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# Effective global regularity and empirical modeling of direct, inverse and mixed demand systems

**Keith R McLaren and K.K. Gary Wong**

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# Effective Global Regularity and Empirical Modeling of Direct, Inverse and Mixed Demand Systems

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## Abstract

In this paper, we utilize the notion of "effective global regularity" and the intuition stemming from Cooper and McLaren (1996)'s General Exponential Form to develop a family of "composite" (product and ratio) direct, inverse and mixed demand systems. Apart from having larger regularity regions, the resulting specifications are also of potentially arbitrary rank, which can better approximate non-linear Engel curves. We also make extensive use of duality theory and a numerical inversion estimation method to rectify the endogeneity problem encountered in the estimation of the mixed demand systems. We illustrate the techniques by estimating different types of demand systems for Japanese quarterly meat and fish consumption. Results generally indicate that the proposed methods are promising, and may prove beneficial for modeling systems of direct, inverse and mixed demand functions in the future.

***JEL classification***: D11, D12

*Keywords*: Effective Global Regularity; Mixed Demands; Conditional Indirect Utility Functions; Numerical Inversion Estimation Method

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## 1 INTRODUCTION

Specification of consumer demand systems typically replies on one of two assumptions: either i) prices (and expenditure) are predetermined or ii) quantities are predetermined. The first (or second) of these assumptions leads to direct (or inverse) demand systems.[1] In between the two polar cases of direct and inverse demands, there exists a whole class of mixed demand systems wherein prices of some goods and quantities of the others are predetermined, so that the respective quantities demanded and prices must adjust to clear the market.[2]

Direct, inverse and mixed demand systems have been the object of many applications, for example, Christensen, Jorgenson and Lau (1975), Deaton and Muelbauer (1980), and Moschini (1998) on direct demands; Moschini and Vissa (1992), Eales and Unnevehr (1994), and Holt (2002) on inverse demands; and Moschini and Rizzi (2006) on mixed demands. The empirical analysis of these systems usually proceeds by selecting a functional form to approximate the indirect utility, direct utility, or the conditional cost function and then deriving the corresponding demand or share functions via simple differentiation, accordingly to Roy's identity, the Hotelling-Wold identity or Samuelson's Envelope Theorem. Note however that there are many functional forms that could be used; selection of these forms is usually conducted in the context of the perennial trade-off between regularity and flexibility.

A demand system is said to be regular if it satisfies the restrictions imposed by the paradigm of rational consumer choice; i.e., the systems must satisfy homogeneity, monotonicity, symmetry and curvature restrictions. The class of known globally regular demand systems however is quite small

[1] The first (or second) assumption would be appropriate in the case of an infinitely elastic (or an inelastic) supply function.
[2] See Samuelson (1965) and Chavas (1984).

(e.g., the Cobb-Douglas Form) and comes at the price of inflexibility.[3] At the other extreme are demand systems such as Deaton and Muellbauer's (1980) Almost Ideal Demand System (AIDS) or Eales and Unnevehr's (1994) Inverse Almost Ideal Demand System (IAIDS) designated as locally flexible, in that they do not put any prior restrictions on elasticities, other than those imposed by the regularity conditions, at the point of approximation. The cost of this flexibility at a point is that these systems usually exhibit small regions of regularity about the point of approximation.

A convenient compromise between these two extremes is the class of "effectively globally regular" (EGR) demand systems, in the sense of Cooper and McLaren (1996). By "EGR" is meant that there exists a price index P(**p**) such that the regularity properties are satisfied for all expenditure ($c$) - price (**p**) combinations satisfying $c \geq \mathrm{P}(\mathbf{p})$. Thus the regularity region is an unbounded region in price-expenditure space, potentially including all points in the sample, and all points corresponding to higher levels of "real income". Examples of these systems include Stone's (1954) Linear Expenditure System (LES), Lewbel's (1987 & 1992) Fractional Demand Systems and Cooper and McLaren's (1996) General Exponential Form (GEF).[4] In spite of the possible benefits associated with the EGR demand systems, there are few empirical applications beyond those considered originally by Cooper and McLaren (1996). To our knowledge no previously published empirical studies have used the notion of "effective global regularity" and the intuition stemming from GEF to develop new models of inverse and mixed demands. We do so here.

The first aim of this paper is to develop parametric representations of the indirect utility, direct utility and conditional indirect utility functions in terms of expenditure, and price and quantity indices, in order to generate direct, inverse and mixed demand systems in the spirit of "effective

[3] For instance, in a Cobb-Douglas direct demand system, income, own price and cross-price elasticities are a priori constrained to be +1, -1 and zero, respectively.

[4] Such systems can be extended to be locally flexible at a point, such as the original Modified AIDS of Cooper and McLaren (1992).

global regularity" and GEF with improved flexibility properties. This is achieved by the use of regular functional forms for price and quantity functions, which are components of different types of utility functions. As will be illustrated, the proposed parametric forms are ideally suited for empirical analysis, since they are potentially locally flexible but have larger regularity regions, since the functional structures are parsimonious in the number of additional parameters, and since they are of potentially arbitrary rank that can better approximate non-linear Engel curves.

The second aim is to introduce a new approach to the specification of empirical mixed demand functions, which is based on parametric representations of the conditional (or partial) indirect utility function used in the area of rationed demand.[5] Recent efforts aimed at modeling estimable mixed demand systems including those by Moschini and Vissa (1993), Matsuda (2004), Brown and Lee (2006), and Moschini and Rizzi (2006) have not focused on the use of conditional indirect utility functions. We provide here the first attempt to do so. Differentiation of a finally chosen conditional indirect utility function with respect to prices and quantities, after some manipulation, yields the systems of conditional mixed demand functions. Whilst these functions are conditioned on an endogenous variable (conditional expenditure), in most cases they do not have an explicit closed-form representation as the Marshallian mixed demand functions i.e. in terms of the exogenous variables such as quantities, prices and total expenditure. As pointed out by McLaren, Rossiter and Powell (2000), the endogeneity problem of conditional expenditure need not hinder estimation. A simple one-dimensional numerical inversion allows us to estimate the parameters of a particular conditional indirect utility function via the parameters of the implied Marshallian mixed demand functions. The formal theory for using a conditional indirect utility function in this context will be developed and illustrated in the next section of this paper.

The remainder of this paper is organized as follows. Section 2 develops the theoretical

[5] See Neary and Roberts (1980) and Chavas (1984).

foundations formally. These include relevant concepts and results from static duality theory as well as the ideas of effective global regularity and the numerical inversion estimation method. Section 3 considers possible specifications for the direct utility, indirect utility and conditional indirect utility functions. Descriptions of the data, estimation method and the empirical application using Japanese data are provided in Section 4. Finally, Section 5 recapitulates and concludes.

## 2 BACKGROUND DEVELOPMENTS

### 2.1 Marshallian Direct and Inverse Demand Functions

Let $\mathbf{x} \in \Omega^{N}$ represent an N-vector of commodities, $\mathbf{p} \in \Omega_{+}^{N}$ the corresponding price vector, and $c > 0$ a level of expenditure, where $\Omega^{N}$ (or $\Omega_{+}^{N}$) is the non-negative (or positive) orthant. Suppose that individual preferences can be represented by a direct utility function u = U(**x**),[6] satisfying the following regularity conditions **RU**:

RU1: U is real;
RU2: U is continuous;
RU3: U is increasing in **x**; and
RU4: U is quasi-concave in **x**.

The Marshallian direct demand functions $\mathbf{X}^{M}(c, \mathbf{p})$ are defined as the solutions to the constrained optimization problem:

(1) $$\text{Max}_{\mathbf{x}} \{U(\mathbf{x}): \mathbf{p}'\mathbf{x} = c\},$$

where the adjective “Marshallian” and the superscript “M” refer to the arguments ($c$, **p**) of the corresponding functions.

Dual to U(**x**) is the indirect utility function defined by:

(2) $$U^{M}(c, \mathbf{p}) = U[\mathbf{X}^{M}(c, \mathbf{p})],$$

[6] The notation u=U(**x**) is indicative of that used in the rest of this paper. Upper case letters denote functions, and the corresponding lower case letters denote the scalar values of those functions.

which gives the maximized value of utility conditional on given expenditure and prices. Under the assumptions that U(**x**) satisfies Conditions **RU**, the indirect utility function will inherit the regularity conditions **RIU**:

RIU1: $U^M$ is real;
RIU2: $U^M$ is continuous;
RIU3: $U^M$ is homogeneous of degree zero (HD0) in ($c$, **p**);
RIU4: $U^M$ is non-increasing in **p**;
RIU5: $U^M$ is non-decreasing in $c$; and
RIU6: $U^M$ is quasi-convex in **p**.

The Marshallian demand functions are related to the indirect utility function via Roy's identity:

(3) $$X_i^M = \frac{-\partial U^M / \partial p_i}{\partial U^M / \partial c}.$$

Duality theory is concerned with the fact that preferences may be represented equivalently by a direct utility function satisfying **RU**, or by an indirect utility function satisfying **RIU**. This argument can be illustrated by using the dual relationship between the direct utility and normalized indirect utility functions:

(4) $$U(\mathbf{x}) = \text{Min}_{\mathbf{p}/c} \{U^M(c/c, \mathbf{p}/c) : (\mathbf{p}/c)'\mathbf{x} = 1\}$$
$$= \text{Min}_{\mathbf{r}} \{U^M(\mathbf{r}) : \mathbf{r}'\mathbf{x} = 1\}.$$

Solving the first order conditions of (4) for the normalized prices yields the Marshallian inverse demand functions:

(5) $$r_i = R_i^{MI}(\mathbf{x})$$

which satisfy the Hotelling-Wold identity:

(6) $$r_i = R_i^{MI}(\mathbf{x}) = \frac{\partial U(\mathbf{x}) / \partial x_i}{\sum_j \left[\partial U(\mathbf{x}) / \partial x_j\right] x_j}.$$

Here the adjective "Marshallian inverse" and the superscript MI refer to the arguments (**x**) of the corresponding functions.

Indeed, expressions (3) and (6) show how the direct and inverse demands might be obtained from the indirect and direct utility functions respectively. Since total expenditure, prices and quantities are observable variables, the empirical analysis of direct (or inverse) demands usually proceeds by specifying an indirect (or direct) utility function which satisfies **RIU (**or **RU)**, exploiting Roy's identity (or the Hotelling-Wold identity) to derive the direct (inverse) Marshallian demand functions, and then statistically estimating the parameters that characterise the Marshallian direct (inverse) demand functions given data on **x**, *c* and **p**.

## 2.2 Marshallian Mixed Demand Functions and the Numerical Inversion Approach

Mixed demand functions are appropriate in the situation where one group of commodities are subject to an infinitely elastic supply, while the remainder are subject to a fixed supply. Mixed demand functions have also been found to be a useful tool for analyzing consumer behavior in a number of other situations including consumer rationing, the distinction between short run and long run consumer behaviour, and the presence of non-market goods. In order to discuss mixed demands, consider the partition of the commodity vector **x** into two sub-vectors $\mathbf{x} = \{\mathbf{x}_A, \mathbf{x}_B\}$ with $\mathbf{x}_A$ containing commodities chosen optimally, and $\mathbf{x}_B$ containing commodities in fixed quantities whose prices are optimally determined.[7] Likewise, the price vector **p** can be partitioned as $\mathbf{p} = \{\mathbf{p}_A, \mathbf{p}_B\}$ with $\mathbf{p}_A$ and $\mathbf{p}_B$ containing the prices of group A and B commodities respectively.

According to Samuelson (1965) and Chavas (1984), mixed demand functions may be derived from the constrained optimization problem:

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[7] The expression "whose prices are optimally determined" is a shorthand for the idea that the corresponding prices are such that at those prices the fixed quantities $\mathbf{x}_B$ are the quantities that would have been chosen according to the consumer optimization (1), and hence are in fact the inverses of Marshallian demands. It is an implication of duality theory that such prices can be viewed as the outcome of a hypothetical optimization problem like (4).

(7) $\text{Max}_{\mathbf{x}_A, \mathbf{p}_B} \; \{U(\mathbf{x}_A, \mathbf{x}_B) - U^M(\mathbf{p}_A, \mathbf{p}_B, c) : \mathbf{p}'_A\mathbf{x}_A + \mathbf{p}'_B\mathbf{x}_B = c\}$

where U and $U^M$ are the direct and indirect utility functions respectively.[8] The solutions to (7) give a Marshallian mixed demand system:

$$x_{Ai} = X^{MM}_{Ai}(c, \mathbf{p}_A, \mathbf{x}_B), \text{ and}$$

$$p_{Bj} = P^{MM}_{Bj}(c, \mathbf{p}_A, \mathbf{x}_B),$$

where $X^{MM}_{Ai}$ and $P^{MM}_{Bj}$ are the Marshallian mixed (direct plus inverse) demand functions, and the adjective "Marshallian mixed" and the superscript "MM" refer to the arguments ($\mathbf{p}_A$, $\mathbf{x}_B$, $c$) of the corresponding functions. While this is a possible way to characterize mixed demands, the specification (7) is not empirically useful, since it requires a compatible specification of the explicit functional forms of both the direct and the corresponding indirect utility functions, and hence does not exploit the power of duality theory.

An alternative derivation of mixed demands begins with conditional demands. Conditional demands may be characterized in terms of the conditional indirect utility function, defined as:

(8) $U^C(c_A, \mathbf{p}_A, \mathbf{x}_B) = \text{Max}_{\mathbf{x}_A} \{U(\mathbf{x}_A, \mathbf{x}_B) : \mathbf{p}'_A\mathbf{x}_A = c_A\}$

$$= U[\mathbf{X}^C_A(c_A, \mathbf{p}_A, \mathbf{x}_B), \mathbf{x}_B],$$

where $U^C$ is the maximized value of utility when ($c_A$, $\mathbf{p}_A$, $\mathbf{x}_B$) are given, the superscript C is to indicate that the function is conditioned on $c_A$, $\mathbf{p}_A$, and $\mathbf{x}_B$, and $\mathbf{X}^C_A$ [the solutions to (8)] are the conditional direct demand functions for group A commodities. Due to the fact that $U^C$ is a dual representation of the direct utility function, it will inherit the regularity conditions $\mathbf{RU^C}$:

$RU^C1$: $U^C$ is real;

[8] In Samuelson (1965) and Chavas (1984), the indirect utility function is represented in terms of normalized prices (**r**). As indicated by Moschini and Rizzi (2006), provided that $c$ is given, the representation in (7) is admissible and simplifies the interpretation of the model.

$RU^C2$: $U^C$ is continuous;

$RU^C3$: $U^C$ is decreasing in $\mathbf{p}_A$;

$RU^C4$: $U^C$ is increasing in $\mathbf{x}_B$;

$RU^C5$: $U^C$ is increasing in $c_A$;

$RU^C6$: $U^C$ is HD0 in $(c_A, \mathbf{p}_A)$;

$RU^C7$: $U^C$ is quasi-convex in $\mathbf{p}_A$; and

$RU^C8$: $U^C$ is quasi-concave in $\mathbf{x}_B$.

These conditional (on $\mathbf{x}_B$ and $c_A$) demands become conditional (on $c_A$) mixed demands if the group B prices are replaced by their shadow prices. Duality theory then allows the conditional (on $c_A$) mixed (direct and inverse) demand functions to be derived from the conditional indirect utility function via simple differentiation, according to the Envelope theorem; i.e.,

(9)
$$X^C_{Ai}(c_A, \mathbf{p}_A, \mathbf{x}_B) = \frac{-\partial U^C / \partial p_{Ai}}{\partial U^C / \partial c_A},$$

$$P^C_{Bj}(c_A, \mathbf{p}_A, \mathbf{x}_B) = \frac{\partial U^C / \partial x_{Bj}}{\partial U^C / \partial c_A},$$

where $X^C_{Ai}$ are the conditional direct demand functions, and the $P^C_{Bj}$ are the conditional inverse demand (or shadow price) functions. These functions can be converted into mixed demands by replacing the conditioning on $c_A$ by a conditioning on $c$. This can be achieved by applying the Envelope theorem to derive the conditional total cost function:

(10)
$$C^C(c_A, \mathbf{p}_A, \mathbf{x}_B) = \sum_i p_{Ai} X^C_{Ai} + \sum_j P^C_{Bj} x_{Bj}$$

$$= -\sum_i p_{Ai}\left(\frac{\partial U^C / \partial p_{Ai}}{\partial U^C / \partial c_A}\right) + \sum_j \left(\frac{\partial U^C / \partial x_{Bj}}{\partial U^C / \partial c_A}\right) x_{Bj},$$

which allows solution for the optimal $c_A$ conditional on given $c$, and hence to relate the conditional and Marshallian mixed demand functions via the identities:

(11) $$X^{MM}_{Ai}[C^{C}(c_A, \mathbf{p}_A, \mathbf{x}_B), \mathbf{p}_A, \mathbf{x}_B] = X^{C}_{Ai}(c_A, \mathbf{p}_A, \mathbf{x}_B)$$

$$P^{MM}_{Bj}[C^{C}(c_A, \mathbf{p}_A, \mathbf{x}_B), \mathbf{p}_A, \mathbf{x}_B] = P^{C}_{Ai}(c_A, \mathbf{p}_A, \mathbf{x}_B)$$

(12) $$X^{C}_{Ai}[C^{MM}_{A}(c, \mathbf{p}_A, \mathbf{x}_B), \mathbf{p}_A, \mathbf{x}_B] = X^{MM}_{Ai}(c, \mathbf{p}_A, \mathbf{x}_B)$$

$$P^{C}_{Bj}[C^{MM}_{A}(c, \mathbf{p}_A, \mathbf{x}_B), \mathbf{p}_A, \mathbf{x}_B] = P^{MM}_{Bj}(c, \mathbf{p}_A, \mathbf{x}_B)$$

where $c_A = C^{MM}_{A}(c, \mathbf{p}_A, \mathbf{x}_B)$ is the Marshallian mixed conditional cost function which may be obtained by inverting the identity function $c = C^{C}(c_A, \mathbf{p}_A, \mathbf{x}_B)$ .

Four approaches to the derivation of Marshallian mixed demand functions may be identified. In the primal approach, the mixed demand functions are derived literally by specifying a direct utility function and solving the constrained optimization problem (1). In particular, the first order conditions for the optimality of all of the $\mathbf{x}$ variables need to be manipulated in such a way as to solve for the $\mathbf{x}_A$ and $\mathbf{p}_B$ as dependent variables, as functions of $c$, $\mathbf{p}_A$, and $\mathbf{x}_B$ as independent variables. The second approach is the symmetric opposite: the mixed demand functions are derived by specifying an indirect utility function and solving the constrained optimization problem (4), or at least manipulating the first order conditions to again solve for the $\mathbf{x}_A$ and $\mathbf{p}_B$ as dependent variables, as functions of $c$, $\mathbf{p}_A$, and $\mathbf{x}_B$ as independent variables. Both of these approaches are subject to the usual problems encountered with a primal approach, in that they require analytical solution of a system of nonlinear equations, and are intractable for all but the simplest specifications. A third approach would be based on solving the constrained optimization problem (7); this approach requires the specification of consistent functional forms for both the direct and indirect utility function, as well as the analytical solution of a system of nonlinear equations, and again the difficulties involved in this practice effectively limit the type of functional forms one could choose from to a small group of restrictive forms. This paper is in the spirit of a fourth approach, which exploits the theory of duality among conditional indirect utility functions, conditional total cost functions and Marshallian mixed demands.

The conditional indirect utility function, together with its derivative properties (9) and (10), provides a convenient vehicle for generating regular Marshallian mixed demand functions. Specifically, for a parametric specification of $U^C$ that satisfies Conditions $\mathbf{RU}^C$, one can obtain the conditional mixed demands ($X^C_{Ai}$ and $P^C_{Bj}$) and total cost functions ($C^C$) via the Envelope theorem. If we could invert $C^C$ explicitly to give the implied Marshallian mixed conditional cost function ($C^{MM}_A$), then the conditional demands could be "unconditioned" by replacing the conditional expenditure $c_A$ by $C^{MM}_A$, as indicated by (12). Clearly, it is not always possible to obtain a closed-form solution for $C^{MM}_A$ for an arbitrary specification of $C^C$; it depends heavily on the particular parametric form of $C^C$, which is itself determined by the particular parametric form of $U^C$. In fact, the class of preferences for which there exists explicit closed-form solutions for both the total cost function and conditional cost functions is quite limited. This paper focuses on the class of total cost functions for which such explicit inversion is not available; that is, solving $c = C^C(c_A, \mathbf{p}_A, \mathbf{x}_B)$ for $C^{MM}_A$ may not be accomplished analytically. Thus, for a given parametric form for the conditional indirect utility function with parameters $\xi$, the Marshallian mixed demand functions have to be expressed implicitly by the set of functions:

(13)
$$X^C_{Ai}(c_A, \mathbf{p}_A, \mathbf{x}_B; \xi) = \frac{-\partial U^C(c_A, \mathbf{p}_A, \mathbf{x}_B; \xi)/\partial p_{Ai}}{\partial U^C(c_A, \mathbf{p}_A, \mathbf{x}_B; \xi)/\partial c_A},$$

$$= X^C_{Ai}[C^{MM}_A(c, \mathbf{p}_A, \mathbf{x}_B; \xi), \mathbf{p}_A, \mathbf{x}_B; \xi],$$

$$= X^{MM}_{Ai}(c, \mathbf{p}_A, \mathbf{x}_B; \xi),$$

$$P^C_{Bj}(c_A, \mathbf{p}_A, \mathbf{x}_B; \xi) = \frac{\partial U^C(c_A, \mathbf{p}_A, \mathbf{x}_B; \xi)/\partial x_{Bj}}{\partial U^C(c_A, \mathbf{p}_A, \mathbf{x}_B; \xi)/\partial c_A},$$

$$= P^{C}_{Bj}[C^{MM}_{A}(c,\ \mathbf{p}_A, \mathbf{x}_B; \xi), \mathbf{p}_A, \mathbf{x}_B; \xi],$$

$$= P^{MM}_{Bj}(c, \mathbf{p}_A, \mathbf{x}_B; \xi),$$

$$C^{C}(c_A, \mathbf{p}_A, \mathbf{x}_B; \xi) = \sum_i \left( \frac{-\partial U^C(c_A, \mathbf{p}_A,\ \mathbf{x}_B;\ \xi)/\partial p_{Ai}}{\partial U^C(c_A, \mathbf{p}_A,\ \mathbf{x}_B;\ \xi)/\partial c_A} \right) p_{Ai} +$$

$$\sum_j \left( \frac{\partial U^C(c_A,\ \mathbf{p}_A,\ \mathbf{x}_B;\ \xi)/\partial x_{Bj}}{\partial U^C(c_A,\ \mathbf{p}_A,\ \mathbf{x}_B;\ \xi)/\partial c_A} \right) x_{Bj},$$

where $C^{MM}_{A}(c, \mathbf{p}_A, \mathbf{x}_B; \xi)$ is the numerical solution of the identity function:

(14) $$c = C^{C}(c_A, \mathbf{p}_A, \mathbf{x}_B; \xi).$$

Provided that $C^C$ is strictly increasing in $c_A$, then it is feasible to numerically invert (14) to express $c_A$ as a function of $c$, $\mathbf{p}_A$, and $\mathbf{x}_B$.

In a maximum likelihood search for the parameters of the mixed demands, explicit solution is not necessary; all that is required is that software capable of solving the identity function (14) be imbedded in the maximum likelihood computer routine. At each iterative step of the maximization of the likelihood function, there is a given set of parameter values. For these parameter values, (14) can be numerically inverted to recover the value of conditional expenditure consistent with the given values of $c$, $\mathbf{p}_A$, and $\mathbf{x}_B$. Then, this value of conditional expenditure can be used to eliminate the value of $c_A$ from the conditional mixed demand system.

# 3 EMPIRICAL SPECIFICATION OF DIFFERENT TYPES OF UTILITY FUNCTIONS

In this section, we choose some specific forms for the indirect utility, direct utility and conditional indirect utility functions, and utilize their derivative properties to derive the corresponding systems of direct, inverse and mixed demands. The procedure to model these functions can be briefly summarized as follows: we first generalize three parametric representations of alternative utility functions in terms of total expenditure, conditional expenditure, and price and quantity indexes; we next consider possible specifications for the price and quantity indexes, which satisfy certain regularity conditions. As can be seen, the proposed parametric forms are ideally suited for empirical applications, since they can be easily constrained to be regular over an unbounded region, and since the functional structures are parsimonious in the number of additional parameters.

## 3.1 The Family of Composite Utility Functions

A system of Marshallian direct (or inverse) demand functions $X_i^M$ (or $R_i^{MI}$) would be described as globally regular if the corresponding indirect (or direct) utility function satisfies the regularity conditions $\mathbf{RU}^M$ (or $\mathbf{RU}$) over the region $\Omega^N$(or $\Omega_+^N$). Similarly, the Marshallian mixed demand functions $X_{Ai}^{MM}$ and $P_{Bj}^{MM}$ (obtained by using the conditional indirect utility function and its derivative properties) would be said to be globally regular if the corresponding conditional indirect utility function satisfies the regularity conditions $\mathbf{RU}^C$ over the region $\Omega^N$. Global regularity can be achieved only for quite simple functional forms, such as Cobb-Douglas. Empirically, local regularity is usually all that can be achieved. One form of local regularity that is empirically very attractive applies if the regularity conditions are satisfied over the whole data set, and for an unbounded region in the direction of increasing $c$ (or $\mathbf{x}$ for inverse demands, or $c$ and $x_B$ for mixed demands). Cooper and McLaren (1996) refer to this concept as "effectively globally regular" since, from the point of view of empirical estimation and policy analysis, it is indistinguishable from global regularity.

Classes of functional forms that achieve effectively globally regularity are the classes of "composite" utility functions, which are constructed from the product of pairs of component functions each having slightly stricter regularity conditions. This class is specified as having the general forms:

(15) $U^M(c, \mathbf{p}) = V_1^M(c, \mathbf{p}) \times V_2^M(c, \mathbf{p})$,

(16) $U(\mathbf{x}) = V_1(\mathbf{x}) \times V_2(\mathbf{x})$, and

(17) $U^C(c_A, \mathbf{p}_A, \mathbf{x}_B) = V_1^C(c_A, \mathbf{p}_A, \mathbf{x}_B) \times V_2^C(c_A, \mathbf{p}_A, \mathbf{x}_B)$

where the component functions $V_L^K$ and $V_L$ (K = M, C, and L = 1, 2) have the following regularity properties in the region $\Omega_+^N$:

Conditions $\mathbf{RV}^M$:

$RV^M1$: $V_L^M$ is positive;

$RV^M2$: $V_L^M$ is increasing in $c$;

$RV^M3$: $V_L^M$ is decreasing in $\mathbf{p}$;

$RV^M4$: $V_L^M$ is HD0 in $(c, \mathbf{p})$; and

$RV^M5$: $V_L^M$ is convex in $\mathbf{p}$, with one of the $V_L^M$ expressible as the reciprocal of a concave function in $\mathbf{p}$.

Conditions $\mathbf{RV}$:

$RV_L1$: $V_L$ is positive;

$RV_L2$: $V_L$ is increasing in $\mathbf{x}$;

$RV_L3$: $V_L$ is concave in $\mathbf{x}$;

Conditions $\mathbf{RV}^C$:

$RV^C1$: $V_L^C$ is positive;

$RV^C2$: $V_L^C$ is decreasing in $\mathbf{p}_A$;

$RV^C3$: $V_L^C$ is increasing in $c_A$ and $\mathbf{x}_B$;

$RV^C4$: $V_L^C$ is HD0 in $(c_A, \mathbf{p}_A)$;

$RV^C5$: $V_L^C$ is convex in $\mathbf{p}_A$ with one of the $V_L^C$ expressible as the reciprocal of a concave function in $\mathbf{p}$; and

$RV^C6$: $V_L^C$ is concave in $\mathbf{x}_B$.

Structures (15) through (17) have two common features that are intuitively appealing. First, the derivative properties of $U^M$, U and $U^C$ (according to Roy's identity, the Hotelling-Wold identity and the Envelope theorem) allow the derivation of demand functions or shares as ratios or fractional forms (ratios of partial derivatives of $U^M$, U and $U^C$), except in the special cases when the denominator collapses to a constant. For instance, Roy's identity (or the Hotelling-Wold identity) applied to (15) [or (16)] generates direct (or inverse) demand functions or shares as ratios of combinations of partial derivatives of $V_1^M$ and $V_2^M$ (or $V_1$ and $V_2$). Second, these structures provide convenient ways to construct regular utility functions from two component functions as the following theorems show.

**Theorem 1**: Provided the two component functions $V_L^M$ (L = 1, 2) satisfy these properties $\mathbf{RV}^M$ on a region Γ, the corresponding indirect utility function $U^M$ defined by (15) will be positive and satisfy the regularity conditions **RIU** of an indirect utility function on the region Γ.

**Theorem 2**: Given that the component functions $V_L$ (L = 1, 2) satisfy regularity conditions **RV** on a region Γ, then the direct utility function constructed as (16) is positive and satisfies the regularity conditions **RU** on the region Γ.

**Theorem 3**: Let the component functions $V_L^C$ (L = 1, 2) satisfy regularity conditions $\mathbf{RV}_L^C$ on a region Γ. Then the conditional indirect utility function defined as (17) is positive and satisfies the regularity conditions $\mathbf{RV}^C$ on the region Γ.[9]

The power of the above constructions follows from the following well-known properties: positive linear combinations of positive, decreasing, and convex functions are positive, decreasing, and convex functions; and positive linear combinations of positive, increasing, and concave, functions are positive, increasing, and concave functions. Henceforth, if $V_{1i}^M$ and $V_{2i}^M$ (or $V_{1i}$ and $V_{2j}$), i = 1,…, n and j = 1,…, m satisfy Conditions $\mathbf{RV}^M$ (or **RV**), and if the constants $\theta_i$ and $\pi_j$ satisfy

[9] Proofs of the theorems are available online as Appendix A at http://au.geocities.com/garywong21/ratio_appendix.pdf .

$1 \le \theta_i, \pi_j \le 0$, then the indirect utility function $U^M$ (or direct utility function U) given by

(18) $$U^M(c, \mathbf{p}) = \left( \sum_{i=1}^{n} \theta_i V_{1i}^M \right) \cdot \left( \sum_{j=1}^{m} \pi_j V_{2j}^M \right)$$

(19) $$\left[ \text{or } U(\mathbf{x}) = \left( \sum_{i=1}^{n} \theta_i V_{1i} \right) \cdot \left( \sum_{j=1}^{m} \theta_j V_{2j} \right) \right]$$

satisfies Conditions **RIU** (or **RU**). Likewise, if $V_{1i}^C$ and $V_{2i}^C$ satisfy Conditions $\mathbf{RV}^C$, and if the constants $\theta_i$ and $\pi_j$ satisfy $1 \le \theta_i, \pi_j \le 0$, then

(20) $$U^C(c_A, \mathbf{p}_A, \mathbf{x}_B) = \left( \sum_{i=1}^{n} \theta_i V_{1i}^C \right) \cdot \left( \sum_{j=1}^{m} \pi_j V_{2j}^C \right)$$

satisfies Conditions $\mathbf{RU}^C$.

It is clear that the specifications (18) through (20) define classes of regular indirect, direct and conditional indirect utility functions, and the above results imply that it is possible to construct direct, inverse and mixed demand systems with arbitrary rank in the sense of Lewbel (1991). For example, if n = 2 and m = 1, then the indirect utility function is of the form:

(21) $$U^M(c, \mathbf{p}) = (\theta_1 V_{11}^M + \theta_2 V_{12}^M) \cdot (\pi_1 V_{21}^M).$$

By Roy's identity, the direct demand system derived from (21) has the rank three ratio form.[10]

The task in the next three sub-sections is to find a parameterization for $V_L^M$, $V_L$, and $V_L^C$ (L=1, 2) that have the structures (15) through (17), and all of them must be parsimonious and restricted by certain properties.

## 3.2 The Composite Indirect Utility Function

The use in (15) of functions $V_L^M$ that satisfy Conditions $\mathbf{RV}^M$ is a sufficient condition to generate a

[10] See Lewbel (1992), pp. 951-952.

regular indirect utility function, and hence provides an attractive means of construction of regular indirect utility functions from more basic regular generating functions. Using the intuition stemming from Cooper and McLaren's (1996) General Exponential Form (GEF), we obtain the following construction:

(22) $$U^{M}(c, \mathbf{p}) = V_{1}^{M}(c, \mathbf{p}) \cdot V_{2}^{M}(c, \mathbf{p})$$

where

$$V_{1}^{M}(c, \mathbf{p}) = \theta\,(c / P1) + (1 - \theta)\,\frac{\left[(c/\tau P2)^{\mu} - 1\right]}{\mu},$$

$$V_{2}^{M}(c, \mathbf{p}) = (c / P3)^{\eta},$$

the parameters θ, τ, μ, and η satisfy $0 \leq \theta, \eta \leq 1$, $\tau > 0$, and $\mu \geq -1$, and Pk (k = 1 to 3) are the price indices satisfying the regularity conditions **RP**:

RP1: Pk is positive;
RP2: Pk is continuous;
RP3: Pk is HD1 in **p**;
RP4: Pk is non-decreasing in **p**; and
RP5: Pk is concave in **p**.

It is shown that when these conditions are satisfied, $U^{M}$ satisfies Conditions $RU^{M}$ over the region $\{(c, \mathbf{p}): c > \tau P2\}$ and hence the corresponding Marshallian direct demand functions are regular over this region.[11] Note moreover that [in the spirit of Lewbel's (1991) definition] the rank of a direct demand system is determined by the minimum number of price indexes in the indirect utility function. Henceforth, when $0 < \theta < 1$ the direct demand functions obtained from (22) is generalized to a rank 3 form, which allows commodities to effectively change classification from luxuries to necessities at different levels of income.

[11] See Appendix A of this paper at http://au.geocities.com/garywong21/ratio_appendix.pdf .

Two nested special cases (consistent with rank 2 preferences) are of particular interest:

Case 1: $\theta = 0$, $\mu = -1$, and $\eta = 1$. In this case, $U^M$ is of the form

$$\frac{c - \tau P2}{P3},$$

which is the Gorman Polar Form (GPF), a generalization of the Linear Expenditure System.

Case 2: $\theta = 0$. In this case $U^M$ is of the General Exponential Form (GEF) introduced by Cooper and McLaren (1996):

$$U^M = \frac{\left[(c/\tau P2)^{\mu} - 1\right]}{\mu}\left(\frac{c}{P3}\right)^{\eta}.$$

Roy's identity applied to (22) gives the regular ratio direct demand system:

$$\text{(23)} \qquad W_i^M(c, \mathbf{p}) = \frac{p_i X_i^M(c, \mathbf{p})}{c} = \frac{-\partial U^M(c,\mathbf{p})/\partial \log(p_i)}{\partial U^M(c,\mathbf{p})/\partial \log(c)}$$

$$= \frac{\theta Z_1(E_{1i} + \eta E_{3i}) + (1-\theta)(1+\mu Z_2)E_{2i} + \eta(1-\theta)Z_2 E_{3i}}{\theta Z_1(1+\eta) + (1-\theta)[1 + (\mu+\eta)Z_2]}$$

where $Z_1 = c / P1$, $Z_2 = \dfrac{\left[(c/\tau P2)^{\mu} - 1\right]}{\mu}$, and $E_{ki} = \partial\log(Pk) / \partial\log(p_i)$. Given Cobb-Douglas and CES specifications for the price functions:

$$P1 = \prod_j p_j^{\gamma_j} \text{ with } \Sigma_j\gamma_j = 1,\ 0 \le \gamma_i \le 1,$$

$$P2 = \left(\sum_j \alpha_j p_j^{\rho}\right)^{1/\rho} \text{ with } \Sigma_j\alpha_j = 1,\ 0 \le \alpha_i \le 1,\ \rho \le 1, \text{ and}$$

$$P3 = \left(\sum_j \beta_j p_j^{\delta}\right)^{1/\delta} \text{ with } \Sigma_j\beta_j = 1,\ 0 \le \beta_i \le 1,\ \delta \le 1,$$

the elasticity terms in (23) take the form:

$$E_{1i} = \gamma_i,\ E_{2i} = \frac{\alpha_i p_i^{\rho}}{\sum_j \alpha_j p_j^{\rho}} \text{ and } E_{3i} = \frac{\beta_i p_i^{\delta}}{\sum_j \beta_j p_j^{\delta}}.$$

### 3.3 The Composite Direct Utility Function

The composite (product) direct utility function is based on a modification by Holt (2002) of the Inverse Almost Ideal Demand System of Eales and Unnevehr (1994), and results in one of the more regular and flexible inverse demand systems. The basic specification of the direct utility function is:

(24) $$U(\mathbf{x}) = V_1(\mathbf{x}) \times V_2(\mathbf{x}) = \left[\theta X1 + (1-\theta)\left(\frac{X2^{\mu} - 1}{\mu}\right)\right] \cdot X3^{\eta}$$

where Xk (k=1 to 3) are three quantity functions satisfying the following regularity conditions (**RX**):

RX1: Xk is non-negative
RX2: Xk is continuous
RX3: Xk is HD1 in **x**
RX4: Xk is non-decreasing in **x**
RX5: Xk is concave in **x**.

For the empirical application, we assume that the quantity functions take the form, respectively:

$$X1 = \prod_j x_j^{\gamma_j} \text{ with } \Sigma_j\gamma_j = 1,$$

$$X2 = \left(\sum_j \alpha_j x_j^{\rho}\right)^{1/\rho} \text{ with } \Sigma_j\alpha_j = 1, \text{ and}$$

$$X3 = \left(\sum_j \beta_j x_j^{\delta}\right)^{1/\delta} \text{ with } \Sigma_j\beta_j = 1.$$

Application of the Hotelling-Wold identity to the utility function defined in (24) results in:

(25) $$W_i^{MI} = R_i^{MI} x_i = \frac{\theta X_3 E_{1i} + (1\text{-}\theta)(1+\mu Z)E_{2i} + \eta V_1 E_{3i}}{\theta X_3 + (1\text{-}\theta)(1+\mu Z) + \eta V_1},$$

where $E_{ki}$ (k = 1 to 3) = $\partial\log(X_k) / \partial\log(x_i)$, $Z = \left(\frac{X_2^{\mu} - 1}{\mu}\right)$, and $V_1 = [\theta X_1 + (1 - \theta)Z]$. Notably, this system is parametrically similar to (23) so that most of the desirable theoretical properties attributed to (23) carry over to (25). Particularly, in the sense of Lewbel (1991), this system is consistent with rank 3 preferences, which allows far more flexible modeling of Engle responses. In addition, when θ = 0 (or 1), U(**x**) is of the form

$$U(\mathbf{x}) = X1 \cdot X_3^{\eta} \left( \text{or } \frac{X2^{\mu} - 1}{\mu} \cdot X3^{\eta} \right)$$

which is consistent with rank 2 preferences. The sufficient conditions to ensure (24) to be a regular direct utility function over the region {(**x**): X2 ≥ 1} are:

(26) $$0 \le \theta, \eta, \gamma_i, \alpha_i, \beta_i \le 1, \mu \ge -1, \rho \le 1, \text{ and } \delta \le 1.$$

### 3.4 The Composite Conditional Indirect Utility Function

Following (22) and Cooper and McLaren (1996,2006), a rank three specification of the conditional indirect utility function is obtained by specifying:

(27) $$V_1^C = [\kappa / F_1 + (1\text{-}\kappa)(F_2^{-\mu} - 1) / \mu] \text{ and } V_2^C = F_3^{-\eta}$$

where κ, μ and η are parameters, and $F_k$ (k = 1 to 3) are functions of $\mathbf{p}_A$, $\mathbf{x}_B$ and $c_A$ satisfying Conditions **RF**:

RF1: $F_k$ are positive;

RF2: $F_k$ are continuous;

RF3: $F_k$ are increasing in $\mathbf{p}_A$;

RF4: $F_k$ are decreasing in $\mathbf{x}_B$ and $c_A$;

RF5: $F_k$ are homogeneous of degree zero (HD0) in $\mathbf{p}_A$ and $c_A$;

RF6: $F_k$ are concave in $\mathbf{p}_A$; and

RF7: $F_k$ are convex in $\mathbf{x}_B$.

Suppose that $F_k$ have the following forms:

$$F_1 = P1_A / (c_A \cdot X1_B),$$

$$F_2 = [\theta(P2_A / c_A)^{\delta} + (1 - \theta)/X2_B^{\delta}]^{1/\delta}, \text{ and}$$

$$F_3 = [\nu(P2_A / c_A)^{\rho} + (1 - \nu)/X3_B^{\rho}]^{1/\rho},$$

where θ, δ, ν, and ρ are parameters, and $Pk_A$ and $Xk_B$ (k = 1 to 3) are functions of $\mathbf{p}_A$ and $\mathbf{x}_B$ satisfying Conditions **RP** and **RX** respectively. The particular form of the composite conditional indirect utility function results when $Pk_A$ and $Xk_B$ are specified as:

$$P1_A = \prod_i p_{Ai}^{\gamma_{Ai}}, \ \Sigma_i\gamma_{Ai} = 1; \qquad X1_B = \prod_j x_{Bj}^{\gamma_{Bj}}, \ \Sigma_j\gamma_{Bj} = 1;$$

$$P2_A = \left(\sum_i \alpha_{Ai} p_{Ai}^{\delta_A}\right)^{1/\delta_A}, \ \Sigma_i\alpha_{Ai} = 1; \qquad X2_B = \left(\sum_j \alpha_{Bj} x_{Bj}^{\delta_B}\right)^{1/\delta_B}, \ \Sigma_j\alpha_{Bj} = 1;$$

$$P3_A = \left(\sum_i \beta_{Ai} p_{Ai}^{\rho_A}\right)^{1/\rho_A}, \ \Sigma_i\beta_{Ai} = 1; \qquad \text{and } X3_B = \left(\sum_j \beta_{Bj} x_{Bj}^{\rho_B}\right)^{1/\rho_B}, \ \Sigma_j\beta_{Bj} = 1.$$

With these specifications, the sufficient conditions for global regularity of the conditional indirect utility function over the region $F_2 < 1$ are:

(28) $$0 \le \kappa, \eta, \nu, \theta, \gamma_{Ai}, \gamma_{Bj}, \alpha_{Ai}, \alpha_{Bj}, \beta_{Ai}, \beta_{Bj} \le 1,$$

$$\mu \ge -1, \rho \le 1, \delta \le 1, \delta_A \le 1, \delta_B \le 1, \rho_A \le 1, \text{ and } \rho_B \le 1.$$[12]

Additionally, when κ = 0 or 1, (27) reduces to a rank 2 specification of a conditional indirect utility function. Therefore, selection between rank 2 or rank 3 models can be based on the statistical testing of κ.

Applying the Envelope Theorem to (27), and after some manipulation, we obtain the regular composite (ratio) mixed demand system:

(29) $$W_{Ai}^{C}(c_A, \mathbf{p}_A, \mathbf{x}_B) = \frac{p_{Ai}X_{Ai}^{C}}{c}$$

[12] See Appendix B of this paper at http://au.geocities.com/garywong21/ratio_appendix.pdf .

$$= \frac{-p_{\mathrm{Ai}} \cdot (\mathrm{U}^{\mathrm{C}}_{\mathrm{Ai}}/\mathrm{U}^{\mathrm{C}}_{\mathrm{C_A}})}{-\sum_{\mathrm{i}} p_{\mathrm{Ai}} \times (\mathrm{U}^{\mathrm{C}}_{\mathrm{Ai}}/\mathrm{U}^{\mathrm{C}}_{\mathrm{C_A}}) + \sum_{\mathrm{j}} (\mathrm{U}^{\mathrm{C}}_{\mathrm{Bj}}/\mathrm{U}^{\mathrm{C}}_{\mathrm{C_A}}) \times p_{\mathrm{Bj}}}$$

$$= \frac{\frac{\kappa}{\mathrm{F_1}} \mathrm{E_{A1i}} + (1-\kappa)(1+\mu \mathrm{R}) \cdot \mathrm{Z_2} \cdot \mathrm{E_{A2i}} + \eta \cdot \mathrm{V_1^C} \cdot \mathrm{Z_3} \cdot \mathrm{E_{A3i}}}{\frac{\kappa}{\mathrm{F_1}} (1 + \sum_{\mathrm{j}} \mathrm{E_{B1j}}) + (1-\kappa)(1+\mu \mathrm{R}) + \eta \mathrm{V_1^C}}$$

$$= \frac{\frac{\kappa}{\mathrm{F_1}} \mathrm{E_{A1i}} + (1-\kappa)(1+\mu \mathrm{R}) \cdot \mathrm{Z_2} \cdot \mathrm{E_{A2i}} + \eta \cdot \mathrm{V_1^{CI}} \cdot \mathrm{Z_3} \cdot \mathrm{E_{A3i}}}{\frac{\kappa}{\mathrm{F_1}} (1 + \sum_{\mathrm{j}} \mathrm{E_{B1j}}) + (1-\kappa)(1+\mu \mathrm{R}) + \eta \mathrm{V_1^{CI}}}, \text{ and}$$

$$\mathrm{W^C_{Bj}}(c_{\mathrm{A}}, \mathbf{p}_{\mathrm{A}}, \mathbf{x}_{\mathrm{B}}) = \frac{\mathrm{P^C_{Bj}} x_{\mathrm{Bj}}}{c}$$

$$= \frac{(\mathrm{U}^{\mathrm{C}}_{\mathrm{Bj}}/\mathrm{U}^{\mathrm{C}}_{\mathrm{c_A}}) \cdot x_{\mathrm{Bj}}}{-\sum_{\mathrm{i}} p_{\mathrm{Ai}} \cdot (\mathrm{U}^{\mathrm{C}}_{\mathrm{Ai}}/\mathrm{U}^{\mathrm{C}}_{\mathrm{c_A}}) + \sum_{\mathrm{j}} (\mathrm{U}^{\mathrm{C}}_{\mathrm{Bj}}/\mathrm{U}^{\mathrm{C}}_{\mathrm{c_A}}) \cdot p_{\mathrm{Bj}}}$$

$$= \frac{\frac{\kappa}{\mathrm{F_1}} \mathrm{E_{B1j}} + (1-\kappa)(1+\mu \mathrm{R})(1-\mathrm{Z_2})\mathrm{E_{B2j}} + \eta \cdot \mathrm{V_1^C}(1-\mathrm{Z_3})\mathrm{E_{B3j}}}{\frac{\kappa}{\mathrm{F_1}} (1 + \sum_{\mathrm{j}} \mathrm{E_{B1j}}) + (1-\kappa)(1+\mu \mathrm{R}) + \eta \mathrm{V_1^C}},$$

where $\mathrm{U^C_{A_i}} = \frac{\partial \mathrm{U^C}}{\partial p_{\mathrm{Ai}}}$, $\mathrm{U^C_{B_j}} = \frac{\partial \mathrm{U^C}}{\partial x_{\mathrm{Bj}}}$, $\mathrm{U^C_{c_A}} = \frac{\partial \mathrm{U^C}}{\partial c_{\mathrm{A}}}$, $\mathrm{E_{Aki}}$ (k = 1 to 3) = $\frac{\partial \log(\mathrm{Pk_A})}{\partial \log(p_{\mathrm{Ai}})}$, $\mathrm{E_{Bkj}}$ (k = 1 to 3) $= \frac{\partial \log(\mathrm{Xk_B})}{\partial \log(x_{\mathrm{Bj}})}$, $\mathrm{V_1^C}$ is defined in (27), R = $(\mathrm{F}^{-\mu}-1)/\mu$, $\mathrm{Z_2} = \frac{\theta(\mathrm{P2_A}/c_{\mathrm{A}})^{\delta}}{\theta(\mathrm{P2_A}/c_{\mathrm{A}})^{\delta} + (1-\theta)/\mathrm{X_{2B}^{\delta}}}$ and $\mathrm{Z_3} = \frac{\nu(\mathrm{P3_A}/c_{\mathrm{A}})^{\rho}}{\nu(\mathrm{P3_A}/c_{\mathrm{A}})^{\rho} + (1-\nu)/\mathrm{X_{3B}^{\rho}}}$. Employing (10) compatibly with $\mathrm{C^C}$ specified as:

(30) $$\mathrm{C^C} = c_{\mathrm{A}} \left[ \frac{\frac{\kappa}{\mathrm{H}}(1 + \sum_{\mathrm{j}} \mathrm{E_{B1j}}) + (1-\kappa)(1+\mu \mathrm{R}) + \eta \mathrm{V_1^C}}{\frac{\kappa}{\mathrm{H}} + (1-\kappa)(1+\mu \mathrm{R})\mathrm{Z_2} + \eta \mathrm{V_1^C} \mathrm{Z_3}} \right],$$

it is impossible to solve (30) explicitly for the value of $c_{\mathrm{A}}$ in terms of parameters, $\mathbf{p}_{\mathrm{A}}$, $\mathbf{x}_{\mathrm{B}}$ and $c$. In order to convert (29) to a Marshallian mixed demand system, the $c_{\mathrm{A}}$ in (29) has to be replaced by the numerical inversion of (30) at $C^C = c$.

# 4 EMPIRICAL IMPLEMENTATION AND RESULTS

## 4.1 Brief Remarks on the Database

For illustrative purposes, budget share systems (23), (25) and (29) were estimated using time series data for Japanese fish and meat consumption and prices. The data consist of 38 types of fish and meat products, and they were aggregated into six categories comprised of:

1. $x_1$ = Salted and dry fish;
2. $x_2$ = Bonito fillets and fish flakes;
3. $x_3$ = Processed meat (including ham, sausages, bacon and other meat products);
4. $x_4$ = Fresh fish;
5. $x_5$ = Fresh meat; and
6. $x_6$ = Shellfish.

In order to fulfill the basic assumptions underlying the applicability of a mixed demand system, we divided the six commodities into two groups according to the following classification:

| | |
|---|---|
| Group A: $x_{A1}$ : Salted and dry fish | Group B: $x_{B4}$ : Fresh fish |
| $x_{A2}$ : Bonito fillets and fish flakes | $x_{B5}$ : Fresh meat |
| $x_{A3}$: Processed meat | $x_{B6}$ : Shellfish. |

Apparently, Group A categories (salted fish, fillets and processed meat) are easily stored so that it is acceptable to treat their prices ($\mathbf{p}_A$) as given in the consumer problem. On the other hand, due to the highly perishable nature and biological production lags, supply of fresh fish, fresh meat and shellfish (Group B Categories) is often inelastic in the short run, which implies that for these categories, equilibrium should be characterized by exogenously determined quantities ($\mathbf{x}_B$) with prices ($\mathbf{p}_B$) adjusting to clear the market. Therefore, it is natural to view these goods as quantity dependent or treat their quantities as given in the consumer problem.

The raw data, gathered from *Annual Report on the Family Income and Expenditure Survey*, consists of monthly data averaged over 8000 households throughout the country. These households keep journals of prices paid (per 100 grams) and expenditures on a large number of fish and meat products and other food commodities. The sample period covers January 1985 through December 2003 for a total of 228 monthly observations. The data were further aggregated to quarterly frequency resulting in 76 usable observations, and were deseasonalized and mean centered prior to estimation.

## 4.2 Estimation and Stochastic Specification

The computation of the maximum-likelihood estimates reported below is feasible because the GAUSS language used to program the estimators handles the implicit representation of functional relationships well. All budget share systems are estimated by using the GAUSS 3.6.27 computer package with the modules NLSYS and CML. The inequality constraints such as (26) and (28) were imposed when estimating the systems.[13] For purposes of estimation, an error term $e_{it}$ is appended additively in all systems. One equation in (23), (25) and (29), which is the budget share equation for fillets, is deleted to ensure non-singularity of the error covariance matrix. As usual, the estimation should be independent of which equation is excluded.

Results of initial estimation revealed that the computed Durbin-Watson statistics (or Box-Pierce $\chi^2_8$ statistics) were low (or high) suggesting significant positive serial correlation. We therefore introduce the forth-order autoregressive scheme based on an order N parameterization of the autocovariance matrix using the full information maximum likelihood algorithm of Moschini and Moro (1994).

[13] Empirical results of the unconstrained general demand models revealed that the required concavity and convexity conditions are violated for some observations. We therefore impose curvature requirements by incorporating those prior restrictions into the likelihood functions. As long as $c \geq \tau P1$, $X2 \geq 1$ and $F_2 \leq 1$, these turn out to be sufficient for the resulting estimates to satisfy all regularity conditions for all observations.

**Table 1: Single Equation and System Measures of Fit**

| Specific Models | Direct | Inverse | Mixed |
|---|---|---|---|
| **No. of Free Parameters** | 22 | 20 | 16 |
| **$R^2$** | | | |
| Salted Fish | 0.952 | 0.963 | 0.952 |
| Processed Meat | 0.949 | 0.935 | 0.945 |
| Fillet | 0.379 | 0.405 | 0.315 |
| Fresh Meat | 0.676 | 0.708 | 0.727 |
| Fresh Fish | 0.873 | 0.871 | 0.880 |
| Shellfish | 0.963 | 0.981 | 0.981 |
| **L** | 1629.31 | 1647.87 | 1642.94 |
| **AII** | 0.021% | 0.019% | 0.020% |
| **SC** | -40.825 | -40.906 | -40.502 |
| **AIC** | -40.129 | -41.602 | -41.198 |
| **HQC** | -40.992 | -41.769 | -41.365 |
| **Residual Diagnostics** | | | |
| **Durbin-Watson Statistics** | | | |
| Salted Fish | 2.257 | 2.395 | 2.485 |
| Processed Meat | 2.586 | 2.813 | 2.596 |
| Fillet | 1.936 | 1.600 | 2.009 |
| Fresh Meat | 2.743 | 2.723 | 2.478 |
| Fresh Fish | 2.627 | 2.869 | 2.806 |
| Shellfish | 2.782 | 2.224 | 2.330 |
| **Box-Pierce $\chi^2$ Statistics** | $\chi^2_{1\%, 8}$ = 20.090 | | |
| Salted Fish | 32.902 | 46.997 | 24.424 |
| Processed Meat | 11.763 | 42.324 | 25.577 |
| Fillet | 7.917 | 6.445 | 6.560 |
| Fresh Meat | 33.367 | 13.263 | 15.617 |
| Fresh Fish | 36.473 | 25.361 | 26.291 |
| Shellfish | 17.948 | 10.770 | 20.719 |

## 4.3 Empirical Results and Their Interpretation

### *Analysis of Measures of Fit*

All demand models were estimated with adding up and homogeneity restrictions imposed. Several single-equation measures of goodness-of-fit and model performance for the general (rank 3) demand models [(23), (25) and (29)] are presented in Table 1.[14] Regarding the single equation fit and performance, results indicate all three systems fit the data reasonably well, even though estimation is in share form and the data employed are quarterly: the share equation $R^2$ values range from 31.5% for fillet (implied by the mixed system) to 98.1% for shellfish (implied by the inverse and mixed systems). Not unexpectedly, the $R^2$ value for the share equation of fillet (for all systems) is the lowest relative to the other share equations. Probably, this exhibits signs of dynamic misspecification. More likely, this may be caused by the failure to allow for imperfect adjustment to quantity changes as the share of fillet has a reasonable high amount of variation.

The serial correlation properties of the error terms as shown in the Durbin-Watson and Box-Pierce $\chi^2_8$ statistics are no longer severely pathological, although there is still evidence of positive and negative serial correlation. Probably, this is the consequence of splicing techniques in the data series. To obtain an improvement here, it would be preferable to revise the data rather than making technical model corrections.

To facilitate meaningful cross-model comparisons, several system-wide measures of goodness-of-fit including the optimized log-likelihood values (L), Theil's Average Information Inaccuracy (AII), Schwartz Criterion (SC), Akaike's Information Criterion (AIC), and Hannan-Quinn Criterion (HQC) are also presented in Table 1. Based upon the values of L, AII, SC, AIC and HQC, we see the inverse system dominates the other systems with the direct system displaying the weakest

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[14] For reasons of brevity, the detailed parameter estimates of the models are not reported below but are available upon request.

performance overall; however the discrepancy between the AII for the inverse and mixed systems is not large. Of interest is that the direct system, while containing six more free parameters than the mixed system, has a lower L value but higher AII, AIC, and HQC values. On prima facie grounds, it might be concluded that the inverse system is preferred to the mixed and direct systems, whereas the mixed system is preferred to the direct system.

### *Nested Tests*

There are a variety of models nested within the general (rank 3) specifications [(23), (25) and (29)], which are of interest and worth discussing. Table 2 provides a summary of the specific model results in which Models 1, 5 and 8 represent the general specifications of the direct, inverse and mixed systems respectively. The following comments are in order. First, Table 2 confirms that Models 2 (GEF), 3 (GDF) and 4 are rejected in favor of their generalization (Model 1). Second, subsequently freeing up $\mu$ and $\eta$ (Model 2) does not lead to a significant improvement once $\theta = 0$; i.e., Model 3 compares favorably with Model 2. Third, with respect to the inverse systems (Models 5 to 7), the freeing up of $\theta$ is of little statistical value, which implies that Models 6 and 7 are not statistically inferior to the model (Model 5) in which they are nested. Fourth, Models 6 and 7 are not nested but strong ground for preferring Model 7 lies in its higher log-likelihood value. Finally, in the mixed systems, while Model 10 is clearly dominated by Model 8, the hypothesis $\kappa = 0$ maintained by Model 9 is not rejected relative to the general specification. Overall, Models 1, 7 and 9 are the preferred direct, inverse and mixed demand models respectively.

**Table 2: Summary of Specific Model Results**

| Specific Model | Functional Form Parameter Restrictions $\theta$ | $\kappa$ | $\mu$ | $\eta$ | Likelihood Value | No. of Free Parameters |
|---|---|---|---|---|---|---|
| The Regular Ratio Direct Demand System | | | | | | |
| 1: The General Model – Rank 3 | Free | — | Free | Free | 1629.31 | 22 |
| 2: The GEF – Rank 2 | 0 | — | Free | Free | 1622.40 | 21 |
| 3: The GPF – Rank 2 | 0 | — | -1 | 1 | 1621.36 | 20 |
| 4: The Nested Model – Rank 2 | 1 | — | Free | Free | 1608.54 | 21 |
| The Regular Ratio Inverse Demand System | | | | | | |
| 5: The General Model – Rank 3 | Free | — | Free | Free | 1647.87 | 20 |
| 6: The Nested Model – Rank 2 | 0 | — | Free | Free | 1644.87 | 19 |
| 7: The Nested Model – Rank 2 | 1 | — | Free | Free | 1647.80 | 19 |
| The Regular Ratio Mixed Demand System | | | | | | |
| 8: The General Model – Rank 3 | — | Free | Free | Free | 1643.73 | 16 |
| 9: The Nested Model – Rank 2 | — | 0 | Free | Free | 1642.94 | 15 |
| 10: The Nested Model – Rank 2 | — | 1 | Free | Free | 1598.64 | 15 |

Nested test of Model 1 against Model 2 (rejected): $\chi^2_1$ (test statistic) = 13.82, $\chi^2_{1,\ 1\%}$ = 6.63 (critical value).

Nested test of Model 1 against Model 3 (rejected): $\chi^2_3$ = 15.90, $\chi^2_{3,\ 1\%}$ = 11.34.

Nested test of Model 1 against Model 4 (rejected): $\chi^2_1$ = 41.54, $\chi^2_{1,\ 1\%}$ = 6.63.

Nested test of Model 3 (rejected) against Model 2: $\chi^2_2$ = 15.90, $\chi^2_{2,\ 1\%}$ = 9.21.

Nested test of Model 5 against Model 6 (rejected): $\chi^2_1$ = 6.38, $\chi^2_{1,\ 1\%}$ = 6.63.

Nested test of Model 5 against Model 7 (rejected): $\chi^2_1$ = 0.14, $\chi^2_{1,\ 1\%}$ = 6.63.

Nested test of Model 8 against Model 9 (rejected): $\chi^2_1$ = 1.58, $\chi^2_{1,\ 1\%}$ = 6.63.

Nested test of Model 8 against Model 10 (rejected): $\chi^2_1$ = 90.18, $\chi^2_{1,\ 1\%}$ = 6.63

**Table 3: Non-Nested Tests (Davidson & MacKinnon's P Test) of the General Demand Models**

| Comparison (Null versus Alternative) | | | Test Statistic |
|---|---|---|---|
| Direct (rejected) | versus | Inverse | 2.958 |
| Inverse (rejected) | versus | Direct | 2.990 |
| Direct (rejected) | versus | Mixed | 3.909 |
| Mixed | versus | Direct (rejected) | -0.824 |
| Inverse (rejected) | versus | Mixed | 4.763 |
| Mixed | versus | Inverse (rejected) | -1.535 |

### ***Non-Nested Tests***

Consider next the formal comparisons of the preferred direct (Model 1), inverse (Model 7) and mixed (Model 9) demand systems. While these systems are not nested, they have identical dependent variables, allowing us to test them against one another using a modification of Davidson and MacKinnon's (1983) p-test.[15 & 16] Results of this test are summarized in Table 3. When testing the direct system against the inverse system, we find that both specifications are decisively rejected when each is in turn viewed as the null model. As can be seen, the computed t statistics far exceed the critical value for the 1% significance level. On the other hand, the direct and inverse systems are rejected by the mixed system, whereas the mixed system is not rejected when it is the null model. It seems that there is a decisive outcome: the mixed demand system (Model 9) is preferred to the direct (Model 1) and inverse (Model 7) systems. Since the p-test indicates that Model 9 is the preferred specification, its parameter estimates are used to compute the welfare changes associated with quantity reductions.

[15] See Eales, Durham and Wessels (1994) p. 1160 for the procedures needed to perform the p-test.

[16] According to Davidson and MacKinnon (1983), p-test requires modification to account for endogeneity of the alternative model's right hand side (RHS) variables. To overcome this problem, we specify instrument sets for the direct, inverse and mixed demand systems. The instruments are fourth order lag of all potentially endogenous RHS variables ($\mathbf{p}$ and $c$ in the direct demands, $\mathbf{x}$ in the inverse demands, and $\mathbf{p}_{\mathbf{A}}$, $\mathbf{x}_{B}$ and $c_{A}$ in the mixed demands), exchange rates in yen per U.S. dollar, yields to subscribers of ten-year interest bearing government bonds, total public debt, total employed people and CPI.

### *Analysis of Estimated Welfare Change*

One motivation for specifying regular mixed demand systems is to obtain accurate and consistent estimates of welfare changes associated with quantity changes. To illustrate, we conclude our application by computing the welfare loss associated with a 10 per cent reduction in supply of fresh meat, fresh fish and shellfish. Given a parametric form of the conditional indirect utility function $U^C$ with parameters ξ, an exact measure of compensating variation (CV) associated with a change in $\mathbf{x}_B$ from $\mathbf{x}_B^0$ to $\mathbf{x}_B^1$ is given by:

$$\text{(31)} \qquad CV = c_A^0 - c_A^1 = C_A^H(u^0, \mathbf{p}_A, \mathbf{x}_B^0; \xi) - C_A^H(u^0, \mathbf{p}_A, \mathbf{x}_B^1; \xi),$$

where $c_A^0$ is defined implicitly from $c = C^C(c_A^0, \mathbf{p}_A, \mathbf{x}_B^0; \xi)$, $u^0 = U^C(c_A^0, \mathbf{p}_A, \mathbf{x}_B^0; \xi)$ is the base utility, and $c_A^1$ is obtained by inverting $u^0 = U^C(c_A^1, \mathbf{p}_A, \mathbf{x}_B^1; \xi)$. Intuitively speaking, CV is defined as the amount of additional expenditure required for consumers to reach the utility level $u^0$ while facing the quantity $\mathbf{x}_B^1$. A positive (negative) value for CV indicates that consumers are worse (better) off while facing quantities $\mathbf{x}_B^1$.

In a similar manner, the equivalent variation (EV) for a change in quantity from $\mathbf{x}_B^0$ to $\mathbf{x}_B^1$ is defined as:

$$\text{(32)} \qquad EV = \tilde{c}_A^0 - \tilde{c}_A^1 = C_A^H(u^1, \mathbf{p}_A, \mathbf{x}_B^0) - C_A^H(u^1, \mathbf{p}_A, \mathbf{x}_B^1),$$

where $\tilde{c}_A^1$ is the solution of the identity function $c = C^C(\tilde{c}_A^1, \mathbf{p}_A, \mathbf{x}_B^1; \xi)$, $u^1 = U^C(\tilde{c}_A^1, \mathbf{p}_A, \mathbf{x}_B^1; \xi)$, and $\tilde{c}_A^0$ is obtained by inverting $u^1 = U^C(\tilde{c}_A^0, \mathbf{p}_A, \mathbf{x}_B^0; \xi)$. Here EV is the amount of additional expenditure that would enable the consumer to maintain the new utility level $u^1$ while facing the initial quantities $\mathbf{x}_B^0$. Similar to CV, a positive (negative) value for EV suggests that consumers are worse (better) off under $\mathbf{x}_B^1$ than under $\mathbf{x}_B^0$.

**Table 4: Compensating and Equivalent Variations for a 10% Reduction in Supply of Fresh Meat, Fresh Fish and Shellfish (Yens for Annual)**

| Fish Category | CV (Yens) | %CV | EV (Yens) | %EV |
|---|---|---|---|---|
| | | **1985** | | |
| Fresh Meat | 7951.51 | 3.97% | 6663.55 | 3.34% |
| Fresh Fish | 8672.37 | 4.34% | 7248.86 | 3.64% |
| Shellfish | 1226.35 | 0.61% | 1184.035 | 0.59% |
| | | **1994** | | |
| Fresh Meat | 7337.00 | 3.85% | 6198.85 | 3.26% |
| Fresh Fish | 8296.30 | 4.37% | 6956.90 | 3.67% |
| Shellfish | 1227.85 | 0.65% | 1185.652 | 0.62% |
| | | **2003** | | |
| Fresh Meat | 5435.63 | 3.57% | 4686.60 | 3.08% |
| Fresh Fish | 6829.67 | 4.48% | 5753.75 | 3.78% |
| Shellfish | 1035.60 | 0.68% | 1004.94 | 0.66% |
| | | **Average** | | |
| Fresh Meat | 7913.58 | 3.83% | 6065.53 | 3.24% |
| Fresh Fish | 8209.37 | 4.40% | 6860.82 | 3.68% |
| Shellfish | 1205.99 | 0.65% | 1162.00 | 0.62% |

Note: The Column titled %CV denotes compensating variation as a percent of total expenditure on meat and fish, while the column headed %EV is similarly defined for equivalent variation.

Evaluating CV and EV at the estimated parameters of Model 9, we obtain the results in Table 4. A number of points are worth making. As expected, the estimated CV and EV are positively small in all instances, indicating that consumers are made slightly worse off after the reduction in harvest of fish, meat and shellfish. For example, the CV for a 10% catch restriction on fresh meat is only 5435.63 yens loss per capita in 2003.[17] Furthermore, within the sample period, the largest (smallest) welfare loss in absolute terms associated with catch restriction is for fresh fish (shellfish). More importantly, the numerical differences between the CV and EV estimates are not large, amounting to no more than 1500 yens in all instances. Lastly, small variations over time in CV and EV estimates as a percentage of total expenditure (CV% and EV%) are observed for each category. In particular, CV% estimates associated with a 10% reduction in fresh fish (or shellfish) catch increases from

[17] Similar interpretations apply for fresh fish and shellfish.

3.64% (or 0.61) in 1985 to 3.78% (or 0.68%) in 2003.

## 5 CONCLUSION

The objective of this paper was twofold. First, we utilized the notion of “effective global regularity” to develop a family of “regular composite” (product or ratio) direct, inverse and mixed demand systems. These new specifications are empirically appealing since it is relatively easy to impose effective global regularity conditions in estimation, and since the general models nest a number of popular demand systems such as the Gorman-Polar Form (GPF) and the General Exponential Form (GEF) as special cases. More importantly, they are of potentially arbitrary rank, which allows more flexible Engel behavior than is possible in GPF and GEF. We illustrated the techniques by estimating systems of direct, inverse and mixed demands for Japanese quarterly meat and fish consumption. The main findings indicate that these systems fit the data well and satisfy the required regularity conditions for all observations in the sample period.

The second objective was to advocate a more general use of the conditional indirect utility function in the specification and estimation of mixed demand systems. Notably, this paper only focused on the type of conditional indirect utility functions for which it is not necessary to have closed functional forms for the Marshallian mixed demands, nor for the Marshallian mixed conditional cost function. The technical aspects on how to estimate the Marshallian mixed demands have been discussed in considerable detail. In particular, a method based on a numerical inversion estimation method first pioneered by McLaren, Powell & Rossiter (2000) was adopted to deal with the endogeneity of the conditional expenditure. The overall results reported in subsection 4.3 indicate that this method is operationally feasible. Therefore, a further avenue has been opened up for deriving estimable systems of mixed demands, which are more flexible and regular than those currently employed in applied demand analysis.

Results of the nested and non-nested tests of different models are of special interest to demand system analysts. In particular, the nested tests indicate that the preferred direct systems are of rank 3 while the preferred inverse and mixed systems are of rank 2. As for the non-nested test, the preferred mixed demand system strongly rejects the preferred direct and inverse demand systems. It might be concluded that the exogenous treatment of processed fish and meat prices, and quantities of fresh fish and meat, is appropriate for the purpose of econometric estimation. The results obtained also show considerable variations in the magnitudes of the CV and EV estimates across species and over time. Overall, the modeling procedures and estimation methods employed here appear promising, and may prove beneficial for price, quantity and welfare analysis in the future when modeling systems of direct, inverse and mixed demand functions.

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## Appendix A

Lemmas 1 and 2 are proved in Cooper and McLaren (2006). As noted there, these results appear to be "well-known", although proofs are not easily accessible. The most accessible reference is probably Mangasarian (1969) who sets the results as problems with hints.

**Lemma 1**: Let $\Gamma \subset \Omega_{+}^{n}$ be convex. For $x \in \Gamma$ define f($x$) = g($x$) / h($x$) where g($x$) is convex and positive on $\Gamma$ and h($x$) is concave and positive on Γ. Then f($x$) is quasi-convex and positive on Γ.

**Lemma 2:** Let g($x$) be a positive concave function on Γ. Then f($x$) = 1/ g($x$) is a positive convex function on Γ.

The following result will also be useful, and would appear to be new.

**Lemma 3**: Let Γ be convex. For $x \in \Gamma$ define f($x$) = g($x$) · h($x$) where both g($x$) and h($x$) are concave and positive on Γ. Then f($x$) is quasi-concave and positive on Γ.

**Proof of Lemma 3**.

Consider 1/f($x$) = 1 / [g($x$) · h($x$)] = $\frac{1}{g(x)}\frac{1}{h(x)} = \frac{e(x)}{h(x)}$. By Lemma 2 e($x$) is positive convex, and hence by Lemma 1 1/f($x$) is quasi-convex and positive on Γ. The reciprocal of a quasi-convex and positive function is quasi-concave and positive.

[Note: Theorem 2 in Cooper and McLaren (2006) is incorrect. Lemma 3 above is the appropriate implication of Lemmas 1 and 2.]

**Proof of Theorem 1:** The positivity and homogeneity of degree zero conditions are obvious. The monotonicity conditions follow from the monotonicity and positivity of the component functions. Curvature conditions follow from Lemma 1. Thus sufficient conditions that define the region of (guaranteed) regularity follow from sufficient conditions for the positivity of the component functions

**Proof of Theorem 2:** As for Theorem 1, except that curvature conditions follow from Lemma 3.

**Proof of Theorem 3**: As for Theorem 1, except that curvature conditions in p follow from Lemma 1, and curvature conditions in x follow from Lemma 3.

## Appendix B

Write (27) as:

$$U^C = [\kappa / F_1 + (1\text{-}\kappa)(F_2^{-\mu} - 1) / \mu] \cdot F_3^{-\eta}$$

where, for example,

$$F_2 = \left[\theta\left(P2_A / c_A\right)^{\delta} + (1-\theta) / X2_B^{\delta}\right]^{1/\delta}$$

$U^C$ is individually decreasing in $F_k$ (k = 1, 2 and 3), which for the given parameter value constraints are themselves are either decreasing in $\mathbf{p}_A$ or increasing in $\mathbf{x}_B$, giving the requisite monotonicity conditions, provided the two component functions are positive. The transformation of $F_2$ is positive for $\mu > -1$ provided $F_2 < 1$. If the data is normalized such that $c_A = 1$, $\mathbf{p}_A = 1$, and $\mathbf{x}_B = 1$, then $P2_A = 1$ and $X2_B = 1$ and $F_2 < 1$ will be implied for all larger values of real income $(c_A / P2_A) > 1$ and larger values of $\mathbf{x}_{B,}$ $\mathbf{x}_B>1$. This defines the region of effective global regularity with regards to sufficient conditions for monotonicity.

Now turn to curvature properties. Given that $P2_A$ and $X2_B$ are concave and increasing functions, and given the functional form for $F_1$, the first term in square brackets in $U^C$ above is convex and decreasing in $\mathbf{p}_A$ , and concave and increasing in $\mathbf{x}_B$. For the second term in square brackets in $U^C$, consider first the curvature properties of $F_2$ in $\mathbf{p}_A$ (or $\mathbf{x}_B$). $P2_A$ and $X2_B$ are concave and increasing functions. Given that $\delta < 1$, then $F_2$ is concave and increasing in

$P2_A$ while $P2_A$ is concave and increasing in $\mathbf{p}_A$. Since an increasing concave function of an increasing concave function is increasing concave, $F_2$ is increasing and concave in $p_A$.

Turn now to curvature in $X2_B$. Rewrite the second term in the bracket in $F_2$ as (1-θ)$\left(X2_B^{-\delta}\right)$. If $\delta < 1$, then (from inspection of the derivatives) $F_2$ is convex and decreasing in X2B, and a decreasing convex function of an increasing concave function is convex and decreasing in $\mathbf{x}_B$. The properties of $F_3$ in $\mathbf{p}_A$ (or $\mathbf{x}_B$) are similar. Now the transformation $(F_2^{-\mu} - 1) / \mu$ is convex and decreasing in $F_2$ for the given parameter values.

In terms of prices, the second term in square brackets in $U^C$ is thus a decreasing convex function of an increasing concave function, and hence a decreasing convex function. The whole term in square brackets is thus the sum of two decreasing convex functions, and hence decreasing convex. Since the expression for $U^C$ can be rewritten as

$$U^C = [\kappa / F_1 + (1-\kappa)(F_2^{-\mu} - 1) / \mu] / F_3^{\eta}$$

with $F_3^{\eta}$ an increasing concave function of $F_3$ and hence of prices, then Lemma 1 applies and the properties in prices are confirmed over the same region of effective global regularity as for monotonicity.

In terms of quantities, however, $(F_2^{-\mu} - 1) / \mu$ is a convex and decreasing function of a convex and decreasing function in $\mathbf{x}_B$. Similarly, $F_3^{-\eta}$ is a convex and decreasing function of a convex and decreasing function in $\mathbf{x}_B$. To apply Lemma 3 requires the result of both of these compositions to be concave increasing, which cannot be assured by known general results. However, a direct evaluation of second derivatives is possible. The second derivative of $(F_2^{-\mu} - 1) / \mu$ with respect to X2 results in 3 terms, one positive and two negative. Collecting common terms, the second derivative can be written as the product of two terms

$$\left\{F_2^{-\mu}(1-\theta)A^{-1}X_2^{-(\delta+2)}\right\}\left\{(\mu+1)A^{-1}(1-\theta)X_2^{-\delta}-(1-\delta)A^{-1}(1-\theta)X_2^{-\delta}-(1+\delta)\right\}$$

where

$$A=\left[\theta\left(\frac{P2}{c}\right)^{\delta}+(1-\theta)X_2^{-\delta}\right].$$

The first term in curly brackets is positive, while the second term can be written as

$$(\mu+1)f-(1-\delta)f-(1+\delta)$$

where $f=\dfrac{(1-\theta)X_2^{-\delta}}{\left[\theta\left(\frac{P2}{c}\right)^{\delta}+(1-\theta)X_2^{-\delta}\right]}$ and hence 0<f<1. Thus sufficient conditions for the concavity of the second term in square brackets in $U^C$ are

$$-1<\mu<\delta\left(\frac{1}{f}-1\right)+\frac{1}{f}.$$

A similar evaluation of the second derivative of $F_3^{-\eta}$ with respect to X2 demonstrates that this function is concave for the given parameter constraints over the same region of effective global regularity as for monotonicity. Hence the above conditions are also sufficient conditions for effective global regularity of $U^C$.

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