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An analysis of a new family of eighth-order optimal methods

Changbum Chun\textsuperscript{a}, Beny Neta\textsuperscript{b,\*}

\textsuperscript{a}Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea
\textsuperscript{b}Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA 93943, United States

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\textbf{Abstract}

A new family of eighth order optimal methods is developed and analyzed. Numerical experiments show that our family of methods perform well and in many cases some members are superior to other eighth order optimal methods. It is shown how to choose the parameters to widen the basin of attraction.

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\section{Introduction}

There are many multistep methods for the solution of nonlinear equations, see e.g. Traub \cite{1}, and the recent book by Petković et al. \cite{2}. The idea of optimality in such methods was introduced by Kung and Traub \cite{3} who also developed optimal multistep method of increasing order. For example, the fourth-order optimal method given in \cite{3} is

\begin{equation}
\begin{aligned}
W_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} &= W_n - \frac{1}{f'(W_n)}.
\end{aligned}
\end{equation}

Based on this method Chun and Neta \cite{4} constructed and analyzed the sixth order method

\begin{equation}
\begin{aligned}
W_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
S_n &= W_n - \frac{1}{f'(W_n)}
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
x_{n+1} &= S_n - \frac{1}{f'(S_n)}.
\end{aligned}
\end{equation}

In this paper we will use the idea of weight function to develop a family of optimal eighth order methods and show how to choose the parameters to obtain the best basins of attraction.

\* Corresponding author.

E-mail addresses: cbchun@skku.edu (C. Chun), bneta@nps.edu (B. Neta).

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2. An optimal eighth-order method

We consider here a generalization of the Chun–Neta sixth order scheme (2). The new family is constructed using the idea of weight functions. The multistep method is given by

\[
\begin{align*}
    w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
    s_n &= w_n - \frac{w_m}{1 - r_n}, \\
    x_{n+1} &= s_n - \frac{f(x_n)}{f'(x_n)}[1 - H(r_n)f(x_n)P(q_n)],
\end{align*}
\]

(3)

where \( r_n = \frac{w_m}{f(x_n)}, \ t_n = \frac{w_m}{x_n}, \ q_n = \frac{w_m}{w_m} \) and \( H(r), J(t), P(q) \) are real-valued weight functions to be determined later.

For the method defined by (3), we have the following analysis of convergence.

**Theorem 2.1.** Let \( \xi \in I \) be a simple zero in an open interval \( I \) of a sufficiently differentiable function \( f : I \rightarrow \mathbb{R} \). Let \( e_n = x_n - \xi \). Then the new family of methods defined by (3) is of optimal eighth-order when

\[
\begin{align*}
    H(0)P(0) &= 2, \\
    H'(0)P(0) &= -1, \\
    H''(0)P(0) &= -1, \\
    H^{(4)}(0)P(0) &= 3, \\
    |H^{(4)}(0)| &< \infty, \\
    f'(0) &= -3f(0)/8, \\
    |f''(0)| &< \infty, \\
    P'(0) &= -P(0)/4, \\
    |P''(0)| &< \infty.
\end{align*}
\]

The error at the \((n + 1)\)th step, \( e_{n+1} \), satisfies the relation

\[
e_{n+1} = c_2 \left[ \left( 2P'(0) - \frac{1}{4} \right) c^3 - c_2 c_3 c_4 + \left( 4 - \frac{3P''(0)}{P'(0)} \right) c_2 c_3^2 + 2c_2 c_4 + \left( \frac{1}{3} \right) P'(0)H^{(4)}(0) - \frac{6P'(0)}{P''(0)} - \frac{29}{4} \right] e_n^2 + O(e_n^3),
\]

(4)

where \( c_i \) are given by

\[
c_i = \frac{f^{(i)}(\xi)}{i!f'(\xi)}, \quad i \geq 1.
\]

(5)

**Proof.** Let \( e_n = x_n - \xi \). \( e_n^m = w_n - \xi \) and \( e_n = s_n - \xi \). Using the Taylor expansion of \( f(x) \) around \( x = \xi \) and taking \( f(\xi) = 0 \) into account, we get

\[
f(x_n) = f'(\xi) [e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9)]
\]

(6)

and

\[
f'(x_n) = f'(\xi) [1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + O(e_n^7)].
\]

(7)

Dividing (6) by (7) gives

\[
u_n = \frac{f'(x_n)}{f'(\xi)} = e_n - c_2 e_n^2 + (2c_3 + 2c_4) e_n^3 + (-3c_4 + 7c_5 c_3 - 4c_3^2) e_n^4 + (10c_4 c_4 - 4c_5 + 6c_3 c_2 - 20c_3 c_2^2 + 8c_4 c_2) e_n^5 + (17c_4 c_3 - 28c_4 c_2^2 + 13c_3 c_2 - 5c_6 - 33c_4 c_3 + 52c_5 c_3 - 16c_5 c_2) e_n^6 + (19c_5 c_4 - 92c_4 c_2 + 22c_5 c_2 - 18c_5 c_2 c_2 + 126c_5 c_2^2 - 128c_4 c_2 c_2 + 2c_6 c_2^2 + 36c_6 c_2 - 3c_7 c_3 - 14c_6, c_3 + 27c_7 c_2 - 44c_7 c_2 + 27c_7 c_2^2 - 135c_4 c_3 c_2^2 + 108c_5 c_3 c_2^2 + 30c_6 c_2^2 - 64c_2 c_2^2) e_n^7 + O(e_n^9).
\]

(8)
From (8), we have
\[
e_n^w = c_2 e_n^w - (2c_2^2 - 2c_2) e_n^w + (3c_4 - 7c_2 c_3 + 4c_2^3) e_n^w + (-10c_2 c_4 + 4c_5 - 6c_3^2 + 20c_2 c_5 - 8c_4^4) e_n^w + (-17c_4 c_3 + 28c_4 c_2^2)
- 13c_2 c_5 + 5c_6 + 33c_2 c_3^2 - 52c_2 c_2^2 + 16c_2^3) e_n^w + (92c_2 c_2 c_4 - 22c_3 c_5 + 18c_3^2 - 126c_3^2 c_2^2 + 128c_2^2 c_4^2 - 12c_4^2
- 72c_4 c_3^2 + 36c_2 c_3^2 + 6c_6 - 16c_2 c_5 + 32c_3 c_2^2 + (64c_4^2 + 7c_8 - 118c_2 c_2 c_3 - 348c_2 c_2 c_4 - 19c_2 c_7 + 64c_2 c_4^2
- 31c_4 c_5 + 75c_4 c_3^2 + 176c_4 c_2^2 - 27c_3 c_5 + 44c_2 c_3^2 - 135c_2 c_3^2 + 408c_2 c_2^2 - 304c_2 c_4^2 + 0(e_n^w).
\]

Writing the Taylor's expansion for \(f(w_n)\) and using (9), we obtain
\[
f(w_n) = f'(\xi)[e_n^w + c_2(e_n^w)^2 + c_3(e_n^w)^3 + c_4(e_n^w)^4 + O((e_n^w)^5)]
= f'(\xi)[c_2 e_n^w + (2c_3 - 2c_2^2) e_n^w + (3c_4 - 7c_2 c_3 + 5c_5 + 33c_2 c_3^2 - 73c_2 c_2^2 + 28c_2^2) e_n^w + (18c_3 - 64c_5 + 6c_7 + 104c_2 c_4)
- 16c_2 c_6 - 22c_3 c_5 + 206c_2 c_3^2 + 12c_4^2 - 104c_2 c_2^2 + 44c_2 c_4^2) e_n^w + (144c_2^2 + 7c_5 + 134c_2 c_5) - 455c_2 c_3^2 - 19c_2 c_7 + 73c_2 c_3^2 - 31c_4 c_5 + 75c_4 c_3^2 + 297c_4 c_2^2 - 134c_5 c_2^2 - 27c_6 c_3 + 54c_6 c_2^2 - 147c_2 c_3^2
+ 582c_2^2 c_4^2 - 552c_2 c_4^2 + 0(e_n^w)].
\]

Dividing (10) by (7) gives
\[
\frac{f(w_n)}{f(x_n)} = c_2 e_n^w + (2c_3 - 4c_2^2) e_n^w + (3c_4 - 14c_2 c_3 + 13c_3^2) e_n^w + (20c_2 c_4 + 4c_5 - 12c_3^2 - 64c_2 c_2^2 - 38c_4^2) e_n^w + (104c_2^2
+ 5c_6 - 26c_2 c_6 - 34c_4 c_3 + 90c_4 c_2^2 + 103c_2 c_3^2 - 240c_2 c_2^2) e_n^w + 0(e_n^w).
\]

Dividing (10) by (6) gives
\[
r_n = \frac{f(w_n)}{f(x_n)}
= c_2 e_n^w + (2c_3 - 3c_2^2) e_n^w + (3c_4 - 10c_2 c_3 + 8c_3^2) e_n^w + (31c_5 - 2c_2 c_5 - 7c_4 c_2 + 21c_4 c_2^2 + 30c_2 c_3^2 - 72c_2 c_2^2) e_n^w
+ (20c_3 - 74c_4 + 88c_3 c_3 c_4 - 4c_2 c_6 - 10c_3 c_5 - 188c_2 c_3^2 + 246c_3 c_2^4 - 6c_2^4 - 100c_2 c_4^2 + 28c_2 c_2^4) e_n^w + 0(e_n^w),
\]

Using (8), (11) and (12), we find
\[
e_n^w = e_n^w - \frac{f(w_n)}{f(x_n)} \frac{1}{(1 - r)^2}
= (-c_2 c_3 + 2c_2^2) e_n^w + (-2c_2 c_4 + 14c_2 c_3^2 - 2c_3^2 - 10c_4^4) e_n^w + (31c_5 - 3c_2 c_5 - 7c_3 c_3 + 21c_4 c_2^2 + 30c_2 c_3^2 - 72c_2 c_2^2) e_n^w
+ (20c_3 - 74c_4 + 88c_3 c_3 c_4 - 4c_2 c_6 - 10c_3 c_5 - 188c_2 c_3^2 + 246c_3 c_2^4 - 6c_2^4 - 100c_2 c_4^2 + 28c_2 c_2^4) e_n^w + 0(e_n^w),
\]

so that, after elementary calculation,
\[
f(s_n) = f'(\xi)[e_n^w + c_2(e_n^w)^2 + c_3(e_n^w)^3 + O((e_n^w)^4)]
= f'(\xi)[(-c_2 c_3 + 2c_2^2) e_n^w + (-2c_2 c_4 + 14c_2 c_3^2 - 2c_3^2 - 10c_4^4) e_n^w + (31c_5 - 3c_2 c_5 - 7c_3 c_3 + 21c_4 c_2^2 + 30c_2 c_3^2 - 72c_2 c_2^2) e_n^w
+ (20c_3 - 74c_4 + 88c_3 c_3 c_4 - 4c_2 c_6 - 10c_3 c_5 - 188c_2 c_3^2 + 246c_3 c_2^4 - 6c_2^4 - 100c_2 c_4^2 + 28c_2 c_2^4) e_n^w + 0(e_n^w)].
\]

An easy calculation then produces
\[
\frac{f(s_n)}{f(x_n)} = (-c_2 c_3 + 2c_2^2) e_n^w + (-2c_2 c_4 + 16c_2 c_3^2 - 2c_3^2 - 14c_4^4) e_n^w + (59c_5 - 3c_2 c_5 - 7c_3 c_3 + 25c_4 c_2^2 + 37c_2 c_2^2 - 110c_3 c_3^2) e_n^w
+ (26c_3 - 192c_2^2 + 112c_2 c_2 c_3 - 4c_2 c_6 - 10c_3 c_5 - 310c_2 c_3^2 + 508c_2 c_2^2 - 6c_2^4 - 158c_4 c_2^2 + 34c_2 c_2^4) e_n^w + 0(e_n^w),
\]

and
\[
t_n = \frac{f(s_n)}{f(x_n)}
= (-c_2 c_3 + 2c_2^2) e_n^w + (-2c_2 c_4 - 2c_3^2 + 15c_2 c_3^2 - 12c_4^4) e_n^w + (-3c_2 c_5 - 7c_3 c_4 + 33c_2 c_3^2 - 89c_3 c_2^2 + 23c_2 c_2^4 + 43c_2^4) e_n^w
+ (98c_2 c_2 c_3 - 4c_2 c_6 - 6c_2^4 - 125c_4 c_2^2 - 10c_3 c_5 + 31c_4 c_2^2 - 236c_2^2 + c_2^4 + 347c_2 c_2^2 + 22c_3^2 + 117c_2 c_2^2) e_n^w
+ (130c_2 c_2 c_3 - 651c_2 c_2 c_3 - 5c_3 c_7 - 17c_3 c_4 - 162c_3 c_2^2 - 13c_6 c_3 + 39c_6 c_2^2 + 72c_2 c_4^2 + 95c_2 c_3^2 + 468c_4 c_2^2 - 266c_2 c_3^2
+ 1087c_2 c_2^2 - 1042c_2 c_2^2 + 266c_2 c_2^2) e_n^w + 0(e_n^w).
\]
\[ q_n = \frac{f(s_n)}{f'(w_n)} = (-c_3 + 2c_2^2)e_n^2 + (-2c_4 + 8c_2c_3 - 6c_3^2)e_n^3 + (-3c_5 + 7c_3^2 - 25c_5c_3 + 11c_2c_4 + 9c_4^2)e_n^4 + (18c_3c_4 - 4c_6 - 30c_3c_2^2 + 14c_2c_3 - 32c_4^2 + 36c_3c_2^2 - 2c_2^3)e_n^5 + (22c_3c_4 - 71c_4c_2c_3 - 5c_2c_4 + 36c_5c_2^2 + 17c_2c_6 + 11c_4^2 + 37c_4c_2^4 - 13c_3^2 + 41c_4c_3^2 + 26c_5c_2^2 - 28c_2^4)e_n^6 + O(e_n^7). \]  

(17)

We now expand \( H(r_n), J(t_n), P(q_n) \) into Taylor series about 0 to obtain

\[ H(r_n) = H(0) + H'(0)r_n + \frac{H''(0)}{2}r_n^2 + \frac{H'''(0)}{6}r_n^3 + O(r_n^4), \]

\[ J(t_n) = J(0) + J'(0)t_n + \frac{J''(0)}{2}t_n^2 + O(t_n^3), \]

\[ P(q_n) = P(0) + P'(0)q_n + \frac{P''(0)}{2}q_n^2 + O(q_n^3). \]  

(18)

Upon using the values

\[ H(0)J(0)P(0) = 2, \]

\[ H'(0)P(0)J(0) = -1, \]

\[ H''(0)P(0)J(0) = -1, \]

\[ H'''(0)P(0)J(0) = 3, \]

\[ J'(0) = -3J(0)/8, \]

\[ P'(0) = -P(0)/4 \]

and Eqs. (11), (16) and (17) we obtain

\[
\frac{1}{1 - H(r_n)J(t_n)P(q_n)} = 1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 \]

\[ + \left[ \frac{2P'(0)}{P(0)} + \frac{1}{3}P'(0)J(0)H''(0) - \frac{9}{4} \right] e_n^2 + 5c_5 - c_2c_4 + \left( \frac{9}{2} - 2 \frac{P'(0)}{P(0)} \right) c_3c_2^2 + \frac{1}{2} \left( \frac{P'(0)}{P(0)} - \frac{1}{2} \right) c_3^2 \]

\[ + O(e_n^5). \]

(19)

Therefore, from (13), (15) and (19), we obtain

\[
e_n = e_n^0 \frac{f(s_n)}{f'(x_n)} \frac{1}{1 - H(r_n)J(t_n)P(q_n)} = c_2 \left[ \frac{2P'(0)}{P(0)} - \frac{1}{4} \right] c_n^3 - c_2c_4c_2 + \left( 4 - \frac{3P'(0)}{P(0)} \right) c_2c_5 + 2c_3c_4 + \left( \frac{1}{3} P'(0)J(0)H''(0) + \frac{6P'(0)}{P(0)} - \frac{29}{4} \right) c_2c_3 \]

\[ + \left( \frac{1}{2} - \frac{2}{3} P'(0)J(0)H''(0) - \frac{4P'(0)}{P(0)} \right) c_3^2 \]

\[ + e_n^0 + O(e_n^5). \]

(20)

this completing the proof. □

Now we can choose

\[ H(r) = \frac{a}{1 + dr + gr^2}, \]

\[ J(t) = \frac{\alpha + \beta t}{1 + \gamma t}, \]

\[ P(q) = \frac{A_q + B_q}{1 + C_q}. \]

It is easy to see that \( B = A_r(C - 1/4), \) \( a = \frac{\lambda}{\lambda^2}, \) \( b = \frac{\lambda - \mu}{\lambda^2}, \) \( \beta = \alpha(\gamma - 3/8), \) \( c = \frac{1}{2} \frac{6g - 1}{\lambda^2}, \) and \( d = 1 - 2g. \) Substituting these values in \( H, J \) and \( P, \) we have

\[ H(r) = \frac{1}{2} \left( 4 + (2 - 8g)r + (8g - 3)r^2 \right) \]

\[ \frac{\alpha(8 + (8\gamma - 3)t)}{1 + \gamma t}, \]

\[ J(t) = \frac{1}{8} \frac{\alpha(8 + (8\gamma - 3)t)}{1 + \gamma t}, \]
\[ P(q) = \frac{1}{4} A(4 + (4C - 1)q) \]"
• The method based on Kung–Traub optimal fourth-order method and Hermite interpolating polynomial (HKT8) [2]

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)} , \]

\[ \tau_n = y_n - \frac{1}{f(x_n)} \frac{f(y_n)}{[1 - f(y_n)/f(x_n)]^2} , \]

\[ x_{n+1} = \tau_n - \frac{f(\tau_n)}{H_3(\tau_n)} , \]

where

\[ H_3(\tau_n) = 2(f[x_n, \tau_n] - f[x_n, y_n]) + f[y_n, \tau_n] + \frac{y_n - \tau_n}{y_n - x_n} (f[x_n, y_n] - f(x_n)) . \]

• The method based on Kung–Traub optimal fourth-order method and Hermite interpolating polynomial replacing the function (HKN8) [5]

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)} , \]

\[ \tau_n = y_n - \frac{1}{f(x_n)} \frac{f(y_n)}{[1 - f(y_n)/f(x_n)]^2} , \]

\[ x_{n+1} = \tau_n - \frac{H_3(\tau_n)}{f(\tau_n)} , \]

Table 1

Comparison of eighth-order iterative schemes.

<table>
<thead>
<tr>
<th>f</th>
<th>KT8</th>
<th>HKT8</th>
<th>HKN8</th>
<th>N8</th>
<th>WM8</th>
<th>OM1</th>
<th>OM2</th>
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<tbody>
<tr>
<td>x_0 = 1.5</td>
<td>f(x_0)</td>
<td>3.5e-69</td>
<td>4.8e-72</td>
<td>4.8e-72</td>
<td>1e-66</td>
<td>2.3e-60</td>
<td>-5.4e-86</td>
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<td>-4.4e-94</td>
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<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>x_0 = 4</td>
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<td>-1.2e-126</td>
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<td>-1.2e-126</td>
<td>3.5e-37</td>
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<td>4</td>
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<td>101</td>
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<td>f(x_0)</td>
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<td>f(x_0)</td>
<td>1.4e-43</td>
<td>2.9e-46</td>
<td>-6.9e-33</td>
<td>9.4e-38</td>
<td>5e-31</td>
<td>-1.1e-52</td>
</tr>
</tbody>
</table>
Table 2
Comparison of eighth-order iterative schemes.

<table>
<thead>
<tr>
<th>f</th>
<th>OMN1</th>
<th>OMN2</th>
<th>OMN3</th>
<th>OMN6</th>
</tr>
</thead>
<tbody>
<tr>
<td>f_1</td>
<td>IT</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>x_0 = 1.5</td>
<td>f(x_0)</td>
<td>-1.0e-62</td>
<td>-8.9e-67</td>
<td>-2.4e-75</td>
</tr>
<tr>
<td>f_2</td>
<td>IT</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>x_0 = 1.37</td>
<td>f(x_0)</td>
<td>9.5e-82</td>
<td>1.6e-83</td>
<td>5.1e-92</td>
</tr>
<tr>
<td>f_3</td>
<td>IT</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>x_0 = 2.5</td>
<td>f(x_0)</td>
<td>-8.9e-121</td>
<td>-6.4e-30</td>
<td>-1.7e-29</td>
</tr>
<tr>
<td>f_4</td>
<td>IT</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>x_0 = 4</td>
<td>f(x_0)</td>
<td>-2.1e-35</td>
<td>0</td>
<td>-1.4e-126</td>
</tr>
<tr>
<td>f_5</td>
<td>IT</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>x_0 = -1.5</td>
<td>f(x_0)</td>
<td>5.5e-105</td>
<td>-1.1e-126</td>
<td>1.2e-126</td>
</tr>
<tr>
<td>f_6</td>
<td>IT</td>
<td>10</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>x_0 = 4</td>
<td>f(x_0)</td>
<td>-6.2e-46</td>
<td>-2.0e-126</td>
<td>-1.2e-38</td>
</tr>
<tr>
<td>f_7</td>
<td>IT</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>x_0 = 2</td>
<td>f(x_0)</td>
<td>2.8e-66</td>
<td>1.6e-71</td>
<td>1.0e-78</td>
</tr>
<tr>
<td>f_8</td>
<td>IT</td>
<td>6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>x_0 = 4</td>
<td>f(x_0)</td>
<td>0</td>
<td>-4.0e-72</td>
<td>-2.1e-96</td>
</tr>
<tr>
<td>f_9</td>
<td>IT</td>
<td>5</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>x_0 = 1</td>
<td>f(x_0)</td>
<td>2.0e-111</td>
<td>-1.1e-52</td>
<td>-3.8e-82</td>
</tr>
<tr>
<td>f_10</td>
<td>IT</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>x_0 = 2.1</td>
<td>f(x_0)</td>
<td>-1.9e-124</td>
<td>0</td>
<td>-3.3e-76</td>
</tr>
<tr>
<td>f_11</td>
<td>IT</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>x_0 = 1</td>
<td>f(x_0)</td>
<td>6.2e-90</td>
<td>2.2e-109</td>
<td>0</td>
</tr>
<tr>
<td>f_12</td>
<td>IT</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>x_0 = 0.5</td>
<td>f(x_0)</td>
<td>2.4e-78</td>
<td>1.1e-86</td>
<td>1.5e-87</td>
</tr>
<tr>
<td>f_13</td>
<td>IT</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>x_0 = 1</td>
<td>f(x_0)</td>
<td>-4.3e-32</td>
<td>-1.3e-69</td>
<td>-8.7e-86</td>
</tr>
<tr>
<td>f_14</td>
<td>IT</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>x_0 = -0.85</td>
<td>f(x_0)</td>
<td>-2.5e-64</td>
<td>-8.6e-116</td>
<td>-4.6e-124</td>
</tr>
<tr>
<td>f_15</td>
<td>IT</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>x_0 = 1.2</td>
<td>f(x_0)</td>
<td>-1.9e-36</td>
<td>-6.1e-35</td>
<td>-7.6e-40</td>
</tr>
</tbody>
</table>

Table 3
The parameters for each member. The first 2 belong to OM and the last 6 are OMN members.

<table>
<thead>
<tr>
<th>Case</th>
<th>g</th>
<th>γ</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>41/36</td>
<td>2</td>
<td>-2/3</td>
</tr>
<tr>
<td>2</td>
<td>41/36</td>
<td>-1</td>
<td>-2/3</td>
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<tr>
<td>3</td>
<td>-4</td>
<td>0</td>
<td>-4</td>
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<tr>
<td>4</td>
<td>-4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>-4</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>2.9</td>
<td>-4</td>
<td>-4</td>
</tr>
</tbody>
</table>

where

\[ H_3(t_n) = \frac{f(x_n) + f'(x_n) + f''(x_n) (t_n - x_n)^2 (t_n - y_n) (y_n - x_n)(y_n + 2y_n - 3t_n)}{f(x_n) + f'(x_n) + (y_n - x_n)(y_n + 2y_n - 3t_n)} - \frac{(t_n - x_n)^3}{(y_n - x_n)(y_n + 2y_n - 3t_n)}. \]  

(25)

- An eighth-order (N8) optimal method proposed by Neta [6] and based on King's fourth order optimal method [7] with \( \beta = 2 \) given by

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \]

(26)

\[ \tau_n = y_n - \frac{f(y_n)}{f'(y_n)} \quad \frac{f(x_n) + \beta f(y_n)}{f'(x_n) + (\beta - 2) f'(y_n)}. \]

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \beta f^2(x_n) - \beta f^3(x_n). \]
### Table 4
Number of EFPs, minimum and maximum values of the absolute value of the real parts of EFPs.

<table>
<thead>
<tr>
<th>Case</th>
<th>Number of EFPs</th>
<th>Min. value</th>
<th>Max. value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>54</td>
<td>5.06e−2</td>
<td>0.507</td>
</tr>
<tr>
<td>2</td>
<td>54</td>
<td>8.6e−3</td>
<td>0.353</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>9.7e−5</td>
<td>3.038</td>
</tr>
<tr>
<td>4</td>
<td>48</td>
<td>1.5e−8</td>
<td>0.528</td>
</tr>
<tr>
<td>5</td>
<td>48</td>
<td>2.85e−2</td>
<td>0.524</td>
</tr>
<tr>
<td>6</td>
<td>48</td>
<td>1.85e−2</td>
<td>3.040</td>
</tr>
<tr>
<td>7</td>
<td>48</td>
<td>1.96e−3</td>
<td>0.495</td>
</tr>
<tr>
<td>8</td>
<td>54</td>
<td>0.161</td>
<td>11.364</td>
</tr>
</tbody>
</table>

Fig. 1. Our method with $g = 41/36$, $\gamma = 2$, $C = -2/3$, $x = 1$, and $A = -3$ for the roots of the polynomial $z^2 - 1$.

Fig. 2. Our method with $g = 41/36$, $\gamma = -1$, $C = -2/3$, $x = 2$, and $A = 1$ for the roots of the polynomial $z^2 - 1$. 
where

\[
\rho = \frac{\phi_y - \phi_1}{F_y - F_i}, \quad \gamma = \phi_y - \rho F_y, \quad F_y = f(y_n) - f(x_n), \quad F_i = f(\tau_n) - f(x_n),
\]

\[
\phi_y = \frac{y_n - x_n}{F_y}, \quad \phi_1 = \frac{\tau_n - x_n}{F_i} - \frac{1}{F_i f'(x_n)}
\]

\[
(27)
\]

- A weight function based eighth order (WM8) optimal method [2] (using the fourth order Maheshwari’s method [8]) given by

Fig. 3. Our method with \( g = -4, \gamma = 0, \ C = -4, \) and any \( \alpha \) and \( A \) for the roots of the polynomial \( z^4 - 1. \)

Fig. 4. Our method with \( g = -4, \gamma = 0, \ C = 0, \) and any \( \alpha \) and \( A \) for the roots of the polynomial \( z^4 - 1. \)
\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \]
\[ \tau_n = x_n - \left[ \frac{f(y_n)}{f'(x_n)} \right]^2 \frac{f(x_n)}{f'(x_n)} \frac{f(x_n)}{f'(x_n)}, \]
\[ x_{n+1} = \tau_n - \left[ \phi \left( \frac{f(y_n)}{f(x_n)} \right) + \frac{f(\tau_n)}{f'(y_n) - af(\tau_n)} + \frac{4f(\tau_n)}{f(x_n)} f'(x_n) \right] \frac{f(\tau_n)}{f'(x_n)} \],

where \( \phi \) is an arbitrary real function satisfying the conditions
\[ \phi(0) = 1, \quad \phi'(0) = 2, \quad \phi''(0) = 4, \quad \phi'''(0) = -6. \]

We have taken \( a = 1 \) and \( \phi(t) = 1 + 2t + 2t^2 - t^3 \).

Fig. 5. Our method with \( g = 0, \gamma = -4, C = 0 \), and any \( x \) and \( A \) for the roots of the polynomial \( z^2 - 1 \).

Fig. 6. Our method with \( g = 2.9, \gamma = -4, C = -4 \), and any \( x \) and \( A \) for the roots of the polynomial \( z^2 - 1 \).
Table 5
Average number of iterations per point.

<table>
<thead>
<tr>
<th>Case</th>
<th>Ex1</th>
<th>Ex2</th>
<th>Ex3</th>
<th>Ex4</th>
<th>Ex5</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.2708</td>
<td>4.7640</td>
<td>7.5009</td>
<td>6.0247</td>
<td>11.6231</td>
<td>33.1835</td>
</tr>
<tr>
<td>2</td>
<td>3.1197</td>
<td>4.3875</td>
<td>6.6839</td>
<td>5.8369</td>
<td>10.7186</td>
<td>30.7466</td>
</tr>
<tr>
<td>3</td>
<td>3.2751</td>
<td>4.3391</td>
<td>5.6630</td>
<td>5.6329</td>
<td>8.2313</td>
<td>27.1414</td>
</tr>
<tr>
<td>4</td>
<td>3.1885</td>
<td>4.1557</td>
<td>5.4185</td>
<td>5.8138</td>
<td>8.5328</td>
<td>27.1093</td>
</tr>
<tr>
<td>5</td>
<td>3.1163</td>
<td>4.3463</td>
<td>6.0809</td>
<td>5.5301</td>
<td>9.9465</td>
<td>29.0201</td>
</tr>
<tr>
<td>6</td>
<td>3.2751</td>
<td>4.3391</td>
<td>5.6630</td>
<td>5.6329</td>
<td>8.2313</td>
<td>27.1414</td>
</tr>
<tr>
<td>7</td>
<td>3.1885</td>
<td>4.1557</td>
<td>5.4185</td>
<td>5.8138</td>
<td>8.5328</td>
<td>27.1093</td>
</tr>
<tr>
<td>8</td>
<td>3.2452</td>
<td>5.0434</td>
<td>8.0473</td>
<td>5.9350</td>
<td>11.9265</td>
<td>34.1974</td>
</tr>
</tbody>
</table>

Fig. 7. Our method with $g = 41/36$, $\gamma = 2$, $C = -2/3$, $x = 1$, and $A = -3$ for the roots of the polynomial $(z - 1)^3 - 1$.

Fig. 8. Our method with $g = 41/36$, $\gamma = -1$, $C = -2/3$, $x = 2$, and $A = 1$ for the roots of the polynomial $(z - 1)^3 - 1$. 
In Table 1 we presented the results for KT8 (21), HKT8 (22), HKN8 (24), N8 (26) with $\beta = 2$, WM8 (28) and our new methods OM (OM1 is with $\alpha = 1$, $A = \frac{1}{C_0}$; $c = \sqrt{2}$ and OM2 is with $\alpha = 2$, $A = 1$, $\gamma = -1$). In Table 2 we also presented the results for our methods OMN1 (case 3, see Table 3), OMN2 (case 4), OMN3 (case 5) and OMN6 (case 8). The number of iterations $\mathit{IT}$ required to converge is given along with the value of the function at the last iteration $f(x)$. It can be observed that for most of the considered test functions our methods show as good performance as the other methods in their convergence speed and also have reasonable smallness of the residuals. In fact, in one case ($f_8$) our methods converged even though WM8 diverged. We also found that in that example OMN6 diverged even though the other methods converged. We will show later that OMN6 is not a good choice. Therefore we can conclude that the new methods (OM1, OM2, OMN1–OMN3) are competitive with other eighth-order schemes being considered for solving nonlinear equations.
Fig. 11. Our method with $g = 0$, $\gamma = -4$, $C = 0$, and any $x$ and $A$ for the roots of the polynomial $(z - 1)^3 - 1$.

Fig. 12. Our method with $g = 2.9$, $\gamma = -4$, $C = -4$, and any $x$ and $A$ for the roots of the polynomial $(z - 1)^3 - 1$.

Fig. 13. Our method with $g = 41/36$, $\gamma = 2$, $C = -2/3$, $x = 1$, and $A = -3$ for the roots of the polynomial $z^4 - 1$. 
In the next section, we analyze the basin of attraction of our eighth order family of methods to find out what is the best choice for the parameters. The idea of using basins of attraction was initiated by Stewart [9] and followed by the works of Amat et al. [10–13], Scott et al. [14], Chun et al. [15], Chicharro et al. [16], Cordero et al. [17], Neta et al. [18] and Chun et al. [19]. The only papers comparing basins of attraction for methods to obtain multiple roots is due to Neta et al. [20] and Neta and Chun [21–23].

4. Basins of attraction

In this section we give the basins of attraction of various members of the families OM and OMN. The 8 members are listed with their parameters in Table 3. The first 2 cases are of OM type and the last 6 cases are of OMN type.

![Fig. 14. Our method with $g = 41/36$, $\gamma = -1$, $C = -2/3$, $\alpha = 2$, and $A = 1$ for the roots of the polynomial $z^4 - 1$.](image1)

![Fig. 15. Our method with $g = -4$, $\gamma = 0$, $C = -4$, and any $\alpha$ and $A$ for the roots of the polynomial $z^4 - 1$.](image2)
The first 2 cases were chosen so that we annihilate two terms in the error constant. The first of those was arbitrarily picked. In order to understand the choice of the parameters in the other cases we discuss the extraneous fixed points. In solving a nonlinear equation iteratively we are looking for fixed points which are zeros of the given nonlinear function. Many multipoint iterative methods have fixed points that are not zeros of the function of interest. Thus, it is imperative to investigate the number of extraneous fixed points, their location and their properties. In the family of methods described in this paper, the parameters \( g \), \( \gamma \), and \( C \) can be chosen to position the extraneous fixed points on or close to the imaginary axis. This idea is due to Neta et al. [18] where they have shown an improvement in King’s method by choosing the parameter that will position the extraneous fixed points on the imaginary axis. The second case was chosen so that we also have the extraneous fixed points as close as possible to the imaginary axis. Similarly, in the next 5 cases we have chosen the extraneous fixed points...
points to be close to the imaginary axis. The last case violates that condition and one can see later that this violation increases the average number of iterations.

The eighth order family of methods discussed here can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n),$$

(29)

where

$$H_f(x_n) = 1 + \frac{r_n}{(1 - r_n)^2} + \frac{t_n}{(1 - H(r_n)J(t_n)P(q_n))^2}. $$

(30)

Fig. 18. Our method with $g = 2.9$, $\gamma = -4$, $C = -4$, and any $x$ and $A$ for the roots of the polynomial $z^4 - 1$.

Fig. 19. Our method with $g = 41/36$, $\gamma = 2$, $C = -2/3$, $x = 1$, and $A = -3$ for the roots of the polynomial $z(z^2 + 1)(z^2 + 4)$.
Clearly the root $\xi$ of $f(x)$ is a fixed point of the method. The points $x \neq \xi$ at which $H_f(x) = 0$ are also fixed points of the family. We have searched the parameter space and found that there are no point on the imaginary axis, but there are 5 cases with the smallest real part (these are denoted cases 3–7). To convince that this is a good choice we have taken a case where the real part of the extraneous roots is the largest (case 8). In Table 4 we have listed the number of extraneous fixed points (EFPs) and range of absolute value of the real parts of the EFPs for each case.

**Example 1.** In our first example we have used the polynomial $z^2 - 1$. The basins of attraction are given in Figs. 1–6. All the results are good. The best one is case 5 (Fig. 5) and the worst is case 3 (Fig. 3). The difference is barely noticeable. One can see it only when computing the average number of iterations per point (see Table 5.) All cases require between 3.1 and 3.3 iterations per point. Notice that cases 6 and 7 were not shown since we found that they yield identical results to cases 3 and 4, respectively.

**Fig. 20.** Our method with $g = 41/36$, $\gamma = -1$, $C = -2/3$, $x = 2$, and $A = 1$ for the roots of the polynomial $z(z^2 + 1)(z^2 + 4)$.

**Fig. 21.** Our method with $g = -4$, $\gamma = 0$, $C = -4$, and any $x$ and $A$ for the roots of the polynomial $z(z^2 + 1)(z^2 + 4)$. 
Example 2. In our next example we used the polynomial
\[ p_2(z) = (z - 1)^3 - 1. \]
The basins are given in Figs. 7–12. In this case the best performer is case 4 (Fig. 10) and the worst is case 8 (Fig. 12).

Example 3. In our third example we have taken the polynomial
\[ p_3(z) = z^4 - 1. \]
The basins are given in Figs. 13–18. Again the best performer is case 4 (Fig. 16) and the worst is case 8 (Fig. 18).
Example 4. In our fourth example we have taken the polynomial
\[ p_4(z) = z(z^2 + 1)(z^2 + 4), \]
whose roots are 0, ±1, ±2i. The basins are given in Figs. 19–24. In this example the best is case 5 (Fig. 23) and the worst is case 1 (Fig. 19).

Example 5. In our last example we have taken the polynomial
\[ p_5(z) = (z^2 - 1)(z^2 + 1)(z^2 + 2i). \]
whose roots are \( \pm 1, \pm i, -1+i, \) and \( 1-\imath \). The basins are given in Figs. 25–30. The best is case 3 (Fig. 28) and the worst is case 8 (Fig. 30).

Based on Table 5, the best case overall is case 4 for which \( g = -4 \) and \( C = \gamma = 0 \) and the worst is case 8 for which \( g = 2.9 \) and \( C = \gamma = -4 \). In general cases 3–5 (OMN) are better than cases 1–2 (OM). Case 8 is also OMN but there we picked parameters that lead to largest (in absolute value) real part of the extraneous fixed points. On the other hand, cases 3–5 have the smallest real part.

**Remarks.**

(1) Note that cases 3 and 6 yield identical results. Similarly cases 4 and 7 are identical. For this reason, we have not shown the results for cases 6 and 7.
(2) Out of the first 2 cases, were we annihilated two terms in the error constant, the best is case 2 were we also chose the extraneous fixed points to be closest to the imaginary axis.

5. Conclusions

In this paper we have developed a new family of optimal eighth order iterative method. The scheme is optimal in the sense that it satisfies the Kung–Traub conjecture. We have compared several members of our family to existing optimal eighth order schemes and found that some members of our family are competitive. We have shown how to choose the parameters of the family to find the best members by evaluating all the extraneous fixed points. We have shown that the
best members have extraneous fixed point close to the imaginary axis. One member (case 8) for which this is not true was the worst perform.

Acknowledgments

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References