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Parametric Pricing of Higher Order Moments in S&P500 Options

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Abstract

A general parametric framework is developed for pricing S&P500 options. Skewness and leptokurtosis in stock returns as well as time-varying volatility are priced. The parametric pricing model nests the Black-Scholes model and can explain volatility smiles and skews in stock options. The data consist of S&P500 options traded on select days in April, 1995, a total sample of over 500,000 observations. A number of performance criteria are used to evaluate the alternative models. The empirical results show that pricing higher order moments yield improvements in the pricing of options over the Black-Scholes model as well as other models.

Key words: Option pricing, volatility smiles and skews, generalised Student t, skewness, kurtosis and time-varying volatility.

JEL classification: C13, G13

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1 Introduction

The Black and Scholes (1973) model represents the most common framework adopted in practice for pricing options. Part of the reason for the popularity of the Black-Scholes model is its analytical tractability as the price is simply the mean of a truncated lognormal distribution. Two key assumptions of the Black-Scholes model are that the distribution of the underlying asset returns is normal and that volatility is constant. In modelling options written on equities, neither assumption is found to be valid; for a review of the empirical literature see Bollerslev, Chou and Kroner (1992). One manifestation of these misspecifications is the occurrence of volatility smiles and skews, whereby implied volatility estimates vary across strike prices written on contracts in the same market; see Hull and White (1987), Corrado and Su (1997) and Hafner and Herwartz (2001), amongst others.

A number of alternative frameworks have been proposed in the literature to correct for the misspecification of the Black-Scholes model; see Jackwerth (1999) for a review. These frameworks can be classified into three broad categories. The first category involves relaxing the constant volatility assumption. Examples are the deterministic volatility model of Dupire (1994), the stochastic volatility models of Hull and White (1987) and Heston and Nandi (2000), and the ARCH models of Engle and Mustafa (1992), Duan (1995) and Hafner and Herwartz (2001). The second category involves relaxing the normality assumption using either parametric or nonparametric methods. Parametric examples are the lognormal mixture model of Melick and Thomas (1997) and the flexible distributional framework adopted in Martin, Forbes and Martin (2001), whilst nonparametric examples are the Edgeworth expansion of Jarrow and Rudd (1982) and Corrado and Su (1997) and the nonparametric density estimator of Ait-Sahalia (1996), Ait-Sahalia and Lo (1998, 2000) and Ait-Sahalia, Wang and Yared (2001). The third category consists of augmenting the mean returns specification. The most popular form involves the inclusion of a Poisson jump process; see for example, Bakshi, Cao and Chen (1997) and Bates (2000), who also allow for stochastic volatility.

The approach adopted in this paper combines elements of the first two approaches. A parametric risk neutral distribution based on the generalised Student t distribution of Lye and Martin (1993, 1994) is proposed which captures the leptokurtosis and skewness in observed in returns distributions.[1] Time varying volatility is modelled by specifying the conditional variance to be a function of the net state returns over the life of the option; see Rosenberg and Engle (1997) and Rosenberg (1998). The option is then priced by evaluating the expected value of the discounted payoff of the option contract. The chosen specifications are appealing in that they lead to a computationally efficient procedure for pricing options based on univariate numerical quadrature. This is in contrast to models priced using Monte Carlo methods, which require computing the expectation as an average of a large number of simulation paths. Another advantage of the proposed framework is that a number of existing parametric models are special cases of the generalised Student t distribution, including the Black-Scholes model. This means that standard procedures can be adopted to test between competing parametric specifications. Other approaches based on lognormal mixture distributions (Melick and Thomas, 1997) and Edgeworth expansions (Jarrow and Rudd, 1982; and Corrado and Su, 1997) are shown to be related to the generalised Student t distribution but not directly nested. For these cases other statistical criteria are adopted to test between the competing models. A final advantage of the proposed framework is empirical, as the generalised Student t is shown to yield prices of S&P500 options which are superior to other models based on normal, symmetric Student t and mixture distributions.

The rest of the paper is structured as follows. Section 2 presents the

[1] This distribution is also used by Lim, Lye, Martin and Martin (1998) in pricing currency options.

framework for pricing options using a general parametric family of distributions and highlights a number of important special cases. The shapes of the risk neutral probability distributions are investigated, the effects on option pricing examined and the presence of volatility smiles and skews discussed. Some further relationships with alternative pricing models are discussed in Section 4. The empirical implications for pricing S&P500 options on selected days in April 1995 are presented in Section 5. In evaluating the competing models, five performance measures are adopted based on significance testing, mispricing, forecastability, hedging errors and volatility skew corrections. The key result of the analysis is that the generalised Student t model produces option prices that are superior to prices produced by all other models considered. A fundamental feature of these results is the importance of modelling skewness in stock returns both to minimise option pricing errors and to establish a consistent framework to price options across the full spectrum of moneyness in a single market. Concluding remarks are contained in Section 6.

2 Parametric Valuation of Options

2.1 General Framework

In this section, a general framework for pricing stock options is presented based on flexible parametric distributions. The distributional model of the stock price generalises the lognormal distribution which underlies the Black-Scholes option price model, by allowing for both skewness and kurtosis in returns.

As options on the S&P500 index are European options, the option price model developed here does not allow for early exercise.[2] Consider valuing a

[2] However, American style options such as options written on the S&P100 index, could be priced from the framework developed here by using the upper and lower bounds that characterise the relationships between European and American options; see Melick and Thomas (1997).

European call stock option at time t maturing at time $T = t + n$, where n represents the length of the contract. Defining S_t as the spot price at time t of the stock index, the price of the option with exercise price X, is given as the expected value of its discounted payoff; see Ingersoll (1987) and Hull (2000)

$$F(S_t) = E\left[e^{-r\tau}\max\left(S_T - X, 0\right)|S_t\right], \tag{1}$$

where the conditional expectation $E\left[.|S_t\right]$, is taken with respect to the risk neutral probability measure, $\tau = n/365$ represents the time until maturity expressed as a proportion of a year, and r represents the risk-free interest rate. An alternative way of writing (1) which is more convenient in developing the generalised forms of the risk neutral probability distribution adopted in this paper, is

$$F(S_t) = e^{-r\tau}\int_X^\infty (S_T - X) g(S_T|S_t) dS_T, \tag{2}$$

where $g(S_T|S_t)$ is the risk neutral probability density function of the stock price at the time of maturity, S_T, conditional on the current value, S_t.

In deriving the form of the risk neutral probability distribution, $g(S_T|S_t)$ in (2), the returns of the stock index over the life of the option contract are assumed to be generated as

$$\ln\left(\frac{S_T}{S_t}\right) = \left(r - \frac{\sigma_{T|t}^2}{2}\right)\tau + \sigma_{T|t}\sqrt{\tau} z_T, \tag{3}$$

where $\sigma_{T|t}$ is the annualised conditional volatility process and z_T is a standardised random variable with zero mean and unit variance. The Rosenberg and Engle (1997) formulation of the conditional volatility process is adopted whereby the volatility is specified as a function of the net state returns over the life of the option

$$\sigma_{T|t} = \exp\left(\beta_1 + \beta_2 \ln\left(S_T/S_t\right)\right). \tag{4}$$

This formulation of the volatility process has the effect of making the volatility a random variable as it is expressed as a function of the terminal price S_T, which is unknown at the time of writing the option contract.

In choosing the form of the distribution function of z_T, the adopted distribution needs to be able to capture the well-known empirical characteristics of stock returns; namely, fat-tailed and sharp-peaked distributions relative to the normal distribution. The distribution adopted here which has these characteristics is the generalised Student t distribution introduced by Lye and Martin (1993, 1994). Formally the generalised Student t distribution is specified as follows. Let w be a generalised Student t random variable with mean

$$\mu_w = \int w f(w)\, dw, \tag{5}$$

variance

$$\sigma_w^2 = \int w^2 f(w)\, dw - \mu_w^2, \tag{6}$$

and density given by

$$\begin{aligned} f(w) &= k \exp\left[\theta_1 \tan^{-1}\left(w/\sqrt{\nu}\right) + \theta_2 \ln\left(\nu + w^2\right) + \theta_3 w + \theta_4 w^2\right] \\ &= k\left(\nu + w^2\right)^{\theta_2} \exp\left[\theta_4 w^2\right] \exp\left[\theta_1 \tan^{-1}\left(w/\sqrt{\nu}\right) + \theta_3 w\right], \end{aligned} \tag{7}$$

where k is the integrating constant given by

$$k^{-1} = \int \exp\left[\theta_1 \tan^{-1}\left(w/\sqrt{\nu}\right) + \theta_2 \ln\left(\nu + w^2\right) + \theta_3 w + \theta_4 w^2\right] dw. \tag{8}$$

For the standardised generalised Student t variate, $z_T = (w - \mu_w)/\sigma_w$, the density is

$$p(z_T) = k\sigma_w \exp\left[\theta_1 \tan^{-1}\left(\frac{\mu_w + \sigma_w z_T}{\sqrt{\nu}}\right) + \theta_2 \ln\left(\nu + (\mu_w + \sigma_w z_T)^2\right) + \theta_3(\mu_w + \sigma_w z_T) + \theta_4(\mu_w + \sigma_w z_T)^2\right], \quad (9)$$

where k is the same normalising constant as defined in (7). Closed form expressions do not exist for k, μ_w and σ_w^2, but these quantities can be computed numerically.

The risk neutral probability density function $g(S_T|S_t)$, is derived from the returns distribution $p(z_T)$ in (7), via

$$g(S_T|S_t) = |J|\, p(z_T), \quad (10)$$

where J is the Jacobian of the transformation from z_T to S_T, given by

$$J = \frac{dz_T}{dS_T} = \frac{1}{S_T \sigma_{T|t}\sqrt{\tau}}\left[1 + \beta_2 \sigma_{T|t}^2 \tau - \beta_2\left(\ln(S_T/S_t) - \left(r - \frac{\sigma_{T|t}^2}{2}\right)\tau\right)\right], \quad (11)$$

and $\sigma_{T|t}^2$ is defined in (4).

Stock options can be priced by using (9) to (11) in (2). This formulation expands the Black-Scholes pricing framework as now both kurtosis and skewness in stock returns as well as conditional volatility, are all priced in the stock option. Apart from some special cases, the integral in (2) will need to be computed numerically.

2.2 Special Cases

An advantage of the structure of the returns distribution in (9) is that it nests the normal and Student t specifications. The power term, $(\nu + w^2)^{\theta_2}$ in (9) is a generalization of the kernel of a Student t density and controls the fatness in the tails of the distribution. The exponential term, $\exp[\theta_4 w^2]$, corresponds to the kernel of a normal density. Imposing the restriction $\theta_4 < 0$, ensures the existence of all moments of the distribution in the same way that all moments of the normal distribution exist. The parameters θ_1 and θ_3, control the level of skewness. For example, letting $\theta_3 = 0$, the distribution is skewed to the left (right) when $\theta_1 < 0$ ($\theta_1 > 0$) and is symmetric when $\theta_1 = 0$. Some specific cases of (9) that are implemented in the empirical section are as follows.

2.2.1 Black-Scholes Option Pricing

The Black-Scholes option price model is based on the assumption that returns are normally distributed. From (9), normality is achieved by imposing the restrictions

$$\theta_1 = \theta_2 = \theta_3 = 0, \theta_4 = -0.5,$$

thereby yielding the standard normal probability density function

$$p(z_T) = ke^{-0.5z_T^2}, \tag{12}$$

with

$$k = \frac{1}{\sqrt{2\pi}},$$

as now $\mu_w = 0$ and $\sigma_w^2 = 1$. Using (12) in (10) gives the risk neutral probability density as

$$g(S_T|S_t) = |J| \exp\left[-\frac{1}{2}\left(\frac{\ln(S_T/S_t) - \left(r - \frac{\sigma_{T|t}^2}{2}\right)\tau}{\sigma_{T|t}\sqrt{\tau}}\right)^2\right], \tag{13}$$

where J is given by (11) and $\sigma_{T|t}$ by (4).

The other assumption underlying the Black-Scholes model is that volatility is constant over the life of the contract. By setting $\beta_2 = 0$ in (4), (13) simplifies to the lognormal density

$$g(S_T|S_t) = \frac{1}{S_T \exp(\beta_1)\sqrt{\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln(S_T/S_t) - \left(r - \frac{\exp(\beta_1)}{2}\right)\tau}{\exp(\beta_1)\sqrt{\tau}}\right)^2\right]. \tag{14}$$

Using (14) in (2), the price of the option is

$$F(S_t) = e^{-r\tau} \int_X^\infty \frac{(S_T - X)}{S_T \exp(\beta_1)\sqrt{2\pi\tau}} \exp\left[-\frac{1}{2}\left(\frac{\ln(S_T/S_t) - \left(r - \frac{\exp(\beta_1)}{2}\right)\tau}{\exp(\beta_1)\sqrt{\tau}}\right)^2\right] dS_T. \tag{15}$$

The price of the option is given as the discounted value of the mean of a truncated lognormal distribution. For this case an analytical solution exists and is given by the standard Black-Scholes stock option pricing equation

$$F(S_t) = BS = S_t N(d_1) - X e^{-r\tau} N(d_2), \tag{16}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_t/X) + \left(r + \frac{\exp(2\beta_1)}{2}\right)\tau}{\exp(\beta_1)\sqrt{\tau}} \\ d_2 &= \frac{\ln(S_t/X) + \left(r - \frac{\exp(2\beta_1)}{2}\right)\tau}{\exp(\beta_1)\sqrt{\tau}}. \end{aligned}$$

2.2.2 Student t Option Pricing

The Student t distribution is obtained by imposing the restrictions

$$\theta_1 = \theta_3 = \theta_4 = 0, \theta_2 = -0.5(1+\nu),$$

in (9) where ν represents the degrees of freedom parameter. The Student t density is

$$p(z_T) = k\sigma_w \exp\left[-0.5(1+\nu)\ln\left(\nu + (\mu_w + \sigma_w z_T)^2\right)\right], \tag{17}$$

where the normalising constant is given by

$$k = \frac{\Gamma\left(\frac{1+\nu}{2}\right)\nu^{\frac{(1+\nu)}{2}}}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}.$$

Defining $\mu_w = 0$ and $\sigma_w^2 = \nu / (\nu - 2)$, the standardised Student t density is

$$p(z_T) = \sqrt{\left(\frac{\nu}{\nu-2}\right)} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)} . \left(1 + \frac{\left(\frac{\nu}{\nu-2}\right) z_T^2}{\nu}\right)^{\frac{-(1+\nu)}{2}} . \quad (18)$$

Using (18) in (10) gives the Student t form of the risk neutral probability distribution

$$g(S_T|S_t) = |J| \sqrt{\left(\frac{\nu}{\nu-2}\right)} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)} . \left(1 + \left(\frac{\ln(S_T/S_t) - \left(r - \frac{\sigma_{T|t}^2}{2}\right)\tau}{\sigma_{T|t}\sqrt{\tau(\nu-2)}}\right)^2\right)^{\frac{-(1+\nu)}{2}} , \quad (19)$$

where J is given by (11) and $\sigma_{T|t}$ by (4).[3] The student t option prices are computed by using (19) in (2) which can be evaluated using standard numerical quadrature procedures. The degrees of freedom parameter ν, controls the degree of leptokurtosis in the risk neutral distribution. As $\nu \to \infty$, this distribution approaches normality and the student t option price approaches the Black-Scholes price.

2.2.3 GST Option Pricing

The standardised Student t distribution in (18) can be generalised to allow for skewness by imposing the restrictions

$$\theta_3 = \theta_4 = 0, \theta_2 = -\frac{(1+\nu)}{2},$$

on (9). The standardised skewed Student t density is given by

$$\begin{aligned} p(z_T) &= k\sigma_w \exp\left[-0.5(1+\nu)\ln\left(\nu + (\mu_w + \sigma_w z_T)^2\right)\right. \\ &\qquad \left. +\theta_1 \tan^{-1}\left(\frac{\mu_w + \sigma_w z_T}{\sqrt{\nu}}\right)\right], \quad (20) \end{aligned}$$

where k is the normalising constant, which unlike in the case of the normal and Student t distributions, does not have a closed form expression. It is

[3] As the distribution in (19) is defined in terms of S_T, the risk neutral distribution in this case could be referred to as the log-Student t distribution.

this simpler parameterisation of the generalised Student t distribution in (9) which is adopted in this paper, as it provides the minimalist parameterisation for modelling the effects of kurtosis (via ν) and skewness (via θ_1) in stock returns on option prices.

Using (20) in (10) and, in turn, in (2) yields option prices which allow for both skewness and kurtosis in stock returns. As with the Student t price, numerical quadrature procedures are used to evaluate the integral in (2). Option prices based on this distributional formulation are referred to hereafter as GST prices.

3 Implications for Pricing Options

3.1 Risk Neutral Distributional Shapes

Some examples of the risk neutral probability distribution $g(S_T|S_t)$, are given in Figures 1 and 2 for various parameterisations. The initial spot price is $S_t = 500$, for a 6 month option, $\tau = 6/12$, with a risk-free rate of interest of $r = 0.05$. The volatility specification is given by (4) with $\beta_1 = -2$, and values of β_2 chosen to control for the relative impact of expected returns over the life of the option on time-varying volatility.

The Black-Scholes model of constant volatility and normality is represented by the case $\beta_2 = 0$, in Figure 1. The other graphs show that the risk neutral probability distribution becomes more positively skewed as β_2 becomes more positive.

The effects of fat-tails and skewness in the stock returns distribution are compared with the Black-Scholes model, $\nu \to \infty$ and $\theta_1 = 0$, in Figure 2. The risk neutral probability distribution becomes relatively more peaked than the lognormal distribution for relatively low degrees of freedom, $\nu = 4$ and $\theta_1 = 0$. For positive values of θ_1, the risk probability distribution becomes even more positively skewed, whilst for negative values of θ_1 the risk neutral probability distribution exhibits relatively less positive skewness than the lognormal distribution, and possibly even negative skewness.

3.2 Option Price Sensitivities

The sensitivity for pricing options under different distributional parameterisations is highlighted in Table 1. The option prices are computed for both one month contracts, $\tau = 1/12$, and six month contracts, $\tau = 6/12$, with strike prices of $X = 450, 500, 550$. The spot rate is $S_t = 500$ with a risk free rate of interest $r = 0.05$.

The Black-Scholes case is represented by the row labeled, Normal $\beta_2 = 0$, that is, normal returns with constant volatility. The effects on the Black-Scholes price of time-varying volatility are highlighted in the next set of rows with values of β_2 increasing from 0.1 to 0.4. Not surprisingly, the price of options increases monotonically as the value of β_2 increases, reflecting that increases in risk caused by increases in volatility are priced at a premium.

The effects of fat tails in the returns distribution are highlighted by the rows labeled, Student t. For the one month contracts the Black-Scholes price exceeds the Student t prices for the at-the-money option, $S_t = X = 500$. This price differential increases as the fatness in the tails of the returns distribution increases, that is as ν decreases from 9 to 4. The opposite result occurs for both out-of-the-money and in-the-money contracts where the Student t prices exceed the corresponding Black-Scholes prices.

The effects of both positive skewness and fat tails in the returns distribution are highlighted by the rows labeled GST in Table 1. Comparing the Student t and GST prices, the results show that positive skewness yields lower option prices that are both in-the-money and at-the-money. For out-of-the-money options, the option prices are relatively higher than corresponding prices based on symmetric returns distributions.

For the $\tau = 6/12$ contracts, increases in both positive skewness and fatness in the tails of the distribution result in Black-Scholes under-pricing in-

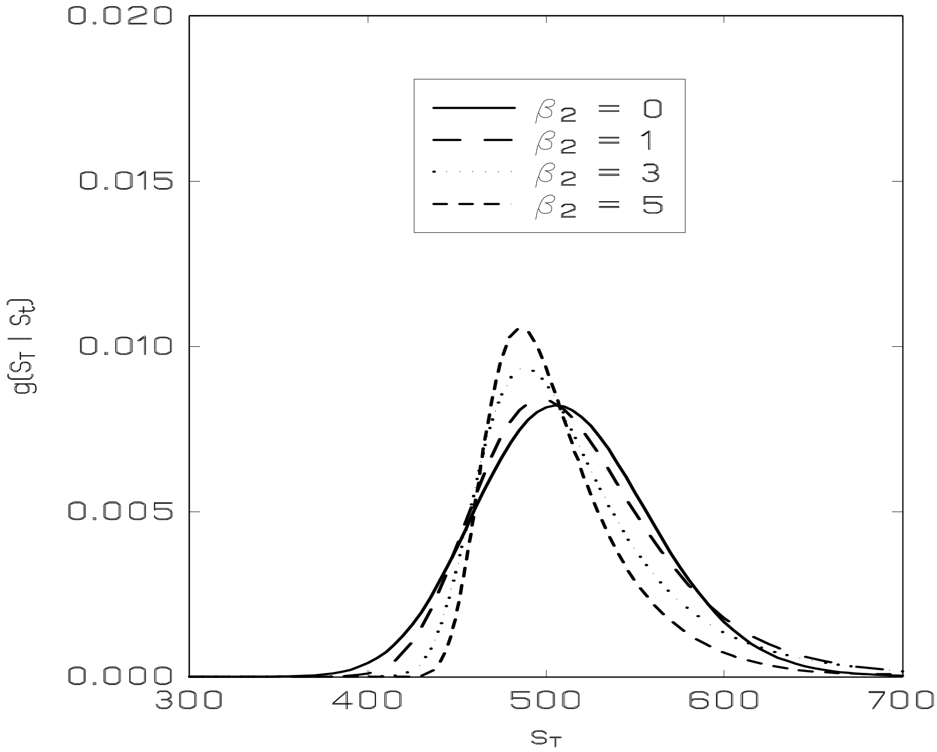

Figure 1: Risk neutral probability distributions for alternative volatility parameterisations assuming normality in stock returns.

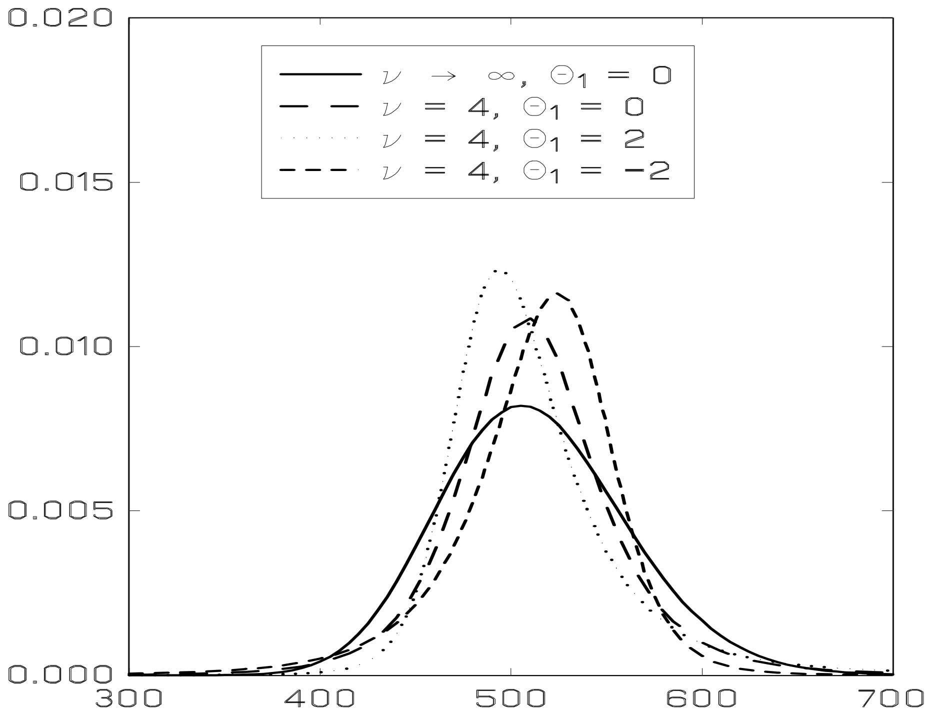

Figure 2: Risk neutral probability distributions for alternative distributional parameterisations with $\beta_2 = 0$.

Table 1:

Sensitivity of option prices to alternative conditional volatility parameterisations and distributional models: $S_t = 500$, $r = 0.05, \beta_1 = -2$.

Distribution Model		$\tau = 1/12$ years			$\tau = 6/12$ years		
	X :	450	500	550	450	500	550
Normal: $\beta_2 = 0.0$		51.88	8.86	0.07	62.89	25.66	6.67
Normal: $\beta_2 = 0.1$		51.96	8.90	0.07	63.25	25.95	6.93
Normal: $\beta_2 = 0.2$		52.03	8.95	0.08	63.62	26.26	7.21
Normal: $\beta_2 = 0.3$		52.11	8.99	0.09	64.00	26.58	7.51
Normal: $\beta_2 = 0.4$		52.19	9.03	0.10	64.79	27.57	8.84
Student t: $\nu = 4, \beta_2 = 0$		52.03	7.99	0.32	63.07	23.73	6.05
Student t: $\nu = 9, \beta_2 = 0$		51.93	8.59	0.16	63.00	25.04	6.49
GST: $\nu = 4, \theta_1 = 1, \beta_2 = 0$		51.93	7.89	0.57	62.49	23.52	7.40
GST: $\nu = 9, \theta_1 = 1, \beta_2 = 0$		51.91	8.58	0.21	62.79	24.99	6.85
GST: $\nu = 4, \theta_1 = 2, \beta_2 = 0$		51.91	7.73	0.76	62.09	23.19	8.27
GST: $\nu = 9, \theta_1 = 2, \beta_2 = 0$		51.89	8.55	0.27	62.58	24.91	7.18

the-money options, but over-pricing at-the-money and out-of-the-money options.

3.3 Volatility Smiles and Skews

The results in Table 1 provide a demonstration of the volatility smile, as the implied volatilities derived from the Black-Scholes model for the $\tau = 1/12$ contracts are relatively higher for both out-of-the-money and in-the-money options than the at-the-money options. For the longer contracts, $\tau = 6/12$, the implied volatilities tend to be relatively higher just for the in-the-money contracts. This last result provides evidence of a volatility smirk.

To highlight the relationship between volatility smiles and mispecification of the returns distribution, the following experiments are performed. The experiments are based on a true volatility parameter value of $\sigma_{T|t} = \sigma = 0.1$ or 10%. The option prices are computed for a three month contract length, $\tau = 3/12$, based on a spot rate of $S = 500$, and strike prices ranging from $X = 400$ to $X = 600$, in steps of 1. The risk free rate of interest is $r = 0.05$.

Two experiments are conducted to highlight the relationship between the volatility smile and nonnormality in returns. In the first experiment, the symmetric Student t distribution risk neutral probability distribution in (19) is used to compute the option price according to (2). The option prices are computed over the range $X/S = 400/500$ to $X/S = 600/500$, with the volatility parameter of $\sigma = 0.1$. Equating this price with the Black-Scholes price gives the value of the implied volatility plotted in Figure 3.[4] The calculations are performed for $\nu = 4, 9$. The results show that the volatility smile is accentuated as the degrees of freedom parameter is reduced. In particular, for contracts that mature approximately at-the-money, the observed price, as based on the Student t distribution, is below the Black-Scholes price using the true volatility value of $\sigma = 0.1$. To equate the two prices the implied volatility is less than the true volatility. For options both deep in-the-money and out-of-the-money, the Black-Scholes price based on the true volatility parameter $\sigma = 0.1$, underpredicts the observed price, thereby resulting in a relatively higher implied volatility parameter value. The degree of underprediction is relatively more severe for deep in-the-money options than it is for the deep out-of-the-money options as the implied volatility is relatively higher for the former class of options.

The second volatility smile experiment is the same as the first, except that the symmetric Student t distribution is replaced by the skewed Student

[4]The GAUSS procedure $NLSYS$ is used to compute the implied volatilities.

t distribution in (20). The range of skewness parameter values are $\theta_1 = \{-1, 0, 1\}$, with $\nu = 4$. The results are presented in Figure 4 and show that negative skewness in the returns distribution accentuates the smirk in the volatility smile.

4 Relationships with Other Models

4.1 The Jarrow and Rudd Model

The Jarrow and Rudd (1982) option pricing model has much in common with the approach adopted here, as both represent augmentations of the Black-Scholes returns distribution through the inclusion of higher order moment terms. The Jarrow Rudd model has recently been implemented by Corrado and Su (1997) and Capelle-Blancard, Jurczenko and Maillet (2001).

To show the relationship between the GST and the Jarrow and Rudd option pricing models, consider expanding the generalised Student t density in an Edgeworth expansion around the normal density. Letting $p(z_T)$ represent the generalised Student t density with distribution function $P(z_T)$, and $n(z_T)$ represent the normal density with distribution function $N(z_T)$, the Edgeworth expansion is

$$\begin{aligned} p(z_T) &= n(z_T) - \frac{(\kappa_1(P) - \kappa_1(N))}{1!}\frac{dn(z_T)}{dz_T} \\ &\quad + \frac{(\kappa_2(P) - \kappa_2(N))}{2!}\frac{d^2 n(z_T)}{dz_T^2} - \frac{(\kappa_3(P) - \kappa_3(N))}{3!}\frac{d^3 n(z_T)}{dz_T^3} \qquad (21) \\ &\quad + \frac{\left(\kappa_4(P) - \kappa_4(N) + 3(\kappa_2(P) - \kappa_2(N))^2\right)}{4!}\frac{d^4 n(z_T)}{dz_T^4} + \varepsilon(z_T), \end{aligned}$$

where $\varepsilon(z_T)$ is an approximating error and κ_i is the i^{th} cumulant of the associated distribution. This expression can be simplified by noting that both returns distributions are standardised to have zero mean ($\kappa_1 = 0$) and

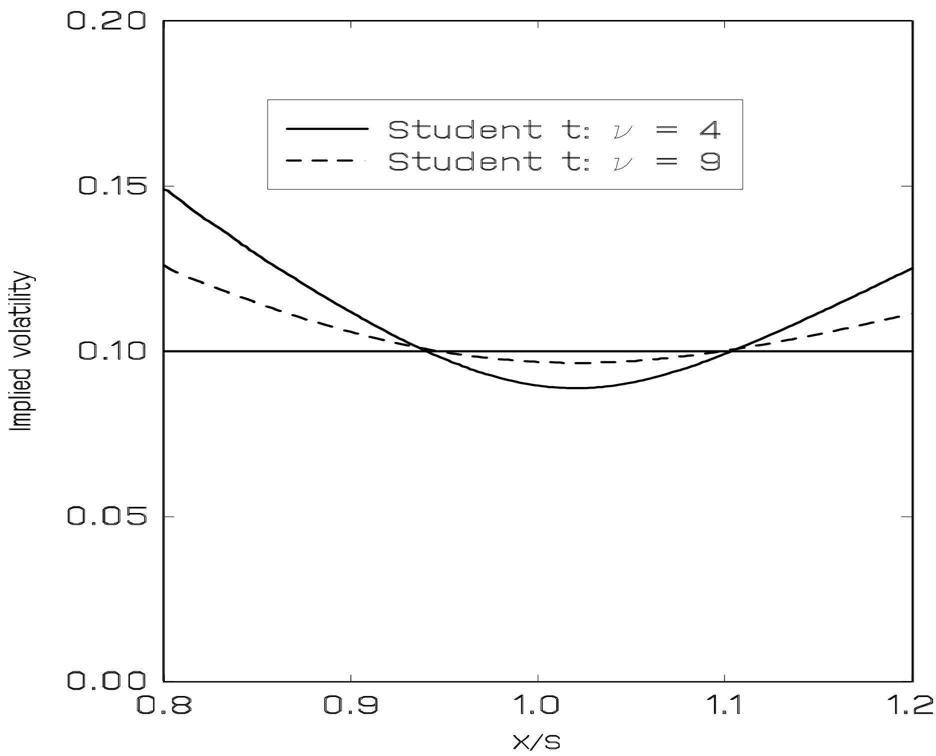

Figure 3: Volatility smiles generated when returns are distributed as Student t with varying degrees of freedom: true volatility is $\sigma = 0.1$.

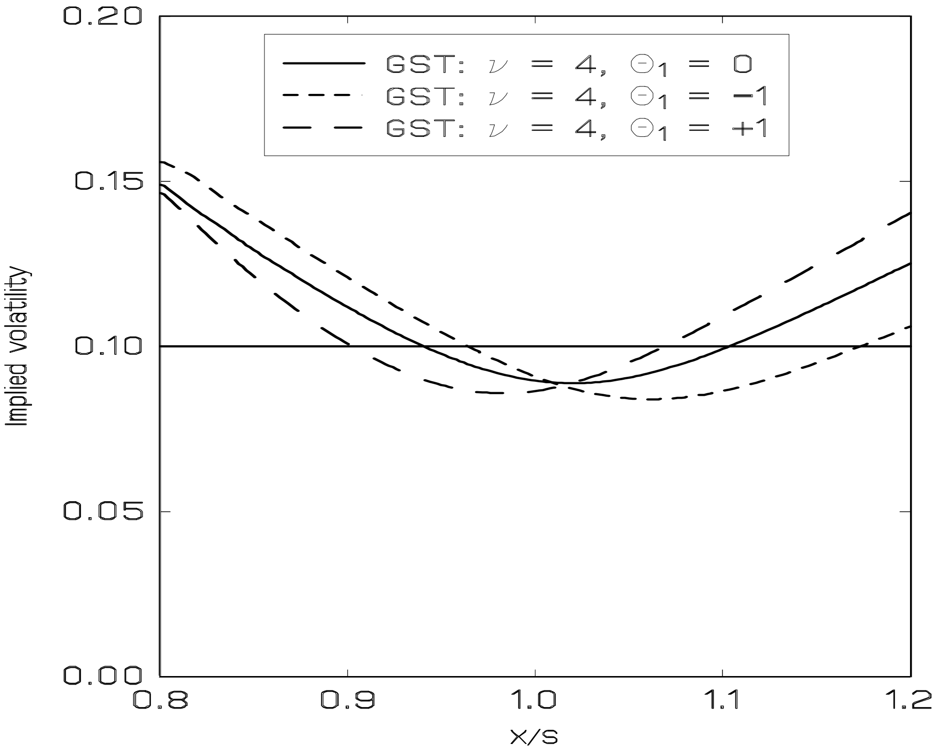

Figure 4: Volatility smiles generated when returns are distributed as Generalised Student t (GST) with $\nu = 4$ and varying skewness parameter, θ_1: true volatility is $\sigma = 0.1$.

unit variance $(\kappa_2 = 1)$,

$$\begin{aligned} p(z_T) &= n(z_T) - \frac{(\kappa_3(P) - \kappa_3(N))}{3!}\frac{d^3 n(z_T)}{dz_T^3} \\ &\quad - \frac{(\kappa_4(P) - \kappa_4(N))}{4!}\frac{d^4 n(z_T)}{dz_T^4} + \varepsilon(z_T). \end{aligned} \tag{22}$$

This expression shows that the difference between the two densities is determined by the third and fourth moments. Substituting (22) into (10) and ignoring the approximation error, gives the approximate risk neutral probability distribution function

$$\begin{aligned} g(S_T|S_t) &= |J|\left[n(z_T) - \frac{(\kappa_3(P) - \kappa_3(N))}{3!}\frac{d^3 n(z_T)}{dz_T^3}\right. \\ &\quad \left. - \frac{(\kappa_4(P) - \kappa_4(N))}{4!}\frac{d^4 n(z_T)}{dz_T^4}\right], \end{aligned} \tag{23}$$

where J is the Jacobian of the transformation from z_T to S_T given in (11) and (3) is used to substitute z_T for S_T. This expression shows that the density $g(S_T|S_T)$ is approximated by the lognormal distribution plus higher order terms which capture skewness and kurtosis. Using (23) in (2) to price options yields the Jarrow and Rudd option pricing model

$$F(S_t) = BS + \lambda_1 Q_3 + \lambda_2 Q_4, \tag{24}$$

where BS is the Black-Scholes price defined in (16), Q_3 and Q_4 capture the third and fourth order moments respectively, and λ_1 and λ_2 are parameters.

The establishment of the relationship between the GST and Jarrow and Rudd models also highlights the risk neutral properties of the GST model. In particular, as the lognormal distribution corresponds to the risk neutral distribution, provided that the mean of the underlying process is set equal to the risk free interest rate, and given that the standardised generalised Student t distribution is constrained to have the same mean, it follows from the arguments in Jarrow and Rudd (1982) that the standardised generalised Student t distribution can be interpreted as the risk neutral probability distribution, at least in the local region around the lognormal distribution.

4.2 Mixture of Lognormals

An alternative approach suggested by Melick and Thomas (1997) to capture departures from normal returns is based on a mixture of lognormal distributions. The option pricing model is

$$F(S_t) = \phi BS\left(\sigma_{1,t}\right) + (1-\phi)\, BS\left(\sigma_{2,t}\right), \tag{25}$$

where $BS\left(\sigma_{i,T|t}\right), i = 1, 2$, is the Black-Scholes price as defined in (16), but with time-varying volatility defined as

$$\begin{aligned} \sigma_{1,T|t} &= \left(\exp\beta_{1,1} + \beta_{1,2}\ln\left(S_T/S_t\right)\right) \\ \sigma_{2,T|t} &= \left(\exp\beta_{2,1} + \beta_{2,2}\ln\left(S_T/S_t\right)\right). \end{aligned} \tag{26}$$

The parameter $0 \leq \phi \leq 1$, is the mixing parameter which weights the two subordinate lognormal distributions.[5]

To highlight the relationships between the lognormal mixture model and the GST model, rewrite (25) as

$$F(S_t) = BS\left(\sigma_{2,t}\right) + \phi\left(BS\left(\sigma_{1,t}\right) - BS\left(\sigma_{2,t}\right)\right),$$

which is of a similar form to the Jarrow-Rudd model in (24), except that the term $\phi\left(BS\left(\sigma_{1,t}\right) - BS\left(\sigma_{2,t}\right)\right)$, takes the place of the skewness $(\lambda_1 Q_3)$ and kurtosis $(\lambda_2 Q_4)$ components.

4.3 Heston's Stochastic Volatility Model

Heston (1993) specifies an option price model based on stochastic volatility which is used by Bates (2000), Bakshi, Cao and Chen (1997), Heston and

[5] In constrast to the specification of (25), in the lognormal mixture model proposed by Melick and Thomas (1997), risk-neutrality is not imposed on the underlying distribution.

Nandi (2000) and Chernov and Ghysels (2000). The general solution is of the form

$$F(S_t) = S_t P_1 - e^{-r\tau} X P_2, \tag{27}$$

where P_1 and P_2 are inverse characteristic functions which are nonlinear expressions of the underlying parameters of the model. To highlight the relationship between the Heston model and the approach presented here, rewrite (2) as

$$F(S_t) = S_t \left[e^{-r\tau} \int_X^\infty \frac{S_T}{S_t} g(S_T|S_t) dS_T \right] - e^{-r\tau} X \left[\int_X^\infty g(S_T|S_t) dS_T \right], \tag{28}$$

with the expressions in the square brackets corresponding respectively to P_1 and P_2 in (27). As noted above, the integrals in (28) are computed easily using standard numerical procedures. In the case of computing the P_1 and P_2 expressions in (27) the inverse of the characteristic function requires integration over the complex plane which is less straightforward.

5 Pricing S&P500 Stock Options

5.1 Data Description

The data set used in the empirical application consists of quotes on call options written on the S&P500 stock index, obtained from the Berkeley Options Database. The quotes relate to options traded in the month of April, 1995. Specifically, the alternative models are estimated using the midpoints of bid-ask quotes for April 4th, 11th and 18th respectively. Predictive and hedging performances are then assessed using data for the remaining days in April. Each sub-sample comprises approximately 40,000 prices, for option contracts extending over the full moneyness spectrum. Defining $S - X$ as the intrinsic value of the call option, options for which $S/X \in (0.97, 1.03)$ are categorised as at-the-money, those for which $S/X \leq 0.97$, as out-of-the-money, and those for which $S/X \geq 1.03$, as in-the-money; see Bakshi, Cao and Chen (1997).

Maturity lengths range from approximately one to five months. Each record in the dataset comprises the bid-ask quote, the synchronously recorded spot price of the index, the time at which the quote was recorded, and the strike price.

As dividends are paid on the S&P500 index, the current spot price, S_t, in (1) and in all subsequent formulae is replaced by the dividend-exclusive spot price, $S_t e^{-d\tau}$, where d is the average rate of dividends paid on the S&P500 index over 1995. This rate is used as a proxy for the rate of dividend payment made over the life of each option. Daily dividend data for 1995, used to construct d, were obtained from Standard and Poors. Only observations for which the average of the bid and ask prices exceeds the lower bound of

$$LB = \max\{0, S_t e^{-d\tau} - e^{-r\tau} X\},$$

and which are recorded between 9.00am and 3.00pm are included in the sample. The first restriction serves to exclude prices which fail to satisfy the no-arbitrage lower bound, whilst the second restriction seeks to minimise the problem of nonsynchroneity between the spot and option prices.

The interest rate, r, is the three month bill rate observed on that day, with interest rate data obtained from Datastream. Tables 2, 3 and 4 summarise the main characteristics of the datasets used in the estimation.

5.2 Model Estimation

Define the theoretical price of the j^{th} option contract at written at time t, as

$$F_{j,t} = F\left(S_t, X_{j,t}, \tau_j, r; \Omega\right).$$

The relationship between $C_{j,t}$, the market price of the j^{th} option contract at time t, and $F_{j,t}$, is given by

$$C_{j,t} = F_{j,t} + \omega e_{j,t}, \tag{29}$$

Table 2:

S&P500 Option Price Dataset: April 4, 1995.

Variable		All Contracts	Moneyness (S/X)		
			< 0.97	$0.97 - 1.03$	> 1.03
Call Price:	$\overline{X}$	\$60.87	\$8.60	\$19.90	\$66.36
	SD	\$31.17	\$2.84	\$5.82	\$28.83
	Min	\$1.04	\$1.04	\$3.38	\$21.63
	Max	\$156.25	\$10.63	\$29.88	\$156.25
	$Number$	43584	511	4519	38554
Maturity (No. of Prices)	May	11600	45	121	11434
	June	16748	39	2047	14662
	Sept.	15236	427	2351	12458
Strike Price:	$\overline{X}$	\$448.65			
	SD	\$34.02			
	Min	\$350.00			
	Max	\$550.00			
	$Number$	30			
Spot: (S&P 500 Index)	$\overline{X}$	\$503.33			
	SD	\$0.56			
	Min	\$502.38			
	Max	\$504.56			

Table 3:

S&P500 Option Price Dataset: April 11, 1995.

Variable		All Contracts	Moneyness (S/X)		
			< 0.97	$0.97 - 1.03$	> 1.03
Call Price:	$\overline{X}$	\$50.00	\$6.31	\$17.13	\$57.01
	SD	\$27.15	\$1.98	\$5.33	\$24.50
	Min	\$0.95	\$0.95	\$1.94	\$19.13
	Max	\$157.63	\$10.07	\$29.00	\$157.63
	$Number$	43509	438	7088	35983
Maturity: (No. of Prices)	May	9609	19	1706	7884
	June	15254	20	1868	13366
	Sept.	18646	399	3514	14733
Strike Price:	$\overline{X}$	\$463.37			
	SD	\$29.96			
	Min	\$350.00			
	Max	\$600.00			
	$Number$	31			
Spot: (S&P 500 Index)	$\overline{X}$	\$506.19			
	SD	\$0.72			
	Min	\$502.29			
	Max	\$508.42			

Table 4:

S&P500 Option Price Dataset: April 18, 1995.

Variable		All Contracts	Moneyness (S/X)		
			< 0.97	$0.97 - 1.03$	> 1.03
Call Price:	$\overline{X}$	\$46.89	\$5.13	\$17.38	\$53.73
	SD	\$25.84	\$2.45	\$5.19	\$23.48
	Min	\$0.60	\$0.60	\$1.10	\$17.38
	Max	\$156.88	\$8.63	\$29.50	\$156.88
	$Number$	38099	670	6274	31155
Maturity: (No. of Prices)	May	7984	25	1297	6662
	June	12579	27	1699	10853
	Sept.	17536	618	3278	13640
Strike Price:	$\overline{X}$	\$465.73			
	SD	\$28.67			
	Min	\$350.00			
	Max	\$550.00			
	$Number$	30			
Spot: (S&P 500 Index)	$\overline{X}$	\$505.65			
	SD	\$0.63			
	Min	\$504.12			
	Max	\$506.71			

where Ω is the vector of parameters which characterise the returns distribution and the volatility specification, and $e_{j,t}$ represents the pricing error with standard deviation ω. In the case of the Black-Scholes option pricing model, for example, $\Omega = \{\beta_1\}$. Following the approach of Engle and Mustafa (1992), Sabbatini and Linton (1998) and Jacquier and Jarrow (2000), the pricing error $e_{j,t}$, is assumed to be a standardised normal random variable; see also the discussion in Clement, Gourieroux and Monfort (2000).[6] Equation (29) can be thought of as a nonlinear regression equation, with the parameter vector, Ω, entering the model nonlinearly.

Letting N represent the number of observations in a pooled data set of time series and cross-sectional prices of option contracts, the logarithm of the likelihood function is defined as

$$\ln L = -\frac{N}{2} \ln \left(2\pi\omega^2\right) - \frac{1}{2} \sum_{j,t} \left(\frac{C_{j,t} - F_{j,t}}{\omega}\right)^2 . \tag{30}$$

This function is maximised with respect to ω and Ω, using the GAUSS procedure MAXLIK. In maximising the likelihood, ω is concentrated out of the likelihood. The numerical integration procedure for computing the theoretical option price $F_{j,t}$, for the various models is based on the GAUSS procedure INTQUAD1. As a test of the accuracy of the integration procedure, both numerical and analytical formulae for the Black-Scholes model were used. Both procedures generated the exact same parameter estimates to at least four decimal points.

5.3 Performance Evaluation

The performance of the alternative pricing models is now investigated. Five procedures are used to assess the performance of the models. The first con-

[6] More general specifications of the pricing error in (29) could be adopted. For example, ω could be allowed to vary across the moneyness spectrum of option contracts, while a more general distributional structure for $e_{j,t}$, could be entertained. Alternatively, the statistical model could be defined in terms of hedging errors; see Bakshi, Cao and Chen (1997).

sists of conducting standard tests of significance on the parameter estimates. The second concentrates on comparing the relative size of mispricing errors of each model. The third focuses on forecasting properties, whilst the fourth procedure compares the relative size of hedging errors from each model. The last procedure examines the ability of the competing models to correct for volatility skews.

5.3.1 Statistical Tests

The parameter estimates of the alternative models are contained in Tables 5 to 7 for the three respective days investigated, with standard errors based on the inverse of the Hessian given in parentheses. The results show clearly that across all three days there is strong statistical evidence of significant negative skewness. This feature highlights the property that the underlying empirical returns distribution is asymmetrical and that models which assume symmetrical distributions such as the symmetric Student t and normal distributions, are misspecified. The skewed and symmetric Student t models also show that the returns distribution exhibits fat-tails, a result which is consistent with empirical results based on direct analysis of stock returns. For example, the April 4th estimate of the degrees of freedom parameter, $\nu = \gamma^2$, from the Student t model is $1.815^2 = 3.294$.

The parameter estimates of β_2 for all empirical models across all days in Tables 5 to 7 are all statistically significant, thereby providing strong evidence of time-varying volatility. This result together with the results discussed above concerning both fat-tails and negative skewness, provide strong evidence against the standard Black-Scholes option price model.

5.3.2 Mispricing

An overall measure of the pricing error is given by the residual variance ω^2, in (29)

$$\omega^2 = \frac{\sum_{j,t} \left(C_{j,t} - F_{j,t}\right)^2}{N}, \tag{31}$$

where $C_{j,t}$ and $F_{j,t}$ are defined above. Estimates of the residual variance for each of the four models across the three days, are given Table 9. A comparison of these estimates across the models shows that the GST model yields large reductions in the amount of mispricing. The mixture model is the next best performer with increases in mispricing over the GST model of between 26% to 119%. The Student t and normal distribution models yield even larger increases in mispricing with increases of over 600% in some cases.

To allow for differences in parameter dimensionality across the models when comparing mispricing properties, the AIC and SIC statistics are also presented in Table 9. These statistics also provide strong evidence in favour of the GST model over the other three models investigated. Comparing the GST and Student t, AIC and SIC values shows that there are significant gains from modelling skewness in stock returns. The smaller values obtained for the AIC and SIC statistics in the GST and Student models over the Normal model also show that there are large gains to be made from modelling the leptokurtosis in stock returns. Comparing the AIC and SIC values for the GST and mixture models shows that the generalised Student t distribution does a better job in modelling the impact of leptokurtosis and skewness on option prices than does the lognormal mixture model.

5.3.3 Forecasting

The forecasting performance of the normal, Student t, GST and lognormal mixture pricing models is compared in Table 10. The approach consists of using the parameter estimates based on the 4th of April data to compute option prices on the 5th, 6th, 7th and 10th of April; see Corrado and Su (1997). The RMSE is computed as

$$RMSE = \frac{\sum_{j,t} \left(C_{j,t} - F_{j,t|\text{April 4th}}\right)^2}{N}, \tag{32}$$

where

$$F_{j,t|\text{April 4th}} = F\left(S_t, X_{j,t}, \tau_t, r; \Omega_{\text{April 4th}}\right),$$

and $\Omega_{\text{April 4th}}$ signifies parameter estimates based on April 4th data. Using data from April 4th in (32) and squaring the result would yield the residual variance estimates reported in Tables 5 to 7.

The procedure is continued for the other two sample periods, with the parameter estimates based on the 11th of April data used to compute option prices for the 12th, 13th and 17th of April, and the parameter estimates based on the 18th of April data used to compute option prices for the 19th to the 21st of April.

The results of the forecasting test show that the GST pricing model overall yields the smallest RMSE. This property consistently occurs over all estimation periods and across all forecast horizons.

5.3.4 Hedging Errors

Consider forming a portfolio that is short in the call option. Normalising the portfolio on a single call option contract, the size of the investment, I_t, to set up the portfolio, P_t, using a delta hedge is

$$P_t = I_t = \Delta_{j,t} S_t - C_{j,t}, \tag{33}$$

where S_t is the spot price at the time the portfolio is constructed, $C_{j,t}$ is the call price on the j^{th} option contract, and $\Delta_{j,t}$ represents the proportion of stocks purchased to delta hedge the portfolio

$$\Delta_{j,t} = \frac{dC_{j,t}}{dS_t}. \tag{34}$$

The value of the portfolio at the start of the next day based on the proportion of stocks purchased in the previous day is

$$P_{j,t+1} = \Delta_{j,t} S_{t+1} - C_{j,t+1}. \tag{35}$$

The value of this portfolio can be compared to investing the amount I_t in (33) at the risk free rate of interest r_t for one day. The value of the investment in period $t+1$, is

$$I_{t+1} = I_t \exp(r_t/365) = (\Delta_{j,t}S_t - C_{j,t}) \exp(r_t/365). \tag{36}$$

The difference between (35) and (36) yields the one day ahead hedging error; see Bakshi, Cao and Chen (1997)

$$\begin{aligned} H_{j,t+1} &= P_{j,t+1} - I_{t+1} \\ &= \Delta_{j,t} (S_{t+1} - S_t \exp(r_t/365)) - (C_{j,t+1} - C_{j,t} \exp(r_t/365)) \end{aligned} \tag{37}$$

The hedging error for k days ahead is calculated as

$$H_{j,t+k} = \Delta_{j,t} (S_{t+k} - S_t \exp(r_t k/365)) - (C_{j,t+k} - C_{j,t} \exp(r_t k/365)).$$

The results of the hedging error experiments for the various option price models across the three sample periods are contained in Tables 11 to 13. The total number of unique contracts that have matching contracts across the forecast period are 574 for April 4th results, 563 for April 11th results and 377 for April 18th results.[7] All values are expressed in dollars whereby a value of $+X$ $(-X)$ means that the portfolio earns $\$X$ more than (less than) would be earned from investing the money at the risk free rate of interest over the pertinent forecast period. The size of the hedging errors are broken-down into moneyness classes, S_t/X_j, as well as being reported for the total class. For comparison the average values of the investment, I_t in (33), across all contracts for each model are also presented in Tables 11 to 13.

The hedging error results in Tables 11 to 13 show that the absolute values of the hedge errors associated with the one-step ahead forecasts are all small relative to the size of the initial investment, being less than 46 cents. There

[7] In computing the hedging errors, the spot rates are those time-stamped with the call option. Hence, the spot rates will vary slightly over the day for different contracts written on the same day.

is a tendency for the in-the-money option contracts to yield portfolios that have lower values than if the money was invested at the risk free rate of interest. Comparing the results across option price models, there is very little to choose between the models as there are no consistent patterns over either moneyness or forecast horizons.

5.3.5 Volatility Skew Corrections

As a final performance measure, the ability of the alternative models to correct for volatility skews are examined. The results are given in Figure 5 for the Black-Scholes, Student t and GST models. using options prices on April 4th 1995, written on May contracts.[8] In computing the option price, $F_{j,t}$, for each model, the point estimates of the distribution parameters in Tables 5 and 8 are used, while the implied volatility parameter is computed by solving

$$C_{j,t} = F_{j,t}(\sigma),$$

for each contract assuming volatility over the life of the contract is fixed, that is, $\sigma_{T|t} = \sigma$. The calculations are performed over the full range of strike prices.[9] To generate a smooth implied volatility surface, the implied volatility estimates presented in Figure 5 are the predictions from regressing the implied volatility values on a constant and a quadratic polynomial in moneyness.

For comparability with Figures 3 and 4, the volatility smiles presented in Figure 5 are plotted against the inverse of moneyness, X/S.[10] The results

[8] Similar qualitative results are obtained for the June and September contracts, as well as from using data on April 11th and 18th, 1995. To save space these results are not presented.

[9] Contracts with the same strike price X, but different moneyness as a result of differences in the spot price S_t, over the day, are all included. This yields a total sample size of over $11,000$ contracts to compute the implied volatility functions for April 4th.

[10] The implied volatilities of the lognormal mixture model are not computed as this model yields two estimates of the volatility parameter. However, constraining the two volatility parameter estimates to be equal is equivalent to the Black-Scholes model.

Table 5:

Maximum likelihood estimates of option price models for the 4th of April 1995: standard errors in brackets, $N = 43584$.

Parameter	GST	Student	Normal
β_1	-1.963 (0.001)	-1.991 (0.001)	-2.123 (0.001)
β_2	0.053 (0.002)	0.142 (0.001)	0.402 (0.001)
$\gamma = \sqrt{\nu}$	1.815 (0.003)	1.890 (0.002)	n.a.
θ_1	-0.750 (0.005)	0.0	0.0
θ_2	$-0.5(1+\gamma^2)$	$-0.5(1+\gamma^2)$	0.0
θ_3	0.0	0.0	0.0
θ_4	0.0	0.0	-0.5
Av. log-likelihood$^{(a)}$	-0.380	-0.800	-1.044

(a) The average log-likelihood is lnL/N, where L is likelihood and N is the sample size.

(b) n.a. = not applicable.

Table 6:

Maximum likelihood estimates of option price models for the 11th of April 1995: standard errors in brackets, $N = 43509$.

Parameter	GST	Student	Normal
β_1	-3.001 (0.011)	-1.935 (0.002)	-2.124 (0.001)
β_2	0.199 (0.006)	0.022 (0.002)	0.420 (0.001)
$\gamma = \sqrt{\nu}$	4.001 (0.004)	1.692 (0.003)	n.a.
θ_1	-24.848 (0.032)	0.0	0.0
θ_2	$-0.5(1+\gamma^2)$	$-0.5(1+\gamma^2)$	0.0
θ_3	0.0	0.0	0.0
θ_4	0.0	0.0	-0.5
Av. log-likelihood$^{(a)}$	-0.281	-0.983	-1.268

(a) The average log-likelihood is lnL/N, where L is likelihood and N is the sample size.

(b) n.a. = not applicable.

Table 7:

Maximum likelihood estimates of option price models for the 18th of April 1995: standard errors in brackets, $N = 38099$.

Parameter	GST	Student	Normal
β_1	-2.344 (0.006)	-1.997 (0.001)	-2.139 (0.001)
β_2	0.520 (0.011)	0.115 (0.002)	0.421 (0.001)
$\gamma = \sqrt{\nu}$	3.523 (0.009)	1.840 (0.002)	n.a.
θ_1	-20.555 (0.075)	0.0	0.0
θ_2	$-0.5(1+\gamma^2)$	$-0.5(1+\gamma^2)$	0.0
θ_3	0.0	0.0	0.0
θ_4	0.0	0.0	-0.5
Av. log-likelihood[(a)]	-0.515	-1.044	-1.197

(a) The average log-likelihood is lnL/N, where L is likelihood and N is the sample size.

(b) n.a. = not applicable.

Table 8:

Maximum likelihood estimates of the mixture of lognormal option price model for various dates: standard errors in brackets.

Parameter	4th of April	11th of April	18th of April
$\beta_{1,1}$	-1.350 (0.003)	-1.515 (0.002)	-1.525 (0.004)
$\beta_{1,2}$	0.253 (0.005)	0.081 (0.003)	0.064 (0.005)
$\beta_{2,1}$	-2.687 (0.001)	-2.924 (0.002)	-2.674 (0.002)
$\beta_{2,2}$	-0.919 (0.025)	0.503 (0.003)	0.472 (0.002)
λ	0.250 (0.001)	0.367 (0.001)	0.311 (0.002)
Av. log-likelihood$^{(a)}$	-0.497	-0.685	-0.895

(a) The average log-likelihood is lnL/N, where L is likelihood and N is the sample size.

(b) n.a. = not applicable.

Table 9:

Estimates of mispricing of alternative models across selected days.

Day	Statistic	Model			
		GST	Student	Normal	Mixture
4th of April	Residual variance[(a)]	0.125	0.290	0.473	0.158
	AIC[(b)]	0.760	1.601	2.088	0.995
	SIC[(c)]	0.760	1.602	2.089	0.996
11th of April	Residual variance	0.103	0.418	0.739	0.230
	AIC	0.563	1.966	2.535	1.370
	SIC	0.563	1.967	2.536	1.371
18th of April	Residual variance	0.164	0.473	0.641	0.351
	AIC	1.029	2.089	2.394	1.790
	SIC	1.030	2.089	2.394	1.791

(a) Based on equation (31).

(b) AIC = -2lnL/N+2k/N, where L is the likelihood, N is the sample size and k is the number of estimated parameters.

(c) SIC = -2lnL/N+ln(N)k/N, where L is the likelihood, N is the sample size and k is the number of estimated parameters.

Table 10:

Forecasting performance of alternative option price models across various days in April 1995: RMSE.

		GST	Student	Normal	Mixture
			4th of April		
Forecast	5th	0.459	0.600	0.701	0.506
	6th	0.498	0.625	0.745	0.525
	7th	0.491	0.624	0.775	0.499
	10th	0.481	0.681	0.816	0.534
			11th of April		
Forecast	12th	0.320	0.660	0.901	0.505
	13th	0.496	0.720	0.832	0.628
	17th	0.799	0.943	0.967	0.947
			18th of April		
Forecast	19th	0.649	0.785	0.872	0.678
	20th	0.440	0.682	0.777	0.606
	21th	0.961	1.049	1.039	1.048

Table 11:

Hedging performance of alternative option price models estimated on 4th of April 1995: expressed in dollars.

Day		Model			
		Normal	Student	GST	Mixture
4th	Investment (I)	407.77	407.36	403.67	426.45
	Moneyness (S/X)				
5th	<0.97	-0.41	-0.45	-0.39	-0.68
	0.97 - 1.00	-0.31	-0.30	-0.24	-0.17
	1.00 - 1.03	0.06	0.09	0.13	0.35
	>1.03	0.18	0.17	0.16	0.21
	Total	0.16	0.15	0.15	0.20
6th	<0.97	-0.17	-0.24	-0.16	-0.60
	0.97 - 1.00	6.06	6.08	6.22	6.31
	1.00 - 1.03	1.27	1.33	1.40	1.76
	>1.03	-3.11	-3.11	-3.12	-3.00
	Total	-2.68	-2.67	-2.67	-2.54
7th	<0.97	14.01	14.00	14.00	13.97
	0.97 - 1.00	13.62	13.64	13.67	13.84
	1.00 - 1.03	5.72	5.74	5.76	5.90
	>1.03	-6.06	-6.06	-6.05	-5.95
	Total	-5.07	-5.06	-5.05	-4.95
10th	<0.97	10.82	10.80	10.78	10.61
	0.97 - 1.00	7.41	7.46	7.50	7.78
	1.00 - 1.03	7.61	7.67	7.71	8.09
	>1.03	-1.79	-1.79	-1.79	-1.66
	Total	-0.91	-0.90	-0.90	-0.76

Table 12:

Hedging performance of alternative option price models estimated on 11th of April 1995: expressed in dollars.

Day		Model			
		Normal	Student	GST	Mixture
11th	Investment (I)	410.29	412.01	408.90	442.07
	Moneyness (S/X)				
12th	<0.97	-0.19	-0.19	-0.20	-0.19
	0.97 - 1.00	0.02	0.01	0.01	-0.03
	1.00 - 1.03	-0.04	-0.06	-0.07	-0.15
	>1.03	-0.04	-0.04	-0.04	-0.05
	Total	-0.04	-0.04	-0.04	-0.06
13th	<0.97	n.a.	n.a.	n.a.	n.a.
	0.97 - 1.00	n.a.	n.a.	n.a.	n.a.
	1.00 - 1.03	-11.03	-11.08	-10.85	-11.09
	>1.03	-0.97	-0.94	-0.95	-0.75
	Total	-1.45	-1.43	-1.43	-1.25
17th	<0.97	5.91	5.85	6.10	5.79
	0.97 - 1.00	2.98	2.99	3.26	3.29
	1.00 - 1.03	5.34	5.49	5.58	6.25
	>1.03	2.01	2.02	1.99	2.22
	Total	2.40	2.43	2.42	2.69

n.a. indicates that the cell does not contain any observations.

Table 13:

Hedging performance of alternative option price models estimated on 18th of April 1995: expressed in dollars.

Day		Model			
		Normal	Student	GST	Mixture
18th	Investment (I)	398.79	401.24	401.42	439.70
	Moneyness (S/X)				
19th	<0.97	0.21	0.27	0.11	0.49
	0.97 - 1.00	0.24	0.22	0.11	0.01
	1.00 - 1.03	0.35	0.29	0.27	-0.03
	>1.03	0.21	0.21	0.24	0.12
	Total	0.24	0.23	0.23	0.09
20th	<0.97	n.a.	n.a.	n.a.	n.a.
	0.97 - 1.00	-11.56	-11.49	-11.70	-11.25
	1.00 - 1.03	-3.99	-4.03	-4.04	-4.22
	>1.03	-9.89	-9.88	-9.84	-9.93
	Total	-9.01	-9.00	-8.98	-9.07
21th	<0.97	n.a.	n.a.	n.a.	n.a.
	0.97 - 1.00	n.a.	n.a.	n.a.	n.a.
	1.00 - 1.03	-11.26	-11.44	-11.05	-12.16
	>1.03	-8.59	-8.57	-8.53	-8.41
	Total	-8.62	-8.60	-8.56	-8.45

n.a. indicates that the cell does not contain any observations.

show the volatility skew associated with the Black-Scholes model, with implied volatility values of in excess of 30% for deep in-the-money contracts, and less than 10% for deep out-of-the-money contracts. The Student t model does a good job in correcting for most of the volatility skew arising from the Black-Scholes model, leaving just a volatility smirk concentrated in the range of the deep in-the-money options. The implied volatility values from the GST model show that the addition of the skewness parameter wipes the smirk from the Student t implied volatility surface, with implied volatility values in the range of 12% to 14%.

6 Conclusions

A general framework for pricing skewness, leptokurtosis and time-varying volatility in S&P500 options was developed. The approach consisted of modelling the returns over the life of the option contract as a generalised Student t distribution. This yielded a parametric form for the risk neutral density function which was used to price options. The parametric pricing model was shown to nest the Black-Scholes model and to capture volatility smiles and skews.

The performance of a range of models were investigated using option contracts written on the S&P500 stock index for selected days in April 1995. The key empirical results were that there were significant gains to be made from pricing skewness and leptokurtosis in stock returns. In particular, the GST option price model corrected for volatility skews and smirks thereby provided a consistent framework to price options in a single market across the full spectrum of moneyness. The generalised Student t modelling framework was also found to be superior to models where the risk neutral distribution was assumed to be a mixture of lognormals.

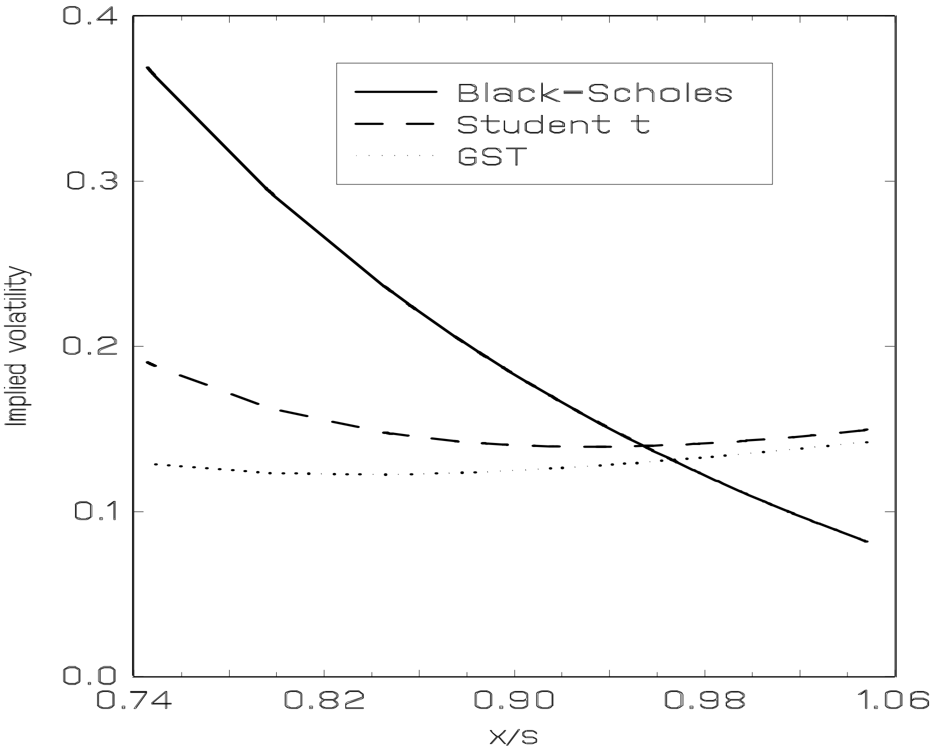

Figure 5: Volatility smiles for alternative models using data from April 4th, 1995, written on May options.

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