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A comparison of maximum likelihood models for fatigue strength characterization in materials exhibiting a fatigue limit

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ABSTRACT

In this study, various probabilistic models were considered to support fatigue strength design guidance in the ultra high-cycle regime (beyond 10^8 cycles), with particular application to Ti-6Al-4V, a titanium alloy common to aerospace applications. The random fatigue limit model of Pascual and Meeker and two proposed simplified models (bilinear and hyperbolic) used maximum likelihood estimation techniques to fit probabilistic stress-life curves to experimental data. The bilinear and hyperbolic models provided a good fit to large-sample experimental data for dual-phase Ti-6Al-4V and were then applied to a small-sample data set for a beta annealed variant of this alloy, providing an initial probabilistic estimate of beta annealed Ti-6Al-4V fatigue strength in the gigacycle regime. The bilinear and hyperbolic models are recommended for use in estimating probabilistic fatigue strength parameters in support of very high-cycle design criteria for metals with clearly defined fatigue limits and fairly constant scatter in fatigue strength.

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1. Introduction

As military and civil systems are being used for longer and longer service lives, the quantification of high-cycle fatigue (HCF) behavior becomes ever more important to ensure safe and affordable operation of systems subject to cyclic loading. The United States Department of Defense has taken a very active role in addressing this issue with the formation of the National HCF Science and Technology Program in 1994, with specific emphasis on turbine-driven jet engines. The primary goal of this program was to further the understanding of HCF behavior and develop methods in order to mitigate the negative impact of HCF on aerospace operations. One of the outputs of this effort was an update to MIL-HDBK-1783B, the Engine Structural Integrity Program (ENSIP). Updated ENSIP guidance stated that all engine parts subjected to HCF should have a minimum life of 10^9 cycles, unless analysis or testing showed that this number of cycles is excessive for a particular component, in which case a lower threshold may be used [1].

This paper is a byproduct of the National HCF Science and Technology Program aimed at developing and analyzing various probabilistic means to characterize the statistical distribution of fatigue strength for alloys commonly used in turbine engines using experimental stress-life behavior. For components subject to HCF, the statistical distribution of the fatigue strength is a key design consideration in order to specify safe operating loads with acceptable risk of fatigue failure. Probabilistic characterization is thus necessary in order to evaluate a component's ability to withstand 10^9 cycles (or any other arbitrary design goal) with a quantifiable degree of risk.

The approach used was first to collect a large-sample set of stress-life data for a common titanium alloy (dual-phase Ti-6Al-4V). Next, the random fatigue limit (RFL) model was applied to this data set to model the probabilistic parameters. The RFL model was developed by Pascual and Meeker in 1999 [2]. Their method was "motivated by the need to develop and present quantitative fatigue-life information used in the design of jet engines" and utilizes maximum likelihood estimation (MLE) techniques. It was found that the RFL model, as presented in the literature, was insufficient in characterizing the dual-phase Ti-6Al-4V data adequately across the entire testing regime. Two other models were thus developed (a bilinear model and a hyperbolic model) in order to better represent the shape of the dual-phase Ti-6Al-4V data set, which is common to many engineering materials. These models also incorporated MLE methods and provided a good fit to the data, allowing a means to characterize the probability of failure of dual-phase Ti-6Al-4V at any given number of cycles over the range of experimental data.

The bilinear and hyperbolic MLE models were then considered for a small-sample data set using a second Ti-6Al-4V alloy, this

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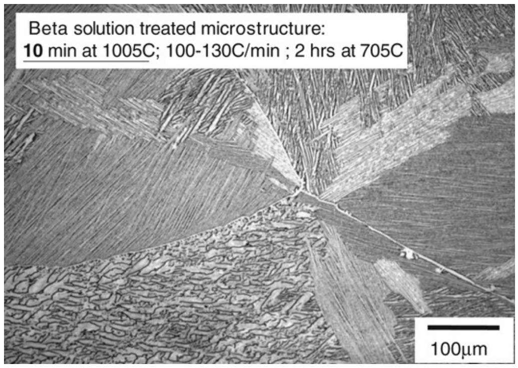

Fig. 1. Beta annealed Ti-6Al-4V microstructure.

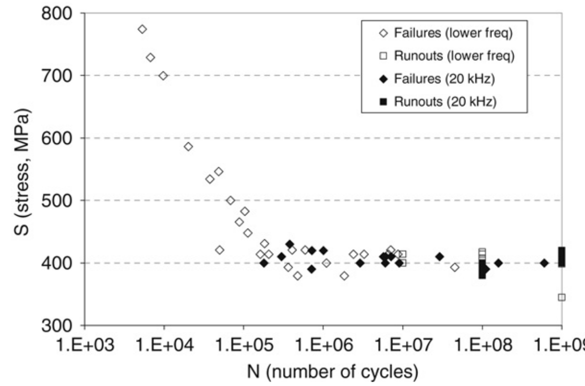

Fig. 2. Dual-phase Ti-6Al-4V fully-reversed fatigue data, collected through the National High Cycle Fatigue Science and Technology Program, 1996–2004.

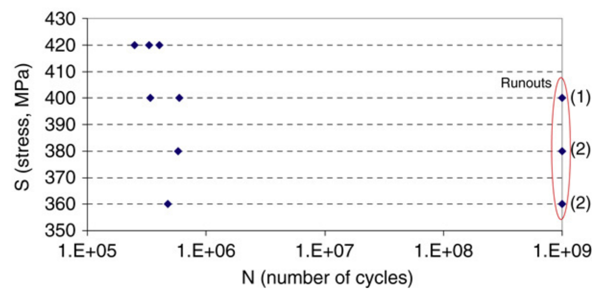

Fig. 3. Beta annealed Ti-6Al-4V fully-reversed fatigue data.

one a beta annealed variant with lamellar microstructure. The ability to characterize stress-life behavior using small-sample testing is important for very high-cycle regimes (such as the 10^9 cycle design goal used in the ENSIP guidance) as experiments take extraordinarily long times using conventional means, and even with a 20 kHz ultrasonic apparatus, a single 10^9 cycle test takes 14 h. Thus, small-sample data sets are common in this very high-cycle regime. Analysis of the small-sample test results suggested that the shape of the stress-life curves for the second Ti-6Al-4V alloy were reasonably similar to the dual-phase alloy. A probabilistic characterization of the beta annealed Ti-6Al-4V stress-life behavior was then accomplished using the bilinear and hyperbolic MLE models. This analysis resulted in an initial estimate of fatigue strength for beta annealed Ti-6Al-4V in the gigacycle regime, with confidence bands.

2. Experimental approach

The material used as a baseline large-sample data set was a dual-phase Ti-6Al-4V titanium alloy, commonly used in turbine engine components. The material was particularly attractive as much of the fatigue testing conducted by the National HCF Science and Technology Program was based on this alloy. The details of the material processing are provided in a number of papers [3–5], but will be summarized here from Morrissey and Nicholas [5]. The material was produced in accordance with Aerospace Material Specification 4928, being forged into flat plates of dimensions 406 mm by 150 mm by 20 mm (approximate). The forged plates were then solution heat treated at 932 °C for 1 h, vacuum annealed at 705 °C for 2 h, and then argon fan cooled. The resulting microstructure is of two-phase design, with approximately 60% by volume consisting of the primary α phase (hexagonally close-packed), with the remaining volume consisting of transformed β phase (body-centered cubic). Average grain size for this solution treated and overaged material was approximately 15–20 μm in each direction. Longitudinal tensile properties at room temperature in an ambient environment were $E = 116$ GPa (Young's modulus), $\sigma_y = 930$ MPa (yield strength), and $\sigma_{\text{UTS}} = 968$ MPa (ultimate tensile stress). For the beta annealed Ti-6Al-4V variant, additional processing of the dual-phase alloy was conducted in order to form a lamellar microstructure. Processing included a 10-min heat treatment at 1005 °C, followed by a rapid quenching at 100–130 °C per minute, and then a final annealing at 705 °C for two hours. The resulting microstructure is shown in Fig. 1. A comparison of dual-phase and beta annealed Ti-6Al-4V alloys using similar materials processing was accomplished by Nalla et al. [6].

All fatigue tests were conducted under fully-reversed loading conditions. About half (37 specimens) of the dual-phase fatigue tests were accomplished at frequencies less than 200 Hz, while the other dual-phase tests (31 specimens) and all of the beta annealed tests (12 specimens) were accomplished at 20 kHz using ultrasonic fatigue testing, based on the work of Mason [7] with modern application described by Bathias and Jingang [8]. Morrissey and Nicholas [5] describe the test setup in more detail. Gage sections for ultrasonic specimens were 6 mm long with a 4 mm diameter.

Experimental data collected for the dual-phase Ti-6Al-4V alloy are aggregated in Fig. 2. No frequency effect on stress-life behavior was observed. Fig. 3 shows the beta annealed Ti-6Al-4V test data.

3. Random fatigue limit (RFL) model

The genealogy of the RFL model traces its lineage through Nelson's work in 1984 [9] to generate probabilistic stress-life (S–N) curves incorporating non-constant standard deviation in fatigue life and using censored data (runouts) through an MLE approach. The RFL development also built upon the work of Hirose in 1993 [10]. Pascual and Meeker developed the RFL model in 1999 [2] to account for the two main trends in many S–N data sets; namely, the increase in fatigue life scatter as stress level is decreased, and the curvature associated with a fatigue limit. The model is shown in (1) using the notation of Annis and Griffiths [11] which is more conventional for fatigue analysis. The fatigue life for each specimen tested is denoted by N and the associated stress level is denoted by S. Fatigue life for specimen i is then modeled by the following equation:

$$\log(N_i) = \beta_0 + \beta_1 \log(S_i - \gamma_i) + \varepsilon_i, \quad S_i > \gamma_i. \tag{1}$$

In this equation, β_0 and β_1 are curve coefficients, γ_i is the fatigue limit of specimen i, ε_i is an error term associated with specimen i, and log denotes natural logarithm. Here, the fatigue limit used in (1) is a random variable. Note that the error term ε_i is the random life variable associated with scatter from specimens which have the same value for fatigue limit γ.

The logarithm of the random variable for fatigue limit γ is also a random variable, and if V is defined such that $V = \log(\gamma)$, then

Pascual and Meeker assume V to be distributed with probability density function (pdf) given by (2):

$$f_V(v,\mu_\gamma,\sigma_\gamma)=\frac{1}{\sigma_\gamma}\phi_V\left(\frac{v-\mu_\gamma}{\sigma_\gamma}\right). \quad (2)$$

In this equation, μ_γ and σ_γ are location and scale parameters for the distribution of γ, respectively, and ϕ_V may be the standardized smallest extreme value (sev) or normal pdf. Next, they let $x=\log(S)$ and $W=\log(N)$ so that x and W are the logarithms of the stress and fatigue life, respectively. Then for $V<x$ (i.e., the fatigue limit is less than the stress level tested), they assume that W given V (denoted as $W|V$) has a pdf of the form in (3):

$$f_{W|V}(w,\beta_0,\beta_1,\sigma,x,v) =\frac{1}{\sigma}\phi_{W|V}\left(\frac{w-[\beta_0+\beta_1\log(\exp(x)-\exp(v))]}{\sigma}\right). \quad (3)$$

In this equation, $\beta_0+\beta_1\log(\exp(x)-\exp(v))$ acts as a location parameter and σ acts as a scale parameter. $\phi_{W|V}$ may be the standardized sev or normal pdf. The marginal pdf of W is then given by (4):

$$f_W(w;x,\theta)=\int_{-\infty}^{x}\frac{1}{\sigma\sigma_\gamma}\phi_{W|V}\left(\frac{w-\mu(x,v,\theta)}{\sigma}\right)\phi\left(\frac{v-\mu_\gamma}{\sigma_\gamma}\right)\mathrm{d}v \quad (4)$$

where $\theta=(\beta_0,\beta_1,\sigma,\mu_\gamma,\sigma_\gamma)$ and $\mu(x,v,\theta)=\beta_0+\beta_1\log(\exp(x)-\exp(v))$. Finally, the marginal cumulative distribution function (cdf) of W (the logarithm of fatigue life) is given by (5):

$$F_W(w;x,\theta)=\int_{-\infty}^{x}\frac{1}{\sigma_\gamma}\Phi_{W|V}\left(\frac{w-\mu(x,v,\theta)}{\sigma}\right)\phi_V\left(\frac{v-\mu_\gamma}{\sigma_\gamma}\right)\mathrm{d}v \quad (5)$$

where $\Phi_{W|V}$ is the cdf of W given V. Pascual and Meeker note that there are no closed-form solutions for the density and distribution functions of the fatigue life, or specifically, $W=\log(N)$. However, numerical means can be used to evaluate these equations.

It is important to note that there are two random variables in the model described by (1) through (5) which have been specified through a probability distribution. The error term ε which represents the scatter in fatigue life can be adequately modeled by the lognormal distribution for many engineering materials (and thus, the logarithm of fatigue life is normal). Then, the conditional distribution for cycles to failure ($W=\log(N)$) given γ ($V=\log(\gamma)$) will be a lognormal distribution with mean $\beta_0+\beta_1\log(S-\gamma)$ and standard deviation σ_ε, such that ε is lognormal$(0,\sigma_\varepsilon)$ [11]. As for the distribution of the random variable γ, the Weibull distribution is an adequate choice for describing the skewed downward (towards lower stress levels) strength distribution of many engineering materials [11]. The Weibull distribution introduces two parameters, namely the location parameter η and the scale parameter β, which correspond to the location and scale parameters (μ_γ and σ_γ) used by Pascual and Meeker. When the RFL model incorporates these assumptions, it includes five total parameters (β_0, β_1, σ_ε, η, and β).

4. RFL model applied to dual-phase Ti-6Al-4V data

When the RFL model was applied to the 68 dual-phase data points, the best-fit (maximum likelihood) parameters were as shown in Table 1. This fit resulted in the probabilistic stress-life characterization shown in Fig. 4. In this case, the RFL model did not adequately characterize the discontinuous change in slope for the experimental data points. An alternative approach was used to better model this stress-life behavior.

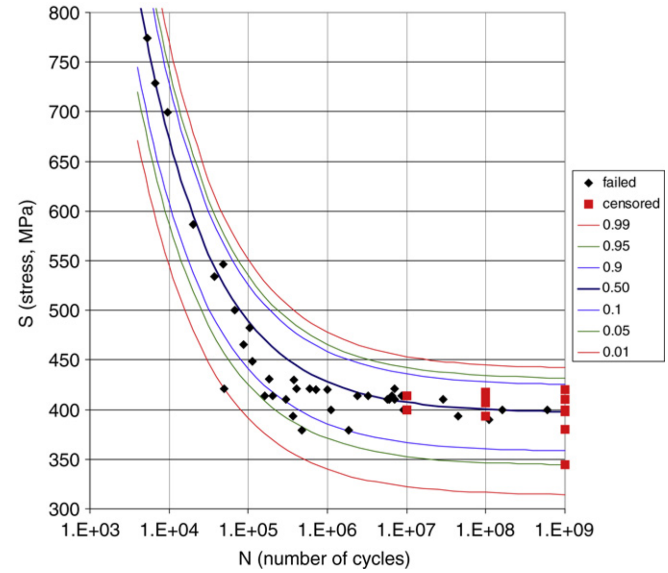

Fig. 4. Random fatigue limit model for dual-phase Ti-6Al-4V data.

Table 1
Best-fit random fatigue limit model parameters for dual-phase Ti-6Al-4V

Parameter	Descriptor	Value
β_0	S–N curve coefficient	4.950
β_1	S–N curve coefficient	−2.110
σ_ε	Standard deviation in lognormal fatigue life	0.16
η	Weibull location parameter for fatigue limit	405
β	Weibull scale parameter for fatigue limit	18

5. Alternative model shapes

Two simplified shapes were used to better represent the Ti-6Al-4V data. The first shape was termed the "bilinear" model, and assumes a constant slope for the S–N curve at lower cycles, followed by a transition to a horizontal fatigue limit at a specified number of cycles. Thus, there are three S–N model parameters: (a) m, the slope of the curve at lower cycles, which is a negative number and expressed in units of stress/log(cycles), (b) FLS, the fatigue limit strength expressed in units of stress, and (c) N^*, the number of cycles at which the curve transitions from sloped to horizontal. The model is thus specified by (6) and (7).

$$S=-m\cdot(\log N^*-\log N)+FLS,\quad \text{for } N<N^* \quad (6)$$

$$S=FLS,\quad \text{for } N\geq N^*. \quad (7)$$

The second shape was based on an adaptation of the Nishijima hyperbolic S–N curve [12] as described by Hanaki et al. [13]. The model is formulated in (8), where A, B, C, and E are parameters which may be varied to create S–N curves ranging from a curved form similar to that exhibited by the RFL model to the more linear form represented by the bilinear model, as shown in Fig. 5.

$$(S-E)(S-A\log N-B)=C. \quad (8)$$

6. Modeling the fatigue strength distribution

The bilinear model was applied to the dual-phase Ti-6Al-4V data. The residuals of the best fit were adequately represented by an extreme value distribution (also known as Fisher–Tippett or log-Weibull). Since no discernible increase in fatigue strength scatter as a function of the number of cycles was observed, the scatter in fatigue strength was modeled as a constant across the test regime. The extreme value distribution is governed by two

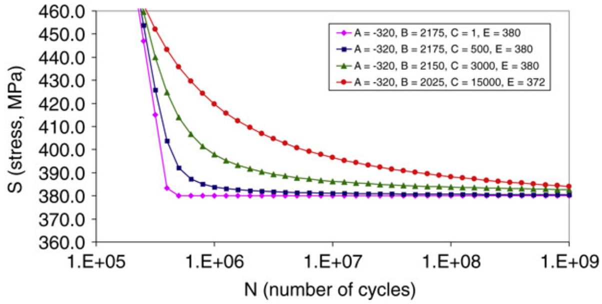

Fig. 5. Sample stress-life (S–N) curves for the Nishijima hyperbolic model.

parameters (location parameter α and scale parameter β) with pdf and cdf shown in (9) and (10).

$$P(x) = \frac{e^{\left(\left(\frac{\alpha-x}{\beta}\right)-e^{\left(\frac{\alpha-x}{\beta}\right)}\right)}}{\beta} \text{ (pdf)} \tag{9}$$

$$D(x) = e^{-e^{\left(\frac{\alpha-x}{\beta}\right)}} \text{ (cdf).} \tag{10}$$

When using this distribution for fatigue strength of the Ti-6Al-4V data, two considerations were made. First, the pdf as defined by (9) has a longer tail in the $+x$ direction. However, residual analysis of the experimental data suggests that the fatigue strength distribution is skewed downwards towards lower values of stress. Thus, the values from the extreme value distribution were subtracted from (not added to) the S–N baseline model in order to represent the fatigue strength scatter about the S–N curve. The second consideration concerns the location parameter α. There was no need to use a location parameter if the baseline S–N curve was modeled as the mode and thus corresponds to the peak in fatigue strength pdf, since the extreme value pdf with $\alpha = 0$ has a peak at $x = 0$. Thus, the fatigue strength distribution when modeled by the extreme value distribution is dependent on only one parameter: the scale parameter β. With the fatigue strength distribution modeled in this manner, there are now four parameters for the bilinear model (m, FLS, N^*, and β) and five parameters for the hyperbolic model (A, B, C, E, and β).

7. Maximum likelihood method for model fitting

To determine the best fit for the bilinear and hyperbolic models, a maximum likelihood approach was used. Each test point from the fatigue testing involves three pieces of data: (1) S_i, the stress level, (2) log N_i, the logarithm (base 10) of the number of cycles tested, and (3) δ_i, a delta function which equals 1 if the specimen failed and 0 if the specimen did not fail (also known as a runout, or censored data). The number of specimens tested is denoted by n. The model parameters are denoted by θ. Thus, for the bilinear model, $\theta = (m, FLS, N^*, \beta)$, and for the hyperbolic model $\theta = (A, B, C, E, \beta)$. Each data point also has a corresponding $\hat{S}_i$ which represents the point on the modeled S–N curve corresponding to log N_i. Thus, given θ and log N_i, Eqs. (6) and (7) are used to determine $\hat{S}_i$ for each test point using the bilinear model, and Eq. (8) is used for the hyperbolic model. Then, x_i is defined by (11).

$$x_i = -\frac{S_i - \hat{S}_i}{\beta}. \tag{11}$$

Thus, x_i represents a scaled residual between the true value of stress as tested (S_i) and the corresponding point on the modeled S–N curve given the specified parameters ($\hat{S}_i$). It is scaled by β, the scale parameter of the extreme value distribution representing fatigue strength scatter. The minus sign in (11) is used because a stress value greater than its modeled companion represents a negative residual since the extreme value distribution is positively skewed downwards to lower stress values.

With the problem as defined thus far, the maximum likelihood method uses a likelihood function of the form in (12).

$$L(\theta) = \prod_{i=1}^{n} [f_S(x_i, \theta)]^{\delta_i} [1 - F_S(x_i, \theta)]^{1-\delta_i} \tag{12}$$

where

$$\begin{aligned} f_S &= \text{fatigue strength pdf} = e^{z-e^z} \\ F_S &= \text{fatigue strength cdf} = e^{-e^z} \end{aligned} \tag{13}$$

and

$$z_i = -\frac{x_i}{\beta}. \tag{14}$$

By the property of logarithms, the likelihood function can be maximized by maximizing its logarithm, so that the log-likelihood function in (15) is used.

$$\mathcal{L}(\theta) = \log[L(\theta)] = \log\left[\prod_{i=1}^{n} L_i(\theta)\right] = \sum_{i=1}^{n} \mathcal{L}_i(\theta) \tag{15}$$

where

$$\mathcal{L}_i(\theta) = \delta_i \log[f_S(x_i, \theta)] + (1 - \delta_i) \log[1 - F_S(x_i, \theta)]. \tag{16}$$

With this formulation in place, the test data can be plotted and initial estimates for the S–N baseline model parameters and distribution scale factor can be made based on a reasonable fit. The log-likelihood function is then maximized by methodically adjusting these parameters until improvements to the fit are no longer possible (or gains are so marginal that the fit is considered "good enough").

8. Analysis of dual-phase Ti-6Al-4V data using bilinear and hyperbolic models

The 68 data points associated with the fully-reversed dual-phase tests were represented by the bilinear model with the extreme value distribution for fatigue strength. The best-fit parameter settings were $\theta = (m, FLS, N^*, \beta) = (-227, 418, 1.80 \cdot 10^5, 13.5)$. Using the results of this analysis, a P–S–N curve (P for probability) can be drawn based on the percentiles of the fatigue strength distribution at each given number of cycles. This P–S–N curve is shown in Fig. 6. If one looks at the failure points only, 34 of the 42 (81.0%) failure points lie within the 10th and 90th percentiles (an 80% band). Likewise, 39 of the points (92.9%) lie within the 5th and 95th percentiles (90% band). Finally, 41 of the points (97.6%) lie within the 1st and 99th percentiles (98% band). Thus, the percentile bands match well with the experimental data.

The hyperbolic model was also applied to the dual-phase Ti-6Al-4V data. Likelihood values showed the hyperbolic best fit was slightly worse than the bilinear best fit for this data set. The P–S–N plot using the hyperbolic model is shown in Fig. 7.

A comparison between the outputs for each model as applied to the dual-phase Ti-6Al-4V data is shown in Table 2. The relatively poor fit using the RFL model resulted in overly conservative estimates for the fatigue strength distribution, with 99% lower bounds at stress levels 12% lower than those calculated using the bilinear model. Although conservatism in design may have advantages, conservatism due to poor model fit is an inefficient use of the available trade space.

Table 2
Comparison of fatigue strength probabilities at 10^9 cycles using the random fatigue limit, bilinear, and hyperbolic models for dual-phase Ti-6Al-4V

Fatigue strength at 10^9 cycles (in MPa)	Random fatigue limit model	Bilinear with extreme value dist'n	Hyperbolic with extreme value dist'n
Median	398	413	410
90% lower bound	359	388	380
95% lower bound	344	378	369
99% lower bound	315	356	343

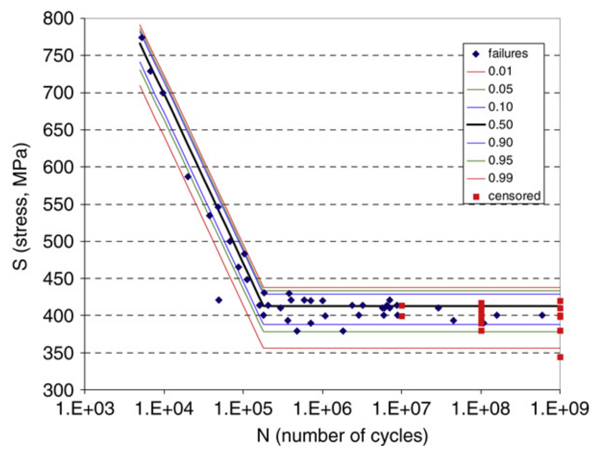

Fig. 6. Bilinear model for dual-phase Ti-6Al-4V data.

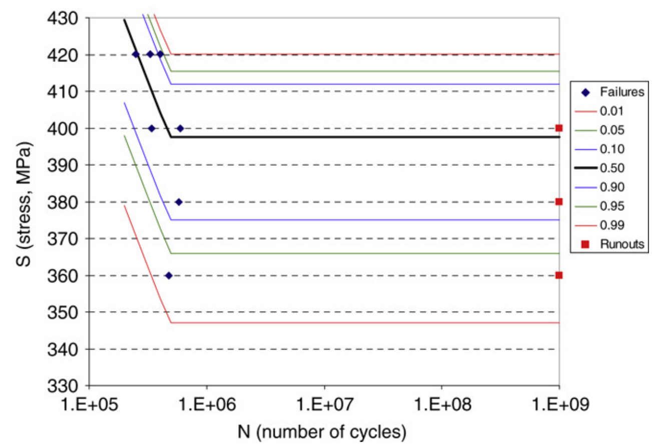

Fig. 8. Bilinear model for beta annealed Ti-6Al-4V data.

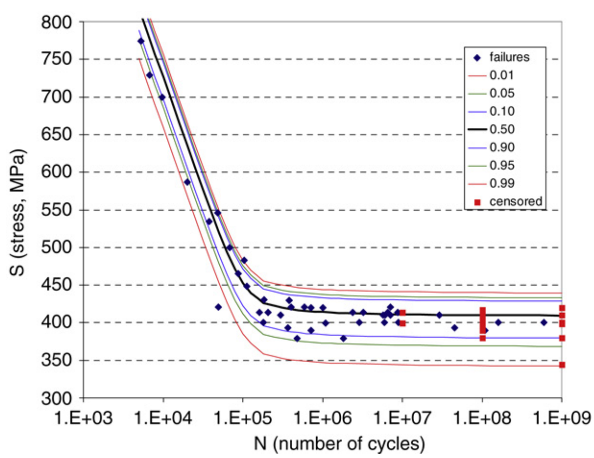

Fig. 7. Hyperbolic model for dual-phase Ti-6Al-4V data.

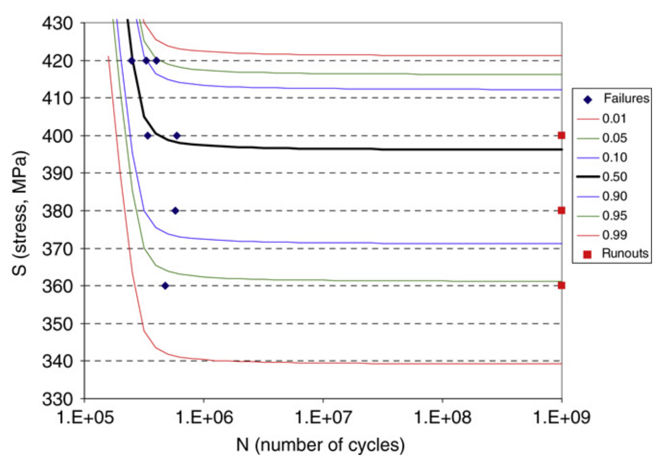

Fig. 9. Hyperbolic model for beta annealed Ti-6Al-4V data.

9. Analysis of beta annealed Ti-6Al-4V fatigue strength at 10^9 cycles

The simplified bilinear and hyperbolic models provided a better fit to the dual-phase Ti-6Al-4V data when compared to the more complex RFL model. Data from the Materials Property Handbook [14] suggested that lamellar Ti-6Al-4V (such as beta annealed microstructure) S–N curve shapes are similar in shape to those of the dual-phase alloy. Thus, the beta annealed data set was also analyzed using the bilinear and hyperbolic models in order to characterize the fatigue strength at 10^9 cycles.

Using the same approach as conducted for the dual-phase Ti-6Al-4V data, the best-fit parameter settings for the bilinear model when applied to the beta annealed data set depicted in Fig. 3 were: $m = -84$ MPa/log(cycle), $FLS = 402$ MPa, $N^* = 4.8 \times 10^5$ cycles, and $\beta = 12.1$. The P–S–N plot based on this model fit is shown in Fig. 8. Based on this fit, the median fatigue strength at 10^9 cycles is 398 MPa, with a 95% lower bound of 366 MPa. Likewise, the best-fit hyperbolic model resulted from parameter settings: $A = -325, B = 2170, C = 250, E = 401$, and $\beta = 13.5$. The P–S–N plot based on this model fit is shown in Fig. 9. This fit yielded a median fatigue strength at 10^9 cycles equal to 396 MPa, with a 95% lower bound at 361 MPa. The RFL model was also applied to this data set, as shown in Fig. 10. The model did not adequately account for the horizontal fatigue limit and was again overly conservative.

10. Summary and recommendations

This paper analyzed a simplified MLE approach (relative to the RFL model) for stress-life curves exhibiting a fatigue limit using bilinear and hyperbolic curve shapes. The approach allowed a probabilistic characterization of the fatigue behavior of beta annealed Ti-6Al-4V under fully-reversed loading at very high cycles using a 12-sample experiment. The approach was validated using a larger data set of dual-phase Ti-6Al-4V fatigue tests. The simplified bilinear and hyperbolic models are recommended for materials which exhibit a sharp transition between constant slope and horizontal fatigue limit behavior with relatively constant fatigue strength scatter over the testing regime. At longer lives, such materials may exhibit a bimodal fatigue limit due to two

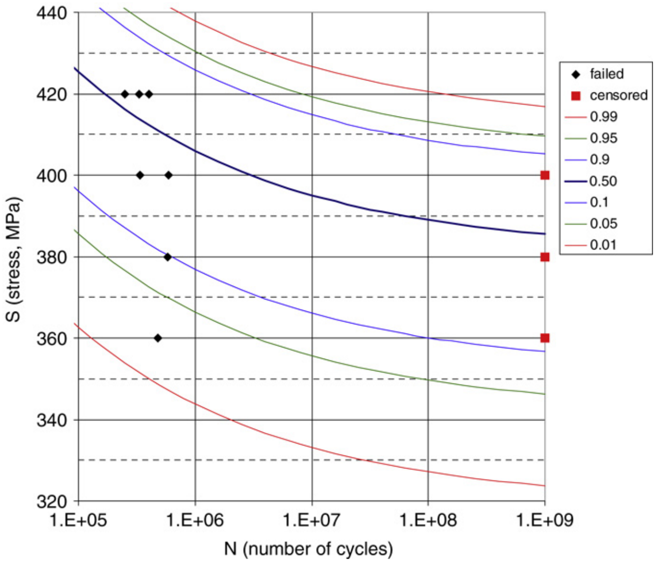

Fig. 10. Random fatigue limit model for beta annealed Ti-6Al-4V data.

failure mechanisms. In such cases, probabilistic stress-life curves may be generated by overlaying two bilinear or hyperbolic models rather than developing a more complex dual-mode RFL model. The improved data fits provided a less conservative estimate of fatigue strength scatter, allowing more flexibility in the design trade space.

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References

[1] Aeronautical Systems Center. Engine structural integrity program. MIL-HDBK-1783B, Wright-Patterson AFB, OH. 2002.
[2] Pascual FG, Meeker WQ. Estimating fatigue curves with the random fatigue-limit model. Technometrics 1999;41:277–89.
[3] Mall S, Nicholas T, Park TW. Effect of predamage from low cycle fatigue on high cycle fatigue strength of Ti-6Al-4V. Int J Fatigue 2003;25:1109–16.
[4] Moshier MA, Nicholas T, Hillberry BM. High cycle fatigue threshold in the presence of naturally initiated small surface cracks. Fatigue Fracture Mech 2002;33:129–46.
[5] Morrissey RJ, Nicholas T. Staircase testing of a titanium alloy in the gigacycle regime. In: Proc of the 3rd int conf on very high cycle fatigue. 2004.
[6] Nalla RK, Boyce BL, Campbell JP, Peters JO, Ritchie RO. Influence of microstructure on high-cycle fatigue of Ti-6Al-4V: Bimodal vs. lamellar structures. Metallurgical Mater Trans 2002;33A:899–906.
[7] Mason WP. Piezoelectric crystals and their application in ultrasonics. New York: Van Nostrand; 1950.
[8] Bathias C, Jingang N. Determination of fatigue limit between 10^5 and 10^9 cycles using an ultrasonic fatigue device. In: Advances in fatigue lifetime predictive technique: Second volume. American Society for Testing and Materials; 1993. STP 1211.
[9] Nelson WB. Fitting of fatigue curves with nonconstant standard deviation to data with runouts. J Testing Eval 1984;12:69–77.
[10] Hirose H. Estimation of threshold stress in accelerated life-testing. IEEE Trans Reliability 1993;42:650–7.
[11] Annis C, Griffiths J. Staircase testing and the random fatigue limit. In: 6th national turbine engine high-cycle fatigue conf. 2001 [presented].
[12] Nishijima S. Trans Japanese Soc Mech Eng 1984;50:451.
[13] Hanaki S, Yamashita M, Uchida H, Zako M, Kurashiki T. On decision of *S*–*N* curve based on evaluation of fatigue strength distribution in ultra high cycle fatigue region. In: Proc of the 3rd int conf on very high cycle fatigue. 2004.
[14] Boyer R, Welsch G, Collings EW, editors. Material properties handbook. Materials Park (OH): ASM International; 1994.
[15] Pollak RD. Analysis of methods for determining high cycle fatigue strength of a material with investigation of Ti-6Al-4V gigacycle fatigue behavior. Ph.D. dissertation, Air Force Institute of Technology; 2005.