Along the washout branch ($0 < \tau^* < \tau^*_V$) we have

$$\text{VSS}_W^* = X_{s,0} \left[1+\frac{X_{s,0}^*}{1+R^*+k_h \tau^*}\right],$$  

which is a decreasing function of the residence time. It's maximum value is when the residence time is zero. For the case of no recycle ($R^* = 0$) we have

$$\text{VSS}_{W,0}^* = X_{s,0}^* + X_{s,0}^* = 163.13.$$  

When there is no settling unit ($R^* = 0$) the volatile suspended solids at the trans-critical bifurcation ($\tau^* = 1.2018$) has decreased to the value $\text{VSS}_W^* = 38.688$.

After the trans-critical bifurcation the volatile suspended solids is given by

$$\text{VSS}_{W,E,W}^* = X_{s,0} + X_{s,0} + \frac{x_{s,0}^*}{1+R^*+k_h \tau^*},$$  

The volatile suspended solids increases to a local maximum $\text{VSS}_{W,E}^* = 40.075$ when $\tau^* = 1.2395$. Thereafter it decreases toward its limiting value. The limiting value, when the residence time is infinity, is given by

$$\text{VSS}_{W,E}^* = \frac{x_{s,0}^*}{1-R^*} + \frac{x_{s,0}^*}{1-R^*+k_h \tau^*} = 56.931.$$  

Using the parameter values in the appendix. The target value of the volatile suspended solids is 77.071.
From figure 3 we see that the intersection point where the volatile suspended solids is equal to the target value is on the washout branch. When this happens the value of the residence time at the intersection point is found from equation (22). We have

\[ \tau_{\text{IW}} = \frac{(1-R^*)[X_{W} + X_{V} - VSS][1-R^*]}{[VSS(1-R^*) - X_{W}]k_{d}} = 0.29093. \]  

(24)

We see that along the no-washout branch the volatile suspended solids is always below the target value. Changing the value of the dimensional target value (12000 mg L^{-1}) changes the location of the horizontal line in figure 3. From figure 3 we can see that there are five generic cases of behavior for the intersection points depending upon the target value.

In case one the target value is larger than the maximum value along the washout branch. Thus there is no intersection point because the volatile suspended solids concentration in the effluent is always below the target value; this case is not realistic.

In case two the target value is larger than the maximum value along the no-washout branch. Thus there is only one intersection point, which is on the washout branch; the value of the residence time at the intersection point is given by equation (24). This is a good case to have because wherever we operate on the no-washout branch the volatile suspended solid is guaranteed to be below the target value. This is the case shown in figure 3.

In the third case the target value is between the volatile suspended solids concentration at the trans-critical bifurcation and the maximum value along the no-washout branch. Thus there are three intersection points: one along the washout branch and two along the no-washout branch. The volatile suspended solids is lower than the target value if the residence time is between the intersection point on the washout branch and the first intersection point on the no-washout branch or larger than the second intersection point on the washout branch. In practice the residence time would need to be larger than the second intersection point on the no-washout branch.

In case four the target value is larger than the asymptotic value along the no-washout branch and smaller than the volatile suspended solids at the trans-critical bifurcation. There is one intersection point, on the no-washout branch. We require the residence time to be larger than the intersection value.

In case five the target value is smaller than both the value at the trans-critical bifurcation and the limiting value when the residence time approaches infinity. There are no intersection points. This is undesirable because the volatile suspended solids can never be reduced below the target value.

There are two non-generic cases. These are when we have two intersection points. These happen when either the target value of the volatile suspended solid is exactly equal to the maximum value along the washout branch or the value at the trans-critical bifurcation.

For the default values in the appendix table 1 shows which case occurs as the effective recycle parameter is varied, case one never happens.

<table>
<thead>
<tr>
<th>Case</th>
<th>R*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case two</td>
<td>0 &lt; R* &lt; 0.46306</td>
</tr>
<tr>
<td>Case three</td>
<td>0.46306 &lt; R* &lt; 0.49803</td>
</tr>
<tr>
<td>Case four</td>
<td>0.49803 &lt; R* &lt; 0.73868</td>
</tr>
<tr>
<td>Case five</td>
<td>0.73868 &lt; R*</td>
</tr>
</tbody>
</table>

Increasing the effective recycle parameter increases the asymptotic value for the volatile suspended solids at infinity, equation (22). Thus there is a critical value of the effective recycle parameter when the volatile suspended solids at infinity is equal to the volatile suspended solids target value.

Setting the leading term of the asymptotic solution equation (22) equal to the target value and rearranging we find

\[ R_{\text{cr},V} = \frac{VSS(1-R^*)k_{d}A_{0}}{VSS} = 0.73868. \]

When the effective recycle parameter is smaller than the critical value \( (R^* < R_{\text{cr},V}) \) then the asymptotic value of the volatile suspended solids is below the target value \( (VSS'(\tau' \to \infty) < VSS^*_t) \). In this case there is at least one intersection point along the no-washout branch. When the effective recycle parameter is larger than the critical value \( (R^* > R_{\text{cr},V}) \) then the asymptotic value of the volatile suspended solids is larger than the target value \( (VSS'(\tau' \to \infty) > VSS^*_t) \). In this case there is no intersection point along the no-washout branch.

**VIII. CONCLUSION**

We examined a simple extension of the standard model. This biochemical model has been proposed before, but not examined in detail. We found the steady state solutions and determined their stability. We use the steady state solutions to investigate how the chemical oxygen demand and the total volatile suspended solids depend upon the residence time and the operation of a settling unit.

We have shown that there is a critical value of the effective recycle parameter. If it is larger than this critical value then the volatile suspended solids can not be reduced below the target value. As we have found analytical expressions for our steady-state solutions, the critical value of the recycle parameter can be readily determined if the target value is changed.
APPENDIX: Values of parameters

Typical parameter values are:

| Table 2. Typical Dimensional Parameter Values. |
|-------------------------------|-------------------------------|----------------|----------------|
| Parameter | Value | Unit | |
| $K_s$ | 5,190 | mg COD L$^{-1}$ | |
| $S_o$ | 10,360 | mg COD L$^{-1}$ | |
| $X_{l,o}$ | 2,810 | mg VSS L$^{-1}$ | |
| $X_{s,o}$ | 22,590 | mg VSS L$^{-1}$ | |
| $f_s$ | 0.8 | g VSS (g VSS $)^{-1}$ | |
| $K_d$ | 0.015 | day$^{-1}$ | |
| $K_m$ | 1.1 | day$^{-1}$ | |
| $\alpha_s$ | 0.3 | mg VSS (mg COD $)^{-1}$ | |
| $\alpha_b$ | 1.88 | mg COD (mg VSS $)^{-1}$ | |
| $\alpha_d$ | 1.42 | mg COD (mg VSS $)^{-1}$ | |
| $\mu_m$ | 0.22 | day$^{-1}$ | |

| Table 3. The Dimensionless Parameter Values. |
|-------------------------------|-------------------------------|----------------|----------------|
| Parameter | Value | Parameter | Value | |
| $S^*$ | 1.996 | $X_{l,o}^*$ | 145.0857 | |
| $X_{s,o}^*$ | 18.0475 | $K_d^*$ | 5 | |
| $\alpha_b^*$ | 0.0564 | $\alpha_s^*$ | 0.0426 | |
| COD$^*$ | 10.1788 | |

ACKNOWLEDGEMENT

Asma Alharbi is grateful to the High Education Ministry of Saudi Arabia for a PhD scholarship.

REFERENCES