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Choosing weight functions in iterative methods for simple roots

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ABSTRACT

Weight functions with a parameter are introduced into an iteration process to increase the order of the convergence and enhance the behavior of the iteration process. The parameter can be chosen to restrict extraneous fixed points to the imaginary axis and provide the best basin of attraction. The process is demonstrated on several examples.

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1. Introduction

There is a vast literature on the solution of nonlinear equations and nonlinear systems, see for example Ostrowski [1], Traub [2], Neta [3] and the recent book by Petković et al. [4] and references therein. Most of the algorithms are for finding a simple root of a nonlinear equation $f(x) = 0$, i.e., for a root α we have $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Methods are generally compared by their efficiency index, defined by

$$I = p^{1/d}, \tag{1}$$

where p is the order of convergence and d is the number of function- (and derivative-) evaluation per step. For example, the well-known Newton's method given by

$$x_{n+1} = x_n - u_n, \tag{2}$$

where

$$u_n = \frac{f(x_n)}{f'(x_n)}, \tag{3}$$

is of second order and requires the computation of f and f' and, thus, its efficiency index is $I = \sqrt{2} = 1.4142$.

Since Newton's Method is of second order, there has been considerable interest in developing iteration procedures that are of higher order. Most of these procedures center on the modification of Newton's method by using more terms in the Taylor Series, which requires the use of higher derivatives or by multipoint methods to approximate the derivative. Both of these techniques have certain drawbacks. The first one can be used if the function is sufficiently differentiable and the differentiation can be performed easily. It is very difficult to develop a general purpose code based on this approach. Thus, most research uses the multipoint approach, which also has a few issues. In solving a nonlinear equation iteratively, we are seeking the fixed points, which are zeros of the given nonlinear function. The multipoint iteration is the preferred approach. Unfortunately, many multipoint iteration methods have fixed points that are not zeros of the function of interest. These points are called extraneous (or free) fixed points (see [5]). The extraneous points could be attractive, which will trap an

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iteration sequence and give erroneous results. Even if the extraneous fixed points are repulsive or indifferent, they can complicate the situation by converging to a root not close to the initial guess. One of the most successful multipoint methods was developed by Jarratt [6] is given by

$$\begin{aligned} y_n &= x_n - \frac{2}{3}u_n, \\ x_{n+1} &= x_n - J_f(x_n)u_n, \end{aligned} \tag{4}$$

where

$$J_f(x_n) = \frac{3f'(y_n) + f'(x_n)}{6f'(y_n) - 2f'(x_n)}, \tag{5}$$

and u_n is given by (3). This method is of order 4 and the error relation is given by

$$\epsilon_{n+1} = (c_2^3 - c_2c_3 + c_4/9)\epsilon_n^4 + O(\epsilon_n^5), \tag{6}$$

where

$$c_i = \frac{f^{(i)}(\alpha)}{i!f'(\alpha)}. \tag{7}$$

In this paper, we introduce parametric weight functions to generalize Jarratt's method to increase the order of convergence to six. The weight functions and their parameter are chosen to increase the order of convergence and to restrict the extraneous fixed points to the imaginary axis.

The sixth order method is given by:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}u_n, \\ s_n &= x_n - q(t_n)u_n, \\ x_{n+1} &= s_n - \frac{f(s_n)}{f(x_n)}w(t_n)u_n, \end{aligned} \tag{8}$$

where $t_n = f'(y_n)/f'(x_n)$. There is flexibility in choosing the weight functions $q(t)$ and $w(t)$ and, in fact, one can find several choices in the literature. We choose

$$q(t) = \frac{3t+1}{6t-2} \tag{9}$$

which matches the first two steps of (4). It is of order 6 if

$$q(1) = 1, \quad q'(1) = -3/4, \quad q''(1) = 9/4, \tag{10}$$

and

$$w(1) = 1, \quad w'(1) = -3/2, \quad |w''(1)| < \infty. \tag{11}$$

Clearly one can choose various forms of the function $w(t)$ as long as we satisfy the relations (11). Kou and Li [7] suggested the following method

$$\begin{aligned} y_n &= x_n - \frac{2}{3}u_n, \\ s_n &= x_n - J_f(x_n)u_n, \\ x_{n+1} &= s_n - \frac{f(s_n)}{\frac{3}{2}J_f(x_n)f'(y_n) + (1 - \frac{3}{2}J_f(x_n))f'(x_n)}. \end{aligned} \tag{12}$$

Petković et al. [4] have shown that method (12) is a special case of (8) for the specific choice of

$$w(t) = \frac{4(3t-1)}{9t^2 + 6t - 7}. \tag{13}$$

Therefore, we use a rational function for $w(t)$ in the form

$$w(t) = \frac{1-3t}{a + bt + ct^2}. \tag{14}$$

Since $w(t)$ should satisfy the relations (11), $a = 4 + c$ and $b = -6 - 2c$. Therefore, the family of weight functions in the third sub-step satisfying (11) is given by

$$w(t) = \frac{1-3t}{4 + c - (6+2c)t + ct^2}. \tag{15}$$

Note that choosing $c = -9/4 = -2.25$ gives Kou–Li's method (12).

The error relation is given by

$$\epsilon_{n+1} = -c_3\left(c_2^3 - c_2c_3 + c_4/9\right)\epsilon_n^6 + O\left(\epsilon_n^7\right). \tag{16}$$

The efficiency index of this method is $I = 6^{1/4} = 1.565$.

In the next two sections, we analyze the basin of attraction of our sixth order family of methods to find out what is the best choice for c. The idea of using basins of attraction was initiated by Stewart [8] and followed by the works of Amat et al. [9–12], Scott et al. [14], Chun et al. [13], Chicharro et al. [15], and Cordero et al. [16]. The only papers comparing basins of attraction for methods to obtain multiple roots is due to Neta et al. [17] and Neta and Chun [18].

2. Corresponding conjugacy maps for quadratic polynomials

Given two maps f and g from the Riemann sphere into itself, an analytic conjugacy between the two maps is a diffemorphism h from the Riemann sphere onto itself such that $h \circ f = g \circ h$. Here we consider only quadratic polynomials.

Theorem 2.1. *For a rational map $R_p(z)$ arising from method (8) with q given by (9) and w given by (15) applied to $p(z) = (z-a)(z-b),\ a \neq b, R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = \frac{-9z^2 + 18 + 8c}{(18 + 8c)z^2 - 9}z^6. \tag{17}$$

Proof. Let $p(z) = (z-a)(z-b),\ a \neq b$ and let M be the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \frac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p\left(\frac{ub-a}{u-1}\right) = \frac{-9u^2 + 18 + 8c}{(18 + 8c)u^2 - 9}u^6. \tag{18}$$

As a special case we see that for Kou–Li's method where $c = -9/4$, we have

$$S(u) = u^8 \tag{19}$$

and for the case that $c = -9/8$ we have

$$S(u) = -u^6. \tag{20}$$

□

3. Extraneous fixed points

As mentioned earlier, in solving a nonlinear equation iteratively we are looking for fixed points which are zeros of the given nonlinear function. Many multipoint iterative methods have fixed points that are not zeros of the function of interest. Thus, it is imperative to investigate the number of extraneous fixed points, their location and their properties. In the method described in this paper, the parameter c can be chosen to position the extraneous points on the imaginary axis.

The sixth order methods discussed here can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}H_f(x_n), \tag{21}$$

where

$$H_f(x_n) = q(t_n) + w(t_n)\frac{f(s_n)}{f(x_n)}. \tag{22}$$

Clearly the root α of $f(x)$ is a fixed point of the method. The points $\xi \neq \alpha$ at which $H_f(\xi) = 0$ are also fixed points of the family, since the second term on the right vanishes.

We have tried several possibilities for the function w and have computed the extraneous fixed points. One would like to have the extraneous fixed points on the imaginary axis which is the boundary between the two roots of the quadratic polynomial. The one we found is given by (15).

First, it is easy to see that $w(1) = 1$ and $w'(1) = -3/2$. When q given by (9) and w given by (15), H_f is given by

$$H_f(z) = \frac{(24c - 9)z^6 - (40c + 405)z^4 + (8c - 171)z^2 + 8c + 9}{16(z^2 + 1)(cz^4 - (2c + 18)z^2 + c)}. \tag{23}$$

Table 1
The six extraneous fixed points for selected values of c.

c	Root 1	Root 2	Root 3
−3.53	±0.335730006160866i	±1.13506250403634i	±1.18898522578128i
−3	±0.2965542022i	±1.621694154i	±0.8948084675i
−2.25	±0.228243474390150i	±0.797473388882404i	±2.07652139657234i
−1.125	0	±0.726763461957526i	±3.07129173294565i

The extraneous fixed points are functions of c. We have searched values of the parameter c so that the extraneous fixed points are on the imaginary axis. We found that $-3.54 < c < -1.12$. Note that for Kou–Li's method $c = -9/4 = -2.25$ is in the interval. In Table 1 we give a list of the extraneous fixed points for selected values of the parameter c, namely $c = -3.53, -3, -9/4, -9/8$. All these points lie on the imaginary axis.

In the next section we plot the basins of attraction for these four cases to find the best performer.

4. Numerical experiments

We have used the four members of the family of methods for six different polynomials.

Example 1. In our first example, we have taken the polynomial to be

$$p_1(z) = z^2 - 1, \tag{24}$$

whose roots $z = \pm 1$ are both real. The results are presented in Figs. 1–4. The darker the shade in each basin, the slower the convergence to the root. Thus at the root the color is white and the shades get darker the more iteration one requires. Therefore, at black points the method did not convergence within 40 iterations. We rate the method qualitatively by looking at the basins of attraction. Based on Figs. 1–4, we can conclude that, for Example 1, Kou–Li's method was best (given a code of 1) and the worst (given a code of 4) is the one with the smallest c value, i.e. $c = -3.53$ (see Fig. 1).

Example 2. In the second example we have taken a cubic polynomial with the 3 roots of unity, i.e.

$$p_2(z) = z^3 - 1. \tag{25}$$

The results are given in Figs. 5–8. Now Kou–Li's method (Fig. 7) is worst as can be seen by the black regions indicating no convergence after 40 iterations. The best method is for $c = -3$ (Fig. 6) followed by the one with $c = -3.53$ (Fig. 5).

Example 3. In the third example we have taken a polynomial of degree 4 with 4 real roots at $\pm 1, \pm 3$, i.e.

$$p_3(z) = z^4 - 10z^2 + 9. \tag{26}$$

The results are given in Figs. 9–12. In this case $c = -3.53$ (Fig. 9) gave the best results followed by Kou–Li (Fig. 11) and the worst is for $c = -3$ (Fig. 10).

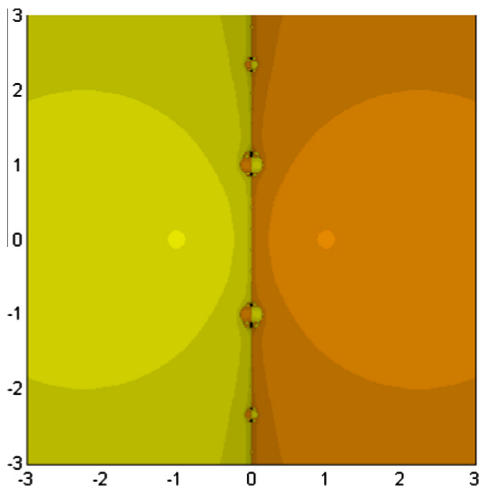

Fig. 1. Our method with $c = -3.53$ for the roots of the polynomial $z^2 - 1$.

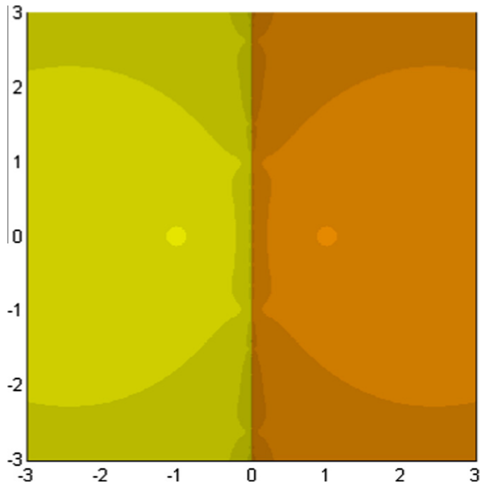

Fig. 2. Our method with $c = -3$ for the roots of the polynomial $z^2 - 1$.

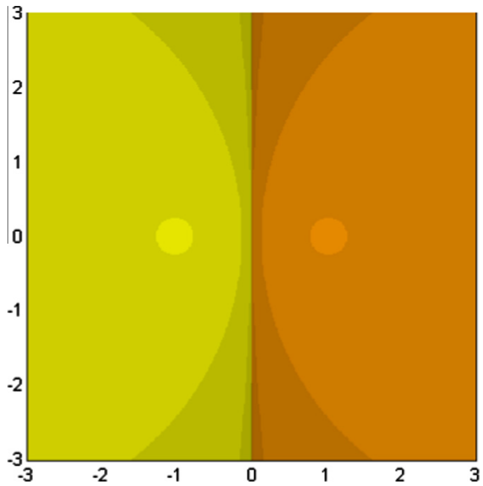

Fig. 3. Kou–Li's method with $c = -9/4$ for the roots of the polynomial $z^2 - 1$.

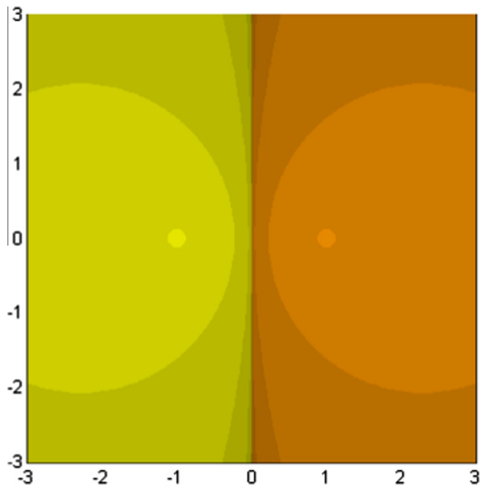

Fig. 4. Our method with $c = -1.125$ for the roots of the polynomial $z^2 - 1$.

Example 4. In the next example we have taken a polynomial of degree 5 with the 5 roots of unity, i.e.

$$p_4(z) = z^5 - 1. \tag{27}$$

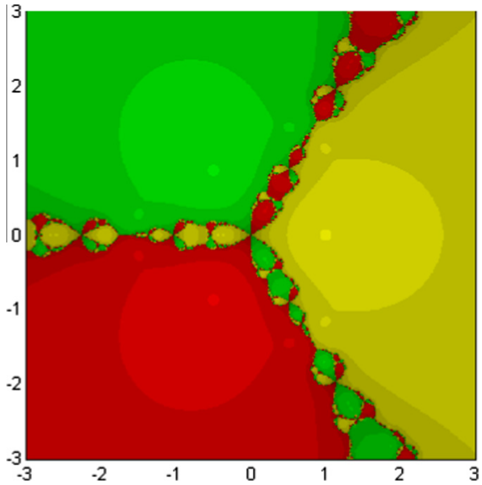

Fig. 5. Our method with $c = -3.53$ for the roots of the polynomial $z^3 - 1$.

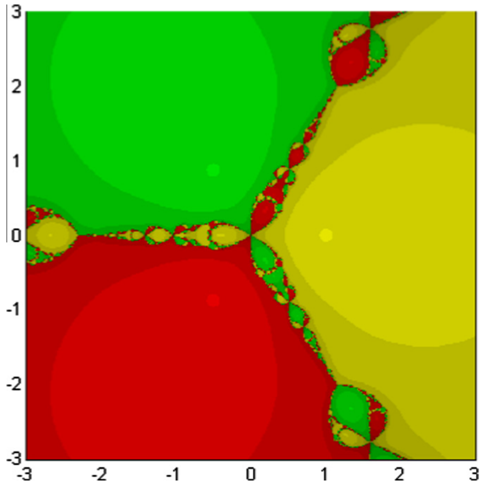

Fig. 6. Our method with $c = -3$ for the roots of the polynomial $z^3 - 1$.

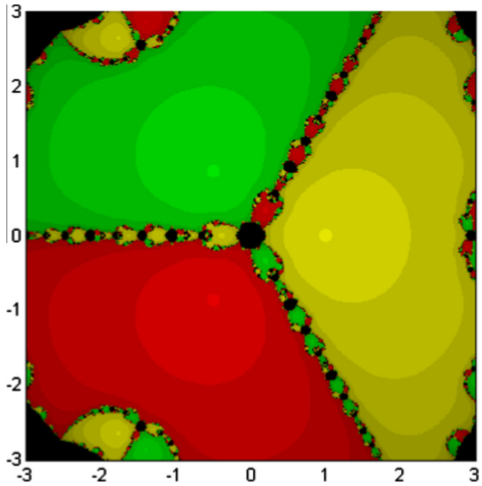

Fig. 7. Kou–Li's method with $c = -9/4$ for the roots of the polynomial $z^3 - 1$.

The results are given in Figs. 13–16. In this case $c = -3.53$ (Fig. 13) again gave the best results followed by $c = -1.125$ (Fig. 16). The worst results are given by Kou–Li (Fig. 15).

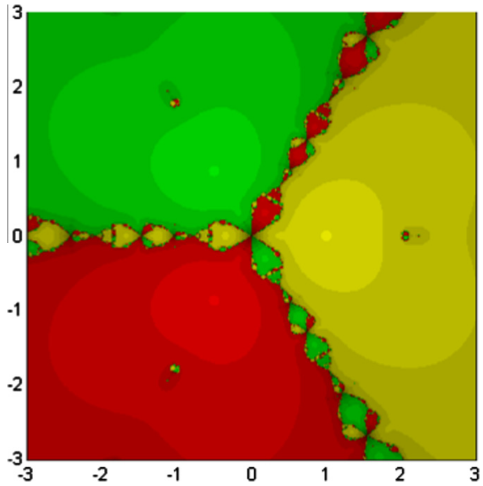

Fig. 8. Our method with $c = -1.125$ for the roots of the polynomial $z^3 - 1$.

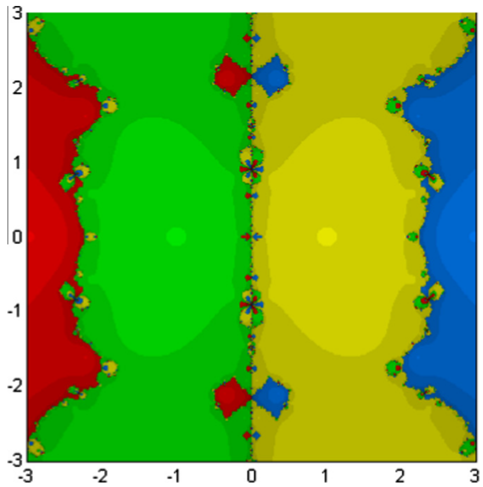

Fig. 9. Our method with $c = -3.53$ for the roots of the polynomial $z^4 - 10z^2 + 9$.

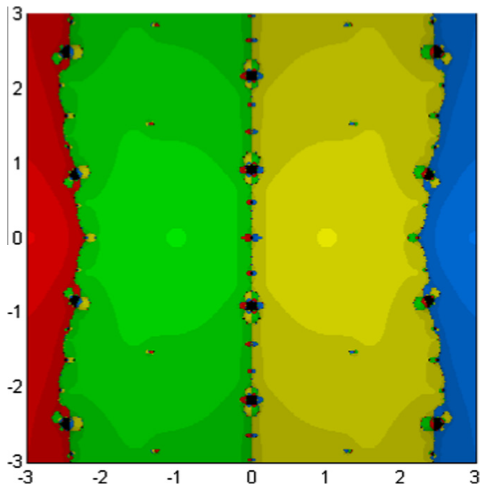

Fig. 10. Our method with $c = -3$ for the roots of the polynomial $z^4 - 10z^2 + 9$.

Example 5. In the next example we took a sixth order polynomial with complex coefficients and complex roots, i.e.

$$p_5(z) = z^6 - \frac{1}{2}z^5 + \frac{11}{4}(i+1)z^4 - \left(\frac{3}{4}i - \frac{19}{4}\right)z^3 + \left(\frac{5}{4}i + \frac{11}{4}\right)z^2 - \left(\frac{1}{4}i + \frac{11}{4}\right)z + \frac{3}{2} - 3i. \tag{28}$$

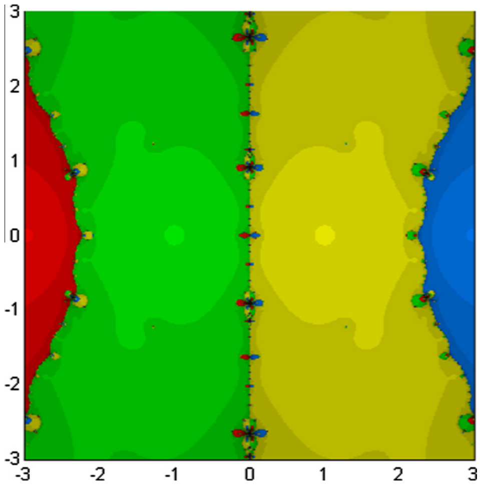

Fig. 11. Kou–Li's method with $c = -9/4$ for the roots of the polynomial $z^4 - 10z^2 + 9$.

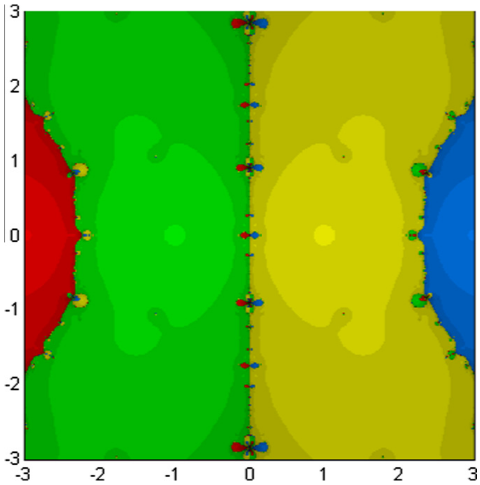

Fig. 12. Our method with $c = -1.125$ for the roots of the polynomial $z^4 - 10z^2 + 9$.

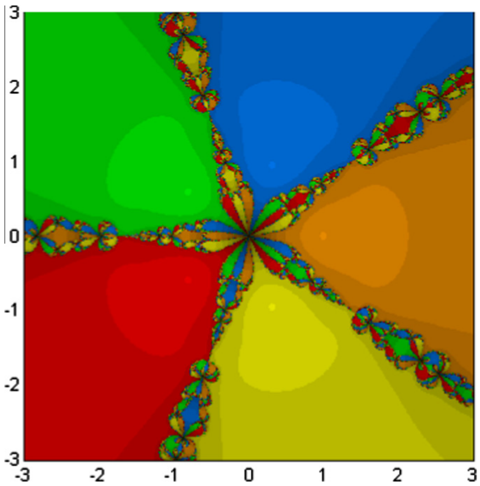

Fig. 13. Our method with $c = -3.53$ for the roots of the polynomial $z^5 - 1$.

The roots are: $z = -1. + 2i, -\frac{1}{2}(1+i), -\frac{3}{2}i, i, 1, 1-i$. The results are given in Figs. 17–20. Now the best methods are with the largest c, i.e. $c = -1.125$ (Fig. 20). The worst is the one with the smallest c, i.e. $c = -3.53$ (Fig. 17).

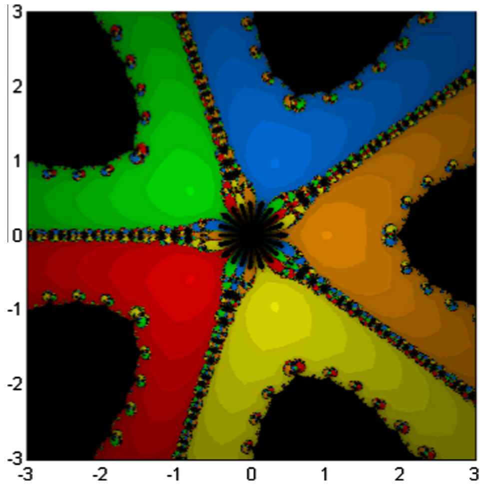

Fig. 14. Our method with $c = -3$ for the roots of the polynomial $z^5 - 1$.

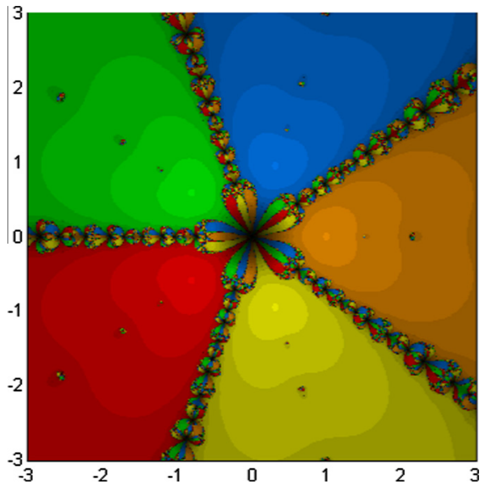

Fig. 15. Kou–Li's method with $c = -9/4$ for the roots of the polynomial $z^5 - 1$.

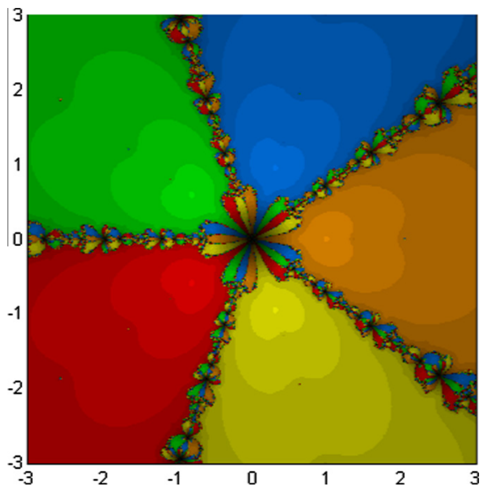

Fig. 16. Our method with $c = -1.125$ for the roots of the polynomial $z^5 - 1$.

Example 6. In the last example we took a polynomial of degree 7 having the 7 roots of unity, i.e.

$$p_6(z) = z^7 - 1. \tag{29}$$

The results are given in Figs. 21–24. The conclusion from these basins are identical to those of the previous example.

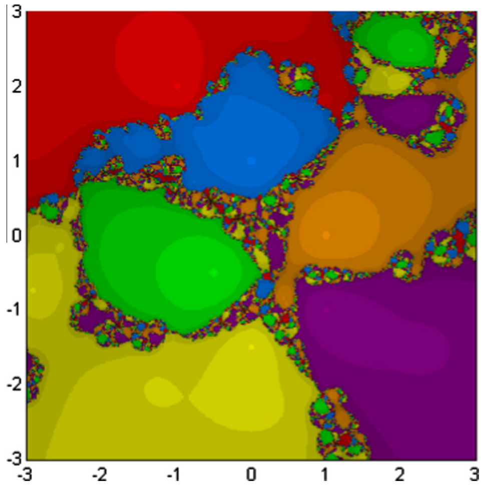

Fig. 17. Our method with $c = -3.53$ for the fifth example.

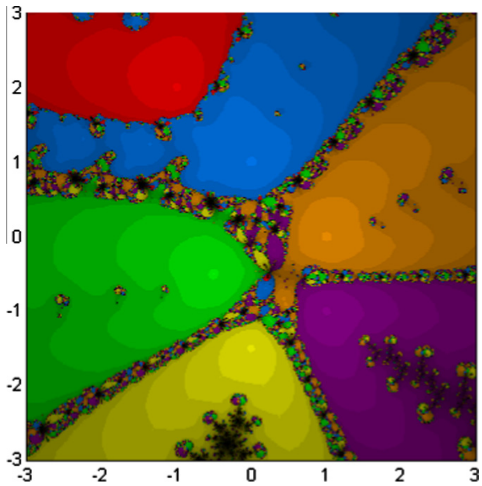

Fig. 18. Our method with $c = -3$ for the fifth example.

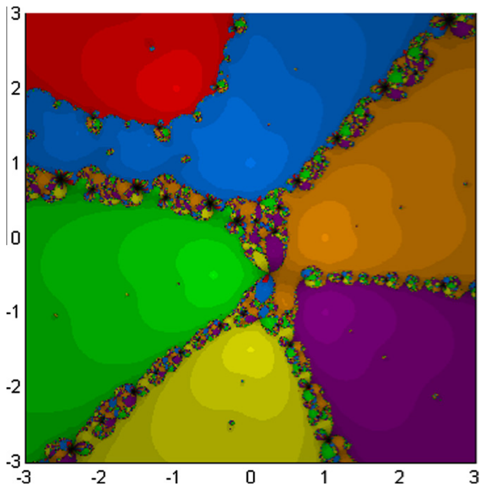

Fig. 19. Kou–Li's method with $c = -9/4$ for the fifth example.

In order to get an overall picture of performance, we assigned a value 1 for best and 4 for the worst performer in each example. In Table 2 we have collected all these results and then computed the total. The best performer is the one with the lowest total score.

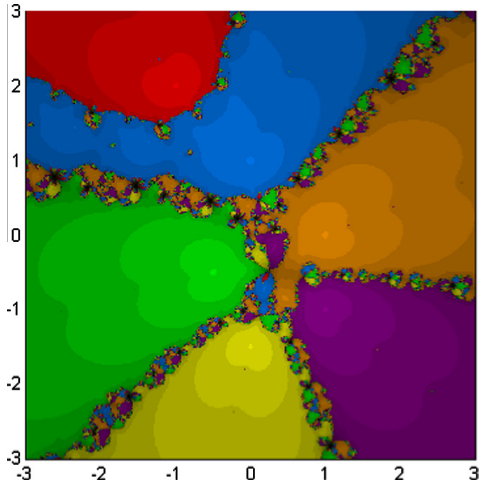

Fig. 20. Our method with $c = -1.125$ for the fifth example.

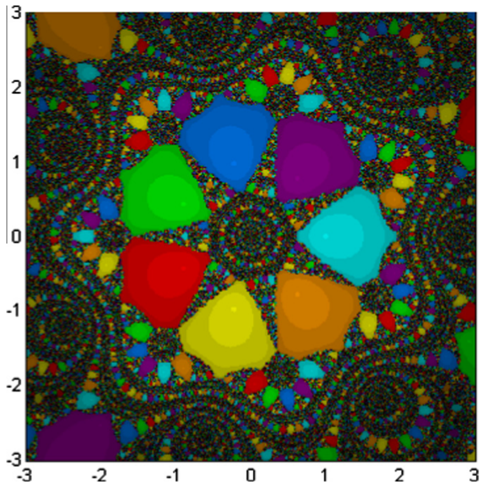

Fig. 21. Our method with $c = -3.53$ for the roots of the polynomial $z^7 - 1$.

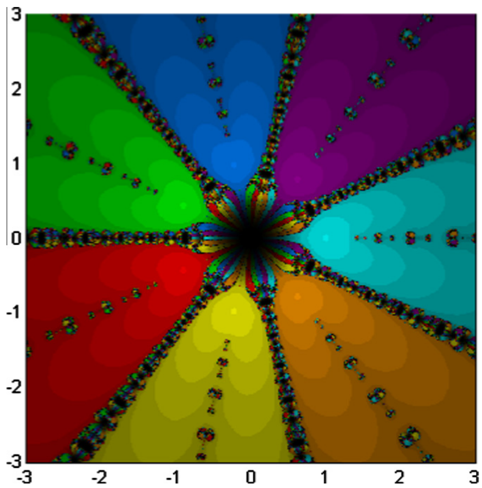

Fig. 22. Our method with $c = -3$ for the roots of the polynomial $z^7 - 1$.

As can be seen in Table 2, the method with the largest c value is best closely followed by Kou–Li's method and the worst are with the smallest c values. The best methods are the only ones to have a polynomial as their conjugacy map.

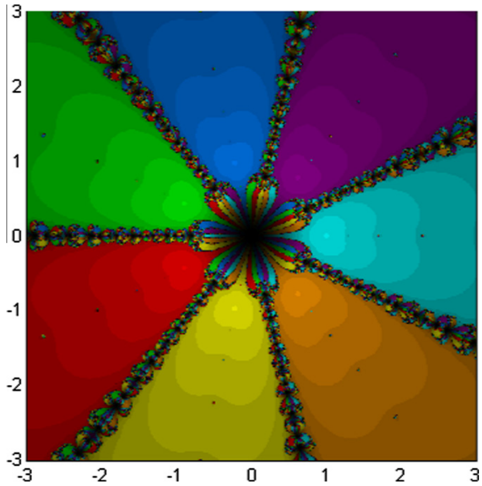

Fig. 23. Kou–Li's method with $c = -9/4$ for the roots of the polynomial $z^7 - 1$.

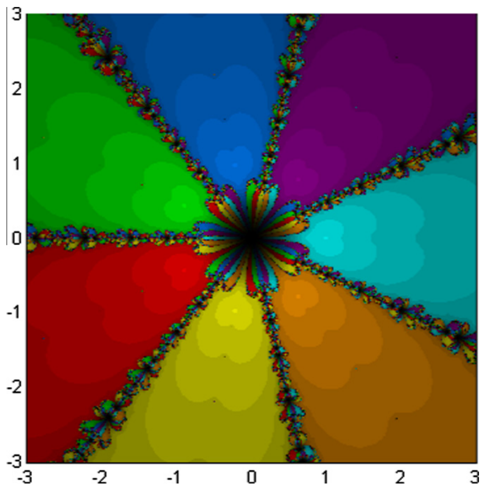

Fig. 24. Our method with $c = -1.125$ for the roots of the polynomial $z^7 - 1$.

Table 2
Ordering the quality of the basins for each example (1–6) and each value of c.

c	Ex1	Ex2	Ex3	Ex4	Ex5	Ex6	Total
-3.53	4	2	1	1	4	4	16
-3	2	1	4	4	3	3	17
-2.25	1	4	2	3	2	2	14
-1.125	3	3	3	2	1	1	13

5. Conclusions

We have analyzed a family of sixth order methods using weight functions. We have shown how the parameter was chosen to restrict extraneous fixed points to lie on the imaginary axis and get the best basin of attraction. Several examples were presented to illustrate the process. Future work will consist of generalizing the process of selecting parameters for both weight functions in this iterative process, and how to develop these for more general weight functions. The best performer is a method for which $c = -1.125$ (largest possible value for the parameter). The second best is the well-known method due to Kou and Li. Even though the total from Table 2 for these two methods is close, Kou–Li's method had many points of no convergence (after 40 iteration) in the second example.

Acknowledgment

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