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LANCHESTER-TYPE MODELS OF WARFARE AND OPTIMAL CONTROL*

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ABSTRACT

The optimization of the dynamics of combat (optimal distribution of fire over enemy target types) is studied through a sequence of idealized models by use of the mathematical theory of optimal control. The models are for combat over a period of time described by Lanchester-type equations with a choice of tactics available to one side and subject to change with time. The structure of optimal fire distribution policies is discussed with reference to the influence of combatant objectives, termination conditions of the conflict, type of attrition process, and variable attrition-rate coefficients. Implications for intelligence, command and control systems, and human decision making are pointed out. The use of such optimal control models for guiding extensions to differential games is discussed.

1. INTRODUCTION

In this paper the structure of optimal fire distribution policies is examined for tactical situations described by Lanchester-type equations of warfare. This is done to provide insight into such important questions as

(1) How should fire be distributed over targets?

(2) Do target priorities change with time?

(3) Does the number of target types affect the optimal distribution of fire?

(4) Do battle termination circumstances affect the optimal allocation policies?

(5) How does the nature of the attrition process affect the optimal distribution of fire?

(6) How does the uncertainty and confusion of combat affect the optimal distribution rules?

A theory of tactical allocation is developed through the examination of a sequence of simplified models. These combat models are too simple to be taken literally but should be interpreted as indicating general principles to serve as hypotheses for subsequent computer simulation studies or field experimentation.

In 1964 Dolansky [9] noted that the Lanchester theory of combat was insufficiently developed in the area of target selection for combat between heterogeneous forces (optimal control/differential games). Even the two references cited by him, Weiss [31] and Isbell and Marlow [14], have been subsequently extended by this author [24], [26]. Since Dolansky's article, no further examples have been published in the literature except for the ones in Isaacs' book [13]. This previous work had never systematically investigated the dependence of optimal tactics upon model form.

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Several idealized fire distribution problems are examined. The combat situations are described by Lanchester-type equations over a period of time with choices of tactics available to one side and subject to change over time. These problems are solved by the mathematical theory of optimal control. A further elaboration on solution development is to be found in the author's report [21].

The body of this paper is organized in the following fashion. First, a sequence of problems is considered, and the effect on the optimal distribution of fire is examined for the following factors: objectives of the combatants, termination conditions of the conflict, number of target types, some special cases of time dependent attrition-rate coefficients, and type of attrition process. Then, two-sided extensions of such problems are discussed, but the value of studying one-sided problems as considered in this paper is pointed out. Finally, various implications of the models are discussed.

2. NOTATION

The symbols which are used in this paper are defined as follows:

$a_1, \ldots, a_n, b_1, \ldots, b_n$ = constant attrition-rate coefficients,

$a_1(t), a_2(t), b_1(t), b_2(t)$ = variable attrition-rate coefficients,

$c_i(t)$ for $i=1, \ldots, n$ = coefficient of ϕ_i in maximization problem (defined by Equation (14)),

C_i for $i=1, 2, 3, 4, 5$ = the ith part of the terminal surface ("target set") as defined in section 3.2.,

$e_i(\tau)$ for $i=1, \ldots, n$ = coefficient of ϕ_i in maximization problem (defined by Equation (23)),

$h(t)$ = variable portion of variable attrition-rate coefficient, e.g., $a_1(t)=k_{a_1}h(t)$,

H = Hamiltonian function,

k = constant of proportionality,

$k_{a_1}, k_{a_2}, k_{b_1}, k_{b_2}$ = constant portions of variable attrition-rate coefficients, e.g., $a_1(t)=k_{a_1}h(t)$,

L = singular "surface" defined by $a_1b_1x_1=a_2b_2x_2$,

L' = line (with equation $a_1px_1=a_2qx_2$) in description of solution to Problem 5,

n = number of X-force target types,

$P^0=(x_1^0, \ldots, x_n^0, y_0)$ = point in the initial state space,

p, q, r = utilities assigned per unit of surviving X_1, X_2, and Y forces, respectively,

$p_i(t)$ for $i=1, \ldots, n+1$ = dual variable corresponding to $x_i(t)\,(x_{n+1}(t)=y(t))$,

$R=a_1b_1/(a_2b_2)$,

R_i for $i=1, \ldots, n-1 = a_i(b_iw_n-b_nw_i)/(a_ib_i-a_nb_n)$,

S_i for $i=1, \underset{i\neq k}{\ldots}, n = a_i(b_iw_k-b_kw_i)/(a_ib_i-a_kb_k)$,

t = time after beginning of battle,

$t_I=T-\tau_I$ = time which separates Phase I of the battle in Problem 5 from Phase II as described in section 3.5.1.

T = total time for the battle,

T_1 = maximum possible duration for battle, i.e., $T\leq T_1$,

$V_k=\sqrt{(R_k^2-w_n^2)a_n/b_n+v^2}$,

$w_1, \ldots, w_n, v$ = utilities assigned per unit of surviving $X_1, \ldots, X_n, Y$ forces, respectively,

$W_k=a_n(b_kw_n-b_nw_k)/(a_kb_k-a_nb_n)$,

$x_1, \ldots, x_n, y$ = average force strengths; with initial values $x_1^0, \ldots, x_n^0, y_0$,

$z=\dfrac{a_1}{q}\dfrac{(b_1q-b_2p)}{(a_1b_1-a_2b_2)}=\dfrac{R-\delta}{R-1}$,

$\alpha = \frac{r}{q}\sqrt{\frac{b_2}{a_2}}$,

$\delta = a_1 p/(a_2 q)$,

δ_{ij} = Kronecker delta = $\begin{cases} 1 \text{ for } i=j, \\ 0 \text{ otherwise,} \end{cases}$

ϕ = fraction of Y-fire directed at X_1,

ϕ_i = fraction of Y-fire directed at X_i,

ϕ^* = optimal control,

τ = "backwards time" from the end of the battle; defined by $\tau = T - t$, i.e. the time remaining before the end of the battle,

τ_1, τ_2, etc. = "backwards time" of the first, second, etc., switch in tactics.

3. SOME FIRE DISTRIBUTION PROBLEMS

A theory of optimal fire distribution is developed by examining a sequence of problems and then contrasting the structures of the optimal fire distribution policies for these problems. In this manner the effect of the model's form on the optimal policy will be illustrated. Five different fire distribution problems are considered in order to study the effect of the model's form on the structure of the optimal policy by contrasting the solutions to these problems. The problems that are considered are for the optimal distribution of fire of a homogeneous force, denoted as Y, in Lanchester combat against heterogeneous forces, denoted as X_1 through X_n. These problems are summarized in Table I. The effects of the following four factors on the optimal allocation policy may be inferred from this study: number of target types, target-type attrition process, time variations in attrition-rate coefficients, and battle termination conditions.

TABLE I. *Summary of Problems Considered to Study Effect of Model Form on Optimal Fire Distribution Policy*

Problem	Number of Target Types	Target-type Attrition Process	Attrition-Rate Coefficients	Battle Termination Conditions
1	2	S	C	PD
2	2	S	C	TC
3	n	S	C	PD
4	2	S	V	PD
5	2	L	C	PD

Explanation of Symbols

Target-type Attrition Process

L = "linear-law" attrition process = attrition rate proportional to product of numbers of firers and targets

S = "square-law" attrition process = attrition rate proportional to only number of firers

Attrition-Rate Coefficients

C = constant

V = variable

Battle Termination Conditions

PD = prescribed duration battle (special case of $x_1 > 0$, $x_2 > 0$, $y > 0$)

TC = terminal control battle (fight-to-the-finish)

3.1. Battle of Prescribed Duration (Two Target Types)

The simplest fire distribution problem is for combat between an X-force of two force types (for example, riflemen and grenadiers) and a homogeneous Y-force (for example, riflemen only). This situation is shown diagrammatically in Figure 1. It is the objective of the Y-force commander to maximize his survivors at the end of battle at time T and minimize those of his opponent (considering weighting factors p, q, and r). This is accomplished through his choice of the fraction of fire, ϕ, directed at X_1. However, this idealized tactical allocation problem may be studied in two different scenarios: (1) a battle lasting a specified length of time (denoted as T_1) or (2) a battle lasting until one side or the other is totally annihilated. Each of these situations will be analyzed separately.

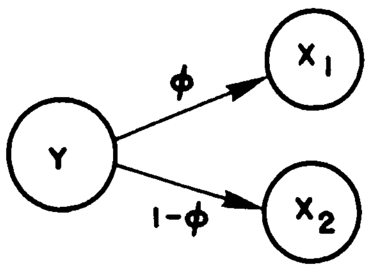

FIGURE 1. *Diagram of fire distribution problem*

Thus, Problem 1 is a prescribed duration battle. (The reader should recall that the definitions of Problems 1 through 5 are given in Table I.) It is stated in mathematical terms below.

(Problem 1) $$\underset{\phi(t)}{\text{maximize}} \{ry(T) - px_1(T) - qx_2(T)\} \text{ with } T_1 \text{ specified,}$$

$$\text{subject to: } \frac{dx_1}{dt} = -\phi a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y,$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2,$$

$$x_1, x_2, y \geq 0, \quad 0 \leq \phi \leq 1, \quad \text{and } T \leq T_1,$$

where all symbols are defined in section 2. above.

3.1.1. Optimal Policy in a Special Case

The battle lasts for $0 \leq t \leq T_1$ unless, of course, one side or the other is annihilated before T_1. To be more precise, the battle terminates under one of the following three conditions:

(1) $x_1(T) = x_2(T) = 0$ and $T \leq T_1$,

(2) $y(T) = 0$ and $T \leq T_1$,

(3) $T = T_1$,

where T denotes the time at which the battle ends. However, to avoid inessential complications only the special case in which $x_1(T) > 0$, $x_2(T) > 0$, $y(T) > 0$, and $T = T_1$ is considered for the comparisons made in this paper. In other words, those subcases in which a state variable is reduced to zero are not considered.* Thus, it is assumed that the initial force levels are such that no force type is annihilated during this prescribed duration battle.

The solution to Problem 1 for the above special case is shown in Table II. A derivation of these results is omitted, since Problem 1 may be considered to be a special case of Problem 3 for which a derivation is provided.

TABLE II. *Solution to Fire Distribution Problem (Problem 1) Battle of Prescribed Duration with Constant Attrition-Rate Coefficients; Special Case in which $x_1(T) > 0$, $x_2(T) > 0$, $y(T) > 0$*

(Nonrestrictive assumption: $a_1b_1 > a_2b_2$)

Case	Optimal Control
A: $a_1p \geq a_2q$	$\phi^*(t) = 1$ for $0 \leq t \leq T$
B: $a_1p < a_2q$	(a) for $\tau_1 \geq T$ $\phi^*(t) = 0$ for $0 \leq t \leq T$
	(b) for $\tau_1 < T$ $\phi^*(t) = \begin{cases} 1 \text{ for } 0 \leq t \leq T - \tau_1 \\ 0 \text{ for } T - \tau_1 \leq t \leq T \end{cases}$

NOTE: The "backwards" switching time is given by

$$\tau_1 = \frac{1}{\sqrt{a_2b_2}} \ln \left\{ \frac{z + \sqrt{z^2 + \alpha^2 - 1}}{1 + \alpha} \right\},$$

where $z = \frac{R - \delta}{R - 1}$, $\delta = a_1p/(a_2q)$, $R = a_1b_1/(a_2b_2)$, and $\alpha = \frac{r}{q}\sqrt{\frac{b_2}{a_2}}$.

3.1.2. Discussion of Structure of Optimal Policy

With reference to Table II, two characteristics of the optimal allocation policies for this particular prescribed duration battle are:

(1) all fire is always concentrated on one target type;

(2) the allocation is not (directly) dependent upon the force levels.

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*In the above problem x_1, x_2, and y are called *state* variables, while ϕ is called a *control* variable. A constraint such as $x_1 \geq 0$ is called a state variable inequality constraint (SVIC) and requires special treatment (see chapter 6 of [19]). Moreover, McIntyre and Paiewonsky [18] remarked in 1967 that "the optimal control problem with state space constraints does not appear to be well understood." The personal experience of this author bears this out [23].

It will be seen later that when there are more than two target types in this scenario, the solution possesses these same characteristics (even when the attrition rates change over time). Both these characteristics, however, are consequences of the assumed model form.

The first characteristic, concentration of effort on one alternative, is a consequence of the "square-law" attrition process for the X-forces. (The attrition of a target type will be referred to as being a "square-law" process when the casualty rate is proportional to the number of enemy firers only and as being a "linear-law" process when it is proportional to the product of the numbers of enemy firers and remaining targets.) It will be shown in section 3.3.2. that this makes the existence of a singular control [15] impossible, and hence the optimal allocation policies are extreme points in the control variable space.

There is, however, a very simple principle which underlies the above mathematical formalities: concentration of effort when constant marginal returns are obtained from the alternatives and the total effort is limited. Constant marginal effect over time per unit of weapon system is a property of the "square-law" attrition process; this is readily seen from considering the X_1-force attrition (when $\phi=1$)

$$\frac{\left(-\dfrac{dx_1}{dt}\right)}{y}=a_1=\begin{pmatrix}\text{rate of casualties produced per}\\ \text{unit of } Y\text{-force weapon system}\end{pmatrix}.$$

Thus there is a constant (or nondiminishing) marginal effect over time. This should be contrasted with the situation for a "linear-law" attrition of the X_1-forces,

$$\frac{\left(-\dfrac{dx_1}{dt}\right)}{y}=a_1x_1=\begin{pmatrix}\text{rate of casualties produced per}\\ \text{unit of } Y\text{-force weapon system}\end{pmatrix}.$$

In this there are diminishing effects over time from allocating a unit of Y-force weapon system against X_1, and a division of total effort (i.e., fraction of fire) may be called for. B. Koopman's 1953 article [17] contains an excellent discussion of such principles which underlie such an optimization problem. Presently, these heuristic arguments will be verified in a mathematically precise fashion when a dynamic model, which considers the interaction of forces over time and in which both X-force target types undergo "linear-law" attrition, is considered. This fundamental difference in the structure of optimal allocation policies based on the nature of target attrition makes the selection of the appropriate attrition process an essential task of analysis.

The second characteristic, the optimal allocation not (directly) dependent upon the force levels, is due to the combination of the "square-law" attrition process for the X-force types and the fixed battle length. T. It is seen that for the special case of this prescribed duration battle in which $x_1(T) > 0$, $x_2(T) > 0$, and $y(T) > 0$ the optimal distribution of fire depends only on the attrition rates of the various force types and relative weights assigned to surviving force types. This is not surprising, since the adjoint differential equations (see section 3.3.2. below) are independent of the state variables, and the values of the dual variables at the end of battle $t=T$ are independent of force strengths. It is recalled that a dual variable represents the rate of change of the payoff (battle outcome as measured by the value of surviving forces at $t=T$) with respect to a particular state variable [2].

It seems appropriate to discuss further the interpretation of the solution shown in Table II. From the above definition of the dual variables,

$$a_1 p_1(t) = \begin{pmatrix}\text{effect on outcome per unit}\\ \text{time for engaging } X_1\end{pmatrix} = \begin{pmatrix}\text{kill rate of}\\ Y \text{ against } X_1\end{pmatrix} \times \begin{pmatrix}\text{effect on outcome per}\\ \text{unit of } X_1 \text{ destroyed}\end{pmatrix}.$$

Hence, the condition $a_1 p < a_2 q$ means that at the end of the battle (recall that $p_1(t=T) = -p$, etc.) there is greater effect on battle outcome (as measured by value of survivors) per unit time per soldier for Y to engage X_2 (short term gain at the end of battle). The value of the dual variable, for example, $p_1(t)$ reflects both the value assigned X_1-force survivors and the dynamic interaction of forces over time through the Lanchester-type equations. Hence, it also accounts for the effectiveness of X_1 against Y. The quantity $a_1 b_1$ may be interpreted as representing the instantaneous rate of destruction of the X_1-force kill rate against the Y-force per unit of Y-force. Then $a_1 b_1 > a_2 b_2$ means that there is greater strategic value for engaging the X_1-force, i.e., more long range return. Thus, Case A of Table II corresponds to when there is both more long range and also short range return for engaging X_1. Case B corresponds to when there is more short term gain at the end of the battle for engaging X_2, but more long range return for engaging X_1. It is easily shown that Case A results when Y values surviving X_1-forces greater than or equal to in direct proportion to their kill rate against the Y-force, i.e., $p/q \geq b_1/b_2$. A switch in tactics (target priority) is seen to occur for this model only when value is not assigned to X_1 survivors (recall that engagement of X_1 always yields more "long range return") greater than or equal to in proportion to their destructive capability (kill rate).

The maximum principle may be interpreted as saying that a target type from several alternatives is engaged when such an engagement yields the greatest favorable effect on battle outcome per unit time. It turns out, though, that the evolution of target engagement return is dependent upon the scenario chosen for the study of the problem. This is clearly seen when we examine the "fight-to-the-finish." This is a special case of a terminal control battle (the combat ends only when the course of battle has been steered to a prescribed end state) and is chosen for mathematical convenience.

3.2. Terminal Control Battle (Two Target Types)

Problem 2 is a terminal control battle (a "fight-to-the-finish") and is stated in mathematical terms below.

(Problem 2)

$$\underset{\phi(t)}{\text{maximize}} \; \{ry(T) - px_1(T) - qx_2(T)\} \text{ with } T \text{ unspecified,}$$

$$\text{subject to: } \frac{dx_1}{dt} = -\phi a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y,$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2,$$

$$x_1, x_2, y \geq 0 \text{ and } 0 \leq \phi \leq 1,$$

where all symbols are defined in section 2. above.

The stopping rule for this battle is that the conflict terminates at $t=T$ defined by

(a) $y(T)=0$,
or (*b*) $x_1(T)=x_2(T)=0$.

Upon further analysis it has been convenient to consider that there are the following five "target sets" for Problem 2:

C_1: $x_1(T)=0$, $x_2(T)>0$, $y(T)=0$,
C_2: $x_1(T)=0$ before $x_2(T)=0$, $y(T)>0$,
C_3: $x_1(T)=0$ after $x_2(T)=0$, $y(T)>0$,
C_4: $x_1(T)>0$, $x_2(T)=0$, $y(T)=0$,
C_5: $x_1(T)>0$, $x_2(T)>0$, $y(T)=0$.

The reader should note that in the above problem statement T is referred to as being undetermined. This is because T is determined by entry to one of the above five target sets. In turn, this depends upon the control used, and hence before an allocation rule is given, it is unspecified.

Problem 2 was first studied by Isbell and Marlow [14] in 1956. However, their solution was not correct for all values of model parameters, since they did not discover the dispersal surfaces (see pp. 132–141 of [13]) present in this problem's solution for a certain range of model parameters. A more complete solution has been given by the author in a previous paper [24]. The solution principles for solving such an optimal control problem may be extended to the special class of terminal control differential games which have pure strategy solutions. This was done by the author in [26] and used to solve the supporting weapon system game of H. K. Weiss [31].

In describing the solution to Problem 2 (see [24] for the details of its development) three cases must be considered

(1) $\delta \geq 1$,
(2) $R-\sqrt{R(R-1)} \leq \delta < 1$,
(3) $0 \leq \delta < R-\sqrt{R(R-1)}$,

where $\delta = a_1 p/(a_2 q)$. The solution for each of these cases is given in [24]. Moreover, these appearingly complex results may be summarized in a particularly simple fashion* (for the nonrestrictive assumption that $R>1$, i.e., $a_1 b_1 > a_2 b_2$). When Y wins, he engages X_1 until depletion before X_2. When Y loses, he may switch from firing at X_1 entirely to firing at X_2 entirely before the X_1 force has been annihilated. This happens in Case (2) ($R-\sqrt{R(R-1)} \leq \delta < 1$) when survivors of force-type X_2 are assigned utility in excess of their Lanchester attrition-rate coefficient as compared with force-type X_1, and certain relationships hold between initial force strengths. Thus, in contrast to the prescribed duration battle (Problem 1), the optimal policy for Problem 2 may depend on initial force levels.

Finally, it seems appropriate to point out that the "backwards" switching time, denoted as τ_1, is different in these two problems. Let τ_1 (Problem 1) denote the "backwards" switching time for an optimal policy in Problem 1. It represents the optimal length of time that Y fires at X_2 before the end o

*Thus, for comparison purposes of the present paper the complete solution need not be given here.

battle at $t=T$ when $a_1p < a_2q$ (for the special case in which $x_1(T)>0$, $x_2(T)>0$, $y(T)>0$, and $\tau_1 \leq T$). It is convenient to define

$$\tau_1(a)=\frac{1}{\sqrt{a_2b_2}}\ln\frac{z+\sqrt{z^2+a^2-1}}{1+a}. \tag{1}$$

From Table II it is seen that

$$\tau_1\ (\text{Problem 1})=\tau_1\left(a=\frac{r}{q}\sqrt{\frac{b_2}{a_2}}\right). \tag{2}$$

It is assumed that r is a strictly positive quantity. Furthermore, in Problem 2 τ_1 denotes the backwards time at which ϕ^* changes from 0 to 1 (in backwards progression) with $x_1(\tau=\tau_1)>0$, i.e., the time of a change in the optimal distribution of fire without the annihilation of a target type. In [24] it was shown that

$$\tau_1\ (\text{Problem 2})=\tau_1(a=0). \tag{3}$$

Then, it is easy to show that when $\delta=a_1p/(a_2q)<1$, it follows that τ_1 (Problem 1) $<\tau_1$ (Problem 2) and in such a case Y fires at X_2 for a longer period of time in Problem 2 than in Problem 1. This is stated as Theorem 1.

THEOREM 1: Assume that $\delta<1$ and $r>0$. Then

$$\tau_1\ (\text{Problem 1})<\tau_1\ (\text{Problem 2}).$$

By observing (2) and (3) and recalling that $q,\ r>0$, we can see that the theorem follows by showing that $\frac{\partial\tau_1}{\partial a}<0$ for $\delta<1$. It is readily computed from (1) that

$$\frac{\partial\tau_1}{\partial a}=\frac{a+1-z^2-z\sqrt{z^2+a^2-1}}{\sqrt{a_2b_2}(1+a)\sqrt{z^2+a^2-1}(z+\sqrt{z^2+a^2-1})}. \tag{4}$$

Now $\delta<1$ (recalling the nonrestrictive assumption $R>1$) implies that $z>1$, so that

$$a<za<z\sqrt{z^2+a^2-1}, \tag{5}$$

since $a>0$. From (5) it follows that

$$a+1-z^2-z\sqrt{z^2+a^2-1}<o, \tag{6}$$

which proves that $\frac{\partial\tau_1}{\partial a}<0$ for $\delta<1$.

3.3. Prescribed Duration Battle with Several Target Types

The first two problems were considered in order to contrast the structures of the optimal allocation policies for different battle termination conditions. Another factor that can be examined is the number of target types. For the prescribed duration battle certain facets which tended to be obscured in the scenario with two target types are brought into sharper focus when n target types are considered. Thus, Problem 3 is a prescribed duration battle against n target types and is stated in mathematical terms below.

(Problem 3)
$$\underset{\phi_i(t)}{\text{maximize}}\ \{vy(T)-\sum_{i=1}^{n} w_i x_i(T)\} \text{ with } T_1 \text{ specified},$$
$$\text{subject to: } \frac{dx_i}{dt}=-\phi_i a_i y \text{ for } i=1,\ldots,n,$$
$$\frac{dy}{dt}=-\sum_{i=1}^{n} b_i x_i,$$
$$x_i,\ y\geq 0,\ \phi_i\geq 0 \text{ for } i=1,\ldots,n \text{ and } \sum_{i=1}^{n}\phi_i=1.$$

3.3.1. Optimal Policy in a Special Case

The battle lasts for $0\leq t\leq T_1$ unless, of course, one side or the other is annihilated before T_1. To be more precise, the battle terminates under one of the following three conditions:

(1) $x_1(T)=\ldots=x_n(T)=0$ and $T\leq T_1$,

(2) $y(T)=0$ and $T\leq T_1$,

(3) $T=T_1$,

where T denotes the time at which the battle ends. However, for the comparisons made in this paper, only the special case in which $x_1(T)>0,\ \ldots\ x_n(T)>0$, $y(T)>0$, and $T=T_1$ is considered. Thus, as done in section 3.1.1. it is assumed that the initial force levels are such that no force type is annihilated during this prescribed duration battle.

The solution to Problem 3 for the above special case is shown in Table III. A derivation of these results is given in the next section. In Table III δ_{ij} denotes the Kronecker delta and is equal to 1 for $i=j$ and zero otherwise. It is seen that the solution to Problem 3 turns out to be a generalization of that to Problem 1. However, certain aspects receive greater emphasis to provide one with a deeper understanding of the phenomena under study. In particlular, two subcases, denoted as Case A and Case B, were considered in the solution to Problem 1 (see Table II). When there are several target types, the generalization of the subcases, which it is convenient to distinguish, is as follows:

Case A, enemy survivors valued in direct proportion to their kill rate against Y-force,

Case B, enemy survivors *not* valued in direct proportion to their kill rate against Y-force.

In the first instance, Case A, target priorities keep their same relative ranking over time. If the highest priority target type is exterminated during such a battle, then fire is merely shifted to the next highest priority target. Hence, when one values enemy survivors in proportion to their kill rate against you, i.e., $w_i=kb_i$, for $i=1,\ldots,n$, the optimal tactic is to concentrate all fire on a single target type until it is entirely destroyed. The sole criterion for target selection in this instance is the quantity a_ib_i, which may be interpreted to be the rate of destruction of enemy attrition capability for his ith force type (see section 3.1.2).

TABLE III. *Solution to Fire Distribution Problem (Problem 3) Battle of Prescribed Duration with* Constant Attrition Rates; Special Case in Which $x_i(T) > 0$ *for* $i = 1, \ldots, n$ *and* $y(T) > 0$

Case	Optimal Control
A: $w_i = kb_i$ for $i = 1, \ldots, n$	$\phi_i^*(t) = \delta_{iJ}$ for $0 \leq t \leq T$ $i = 1, \ldots, n$
B: $w_i \neq kb_i$ for at least one index i	(a) for $\tau_1 \geq T$ $\phi_i^*(t) = \delta_{in}$ for $0 \leq t \leq T$ $i = 1, \ldots n$
	(b) for $\tau_2 \geq T > \tau_1$ $\phi_i^*(t) = \begin{cases} \delta_{ik} & \text{for } 0 \leq t \leq T - \tau_1 \\ \delta_{in} & \text{for } T - \tau_1 < t \leq T \end{cases}$ $i = 1, \ldots, n$
	(c) for $\tau_3 \geq T > \tau_2$ $\phi_i^*(t) = \begin{cases} \delta_{ij} & \text{for } 0 \leq t \leq T - \tau_2 \\ \delta_{ik} & \text{for } T - \tau_2 < t \leq T - \tau_1 \\ \delta_{in} & \text{for } T - \tau_1 < t \leq T \end{cases}$ $i = 1, \ldots, n$
	etc.

NOTES:

(1) J is index such that $a_J b_J = \max(a_1 b_1, \ldots, a_n b_n)$.

(2) n is index assigned so that $a_n w_n = \max(a_1 w_1, \ldots, a_n w_n)$.

(3) k is index such that $R_k = \min\limits_{\substack{R_i > 0 \\ a_i b_i > a_n b_n}} (R_1, \ldots, R_{n-1})$ where $R_i = \dfrac{a_i(b_i w_n - b_n w_i)}{a_i b_i - a_n n_n}$ for $i = 1, \ldots, n-1$.

(4) τ_1 is given by $\tau_1 = \dfrac{1}{\sqrt{a_n b_n}} \ln \left\{ \dfrac{\left(\dfrac{R_k}{w_n}\right) + \sqrt{\left(\dfrac{R_k}{w_n}\right)^2 + \left(\dfrac{v}{w_n}\right)^2 \left(\dfrac{b_n}{a_n}\right) - 1}}{1 + \dfrac{v}{w_n}\sqrt{\dfrac{b_n}{a_n}}} \right\}$.

(5) j is index such that $S_j = \min\limits_{\substack{S_i > 0 \\ a_i b_i > a_k b_k \\ i \neq k}} (S_1, \ldots, S_n)$ where $S_i = \dfrac{a_i(b_i w_k - b_k w_i)}{a_i b_i - a_k b_k}$ for $i = 1, \ldots, n$, $i \neq k$.

(6) τ_2 is given by $\tau_2 = \tau_1 + \dfrac{1}{\sqrt{a_k b_k}} \ln \left\{ \dfrac{\left(\dfrac{S_j}{W_k}\right) + \sqrt{\left(\dfrac{S_j}{W_k}\right)^2 + \left(\dfrac{V_k}{W_k}\right)^2 \left(\dfrac{b_k}{a_k}\right) - 1}}{1 + \dfrac{V_k}{W_k}\sqrt{\dfrac{b_k}{a_k}}} \right\}$.

(7) τ_3 is given by expression similar to those for τ_1 and τ_2 above.

With reference to Table III in Case B it is seen that there may be one or more switches in target priorities if the battle lasts long enough. For example, in subcase (b) of Case B of Table III a switch in the optimal tactic of concentrating all fire on one target type occurs, and fire is shifted from target type k to target type n. It will be shown that necessary conditions for fire to be switched from target type k to n are that $a_k b_k > a_n b_n$ and $\dfrac{b_k}{b_n} > \dfrac{w_k}{w_n}$, i.e., fire is shifted from a target type which causes attrition in a greater proportion than the ratio of values placed upon survivors to the target type which yields the greatest direct return at the end of battle. Additionally, in Table III explicit expressions are given for "switching times" as well as for the determination of the target type upon which all fire is concentrated.

It should be noted that when $n=2$ the results of Table III reduce to those given in Table II. To see this one sets $n=2$ and makes the following identifications: w_1 in Problem 3 is replaced by p in Problem 1, and w_2 by q.

3.3.2. Development of Optimal Fire Distribution Policy

For $y>0$ and $x_i>0$ for $i=1, \ldots, n$, the Hamiltonian for Problem 3 is given by [8]*

$$H(t, x_i, p_i, \phi_i) = -y \sum_{i=1}^{n} a_i p_i(t) \phi_i - p_{n+1} \sum_{i=1}^{n} b_i x_i, \tag{7}$$

where $p_i(t)$ for $i=1, \ldots, n$ denotes the dual variable corresponding to x_i and $p_{n+1}(t)$ denotes the dual variable corresponding to y. According to the maximum principle, the optimal control (there is only one extremal) is determined by the (trivial) linear program

$$\begin{aligned} &\underset{\phi_i}{\text{maximize}}\ H(t, x_i, p_i, \phi_i) \\ &\text{subject to: } \sum_{i=1}^{n} \phi_i = 1, \\ &\phi_i \geq 0, \end{aligned}$$

which in turn leads to

$$\begin{aligned} &\underset{\phi_i}{\text{maximize}} \sum_{i=1}^{n} a_i(-p_i(t))\phi_i \\ &\text{subject to: } \sum_{i=1}^{n} \phi_i = 1, \\ &\phi_i \geq 0. \end{aligned} \tag{8}$$

By inspection the solution to (8) is easily seen to be

$$\phi_i^*(t) = \delta_{ij}(t), \tag{9}$$

where δ_{ij} is the Kronecker delta and is equal to 1 for $i=j$ and zero otherwise and $j(t)$ is the index such that

$$a_j p_j(t) = \text{minimum } (a_1 p_1, a_2 p_2, \ldots, a_n p_n).$$

To trace the history of ϕ_i^* over time, one must consider the adjoint system of differential equations given by

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = p_{n+1} b_i \text{ with } p_i(t=T) = -w_i \text{ for } i=1, \ldots, n,$$

*There is a difference in sign between the version of the maximum principle used by Pontryagin et al. [19] and an equivalent version commonly used in the control theory literature of this country (see p. 108 of [8]).

and

$$\frac{dp_{n+1}}{dt} = -\frac{\partial H}{\partial y} = \sum_{i=1}^{n} \phi_i^* a_i p_i \text{ with } p_{n+1}(t=T) = v. \tag{10}$$

It may be that the index $j(t)$ is not unique, i.e., the linear program (8) has alternate optima. This causes no difficulty unless this situation continues for a finite interval of time. When this happens, the corresponding segment of the battle trajectory is called a *singular subarc* [15]. However, it is easily shown that it is impossible to have a singular solution of Problem 3. If $j(t)$ were not unique for a finite interval of time, then (for example) one would have $a_j p_j(t) = a_k p_k(t)$ for $t_1 \leq t \leq t_2$. If this were to occur, then one must have

$$a_j \frac{dp_j}{dt} = a_k \frac{dp_k}{dt},$$

or using (10)

$$p_{n+1}(t)(a_j b_j - a_k b_k) = 0. \tag{11}$$

Since $p_{n+1}(t) > 0$ for $0 \leq t \leq T$, Equation (11) implies that $a_j b_j = a_k b_k$, which, in general, is not true. Hence, there is no singular solution to Problem 3 and $\phi_i^*(t)$ is either 0 or 1 (almost everywhere).

Considering (10), it is easily seen that

$$\frac{dp_i}{dp_n} = \frac{b_i}{b_n},$$

so that

$$p_i(t) = \frac{b_i}{b_n}\{p_n(t) + w_n\} - w_i. \tag{12}$$

Substituting (12) into (8), one obtains after some manipulation that the optimal control is determined by

$$\begin{aligned} &\underset{\phi_i}{\text{maximize}} \sum_{i=1}^{n} c_i(t)\phi_i \\ &\text{subject to: } \sum_{i=1}^{n} \phi_i = 1, \\ &\qquad \phi_i \geq 0, \end{aligned} \tag{13}$$

where

$$c_i(t) = \frac{(-p_n(t))}{b_n} a_i b_i \left[1 + \frac{w_i}{(-p_n(t))}\left\{\frac{b_n}{b_i} - \frac{w_n}{w_i}\right\}\right]. \tag{14}$$

Hence, it is seen that if $w_i = kb_i$ for $i = 1, \ldots, n$, i.e., Y values enemy survivors in direct proportion to their kill rate against his forces, the above problem is equivalent to

$$\begin{aligned} &\underset{\phi_i}{\text{maximize}} \sum_{i=1}^{n} \phi_i a_i b_i \\ &\text{subject to: } \sum_{i=1}^{n} \phi_i = 1, \\ &\qquad \phi_i \geq 0, \end{aligned}$$

where use has been made of the easily verified fact that $p_n(t) < 0$ for all time. Thus the optimal control is given by

$$\phi_i^*(t) = \delta_{iJ} \text{ for } 0 \leq t \leq T, \tag{15}$$

where J is the index such that

$$a_J b_J = \text{maximum } (a_1 b_1, \ldots, a_n b_n).$$

Hence, for this problem when one values enemy survivors in direct proportion to their kill rate against you, the optimal tactic is to concentrate all fire on a single target type until it is entirely destroyed. The result (15) is given in Table III.

The more complex case in which one does not value enemy survivors in direct proportion to their kill capability (measured by a Lanchester attrition-rate coefficient) against you will now be considered. Since the solution to this problem is developed by working backwards from the end $t = T$, it is convenient to introduce the "backwards time" variable τ defined by $\tau = T - t$. It is assumed that the enemy target types have been indexed so that n is the index such that

$$a_n w_n = a_n(-p_n(\tau = 0)) = \text{maximum } (a_1 w_1, \ldots, a_n w_n). \tag{16}$$

By (8) it is easily seen that

$$\phi_i^*(t = T) = \delta_{in}. \tag{17}$$

By straightforward continuity arguments, it is readily seen that

$$\phi_i^*(\tau) = \delta_{in} \text{ for } \tau \epsilon [0, \tau_1), \tag{18}$$

where τ_1 is the "backwards time" of the first switch in target selection. Giving consideration to (18) and observing that $\frac{d}{dt} = -\frac{d}{d\tau}$, it is seen that for $\tau \epsilon [0, \tau_1]$ one need only consider the following equations from the adjoint system (10)

$$\frac{dp_n}{d\tau} = -b_n p_{n+1} \text{ with } p_n(\tau = 0) = -w_n,$$

$$\frac{dp_{n+1}}{d\tau} = -a_n p_n \text{ with } p_{n+1}(\tau = 0) = v, \tag{19}$$

and it is recalled that (rewriting (12))

$$p_i(\tau) = \frac{b_i}{b_n} \{p_n(\tau) + w_n\} - w_i \text{ for } i = 1, \ldots, n. \tag{20}$$

The above initial value problem (19) is routinely solved to yield

(21) $$p_n(\tau) = -w_n \cosh \sqrt{a_n b_n}\,\tau - v\sqrt{\frac{b_n}{a_n}} \sinh \sqrt{a_n b_n}\,\tau.$$

It will now be determined what conditions are necessary for a change in target selection and the time at which the change occurs, τ_1. To do this it is convenient to rewrite (13) and (14) as

$$\underset{\phi_i}{\text{maximize}} \sum_{i=1}^{n} e_i(\tau)\phi_i$$

$$\text{subject to: } \sum_{i=1}^{n} \phi_i = 1,$$

(22) $$\phi_i \geq 0,$$

where

(23) $$e_i(\tau) = a_i w_i\left[1 + \frac{b_i}{w_i b_n}\{(-p_n(\tau)) - w_n\}\right].$$

A switch in the optimal distribution of fire occurs at the smallest τ for which

(24) $$a_i w_i\left[1 + \frac{b_i}{w_i b_n}\{(-p_n(\tau)) - w_n\}\right] = a_n(-p_n(\tau)),$$

where $i=1, \ldots, n-1$ and certain other conditions (to be determined presently) are met. Let k be the index of the target type to which fire is first shifted in "backwards time." Observe that at $\tau=0$ one has

(25) $$a_i w_i < a_n w_n,$$

for $i=1, \ldots, n-1$, since the index n has been defined by (16). Then for $\tau_1 < \tau < \tau_2$, where τ_2 is the "backwards time" of the second switch in target selection, one has that $\phi_i^*(\tau) = \delta_{ik}$, and thus by (22) and (23) the following inequality must hold

$$a_k w_k\left[1 + \frac{b_k}{w_k b_n}\{(-p_m(\tau)) - w_n\}\right] > a_n(-p_n(\tau)),$$

which may be rearranged to yield

(26) $$a_k(b_k w_n - b_n w_k) < (a_k b_k - a_n b_n)(-p_n(\tau)).$$

It will now be shown that a necessary condition for fire to be shifted from target type n to target type k when one works backwards from the end is that $a_k b_k > a_n b_n$. The proof is as follows. It will be shown that $a_k b_k \leq a_n b_n$ leads to a contradiction. First, consider the special case when $a_k b_k = a_n b_n$. In this case (26) reduces to

$$\frac{w_n}{w_k} < \frac{b_n}{b_k} = \frac{a_k}{a_n},$$

or

$$a_n w_n < a_k w_k.$$

But this is a contradiction to (25) which must hold with $i = k$. In the case when $a_n b_n > a_k b_k$, then using the fact that $(-p_n(\tau)) > w_n$ for $\tau > 0$, we may write (26) as

$$\frac{a_k(b_n w_k - b_k w_n)}{(a_n b_n - a_k b_k)} > (-p_n(\tau)) > w_n,$$

but this leads to $a_k w_k > a_n w_n$ which is a contradiction to (25) as before.

Thus, $a_k b_k > a_n b_n$ and the switch in target selection occurs at

$$\frac{a_k(b_k w_n - b_n w_k)}{(a_k b_k - a_n b_n)} = (-p_n(\tau = \tau_1)) > 0, \tag{27}$$

so that a second necessary condition is

$$\frac{b_k}{b_n} > \frac{w_k}{w_n}.$$

In other words, all fire is concentrated at earlier (forward) times in the battle on the target type which causes attrition proportionally more than the ratio of values placed on survivors and then is switched later to the target type which yields the greatest direct return at the end of battle.

To recapitulate the above, the target to which fire is first shifted (working backwards from the end of battle) has index k determined by

$$R_k = \underset{\substack{R_i > 0 \\ a_i b_i > a_n b_n}}{\text{minimum}} (R_1, \ldots, R_{n-1}), \tag{28}$$

where

$$R_i = \frac{a_i(b_i w_n - b_n w_i)}{(a_i b_i - a_n b_n)} \qquad \text{for } i = 1, \ldots, n-1. \tag{29}$$

The time of switch, τ_1, of fire to the kth target type is determined by the equation

$$w_n \cosh \sqrt{a_n b_n}\, \tau_1 + v \sqrt{\frac{b_n}{a_n}} \sinh \sqrt{a_n b_n}\, \tau_1 = \frac{a_k(b_k w_n - b_n w_k)}{a_k b_k - a_n b_n}, \tag{30}$$

which may be solved to yield the expression for τ_1 given in Table III.

The general pattern of when and to which target types fire is shifted as one works backwards from the end of battle does not emerge until one has considered the second shift in target selection. Since this is dependent upon the evolution of target worth, the backwards integration of the adjoint system of differential equations must be further considered. From above, one has that

$$\phi_i^*(\tau)=\delta_{ik} \qquad \text{for } \tau\epsilon(\tau_1, \tau_2), \tag{31}$$

where τ_2 is the "backwards time" of the second switch in target selection. Giving consideration to (31), it is seen that for $\tau\epsilon[\tau_1, \tau_2]$ one needs only to consider the following equations from the adjoint system (10)

$$\frac{dp_k}{d\tau}=-b_k p_{n+1} \qquad \text{with } p_k(\tau=\tau_1)=-W_k,$$

$$\frac{dp_{n+1}}{d\tau}=-a_k p_k \qquad \text{with } p_{n+1}(\tau=\tau_1)=V_k, \tag{32}$$

where

$$W_k=\frac{a_n(b_k w_n-b_n w_k)}{a_k b_k-a_n b_n} \tag{33}$$

$$V_k=\sqrt{\frac{a_n}{b_n}(R_k{}^2-w_n{}^2)+v^2}. \tag{34}$$

Equation (33) follows from the fact that by (8) and (9) at $\tau=\tau_1$ we have

$$a_k(p_k(-\tau=\tau_1))=a_n(-p_n(\tau=\tau_1)), \tag{35}$$

which may be combined with (27), (28), and (29) to yield the desired result. Equation (34) is readily deduced by observing that according to (19) a "square law" relates the dual variables $p_n(\tau)$ and $p_{n+1}(\tau)$ for $0\leq\tau\leq\tau_1$

$$a_n\{p_n{}^2(\tau)-w_n{}^2\}=b_n\{p_{n+1}{}^2(\tau)-v^2\}, \tag{36}$$

whence follows (34) by use of (27), (29), and (32). It should be noted that all the dual variables may be expressed in terms of $p_k(\tau)$ (let $n=k$ in (12))

$$p_i(\tau)=\frac{b_1}{b_k}\{p_k(\tau)+w_k\}-w_i \qquad \text{for } i=1, \ldots, n. \tag{37}$$

Again, the Equations (32) are routinely solved to yield for $\tau\epsilon[\tau_1, \tau_2]$

$$p_k(\tau)=-W_k \cosh \sqrt{a_k b_k}\,(\tau-\tau_1)-V_k\sqrt{\frac{b_k}{a_k}} \sinh \sqrt{a_k b_k}\,(\tau-\tau_1). \tag{38}$$

Subsequent arguments are now similar to those given for the first switch in tactics. Let j be the index of the target type to which fire is shifted secondly in "backwards time." Then, it may be shown by similar arguments to above that necessary conditions for fire to be shifted to the jth target type are that

(39) $$a_j b_j > a_k b_k > a_n b_n,$$

and

(40) $$\frac{b_j}{b_k} > \frac{w_j}{w_k}.$$

However, more insight may be gained by rewriting (40) as

(41) $$\frac{b_j}{w_j} > \frac{b_k}{w_k} > \frac{b_n}{w_n}$$

It also seems appropriate to point out the military interpretation of the ratio $\frac{b_i}{w_i}$. Recall that

$$w_i = \text{value per unit of } X_i \text{ surviving at } t = T,$$
$$b_i = \text{kill rate per unit of } X_i \text{ against } Y.$$

Then

$$\frac{b_i}{w_i} = \frac{\text{kill rate per unit of } X_i}{\text{value per unit of } X_i \text{ survivors}}.$$

Thus, it is seen that as one progresses backwards from the end of battle that fire is always shifted to target types with larger ratios of kill rate per unit of weapon system per unit value of survivors.

Using an argument similar to the one used to develop (28) and (29) for the first shift in fire, it may be shown that the target to which fire is shifted secondly (working backwards from the end of battle) has index j determined by

(42) $$S_j = \underset{\substack{S_i > 0 \\ a_i b_i > a_k b_k \\ i \neq k}}{\text{minimum}} (S_1, \ldots, S_n),$$

where

(43) $$S_i = \frac{a_i(b_i w_k - b_k w_i)}{a_i b_i - a_k b_k} \quad \text{for } i = \underset{i \neq k}{1}, \ldots, n.$$

The time of switch, τ_2, of fire to the jth target type is determined by the transcendental equation

(44) $$W_k \cosh \sqrt{a_k b_k}\ (\tau_2 - \tau_1) + V_k \sqrt{\frac{b_k}{a_k}} \sinh \sqrt{a_k b_k}\ (\tau_2 - \tau_1) = \frac{a_j(b_j w_k - b_k w_j)}{a_j b_j - a_k b_k},$$

which may be solved to yield the expression for τ_2 given in Table III. Further shifts in fire follow the pattern established above.

3.3.3. Discussion of Structure of Optimal Policy

Considering Table III, it is seen that the optimal allocation policy for Problem 3 has the same structure as that for Problem 1 with just two target types:

(1) fire is always concentrated on one target type,

(2) the allocation is not directly dependent upon the force levels.

The addition of more target types has not changed the nature of the problem: its explicit solution is a generalization of that with two X-force target types.

It is of interest to ask whether the optimal tactic will always be to concentrate fire on only one target type (bang-bang optimal control). The answer to this question turns out to be "no" as consideration of Problem 5 with a "linear-law" attrition process for the X-force target types will show. The reader is referred to section 3.1.2. for a further heuristic discussion of the structure of the optimal policy for the distribution of fire over target types which undergo attrition at a rate proportional to only the number of firers.

3.4. Some Special Cases of Time Dependent Attrition-Rate Coefficients

In the previous idealizations of combat that have been considered above, it has been assumed that all the Lanchester attrition-rate coefficients were constant. In reality, such a coefficient depends upon numerous factors some of which are as follows: hit probabilities, weapon system projectile-target lethality characteristics, rates of fire, rate of target acquisition. These factors themselves may be range dependent or change over time. S. Bonder [5], [6] has developed explicit formulas for relating the Lanchester attrition-rate coefficient to weapons system performance characteristics such as those mentioned above.

Thus, it seems appropriate to examine idealized combat situations in which the attrition-rate coefficients are time dependent. Moreover, this is facilitated by the author's research results on solutions to variable-coefficient Lanchester-type equations for "square-law" attrition processes [22], [27]. A key result is that there is a class of variable-coefficient Lanchester-type equations (combat between two homogeneous forces when the attrition-rate coefficents are variable provided that their ratio is constant) which possess a solution no more complicated than the solution to the constant coefficient case [22]. This type of property (reflecting the physical situation in which two weapon systems cause attrition in a proportional fashion at all times) will now be exploited in an optimal control problem.

Thus, Problem 4 is a prescribed duration battle and is stated in mathematical terms below.

(Problem 4)
$$\underset{\phi(t)}{\text{maximize}}\ \{ry(t)-px_1(T)-qx_2(T)\} \text{ with } T_1 \text{ specified,}$$

$$\text{subject to: } \frac{dx_1}{dt}=-\phi a_1(t)y,$$
$$\frac{dx_2}{dt}=-(1-\phi)a_2(t)y,$$
$$\frac{dy}{dt}=-b_1(t)x_1-b_2(t)x_2,$$
$$x_1,\ x_2,\ y\geqslant 0, \quad 0\leqslant\phi\leqslant 1, \quad \text{and } T\leqslant T_1.$$

It is assumed that both X-force weapon systems are such that

$$b_1(t)=k_{b_1}h(t) \quad \text{and } b_2(t)=k_{b_2}h(t). \tag{45}$$

As done above for Problems 1 and 3, only the special case in which $x_1(T)>0$, $x_2(T)>0$, $y(T)>0$, and $T=T_1$ is considered for the comparisons made in this paper. In the further special case in which the

TABLE IV. *Solution to Fire Distribution Problem (Problem 4) Battle of Prescribed Duration with Variable Attrition-Rate coefficients; Special Case in Which* $x_1(T) > 0$, $x_2(T) > 0$, $y(T) \geq 0$

Special assumption: $a_1(t)/a_2(t) = k_{a_1}/k_{a_2}$, $b_1(t)/b_2(t) = k_{b_1}/k_{b_2}$, and $a_1(t)/b_1(t) = k_{a_1}/k_{b_1}$
Nonrestrictive assumption: $k_{a_1}k_{b_1} > k_{a_2}k_{b_2}$

Case	Optimal Control
A: $k_{a_1}p \geq k_{a_2}q$	$\phi^*(t) = 1$ for $0 \leq t \leq T$
B: $k_{a_1}p < k_{a_2}q$	(a) for $\tau_1 \geq T$ $\phi^*(t) = 0$ for $0 \leq t \leq T$
	(b) for $\tau_1 < T$ $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq T-\tau_1 \\ 0 & \text{for } T-\tau_1 \leq t \leq T \end{cases}$

NOTE: The "backwards" switching time is given by

$$\int_0^{\tau_1} h(\tau)d\tau = \frac{1}{\sqrt{k_{a_2}k_{b_2}}} \ln \frac{z+\sqrt{z^2+\alpha^2-1}}{1+\alpha},$$

where

$$z = \frac{R-\delta}{R-1}, \ \delta = k_{a_1}p/(k_{a_2}q), \ R = k_{a_1}k_{b_1}/(k_{a_2}k_{b_2}), \quad \text{and } \alpha = \frac{r}{q}\sqrt{\frac{k_{b_2}}{k_{a_2}}}.$$

Y-force values surviving X-force types in direct proportion to their kill rates (as measured by the Lanchester attrition-rate coefficients) against the Y-force, i.e., $p/q = k_{b_1}/k_{b_2} = b_1(t)/b_2(t) = b_1(t=T)/b_2(t=T)$, the optimal control law takes a particularly simple form

$$\phi^*(t) = \begin{cases} 1 & \text{for } a_1(t)b_1(t) > a_2(t)b_2(t), \\ 0 & \text{for } a_1(t)b_1(t) < a_2(t)b_2(t). \end{cases} \tag{46}$$

In this instance target selection depends only on the product of attrition-rate coefficients which may be interpreted as the rate of destruction of enemy kill-rate capability. All fire is concentrated on one of the target types depending on which target type has the larger product of attrition-rate coefficients. Target priority is subject to change over time as the ranking of the target types on this decision criterion changes. It is conceivable that the optimal tactic may be to shift fire from one target type to the other several times over the course of battle with the duration of battle not having any effect. Observe that no assumptions at all have been made on the Y-force attrition rates against X_1 and X_2, i.e., $a_1(t)$ and $a_2(t)$.

It is now further assumed that $a_1(t) = k_{a_1}h(t)$ and $a_2(t) = k_{a_2}h(t)$. This means that not only is the ratio of the X-force weapon system attrition rates against the Y-force constant, but also the ratio of the Y-force effectiveness against each of the two X-force types. Furthermore, all four attrition-rate coefficients have the same time dependence except for constant factors. The solution is shown in Table IV. In this special case it is seen that the structure of the optimal allocation policies when the attrition-rate coefficients are variable is essentially identical to that in which they are constant. Only the time scale has been transformed (in [22] this type of observation was first made by the author (see also [25])).

The development of the above results is omitted due to considerations of the length of the paper at hand. Their development and that of further such results are to be found in [28]. Moreover, it has been important to include the above results here, since the purpose of the paper at hand is to contrast the solutions for a sequence of related problems.

Finally, it should be noted that when $h(t) = 1$ the results of Table IV reduce to those for Problem 1 given in Table II. To see this one sets $h(t) = 1$ and makes the following identifications: k_{a_1} in Problem 4 is replaced by a_1 in Problem 1, k_{a_2} by a_2, k_{b_1} by b_1, and k_{b_2} by b_2.

3.5. Fire Distribution for Targets Undergoing "Linear-Law" Attrition

So far in the state equations describing combat the attrition rate of each X-force target type has been proportional to only the number of Y-force firers. Considering Equations [7], [9], and [30] which give rise to the classical Lanchester square law, this may be referred to (somewhat imprecisely) as a "square-law" attrition process of target types.* H. Weiss [30] has given a thorough discussion of the conditions which lead to such an attrition process. These conditions include that "each unit is informed about the location of the remaining opposing units so that when a target is destroyed, fire may be immediately shifted to a new target." It is thus noted that the control theory models which we have considered so far have implicitly assumed perfect information in the above sense.

Another model for target type attrition is one in which the attrition rate (of each X-force target type) is proportional to the product of the numbers of targets and firers. This may be referred to (again somewhat imprecisely) as a "linear-law" attrition process of target types. Such an attrition process can arise under two different general circumstances: (1) fire is uniformly distributed over a constant target area ("area fire"), or (2) the mean time of target acquisition is much larger than target destruction time and is inversely proportional to target density. Again, quoting Weiss [30], it is assumed that units are informed about the general areas in which opposing units are located, but are not informed about the consequences of their own fire. Brackney [7] has shown that "aimed fire" may lead to a linear-law attrition process of targets when target acquisition times are considered and are as postulated above.

Thus, Problem 5 is a battle in which the attrition of each X-force target type is a linear-law process and is stated in mathematical terms below.

(Problem 5) $\underset{\phi(t)}{\text{maximize}}\{ry(T) - px_1(T) - qx_2(T)\}$ with T_1 specified,

$$\text{subject to: } \frac{dx_1}{dt} = -\phi a_1 x_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 x_2 y,$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2,$$

$$x_1, x_2, y \geq 0, \quad 0 \leq \phi \leq 1, \quad \text{and } T \leq T_1.$$

The analysis details upon which the summary given below is based are given in a companion paper [29] (see also pp. 91–105 of [21]) due to their rather lengthy nature. Moreover, it is important that these results be given here so that one can see the effect of the combat attrition model on the structure of the optimal allocation policy by contrasting, for example, the solution to Problem 1 with that for Problem 5.

*The reader should keep in mind that the Y-forces are faced with the problem of determining the optimal distribution of fire over X-force target types.

3.5.1. Description of Optimal Fire Distribution Policy

There is a fundamental difference between the solution to Problem 5 and those considered previously: the optimal allocation, ϕ^*, may be other than 0 or 1. In contrast to those for Problems 1 through 4, the optimal allocation policy does not have to be an extreme point of the control variable space at all times: one may have a singular solution [15] for which the necessary condition of maximizing the Hamiltonian (with respect to the control variable) does not provide a well-defined expression for the extremal control. That part of an optimal trajectory on which the maximum principle does not determine the control is called a singular subarc, and the term "singular solution" will be used to refer to any optimal trajectory which contains one or more singular subarcs.

Singular solutions usually occur when the Hamiltonian (denoted as H) is a linear function of the control variable(s). According to the notational conventions adopted in this paper, when this happens, then if $\frac{\partial H}{\partial \phi}=0$ for a finite interval of time, the maximum principle does not determine the control. Observe that when $\frac{\partial H}{\partial \phi}=0$ and H is a linear function of ϕ, *all* feasible values of ϕ are optimal. All problems, however, for which the Hamiltonian is a linear function of the control variables do not have singular subarcs in their solution. In particular, the reader should note that it has been shown above that it is impossible to have a singular solution to Problem 1 through Problem 4. This was done, for example, for Problem 4 by showing that it is impossible for $\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right)=0$ for a finite interval of time when $\frac{\partial H}{\partial \phi}=0$. Moreover, there is a special second order necessary condition of local optimality (generalized Legendre-Clebsch condition) [16] which must be satisfied in order that a singular subarc can yield a maximum return. This is satisfied for the problem at hand [29].

The optimal battle trajectories are constructed by working backwards from all possible end points of this idealized battle [29]. Consideration is given to both the optimal control at the end of battle and also how the variables upon which it depends vary over time. Based upon such considerations, there are three cases to be considered:

Case (a) $\quad \frac{p}{q}=\frac{b_1}{b_2},$

Case (b) $\quad \frac{p}{q}>\frac{b_1}{b_2},$

Case (c) $\quad \frac{p}{q}<\frac{b_1}{b_2}.$

Consider Case (a) first. The solution for this case is shown in Figure 2. Even though explicit expressions have not been obtained for certain model parameters, the dependence of the optimal control upon these quantities can still be qualitatively discussed. The optimal control depends on the state variables x_1 and x_2 (and also the attrition coefficients) in each "decision region." Above the line $a_1b_1x_1=a_2b_2x_2$, denoted as L, the optimal control $\phi^*=0$ is used until this line is encountered. When L is reached the singular control $\phi^*=\frac{a_2}{a_1+a_2}$, which keeps the trajectory on L, is used until the end of the battle at $t=T$. That portion of an optimal trajectory which lies on L (for a finite interval of time) is a singular subarc. The

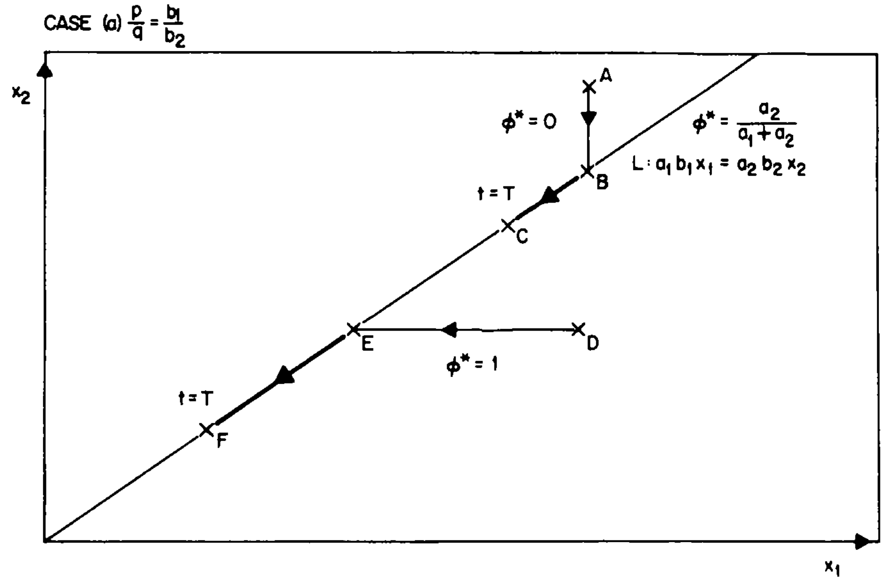

FIGURE 2. *Optimal allocation for linear-law attrition (Case a)*

time history of the optimal control is traced for two particular initial force ratios denoted as points A and D in Figure 2. At point D, $\frac{x_1^0}{x_2^0} > \frac{a_2 b_2}{a_1 b_1}$ and $\phi^* = 1$ is used until the line L is encountered at point E.

The solution for Case (b) is shown diagrammatically in Figure 3. It is similar to the preceding case except that another line, L' with equation $a_1 p x_1 = a_2 q x_2$, plays a role in the solution in addition to the singular "surface" denoted as L. This line L' appears above, on, or below the line L (with equation $a_1 b_1 x_1 = a_2 b_2 x_2$) depending upon whether $\frac{p}{q}$ is greater than, equal to, or less than $\frac{b_1}{b_2}$.

The significance of the line L' and its relationship to the line L is as follows. The battle is divided into two time phases: Phase I for $0 \leq t \leq t_I = T - \tau_I$ and Phase II for $T - \tau_I = t_I \leq t \leq T$. During Phase I

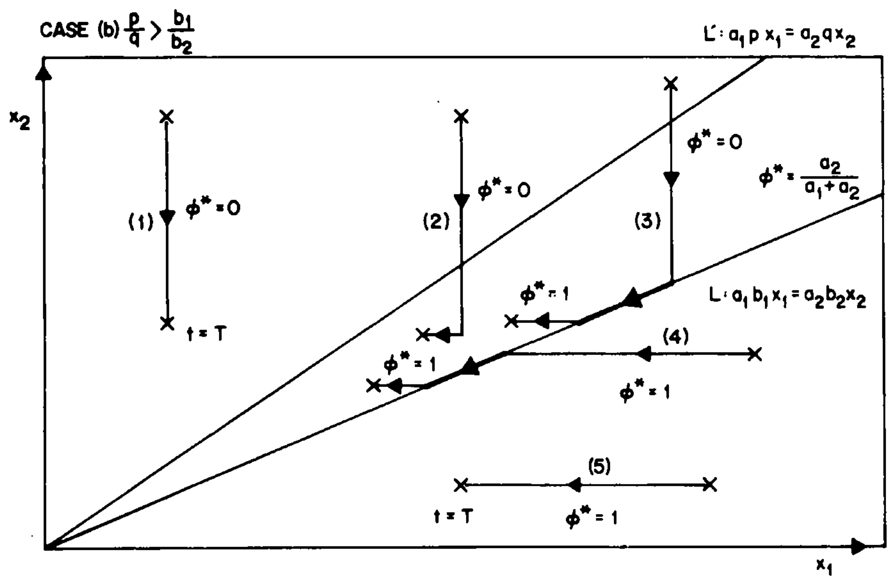

FIGURE 3. *Optimal allocation for linear-law attrition (Case b)*

the optimal target engagement policy at a point in time is determined by the location of the point on the battle trajectory with respect to the line L, which is also the singular "surface." Above L, $\phi^*(t)=0$; while below L, $\phi^*(t)=1$. When a battle trajectory reaches L, it remains on the singular surface through use of the singular control $\phi^*=\frac{a_2}{a_1+a_2}$. During Phase II the optimal target engagement policy is to use $\phi^*(t)=1$ below L' and $\phi^*(t)=0$ above L'. The proof of the above statements is given in [29].

Thus for any optimal trajectory which lies below L' (such as those denoted in Figure 3 as (2), (3), and (4)) the optimal control is $\phi^*=1$ during Phase II. Moreover, the time $t_l=T-\tau_l$ appears in Figure 3, for example, as when the optimal control for (3) switches from $\phi^*=a_2/(a_1+a_2)$ to $\phi^*=1$. Any optimal trajectory which lies entirely above L', such as (1), has a corresponding optimal control of $\phi^*(t)=0$ for $0\leq t\leq T$, whereas a similar remark holds for any one that lies entirely below L, such as (5). Case (c) is symmetric to Case (b).

3.5.2. Discussion of Structure of Optimal Policy

As noted above (see sections 3.1.2. and 3.3.1.), the structure of the optimal allocation policies in these tactical allocation problems is dependent upon how the Y-force values the surviving X-force types relative to their kill rate against the Y-force. Case (a): $\frac{p}{q}=\frac{b_1}{b_2}$ above is when Y assigns utility to surviving X-force types in exact proportion to their destructive capability against Y. In this case, the optimal target selection tactic depends only upon the state of the system as is seen with reference to Figure 2. The optimal tactic is to use $\phi^*(t)=0$ above the line L with equation $a_1b_1x_1=a_2b_2x_2$. The line L also represents an "equilibrium" trajectory which the system follows whenever this line is reached. Case (b): $\frac{p}{q}>\frac{b_1}{b_2}$ is when Y assigns a greater value to surviving X_1's than in proportion to their kill rate against Y relative to that of X_2. Again, the optimal tactic depends upon the state of the system, only this dependence itself depends upon the "time phase" of the fixed length battle.

Based on the above examination of Problem 5, it is seen that the structure of the optimal allocation policies for targets which undergo a linear-law attrition process has the following characteristics:

(1) fire may be divided between target types,

(2) the allocation is (directly) dependent upon the force levels.

These characteristics should be contrasted with those previously observed when target types undergo a square-law attrition process (see sections 3.1.2. and 3.3.3.). When there is a linear-law attrition process of target types, the optimal allocation policy may be other than 0 or 1. Also, the allocation depends upon the force levels of target types. An explanation for this structure of optimal allocation policies in terms of the nature of the attrition process has been given in section 3.1.2.

4. EXTENSIONS TO DIFFERENTIAL GAMES

Even though it is certainly true that combat is an environment of conflicting interests in which the potential actions of both friendly and enemy forces must be considered, there is much to be learned from one-sided dynamic optimization models. The author views these simplified idealizations presented here as "building blocks" for more sophisticated scenarios. It is felt, moreover, that an understanding of the structure of optimal tactics for these initial models is essential before one continues his examination of a sequence of models of greater and greater complexity. Hence, it seems appropriate to review the intimate connection between optimal control theory and differential games.

It has been stated that optimal control problems may be viewed as one-sided differential games for which the roles of all but one of the competing players have been suppressed [2]. Conversely, differential games may be viewed as two-sided optimal control problems [12]. A concise discussion of the interrelationships between these two subjects is contained in Y. C. Ho's [11] excellent review of Isaacs' book [13] (see also chapter 9 of [8]).

It should be recalled (see [24]) that the existing theory of (zero-sum deterministic) differential games is only applicable to problems for which the criterion functional has a saddle point in pure strategies. However, it may be shown (considering the results of A. Friedman (see chapters 5 and 6 of [10])) that the structure of two-sided fire distribution problems in the Lanchester theory of combat as formulated by Isbell and Marlow [14] and Weiss (see pp. 94–95 of [30]) guarantees the existence of pure strategy solutions. This need not be true for other dynamical structures. For example, when defensive capabilities were considered in the attrition process in a tactical air war game extensively studied at RAND, the resulting model did not possess a solution in pure strategies [1], [3], [4].

The author has therefore, used these optimal control problems to study many aspects of such corresponding differential games (two-sided variational problems): the effect of different boundary conditions, devising solution procedures, study of singular behavior, differences in the structure of optimal allocation policies for various model forms. Most solution aspects of the one-sided problem are present in the two-sided one. (There are exceptions, however (see [26]).) In solving [26] the supporting weapon system game of H. K. Weiss [31], the author made use of his knowledge of a related optimal control problem [24].

5. IMPLICATIONS OF MODELS

It seems appropriate to briefly discuss the general implications of the models examined in this paper to the following areas:

(1) optimal tactical allocation,
(2) intelligence,
(3) command and control systems,
(4) human decision making.

The discussion of these areas is not mutually exclusive.

Of interest to the military tactician is whether optimal fire distribution rules evolve dynamically during the course of battle. Are target priorities static or do they evolve dynamically with the course of battle? With respect to optimal control models, this may be mathematically stated as whether there are transition (switching) surfaces in the solution. It has been seen in the idealized and simplified models studied here that target priorities do change. This is related to the evolution of marginal return of target destruction (value of dual variable). It has been seen that this evolution depends on the goals of the combatants (utility assigned to surviving force types at the end of the battle) and also the conditions which terminate the battle. In the terminal control problem studied here, a shift in target priorities is present only in a losing case, whereas in a fixed duration battle such a switch is sometimes independent of winning or losing and then depends only on weapon system capabilities and the prescribed duration of battle.

Schreiber [20] has proposed an idealized and simple, but yet illuminating, way of quantitatively showing the value of intelligence and command control capabilities. He introduces the concept of "command efficiency," which is measured by the fraction of the enemy's destroyed units from which fire has been redirected. The effect of poor intelligence and poor capabilities for redirecting fire from destroyed targets is to produce "overkill." Schreiber's equations for combat involved this fraction called "command efficiency," and they reduce to Lanchester-type equations for area fire when the fraction is 0 and aimed fire for a value of 1. It has been seen that the optimal tactics are quite different for these two cases. When intelligence and command control systems are very efficient, the optimal tactic is seen always to be concentration of fire on a specific target type. When capability for redirection of fire from destroyed targets is poor (either through damage assessment or constraints on new target acquisition), the optimal tactic may be to allocate fire in a proportional fashion over target types in a way that holds the ratios of target density in each target area to be constant. Thus, these models indicate that the optimal tactics of fire distribution vary with command and control capabilities.

These models also show the importance of intelligence in devising the "best" tactics in combat. Intelligence on enemy weapon system capabilities (kill rates including target acquisition rates) and potential length of engagement play a central part. It has also been seen that for fights-to-the-finish and linear-law attrition cases intelligence on enemy force levels is also required. For artillery fire support missions against various troop concentrations, knowledge of troop densities is essential in the assignment of target priorities. Particularly dense concentrations where the initial kill potential is high are seen to be cases where the optimal tactic is to concentrate fire on one target for awhile.

These models may be interpreted to show the value of human judgment in combat. They indicate, as does common sense and experience, that in battle a commander must use his judgment to ascertain to what end can the course of battle be steered so that he may devise his strategy accordingly. The demonstrated sensitivity of these models to many factors shows the importance of human assessment of a situation and the importance of good judgment in assigning utility to forces surviving the battle at hand.

6. SUMMARY

The results of this paper may be summarized as follows:

(1) a sequence of one-sided models has been presented which shows that the tactics of fire distribution are sensitive to force levels, target acquisition process, the type of attrition process, and the termination conditions of combat,

(2) tactics for target selection are heavily dependent upon "command efficiency,"

(3) concentration of fire always on one target type among many occurs as an optimal tactic only when target acquisition is not subject to diminishing returns,

(4) target priorities do not change over time when one assigns a worth to surviving target types in direct proportion to their kill rate against you.

7. ACKNOWLEDGMENT

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