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On the generation of large-scale structures in a homogeneous eddy field

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An analytical theory is developed which illustrates dynamics of the spontaneous generation of large-scale structures in the unforced two-dimensional eddying flows. The eddy field is represented by the closely packed array of standing coherent vortices whose intensity is weakly modulated by the long-wavelength perturbations introduced into the system. The asymptotic multiscale analysis makes it possible to identify instabilities resulting from the positive feedback of the background eddies on large-scale perturbations. Initially, these instabilities amplify at a rate proportional to the square root of their wavenumber. Linear growth is arrested when the amplitude of the long-wavelength perturbations reaches the level of background eddies. The subsequent evolutionary pattern is characterized by the emergence of relatively sharp features in the large-scale streamfunction field – features suggestive of the coherent jets commonly observed in eddying geophysical flows. The proposed solutions differ substantially from their counterparts in forced-dissipative systems, exemplified by the canonical model of Kolmogorov flow. The asymptotic model is successfully tested against numerical simulations.

Key words: instability, ocean processes, vortex interactions

1. Introduction

Spontaneous generation of large-scale structures in eddying geophysical flows is commonly interpreted as a manifestation of the inverse cascade of energy in two-dimensional turbulence (Rhines 1994). Specific examples include the emergence of isolated coherent vortices in turbulent flow (McWilliams 1984). In the presence of the planetary vorticity gradient, eddies are responsible for another spectacular phenomenon – appearance of the long-lived coherent jets emerging from the disorganized eddy field. These structures are ubiquitous in the ocean (Berloff, Kamenkovich & Pedlosky 2009a, b; Kamenkovich, Berloff & Pedlosky 2009a, b), Earth’s atmosphere (Baldwin et al. 2007; Dritschel & McIntyre 2008) and in giant gaseous planets (Galperin et al. 2004; Kaspi & Flierl 2007). Eddy-driven jets are detectable in oceanic observations (Hogg & Owens 1999; Maximenko, Bang & Sasaki 2005), numerical simulations (Treguier & Panetta 1994; Richards et al. 2006) and laboratory experiments (Whitehead 1975; Read et al. 2007). The existing evidence indicates that the transfer of energy from forcing scales to larger wavelengths is a robust effect, which is not particularly sensitive to the precise type of forcing: baroclinic instability (Panetta 1993), decaying turbulence (Huang & Robinson 1998) and random stirring.
models (Vallis & Maltrud 1993) all exhibit emergence of the long-lived coherent structures. These phenomena, however, do not arise in low-resolution simulations that represent eddies by the down-gradient momentum flux – a deficiency which motivates development of more elaborate and realistic eddy parameterizations for general circulation models.

An intriguing aspect of eddy interaction theory concerns the spectrally non-local character of energy transfer. While the classical spectral models (Kraichnan 1967; Kraichnan & Montgomery 1980) envision the sequential coalescence of eddies into larger and larger structures, numerical simulations (Shepherd 1988; Huang & Robinson 1998) suggest the existence of a direct pathway of energy from small to large scales. The interaction between vastly different scales implies that eddies whose strength is modulated on long wavelengths can exert a positive feedback on large-scale perturbations – an effect that is often rationalized by invoking the concept of negative eddy viscosity (Starr 1968). In this regard, the eddy transport of momentum is very different from the transport of passive tracers, which generally tend to mix down-gradient in physical space and towards small scales spectrally. The somewhat counter-intuitive tendency for the up-gradient eddy transfer of momentum is a consequence of the interplay between vorticity and momentum perturbations (e.g. Stern & Radko 1997).

The wealth of numerical and observational examples illustrating emergence of large-scale structures motivate development of analytical deterministic solutions, which attempt to explain and quantify these effects from first principles. Particularly popular are solutions based on the Kolmogorov model – a parallel shear flow with a sinusoidal velocity profile, maintained against viscous dissipation by the external forcing (Meshalkin & Sinai 1961; Sivashinsky 1985). The Kolmogorov model is widely used to represent, in a highly idealized and abstract manner, the interaction of small-scale eddies with large-scale structures. Explicit analytical solutions based on this model indicate that the background flow is unstable with respect to long-wavelength perturbations, indicating the propensity of the system to produce large-scale patterns. Particular problems, conveniently phrased in terms of the Kolmogorov model, include formation of zonal jets (Frisch, Legras & Villone 1996; Manfroi & Young 1999, 2002; Legras & Villone 2009) and the effects of density stratification on the inverse cascade in the vertical plane (Balmforth & Young 2002, 2005). Of course, questions could be raised whether the Kolmogorov model, interesting as it may be in its own right, is an adequate analogue of eddies in nature. Geophysical flows are often rich with roughly isotropic vortices, exemplified by the Gulf Stream rings in the ocean and weather systems in the atmosphere. There is also a related concern that the highly anisotropic background flow in the Kolmogorov model results in anisotropic eddy diffusivities even when environmental conditions are perfectly isotropic. Therefore, attempts have been made to move beyond the parallel flow model of the background eddy field. Novikov & Papanicolaou (2001), for instance, examined modulational stability of the cellular two-dimensional flows with respect to long-wavelength perturbations. Gama, Vergassola & Frisch (1994) considered a variety of basic planforms and used the multiscale method to formulate explicit solutions characterized by isotropic and negative eddy viscosity. The analytical development in these models, however, was guided by the asymptotic scaling of the Kolmogorov model, whereas our study demonstrates that fundamentally different solutions may exist for the two-dimensional background states.

For reasons of tractability, many deterministic theories have adopted the conventional weakly nonlinear approach and focused on marginally unstable regimes.
These solutions were characterized by the order one Reynolds numbers (based on eddy scales) and by the leading-order balance between the external forcing and viscous dissipation. However, the interpretation and relevance of the fundamentally viscous models may be questionable in the geophysical context. Mesoscale variability in the ocean is characterized by the spatial scales of 10–100 km and velocities of 0.01–1 m s\(^{-1}\), which corresponds to extremely high Reynolds numbers \( (Re \sim 10^8–10^{11}) \) based on the molecular viscosity \( v \approx 10^{-6} \text{ m}^2 \text{s}^{-1} \). Difficulties arise even when the viscous dissipation of eddies in the model is attributed to submesoscale processes. The magnitude and mechanisms of energy dissipation in the ocean are sources of great uncertainty (Wunsch & Ferrari 2004). The application of modulational theories to geophysical flows is further complicated by the sensitivity of the solutions to the explicit dissipation (e.g. Novikov & Papanicolaou 2001; Novikov 2003), motivating a different approach to the eddy/long wave interaction problem – the approach based on fully inviscid dynamics. The inviscid stability studies for periodic flows are relatively rare. Notable examples include analyses of the strongly anisotropic flows representing the Rossby waves (e.g. Lorentz 1972; Gill 1974) and internal gravity waves (e.g. Drazin 1977).

The goal of the present study is to formulate solutions that retain the tractability and transparency of Kolmogorov-type models. At the same time, we strive to avoid some of their debatable assumptions: viscous dynamics and strong anisotropy of the background field. Our view is more nonlinear, unforced and inviscid; eventually, it reveals a very different physical picture. The growth rate \( (\lambda) \) of the unstable modes in forced-dissipative models rapidly increases with increasing wavenumber \( (k) \), typically \( \lambda \propto k^4 \) (e.g. Sivashinsky 1985) or \( \lambda \propto k^2 \) (e.g. Gama et al. 1994; Novikov & Papanicolaou 2001). Our solutions, on the other hand, demonstrate the existence of unstable modes with \( \lambda \propto \sqrt{k} \), which, for long wavelengths \( (k \rightarrow 0) \), correspond to much more rapid amplification. These fast-growing solutions will be hereafter referred to as the ‘eddy-explosion’ modes. It is not surprising that the numerical simulations in this paper reflect the appearance of eddy-explosion waves very clearly: after all, the evolution of systems affected by several instabilities is usually controlled by the fastest-growing mode.

This paper is organized as follows. In §2, we present a linear analysis of the eddy field represented by the closely packed array of standing coherent vortices, which reveals the existence of long wave instabilities. Section 3 offers a simple physical interpretation of the model solutions. In §4, we examine the nonlinear equilibration of the unstable modes by deriving, and numerically solving, the amplitude equations for the large-scale field. In §5, we perform simulations of the original fully nonlinear vorticity equation and successfully test our asymptotic theory. Section 6 presents various generalizations of the model, which include effects of viscosity and the aspect ratio of the background eddies. We discuss the key findings in §7.

2. Linear instability

Consider a two-dimensional incompressible inviscid flow on the \( f \)-plane, governed by the vorticity equation as

\[
\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) = 0.
\]
where \( \psi \) is the streamfunction. An exact solution of (2.1) is given by the Taylor–Green flow
\[
\tilde{\psi} = A_e \sin \left( \frac{x}{L_e} \right) \sin \left( \frac{y}{L_e} \right),
\]
which represents a doubly periodic array of stationary vortices. Solution (2.2) will be regarded as an idealized model for the vigorous eddy field of an unspecified origin. Variables in (2.1) are non-dimensionalized by using \( L_e, A_e \) and \( L_e^2 / A_e \) as the scales of length, streamfunction and time, respectively. The vorticity equation (2.1) is invariant with respect to this non-dimensionalization, whereas the basic solution reduces to
\[
\tilde{\psi} = \sin(x) \sin(y).
\]

Our first step is to examine the linear stability of the solution (2.3) with respect to the long-wavelength perturbations. The streamfunction is separated into the steady basic field \( \tilde{\psi} \) and a weak perturbation \( \psi' \). The linearization of the vorticity equation (2.1) about the basic state yields
\[
\frac{\partial}{\partial t} \nabla^2 \psi' + \cos(x) \sin(y) \frac{\partial}{\partial y} (\nabla^2 \psi' + 2\psi') - \sin(x) \cos(y) \frac{\partial}{\partial x} (\nabla^2 \psi' + 2\psi') = 0.
\]
We are concerned by the ability of the eddy field to affect the slow evolution of large-scale structures, and therefore, we introduce new spatial and temporal scales over which the basic field is modulated as
\[
(X, Y) = \varepsilon^2(x, y), \quad T = \varepsilon t,
\]
where \( \varepsilon \) is small. It should be noted that scaling (2.5) is different from that which is a priori specified in most multiscale modes. The earlier studies (Gama et al. 1994; Novikov & Papanicolaou 2001) focused their enquiry, by analogy with the viscous Kolmogorov flow, on the parameter sector \( (\partial/\partial t, \partial/\partial x) \sim (\varepsilon^2, \varepsilon) \). As we shall see shortly, the inviscid problem permits alternative balanced solutions in sector (2.5) – the eddy-explosion modes – which are characterized by more rapid amplification.

The subsequent development follows the conventional approach in multiscale problems (Kevorkian & Cole 1996): (i) \( (x, y, X, Y, T) \) are treated as independent variables, (ii) on short scales \( (x, y) \) we impose the same periodicity as in the basic flow and (iii) derivatives in the original equation (2.4) are replaced as follows:
\[
\frac{\partial}{\partial t} \rightarrow \varepsilon \frac{\partial}{\partial T}, \quad \frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \varepsilon^2 \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial y} \rightarrow \frac{\partial}{\partial y} + \varepsilon^2 \frac{\partial}{\partial Y}.
\]
The solution is sought as a series in \( \varepsilon \) as follows:
\[
\psi' = \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \cdots.
\]
In this section, we analyse the linear equation (2.4) and therefore the order of the leading term in (2.7) is irrelevant; it is arbitrarily chosen to be \( O(1) \). Since our primary goal is to understand the evolution of large-scale flows, we consider the leading-order term which only varies on long spatial scales \( (X, Y) \):
\[
\psi_0 = \psi_0(X, Y, T).
\]
Next, we substitute (2.6) and (2.7) in the governing equation (2.4), collect terms of the same order in \( \varepsilon \), and sequentially solve the resulting hierarchy of equations until the solution for \( \psi_0 \) is found.
Guided by considerations of tractability and transparency, we now introduce yet another approximation – truncation of $\psi_i$ in terms of Fourier harmonics in $x$ and $y$:

$$
\psi_i = \sum_{k,l} [A_{k,l,i}(X, Y, T) \cos(kx + ly) + B_{k,l,i}(X, Y, T) \sin(kx + ly)],
$$

where $k$ and $l$ are integer numbers. The approximate solution is found by using the Galerkin method: the truncated series (2.9) is substituted in each asymptotic equation and the resulting error is required to be orthogonal to the harmonics used in these series. An interesting property of our system is that the solution requires only Fourier components that are characterized by even values of $|k| + |l|$. Therefore, after considering several patterns of truncation, we found it most convenient to truncate the Fourier series by retaining components that satisfy

$$
|k| + |l| \leq 2N,
$$

where $N$ is a positive integer number. The minimal representation of the eddy/long-wave interaction is obtained for $N = 1$, and therefore, our model is described next by using this simple heavily truncated example. However, the formulation can be readily extended to higher values of $N$. The $N = 2$ solutions, more accurate but less transparent, will be presented as well.

The zero-order balance of (2.4) is trivially satisfied and at the first order in $\varepsilon$ we arrive at

$$
\cos(x) \sin(y) \frac{\partial}{\partial y} (\nabla^2 \psi_1 + 2\psi_1) - \sin(x) \cos(y) \frac{\partial}{\partial x} (\nabla^2 \psi_1 + 2\psi_1) = J[\sin(x) \sin(y), \nabla^2 \psi_1 + 2\psi_1] = 0.
$$

Noticing that this requirement is satisfied by any $\psi_1$ that reproduces the pattern of the basic field (2.3), we solve (2.11) by using

$$
\psi_1 = a_{10} + a_{11} \sin(x) \sin(y) + a_{12} \sin(x) \cos(y) + a_{13} \cos(x) \sin(y) + a_{14} \cos(x) \cos(y),
$$

where $a_{ij}$ depend on $(X, Y, T)$. At the second order in $\varepsilon$, we expect the interaction of (2.12) with the basic field to generate higher harmonics. Thus, for $N = 1$, we adopt the following form for the second-order component:

$$
\psi_2 = a_{20} + a_{21} \cos(x + y) + a_{22} \sin(x + y) + a_{23} \cos(x - y) + a_{24} \sin(x - y)
$$

$$
+ a_{25} \cos(2x) + a_{26} \sin(2x) + a_{27} \cos(2y) + a_{28} \sin(2y),
$$

substitution of which in the second-order balance of (2.4) yields, after collecting coefficients of various Fourier harmonics, the following expressions for the second-order coefficients:

$$
a_{25} = -\frac{\partial a_{13}}{\partial T} - \frac{\partial \psi_0}{\partial X}, \quad a_{27} = \frac{\partial a_{12}}{\partial T} - \frac{\partial \psi_0}{\partial Y}, \quad a_{26} - a_{24} = \frac{\partial a_{14}}{\partial T},
$$

as well as a solvability condition for the first-order components, $a_{11} = 0$. Similarly, at the third order in $\varepsilon$, we use

$$
\psi_3 = a_{30} + a_{31} \cos(x + y) + a_{32} \sin(x + y) + a_{33} \cos(x - y) + a_{34} \sin(x - y)
$$

$$
+ a_{35} \cos(2x) + a_{36} \sin(2x) + a_{37} \cos(2y) + a_{38} \sin(2y),
$$

substitution of which in the third-order balance of (2.4) yields, after collecting coefficients of various Fourier harmonics, the following expressions for the third-order coefficients:
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and the corresponding balances of (2.4) yield a series of solvability conditions on lower-order modes

\[
\begin{align*}
4 \frac{\partial a_{25}}{\partial T} + \frac{\partial a_{12}}{\partial Y} + \frac{\partial a_{13}}{\partial X} &= 0, \\
\frac{\partial a_{14}}{\partial X} + 4 \frac{\partial^2 a_{13}}{\partial T^2} + 4 \frac{\partial^2 \psi_0}{\partial X \partial T} &= 0, \\
4 \frac{\partial^2 a_{14}}{\partial T^2} + 4 \frac{\partial a_{25}}{\partial T} &= \frac{\partial a_{12}}{\partial Y} + \frac{\partial a_{13}}{\partial X}, \\
\frac{\partial a_{14}}{\partial Y} + 4 \frac{\partial^2 a_{12}}{\partial T^2} &= 4 \frac{\partial^2 \psi_0}{\partial Y \partial T}.
\end{align*}
\] (2.16)

Sequentially eliminating \(a_{12}, a_{13}\) and \(a_{25}\) in (2.16), we arrive at

\[
\frac{\partial^2 a_{14}}{\partial X^2} + 4 \frac{\partial^3 \psi_0}{\partial X^2 \partial T} + 8 \frac{\partial^4 a_{14}}{\partial T^4} - 4 \frac{\partial^3 \psi_0}{\partial Y^2 \partial T} + \frac{\partial^2 a_{14}}{\partial Y^2} = 0.
\] (2.17)

Next, we average the governing equation (2.4) over the short \((x, y)\) scales and, at the leading order in \(\varepsilon\) (which happens to be the fifth order), we obtain

\[
\frac{\partial a_{14}}{\partial X^2} + 2 \frac{\partial^3 \psi_0}{\partial X^2 \partial T} + 2 \frac{\partial^3 \psi_0}{\partial Y^2 \partial T} - \frac{\partial a_{14}}{\partial Y^2} = 0.
\] (2.18)

Our final step is to eliminate \(a_{14}\) between (2.17) and (2.18), which results in a linear partial differential equation (PDE) for \(\psi_0\)

\[
6 \frac{\partial^4 \psi_0}{\partial X^2 \partial Y^2} + 8 \frac{\partial^6 \psi_0}{\partial T^4 \partial X^2} + 8 \frac{\partial^6 \psi_0}{\partial T^2 \partial Y^2} = \frac{\partial^4 \psi_0}{\partial X^4} + \frac{\partial^4 \psi_0}{\partial Y^4}.
\] (2.19)

Substitution of normal modes \(\psi_0 \propto \exp(iKX + iLY + \lambda T)\) in (2.19) yields the growth rate equation

\[
-6K^2L^2 + K^4 + L^4 + 8\lambda^4(K^2 + L^2) = 0,
\] (2.20)

which is conveniently rewritten in polar coordinates \((\kappa, \theta)\) such that \(K = \kappa \cos \theta,\ \ L = \kappa \sin \theta\) as:

\[
\kappa^2 \cos(4\theta) + 8\lambda^4 = 0,
\] (2.21)

where \(\kappa = \sqrt{K^2 + L^2}\) is the two-dimensional wavenumber. Equation (2.21) indicates that the growth rate is proportional to \(\sqrt{\kappa}\) and the dependence of \(\lambda\) on \(\theta\) is shown in figure 1, where we plot the maximum real part of \(\lambda\), normalized by \(\sqrt{\kappa}\). For \(\theta = \pi/2\) – the large-scale flow oriented in the \(x\)-direction – the maximum growth rate is \(\lambda_r = (2^{3/4}/4)\sqrt{\kappa}\) and the corresponding imaginary component is \(\lambda_i = \pm(2^{3/4}/4)\sqrt{\kappa}\). Reproducing the foregoing derivation for \(N = 2\) results in a minimal change in the growth rate. For \(\theta = \pi/2\), the maximum growth rate is \(\lambda_r = (126^{1/4}/8)\sqrt{\kappa}\) and the imaginary component is \(\lambda_i = \pm(126^{1/4}/8)\sqrt{\kappa}\) – the difference relative to the \(N = 1\) calculation is only 0.4%.

3. Physical interpretation

The finite imaginary component of the growth rate \((\lambda_i)\) indicates that the system is overstable: the evolution of growing modes is characterized by periodic oscillations of ever-increasing magnitude. Analysis of the interaction between the basic state and the large-scale perturbation in the analytical model suggests the following physical interpretation.
Consider the non-uniform parallel large-scale flow in the $x$-direction impinging on a chessboard array of vortices of alternating sign (corresponding to the basic field (2.3)) as indicated in figure 2. The immediate consequence of their interaction would be the differential advection of eddies (figure 2a). Eddies located near the maximal large-scale current (the B row of eddies in figure 2a) are advected farther in $x$ than eddies in the weaker part (rows A and C). This process essentially reflects the first-order asymptotic balances, namely the generation of the mode proportional to $u(Y)(\partial/\partial x)[\sin(x)\sin(y)] = u(Y)\cos(x)\sin(y)$ – the key player in our theory.

The second stage of the interaction, represented by (2.13), is characterized by the transverse displacement of the individual vortices. The differential advection of vortices in the $x$-direction perturbs the perfect balance of the basic state. Consider, for instance, vortex $B_2$ in figure 2(b) and its interaction with the adjacent vortices. Originally (figure 2a), vortices $A_1$ and $C_3$ had compensating influences on vortex $B_2$, as vortices $A_3$ and $C_1$ had. This is no longer the case for the perturbed state (figure 2b), in which vortex $B_2$ is located closer to vortices $A_3$ and $C_3$ than to vortices $A_1$ and $C_1$. Thus, the net advection of $B_2$ by the vorticity field of these vortices $(a_1 + a_3 + c_1 + c_3)$ is now controlled by the contribution from $a_3 + c_3$, which tend to move $B_2$ in the negative $y$-direction. Applying these considerations to other vortices, we discover the tendency for all positive (cyclonic) vortices to move in the negative $y$-direction. The negative (anticyclonic) vortices are displaced in the positive $y$-direction. The cumulative effect of these displacements is indicated in figure 2(c). Since cyclones are displaced to the right of the current and anticyclones to the left, the large-scale pattern develops a perturbation of dipolar structure: positive (negative) vorticity in the $Y<0$ ($Y>0$) regions, respectively. This dipolar pattern has a strong adverse effect on the flow – it accelerates in the negative $x$-direction. This adverse tendency persists until the $x$-displacement of the individual vortices is reversed. During this
period, eddies not only change the direction of the large-scale current, but accelerate it to the level exceeding its original magnitude, which explains the origin of overstable oscillations identified by linear stability analysis in §2.

4. Weakly nonlinear expansion

To gain insight into the nonlinear dynamics of the system, we extend the analysis by retaining the nonlinear terms in (2.1). Aside from their inclusion, the foregoing (§2) derivation requires only minimal modifications. Nonlinear effects enter the system when the leading-order streamfunction is $O(1)$, and therefore, (2.7) can be applied
directly. The resulting set of asymptotic equations (for \( N = 2 \) truncation) is given by

\[
\begin{align*}
\frac{\partial a_{14}}{\partial X^2} + 2 \frac{\partial^3 \psi_0}{\partial X^2 \partial T} + 2 \frac{\partial^3 \psi_0}{\partial Y^2 \partial T} - \frac{\partial a_{14}}{\partial Y^2} &= 0, \\
35 \frac{\partial^3 \psi_0}{\partial Y^2 \partial T} - \frac{245}{32} \frac{\partial^2 a_{14}}{\partial Y^2} - 168 a_{14} \left( \frac{\partial a_{14}}{\partial T} \right)^2 - 84 a_{14}^2 \frac{\partial^2 a_{14}}{\partial T^2} - 80 \frac{\partial^4 a_{14}}{\partial T^4} &= 0, \\
-35 \frac{\partial \psi_0}{\partial X^2 \partial T} - \frac{245}{32} \frac{\partial^2 a_{14}}{\partial X^2} &= 0.
\end{align*}
\]

To simplify analysis, consider one-dimensional large-scale flows \( \psi_0 = \psi_0(T, Y) \), \( a_{14} = a_{14}(T, Y) \), in which case system (4.1) reduces to

\[
\begin{align*}
\frac{\partial \psi_0}{\partial T} &= a_{14}, \\
315 \frac{\partial^2 a_{14}}{\partial Y^2} - 168 a_{14} \left( \frac{\partial a_{14}}{\partial T} \right)^2 - 84 a_{14}^2 \frac{\partial^2 a_{14}}{\partial T^2} - 80 \frac{\partial^4 a_{14}}{\partial T^4} &= 0.
\end{align*}
\]

Before proceeding with the numerical solutions of this system, we should mention a considerable difficulty in treating the large-scale equations (4.2): the increase and divergence of the linear growth rate with the wavenumber implies that the problem is mathematically ill-posed. However, this ultraviolet catastrophe in the model may be unphysical, since the stability analysis itself is valid only for wavelengths exceeding the characteristic eddy scale.

To surmount the numerical difficulties associated with ill-posedness, it becomes necessary to modify the governing equations (4.2) by introducing a selective dumping of high harmonics which has a minor influence on low ones. For instance, in the following simulation, we use a diffusive operator \( \mu (\partial^2 / \partial Y^2) \) with \( \mu = 10^{-4} \), which was applied to \( a_{14} \) and its derivatives. The numerical solution of (4.2) is based on the pseudo-spectral method, analogous to that in Radko (2005). The computational interval \( (0 < Y < 2\pi) \) is resolved by 128 elements. We impose the periodic conditions at \( Y = 0, 2\pi \) and integrate the resulting system in time. Figure 3 presents the numerical solution of (4.2) for the initial condition

\[
\psi_0 \big|_{T=0} = 0, \quad a_{14} \big|_{T=0} = 10^{-5} \sin(Y), \quad \frac{\partial a_{14}}{\partial T} \big|_{T=0} = \frac{\partial^2 a_{14}}{\partial T^2} \big|_{T=0} = \frac{\partial^3 a_{14}}{\partial T^3} \big|_{T=0} = 0.
\]

Initially, the evolutionary pattern \( (T \leq 30) \) is characterized by the growth and statistical equilibration of the individual harmonics, whereas the second stage of the simulation \( (T \geq 30) \) involves the irregular slow oscillations of the system about a quasi-steady equilibrium level. Typical patterns of the streamfunction in figure 4 indicate the propensity of the system to produce well-defined systems of spatially alternating currents. These structures retain complex and non-periodic time dependence throughout the experiment. At any given point in space, the amplitude and even the direction of the flow varies in an incoherent and seemingly unpredictable manner. Note, however, that the complete reorganizations of the flow occur on relatively long time scales, much slower than the primary large-scale instability.

We experimented with several other damping schemes and discretizations, and detected no qualitative differences in results. Of course, our model is susceptible to the so-called sensitive dependence on initial conditions – a common feature of all chaotic systems (e.g. Lorenz 1963) – and therefore specific realizations are affected by even slightest changes in formulation and/or initialization. Fortunately, the general
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Figure 3. Solution of the large-scale equation (4.2). (a) The amplitude of the sin(Y) harmonic of the large-scale streamfunction ($B_1$) is plotted as a function of time. (b) The absolute value of $B_1$ is plotted on the logarithmic scale during the initial period of linear growth. Periodic oscillations modulating the exponential growth are associated with the finite imaginary part of the growth rate and reflect the overstable dynamics of the system.

The minimal description of this nonlinear system is given by its single mode Fourier truncation: $\psi_0 = \Psi \sin(Y)$ and $a_{14} = A \sin(Y)$. Denoting the first, second and third derivatives of $A$ in time as $B$, $C$ and $D$, we reduce the truncated form of (4.2) to a set of ordinary differential equations

$$\begin{align*}
\frac{dA}{dt} &= B, \\
\frac{dB}{dt} &= C, \\
\frac{dC}{dt} &= D, \\
\frac{dD}{dt} &= -\frac{63}{512} A - \frac{21}{10} AB - \frac{21}{10} A^2 C.
\end{align*}$$

While this heavily truncated system is not formally justified in terms of its connection with the multimode solutions in figure 4, its analysis could bring some insight into the dynamics predicted by the large-scale equations. In figure 5, we present results of the numerical integration of (4.4) with an initial condition $(A, B, C, D) = (0.1, 0, 0, 0)$. The patterns of various projections of the orbit (figure 4a–d) are indicative of the chaotic nature of our solutions – a feature that is, apparently, retained by the multimode counterparts of (4.4). The solution spends extensive periods spiralling in an isolated region and then suddenly moves to a different location. These transitions reflect changes in the direction of the current. However, such events are
rather infrequent; during the periods in between these transitions, the large-scale flow pattern can be regarded as persistent and statistically steady. A more detailed analysis of this nonlinear system can be carried out by using conventional methods of dynamical systems theory (e.g. Drazin 1992).

5. Numerical simulations

To determine how well the foregoing asymptotic predictions are realized in the fully nonlinear context, we now turn to numerical simulations of the original vorticity equation (2.1). The numerical model is based on a pseudospectral method (Canuto et al. 1987). The size of the computational domain used in the following calculation was $L_x = L_y = 32\pi$, and the integration was initiated using the basic solution (2.3), slightly perturbed by the fundamental harmonic in $y$

$$\psi = \sin(x) \sin(y) + 10^{-5} \sin \left( 2\pi \frac{y}{L_y} \right). \quad (5.1)$$

The first term in (5.1) represents the basic eddy field (2.3) and the second is the small amplitude large-scale perturbation.

The flow was represented by $N_x = N_y = 64$ non-zero Fourier harmonics, which means that the wavelength of shortest harmonics used in the spectral model (in $x$ and in $y$) is only $L_x/2$. The individual eddies in (2.3) are resolved and so the wavelengths are half as short – these scales appear naturally through nonlinear
interactions – but smaller wavelengths are excluded. Use of the short wave filtering is intentional. Our preliminary highly resolved experiments (not shown) indicate that the late stage evolution of the flow field exhibits instabilities with wavelengths much less than the periodicity of eddies themselves. These super-small-scale instabilities are essential in the downscale cascade of vorticity, associated upscale cascade of energy (Kraichnan 1967), and the destruction of the original array of vortices (∼Le). These inherently two-dimensional effects are of secondary interest for our purpose. They complicate the direct comparison with the analytical model, making it desirable to exclude super-small-scale instabilities from the analysis. Furthermore, it should be noted that, unlike the pure two-dimensional turbulence, the ocean is full of intense and long-lived mesoscale eddies, capable of surviving thousands of revolutions. In order to represent such robust coherent structures in our idealized calculations, it seems sensible to extend the lifespan of the original vortices by suppressing their small-scale instabilities.
Figure 6. Direct numerical simulation of the two-dimensional vorticity equation (2.1). (a) The amplitude \( B_1 \) of the fundamental \( \sin(Y) \) harmonic of the \( x \)-averaged streamfunction is plotted as a function of time. (b) The absolute value of \( B_1 \) is plotted on the logarithmic scale during the initial period of linear growth. The evolutionary pattern of the fundamental harmonic is consistent with its asymptotic description in figure 3.

To connect model results with our asymptotic formulations, we define the large-scale spatial variables as \( (X, Y) = (2\pi/L_y)(x, y) \), which implies that \( \varepsilon = 1/4 \). Similarly, we introduce the corresponding large-scale time variable \( T = \varepsilon t \). In figure 6, we present the amplitude of the fundamental harmonic of the streamfunction field \([B_1 \sin(Y)]\) as a function of time. First, \( B_1 \) exhibits a series of overstable oscillations before equilibrating (at \( T \sim 30 \)) at the level comparable to the amplitude of the background eddies. The subsequent evolution of this harmonic is characterized by a series of irregular oscillations about the quasi-equilibrium magnitude – a pattern that is qualitatively consistent with the asymptotic model in figure 3. The typical spatial patterns of the \( x \)-averaged streamfunction \( \langle \psi \rangle \) in figure 7 are similar to those predicted by the asymptotic model (figure 4), although the magnitudes realized in the direct simulations are generally lower.

To be more quantitative in our comparison of the direct simulations and the asymptotic model, we now compute the linear growth rates of the large-scale instabilities. We performed a series of simulations in which we varied the length of the large-scale perturbation \( (L_y = 16\pi, 32\pi, 64\pi, 128\pi, L_x = L_y) \). The amplitude of the fundamental harmonic \( (B_1) \) was recorded during the period of initial linear growth and the growth rate was evaluated from the best fit of \( B_1(t) \) and the normal mode \( a \exp(\lambda r t) \sin(\lambda_1 t + \varphi) \) with \( (a, \lambda_r, \lambda_1, \varphi) \) as parameters. The theoretical growth rate of the \( x \)-oriented \( (\theta = \pi/2) \) large-scale modes was estimated in §2 to be \( \lambda_r = (1261/4/8)\kappa^{1/2} \). Note that this estimate was made in terms of large-scale variables...
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Figure 7. Profiles of the $x$-averaged streamfunction $\langle \psi \rangle$ at various times obtained from the numerical solution of (2.1). Compare with the asymptotic counterparts in figure 4.

$(T,Y)$. This growth rate can be readily expressed in terms of our primary (small-scale) variables by applying the transformation

$$
\begin{align*}
\lambda^*_r &= \varepsilon \lambda_r, \\
\kappa^* &= \varepsilon^2 \kappa.
\end{align*}
$$

The asterisks are used hereafter to denote the large-scale quantities measured in terms of small-scale spatial and time units. As expected, we find that the expression for the growth rate is invariant with respect to this change of variables

$$
\lambda^*_r = \frac{126^{1/4}}{8} \sqrt{\kappa^*},
$$

which makes it possible to readily compare our theoretical prediction with the direct simulations. The results are presented in figure 8, where we plot the growth rates of unstable modes as a function of their wavenumbers in logarithmic coordinates. The numerical simulations (indicated by plus signs) are generally consistent with the theoretical prediction (solid line), exceeding it by only 10–30%. The variation in the growth rate with wavenumber closely follows the theoretical $\sqrt{\kappa^2}$ pattern. All these similarities indicate that our asymptotic model captures the essence of the interaction between eddies and the large-scale field.
Figure 8. Growth rates of the unstable large-scale modes in the direct simulations (plus signs) are plotted as a function of the wavenumber, along with the theoretical prediction indicated by the straight line.

6. Generalizations

The foregoing analysis revealed the existence of rapidly growing ($\lambda \propto \sqrt{\kappa}$) instabilities – the eddy-explosion modes – which have not been observed in earlier theoretical models, largely focused on anisotropic and/or viscous flows. To clarify the relation of our model to the previous work, we now examine various extensions. The effects of viscosity, stretching of water columns and aspect ratio of eddies are taken into account and the broader parameter space is explored in an attempt to identify regions susceptible to eddy-explosion instabilities. Thus, the governing equation (2.1) is generalized to

$$\frac{\partial}{\partial t} (\nabla^2 \psi - \alpha \psi) + J(\psi, \nabla^2 \psi) + \frac{1}{Re} \nabla^4 (\psi - \bar{\psi}) = 0,$$

(6.1)

where the parameter $\alpha = L_e^2/R_d^2$ quantifies the effect of stretching and squeezing water columns in the quasi-geostrophic approximation (e.g. Pedlosky 1987), $R_d$ is the radius of deformation, and $Re = A_e/v$ is the Reynolds number. The basic field (2.3) is also generalized as

$$\bar{\psi} = \sin(x) \sin(ry),$$

(6.2)

where $r$ is the aspect ratio of the background eddies. Without loss of generality, we consider $r < 1$. An attempt to reproduce the derivation in §2 with the finite $O(1)$ Reynolds number leads to a set of equations which do not allow for the unstable modes in the eddy-explosion sector (2.5). This profound effect of viscosity on our solutions is consistent with the numerical Floquet-based calculations (see the Appendix). In order to gain some insight into the effect of weak friction, we re-scale the Reynolds number as follows:

$$\frac{1}{Re} = \varepsilon \frac{1}{Re_0}.$$

(6.3)
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The solutions with finite and infinite radius of deformation require different treatment and therefore these cases are discussed separately below.

6.1. Barotropic model

The solution of the generalized barotropic ($\alpha = 0$) problem proceeds in the same manner as in the particular case described in §2. The governing equation (6.1) is linearized with respect to the basic state (6.2) and the solution is sought in terms of series (2.7). For the analysis in this section, we consider the $N = 2$ truncation of the individual terms (2.9), as a compromise between the accuracy and tractability. The expansion opens up with the streamfunction component (2.8) that varies only on long spatial and temporal time scales. Substitution of the asymptotic series in the linearized governing equation yields a system of PDEs, which are analysed by using the normal modes. The result is a lengthy algebraic expression relating the growth rate to the wavenumber, taking the form

$$FBT_2(\lambda, \kappa, \theta, r, Re) = 0.$$  \hspace{1cm} (6.4)

The expression for the growth rate simplifies considerably in the following limiting cases.

6.1.1. Effects of the anisotropy

To examine the effects of the anisotropy, we consider the inviscid limit ($Re = \infty$), and vary the aspect ratio of the background eddies ($r$). In this case, the growth rate equation reduces to

$$\lambda^4 + \kappa^2 \left[ f_0(r) + f_2(r) \cos 2\theta + f_4(r) \cos 4\theta \right] = 0,$$  \hspace{1cm} (6.5)

where

$$f_0 = \frac{r^4(3r^8 - 79r^6 + 348r^4 - 79r^2 + 3)}{4(r^2 + 1)(r^2 - 9)(9r^2 - 1)(3r^4 - 34r^2 + 3)},$$

$$f_2 = -\frac{r^2(r^2 - 1)(27r^4 - 266r^2 + 27)(3r^8 - 79r^6 + 348r^4 - 79r^2 + 3)}{64(r^2 - 9)(9r^2 - 1)(3r^4 - 34r^2 + 3)(r^2 + 1)^2},$$

$$f_4 = -\frac{r^2(3r^8 - 79r^6 + 348r^4 - 79r^2 + 3)}{32(r^2 + 1)(3r^4 - 34r^2 + 3)}.$$  \hspace{1cm} (6.6)

The structure of (6.5) indicates that, for any given $r$ and $\theta$, the growth rate is proportional to $\sqrt{\kappa}$. Figure 9 presents the maximum growth rate, normalized by $\sqrt{\kappa}$, as a function of $\theta$ for various values of $r$. As $r$ decreases, the pattern gradually changes and the average growth rates monotonically decrease. Note that for $r \to 0$, roots of the growth rate equation (6.5) uniformly converge to zero. This suggests that the highly anisotropic background patterns, exemplified by the Kolmogorov flow, may not be susceptible to the eddy-explosion instabilities.

Figure 10 attempts to validate the asymptotic theory by comparing its predictions to the numerical calculations based on the Floquet theory, described in the Appendix. To facilitate the comparison, the growth rates ($\lambda_r$) and wavenumbers ($\kappa$) of the unstable modes are expressed in terms of the primary (small scale) units by applying the transformation (5.2). As previously, we find that the growth rate equation (6.5) is invariant with respect to this transformation. To be specific, we consider $\theta = \pi/2$, representing the large-scale flow oriented in the $x$-direction, and plot $\lambda_r$ as a function of $\kappa^*$ for various values of $r$ in logarithmic coordinates. The theoretical relations indicated by the dashed lines in figure 10 agree well with the corresponding numerical
Figure 9. The growth rates of the unstable long-wavelength modes, normalized by \( \sqrt{\kappa} \) as a function of their polar angle, \( \theta \), for various values of the aspect ratio (\( r \)).

Figure 10. Effects of aspect ratio. Growth rates of the unstable large-scale modes deduced from the numerical Floquet-based calculations (solid lines) and from the analytical multiscale model (dashed lines) are plotted as a function of the wavenumber (\( \kappa^* \)) in logarithmic coordinates for various values of the aspect ratio (\( r \)).
Viscous effects. The growth rates of the unstable large-scale modes deduced from the numerical Floquet-based calculations (solid curves) and from the analytical multiscale model (dashed curves) are plotted as a function of the wavenumber for various values of the Reynolds number along with the prediction of the inviscid theory (black line). For $Re \geq 1000$, the growth rates are close to the prediction of the inviscid theory. However, the agreement rapidly deteriorates with decreasing $Re$.

relations (solid lines). In terms of the absolute values, the agreement deteriorates for low values of $r$, although the numerical data remain aligned along straight lines with slope corresponding to the eddy-explosion power law $\lambda_r^* \propto \sqrt{\kappa^*}$.

6.1.2. Viscous effects

The effects of viscosity on long-wavelength instabilities are illustrated using an example of the background eddies with unit aspect ratio ($r = 1$). The growth rate equation (6.4) in this case reduces to

$$61952\lambda R e_0^{-4} + 84480 R e_0^{-3} \lambda^2 + 40064 \lambda^3 R e_0^{-2} + 308 R e_0^{-1} \kappa^2 + 7680 \lambda^4 R e_0^{-1}$$

$$+ 7 \lambda \kappa^2 + 512 \lambda^5 + \kappa^2 (308 R e_0^{-1} + 56 \lambda) \cos(4\theta) = 0,$$

which further simplifies when we consider the $x$-oriented modes ($\theta = \pi/2$) and revert to the primary (small-scale) units as

$$61952\lambda^* R e^{-4} + 84480 R e^{-3} \lambda^{*2} + 40064 \lambda^{*3} R e^{-2} + 616 R e^{-1} \kappa^{*2} + 7680 \lambda^{*4} R e^{-1}$$

$$+ 63 \lambda^{*2} \kappa^{*2} + 512 \lambda^{*5} = 0.$$

Figure 11 shows the theoretical $\lambda_r^*(\kappa^*)$ relations obtained by solving (6.8), indicated by the dashed curves, along with the corresponding numerical Floquet-based calculations (solid curves). The results reveal a surprisingly strong sensitivity of our system to viscous dissipation. For $Re \geq 1000$, the growth rates and their
dependence on wavenumber are close to the prediction of the inviscid eddy-explosion scaling \( \lambda^* \propto \sqrt{\kappa^*} \), indicated by the black solid line. However, the agreement rapidly deteriorates with decreasing \( Re \); the growth rates monotonically decrease. For \( Re = 200 \), the growth rates differ from the inviscid limit by as much as an order of magnitude and their dependence on \( \kappa^* \) bears little resemblance to the corresponding inviscid relation. The weakly viscous theory also has only limited success in predicting the growth rates. The general character of the theoretical \( \lambda^*(\kappa^*) \) relations is consistent with the numerical results: both converge to the inviscid relation for relatively large \( \kappa^* \) and both are characterized by the steep decrease of the growth rate for small \( \kappa^* \). However, the weakly viscous theory consistently overestimates the growth rates. The mismatch between the theory and numerics increases with the decreasing Reynolds number, which is an expected consequence of the model formulation – the weakly viscous theory explicitly assumes large \( Re \) in (6.3).

6.2. Equivalent-barotropic model

Our final extension attempts to gain some preliminary insight into the three-dimensional dynamics associated with stretching/squeezing of fluid columns. In oceanic flows, the vertical stretching is driven by the vertical excursions of the free surface and density interfaces. While almost all multiscale models are focused on purely two-dimensional dynamics, the possibility that the stretching effect could substantially modify the stability properties of our system cannot be ruled out \textit{a priori}.

The minimal representation of the stretching/squeezing of water columns is given by the quasi-geostrophic approximation (6.1) (e.g. Pedlosky 1987). The magnitude of the stretching effect associated with the elevation of the free surface is controlled by the barotropic radius of deformation \( R_d = \sqrt{gH/f} \), thus affecting only very long wavelengths \( \sim 2\pi R_d \). For typical oceanic conditions (\( g \approx 9.8 \text{ m s}^{-2} \), \( f \sim 10^{-4} \text{ s}^{-1} \), \( H \sim 3 \text{ km} \)), the free-surface variation affects wavelengths of the order of 10 000 km or more, thus being largely irrelevant for the process under consideration – formation of jets in an eddying flow. The variation in the depth of density interfaces is potentially more significant. The equivalent-barotropic model, which represents an active homogeneous layer of light fluid overlying the deep passive dense layer, is still described by (6.1), although \( R_d \) in this case represents the baroclinic radius of deformation \( R_d = \sqrt{g' H/f} \), where \( g' \) is the reduced gravity – the product of gravity and relative variation in fluid density. The baroclinic radius of deformation (40 km) is considerably less and therefore its inclusion can lead to substantial modification of two-dimensional results.

An important distinction of the equivalent barotropic model becomes apparent after consideration of the spatially averaged vorticity equation (6.1):

\[
\frac{\partial}{\partial t}(\nabla^2 \hat{\psi} - \alpha \hat{\psi}) + J(\hat{\psi}, \nabla^2 \hat{\psi}) + \frac{1}{Re} \nabla^4 \hat{\psi} = 0,
\]

(6.9)

where hats denote averaging over the scales of the periodicity of the background field. Whereas the leading-order term of (6.9) in the barotropic (\( \alpha = 0 \)) model was \( (\partial/\partial t) \nabla^2 \hat{\psi} \sim \epsilon^5 \hat{\psi} \), the addition of the equivalent-barotropic component makes the second term in (6.9) a dominant one: \( \alpha (\partial/\partial t) \hat{\psi} \sim \epsilon \hat{\psi} \). This suggests that the eddy-explosion scaling (2.5) can be applied to the equivalent-barotropic model provided that the term (2.8) varying on long scales is shifted from its position as the leading component of the asymptotic expansion to \( O(\epsilon^4) \).

Aside from this caveat, the model development proceeds in the same way as previously. The governing equation (6.1) is linearized with respect to the basic state.
Figure 12. Equivalent-barotropic model. The growth rates of the unstable large-scale modes deduced from the numerical Floquet-based calculations (solid lines) and from the analytical multiscale model (dashed lines) are plotted as a function of the wavenumber for various values of the radius of deformation, along with the theoretical prediction of the barotropic model indicated by the heavy black line.

(6.2) and the solution is sought in terms of series (2.7). The representation of the individual terms in the expansion is simplified by using the $N = 2$ truncation and the substitution of the asymptotic series in the governing equation yields a system of PDEs, which are analysed by using the normal modes. The result is again a lengthy algebraic expression relating the growth rate to the wavenumber, this time taking the form

$$F_{EB2}(\lambda, \kappa, \theta, r, Re, \alpha) = 0.$$  \hfill (6.10)

To be specific, we consider the inviscid model with unit aspect ratio of background eddies ($Re = \infty, r = 1$), in which case expression (6.10) reduces considerably to

$$841\lambda^4\alpha^4 + 10788\lambda^4\alpha^3 + 49444\lambda^4\alpha^2 + 95232\lambda^4\alpha - 6272\kappa^2 + 65536\lambda^4 = 0.$$  \hfill (6.11)

It is interesting that the growth rate in this limit becomes independent of the orientation of the large-scale modes ($\theta$). The structure of the growth rate equation (6.11) also implies that, for any given $\alpha$, the growth rates are proportional to $\sqrt{\kappa}$, as we have seen in all foregoing inviscid eddy-explosion models. In figure 12, we attempt to validate the asymptotic prediction by comparing it with the numerical Floquet-based calculations. We first revert the expression (6.11) to the small-scale units ($\lambda^*, \kappa^*$) and then systematically vary $\alpha$ and plot the resulting theoretical $\lambda^*(\kappa^*)$ dependencies (dashed lines in figure 12) in the logarithmic coordinates. In the same figure, we also plot the corresponding numerical results, indicated by the solid lines. The analytical and numerical results agree well over a wide range of $\alpha$ in figure 12. The numerical data are consistent with eddy-explosion power law $\lambda^*_r \propto \sqrt{\kappa^*}$. The coefficient of this power law decreases with $R_d$, reflecting the stabilizing tendency of three-dimensional
dynamics, and this trend is adequately captured by the analytical theory. In figure 12, we also plot the corresponding theoretical barotropic ($\alpha = 0$) relation, indicated by the heavy black line. For small $\alpha$ cases ($\alpha = 0.1$ and 1), the barotropic and equivalent barotropic are sufficiently close – the growth rates differ by less than 20%. However, for larger $\alpha$, the difference can exceed an order of magnitude.

7. Discussion

This paper presents a mechanistic model for the generation of large-scale structures in an eddying flow. The eddy field is represented by the closely packed array of standing coherent vortices, intensity of which is weakly modulated by the long-wavelength perturbations introduced into the system. While our solutions are conceptually analogous to the Kolmogorov-type models, there are fundamental differences in formulation and in results. We exclude the external forcing and viscous dissipation, largely because of the uncertainties with regard to their roles in the dynamics of geophysical eddies. Our asymptotic solutions are based primarily on scale separation rather than on the proximity to the marginally stable balance between forcing and dissipation – an approach used in many earlier models. We also abandon the parallel flow model of the background eddies in favour of a more isotropic formulation (2.2), which seems to be more relevant for atmospheric and oceanic vortices.

Still, we readily admit that the chosen pattern of eddies remains highly idealized and somewhat arbitrary. The long-term evolution of our system will be characterized by the cascade of energy (enstrophy) to progressively larger (smaller) scales, ramifications of which we have not considered. Our interests and intensions lie in a different direction. We believe that simple analytical solutions can clarify, on a very basic and intuitive level, the dynamics of the interplay between eddies and the long-wave perturbations. While eddies observed in geophysical systems are irregular and chaotic, several aspects of their collective evolution could be governed by the same principles as a tidy array of standing eddies in our model. We have striven for a compromise between realism and tractability, an approach taken by other deterministic eddy-interaction theories.

Using the multiscale expansion, we have discovered a new class of rapidly growing long-wavelength instabilities – the eddy-explosion modes – which do not arise in forced-dissipative models. This theoretical prediction is supported by the numerical linear Floquet-based calculations (see the Appendix) and by the initial-value fully nonlinear simulations (§4). We argue that the eddy-explosion instabilities could play a critical role in the spontaneous generation of the large-scale structures, which are frequently observed in geophysical flows. These instabilities persist for anisotropic background eddies, provided that their aspect ratio is finite, and within the equivalent-barotropic framework. However, our solutions are extremely sensitive to viscous dissipation. For $Re \geq 1000$, the growth rates are close to the prediction of the inviscid theory. However, the agreement rapidly deteriorates with decreasing $Re$. For $Re = 200$, the growth rates differ from the inviscid limit by as much as an order of magnitude and their dependence on the wavenumber bears little resemblance to the corresponding inviscid relation. Perhaps this sensitivity to dissipation precluded detection of eddy-explosion instabilities in models with relatively low but finite viscosity (e.g. Novikov & Papanicolaou 2001).

Weakly nonlinear analysis indicates that the linear growth of eddy-explosion modes is arrested when the amplitude of the perturbation reaches the level of background
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eddies. Solutions of the nonlinear amplitude equations are characterized by the development of sharp fronts separating the relatively quiescent zones, features that are suggestive of the oceanic patterns in eddy rich areas. The time-dependent evolution of the large-scale field persists after the equilibration of its linear growth and reveals signs of the inherently chaotic behaviour. However, typical periods of the nonlinear reorganizations and current reversals are much longer than the time scale of the primary large-scale instability, which suggests interpretation of our solutions as quasi-steady coherent structures.

To isolate the essential physics of the phenomenon, we have chosen the simplest possible framework: the largely inviscid flow on the $f$-plane described by the barotropic and equivalent-barotropic models. However, the broader significance of the proposed theoretical approach lies in the possibility of applying it to other, more complex systems, which include effects of the vertical density stratification and the background large-scale circulation. We are particularly optimistic with regard to the prospects of analytical explorations of the eddy-explosion effect in the presence of the planetary vorticity gradient. Its inclusion generally tends to suppress motion in the meridional direction relative to the zonal displacements. Therefore, in this case we anticipate the tendency of large-scale structures to align in the direction normal to the planetary vorticity gradient and to form pronounced zonal jets, such as frequently occur in nature.

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Appendix. Floquet-based calculations

To examine the stability properties of system (6.1) numerically, we first linearize it with respect to the basic state (6.2) and then compute growth rates by using a numerical method based on the Floquet theory. The basic state is written as

\[ \bar{\psi} = \frac{1}{2} \cos(x') - \frac{1}{2} \cos(y'), \]  

(A1)

where

\[ \xi = (\psi_{-M, -M}, \psi_{-M, -M+1}, \ldots, \psi_{M, M}). \]  

(A2)

The perturbation is sought in the following form:

\[ \psi' = \exp(iK^*x + iL^*y + \lambda t) \sum_{n_x = -N}^{N} \sum_{n_y = -N}^{N} \psi_{n_x, n_y} \exp(in_x x' + in_y y'), \]  

(A3)

where \( \lambda \) is the growth rate and (\( K^* \), \( L^* \)) are the Floquet coefficients, which control the long-wavelength components. These coefficients are directly related to wavenumbers (\( K, L \)) of the large-scale eddy-explosion modes discussed in this paper, and the asterisks emphasize that the Floquet-based analysis is performed in the primary small-scale units. For convenience, the two-dimensional Floquet factor is written in polar coordinates as:

\[ K^* = k^* \cos \theta, \quad L^* = k^* \sin \theta. \]  

(A4)

Substituting (A3) into the linearized governing equation (6.1) and collecting the individual Fourier components allows us to express the problem in matrix form,

\[ \lambda \xi = A \xi, \]  

(A5)
where $A$ is a square matrix of order $(2M + 1)^2 \times (2M + 1)^2$, whose elements are functions of $\kappa^*, \theta, Re, \alpha, r, M$ and

$$\xi = (\psi_{-M,-M}, \psi_{-M,-M+1}, \ldots, \psi_{M,M}).$$

The linear growth rates of normal modes correspond to the eigenvalues of matrix $A$. For each set of governing parameters, we determine the eigenvalue with maximum real part, which represents the fastest-growing mode. Experiments quantifying the impact of spectral resolution ($M$) on accuracy (not shown) indicate that $M = 8$ is sufficient to limit the relative error in growth rates to $O(10^{-3})$. This resolution is used in all calculations presented in this paper. The Floquet-based calculations carry a triple benefit of (i) supporting the analytical theory for the eddy-explosion modes, (ii) determining its region of validity and (iii) gaining preliminary insight into the significantly viscous and anisotropic regimes. The $\lambda'(\kappa^*)$ relations deduced from the Floquet calculations and their variation with $r$, $Re$ and $\alpha$ are illustrated in figures 10, 11 and 12, respectively.

REFERENCES


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McWilliams, J. C. 1984 The emergence of isolated coherent vortices in turbulent flow. *J. Fluid Mech.* **146**, 21–43.


