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Applications of the Lambert *W* Function in Electromagnetics

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1. Introduction

The Lambert W function, although not familiar to many engineers, has its origin in the work of Lambert and Euler in the late 1700s. $W(z)$ is the function that solves the equation

$$W(z)e^{W(z)} = z \tag{1}$$

where, in general, z is a complex number. Many of the equations encountered in electromagnetics can be cast into the form of Equation (1). Since most of the mathematics and engineering software packages have some variant of the W function included, many of the traditional graphical-solution methods can be eliminated, and the equations programmed efficiently in *Maple*, *Macsyma*, *Mathematica*, or *Matlab*. (In *Mathematica*, W is the `ProductLog` function.)

Equation (1) is a multivalued function with an infinite number of solutions, most of them complex [1]. Thus, the notation $W_p(z)$ is used, where $p = 0,\ \pm 1,\ \pm 2, \ldots$ denotes the branch. If z is real, which is often the case, then there may be

1. no solution

2. two real solutions, $W_0(z)$ and $W_{-1}(z)$

3. one real solution, $W_0(z)$ (with $W_{-1}(z)$ complex)

As stated in [2], even if z is real, the branches other than $p = 0$ and $p = -1$ are always complex.

Many other forms of Equation (1) can be derived [1]. For example, using the definition of the natural logarithm, the equation

$$xb^x = a \tag{2}$$

can be written as $x\left(e^{\ln b}\right)^x = a$. Multiply both sides by $\ln b$ to get

$$(x \ln b)\ e^{x \ln b} = a \ln b \underset{x' = x \ln b}{\rightarrow} x' e^{x'} = a \ln b,$$

which is Equation (1) with $W(x')e^{W(x')} = a \ln b$. The solution for x is

$$x = W(a \ln b)/\ln b. \tag{3}$$

Two other forms of Equation (1) and their solutions are

$$x^{x^a} = b \rightarrow x = e^{W(a \ln b)/a} \tag{4}$$

$$a^x = x + b \rightarrow x = -b - W\left(-a^{-b} \ln a\right)/\ln a. \tag{5}$$

Several EM applications of the Lambert W function are given in the following sections.

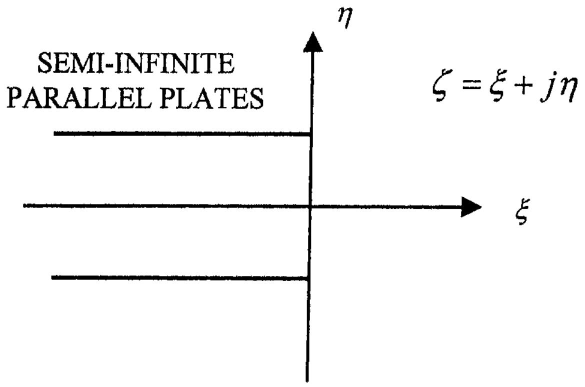

Figure 1a. The original problem: semi-infinite parallel plates at potentials $\pm V_o$.

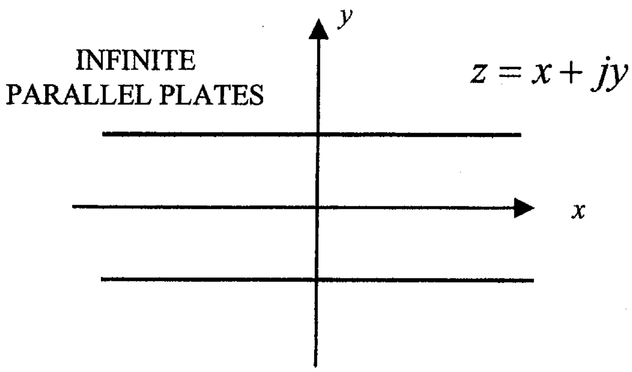

Figure 1b. The transformed problem: infinite parallel plates in the z plane.

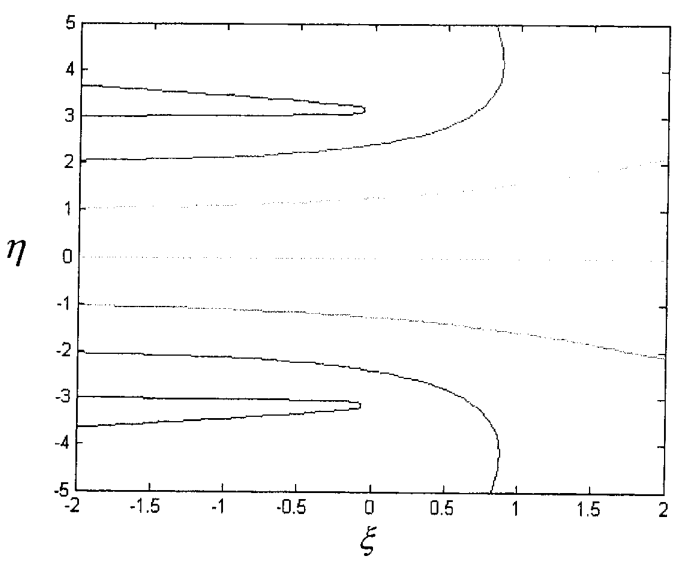

Figure 2. Equipotential lines of the parallel-plate capacitor with plates located at $\eta = \pm 3$.

2. Parallel Plate Capacitor

The Schwartz-Christoffel transformation provides a mapping of the interior of a closed polygon in the complex ζ plane to infinite parallel plates in the z plane, as shown in Figure 1 [2, 3]. A conformal mapping of this type can be used to derive an approximate formula for the potential of the parallel plates, which includes the edge effect. The potential at points in the z plane is easily calculated, and transformed back to the ζ plane to generate equipotentials. Following the derivation in [2], the desired transformation is

$$\zeta = 1 + z + e^{z}.$$

In terms of the Lambert W function,

$$z = \zeta - 1 - W_{K(\zeta)}\left(e^{\zeta - 1}\right),$$

where $K(\zeta) = \left\lceil \frac{\operatorname{Im}(\zeta) - \pi}{2\pi} \right\rceil$, $\lceil\ \rceil$ signifies the ceiling function, and Im is the imaginary part. The potential at a point ζ is

$$V(\zeta) = \frac{V_0}{\pi} \operatorname{Im}\left\{\zeta - 1 - W_{K(\zeta)}\left(e^{\zeta - 1}\right)\right\}. \quad (6)$$

Several equipotential contours are plotted in Figure 2.

3. Surface-Wave Transmission Lines

Surface-wave transmission lines were first investigated by Goubau in the 1950s [4]. He presented a set of curves that can be used to determine the propagation constant as a function of the conductor characteristics. For a circular conductor of radius a, conductivity σ_c, and permeability μ_c (Figure 3), he showed that for the conditions frequently encountered in practice,

$$\gamma a = 1.12\sqrt{|\xi|}\ e^{j(\alpha + \pi)/2} \quad (7)$$

where

$$|\xi| \ln|\xi| = -|\eta|, \quad (8)$$

with

$$|\eta| = 2(0.89)^2 \frac{k^2 a \mu_c}{|k_c| \mu_0},$$

$$\alpha = -\frac{\pi}{4}\left(1 - \frac{1}{\ln|\xi| + 1}\right),$$

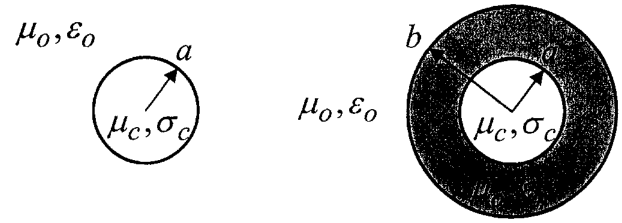

Figure 3. Cross sections of surface-wave transmission lines: a bare conductor (left), and a dielectric-coated conductor (right).

$$k_c = 1.12\sqrt{\omega\mu_c\sigma_c}e^{-j\pi/4},$$

$$k = \omega\sqrt{\varepsilon_0\mu_0},$$

$$\gamma^2 = k^2 - h^2.$$

The quantity $|\eta|$ is known from the conductor geometry and material parameters, and thus Equation (8) must be solved for $|\xi|$. In [4] the solution is done graphically, but Equation (8) can be recast into the form of Equation (4):

$$e^{|\xi|\ln|\xi|} = e^{-|\eta|} \rightarrow \left(e^{\ln|\xi|}\right)^{|\xi|} = e^{-|\eta|} \rightarrow |\xi|^{|\xi|} = e^{-|\eta|}.$$

The solution is $|\xi| = e^{W(-|\eta|)}$, and therefore $|\xi|$ can be determined by calculating W. The equations for the propagation constant can be solved without resorting to curves. Once γ is known, the attenuation constant and phase velocity can be determined. Figure 4 shows the loss in dB per 100 feet as a function of wire radius.

A similar approach can be used for a dielectric-coated conductor (also shown in Figure 3), where b is the radius of the outer surface of the dielectric layer, and ε_r the relative dielectric constant. This problem has been treated by Goubau [3], Harrington [5], Collin [6], and Balanis [7]. Using the notation of Balanis, the eigenequation for the attenuation constant becomes

$$\varepsilon_r b\left(\alpha_\rho^0\right)^2 \ln\left(0.89\alpha_\rho^0 b\right) \approx \left[-\beta_0^2(\varepsilon_r - 1) + \left(\alpha_\rho^0\right)^2\right]t, \quad (9)$$

where $\beta_0 = \omega\sqrt{\varepsilon_0\mu_0}$ and it has been assumed that $a \ll \lambda$, and that the thickness of the coating is $t = (b-a) \ll a$. Letting $x = 0.89\alpha_\rho^0 b$, Equation (9) takes the form

$$Bx^2 \ln x = A + x^2, \quad (10)$$

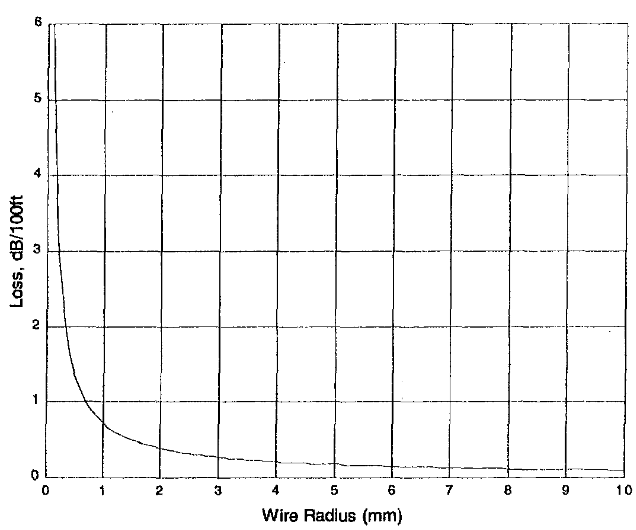

Figure 4. The attenuation in dB per 100 feet as a function of the wire radius (3 GHz).

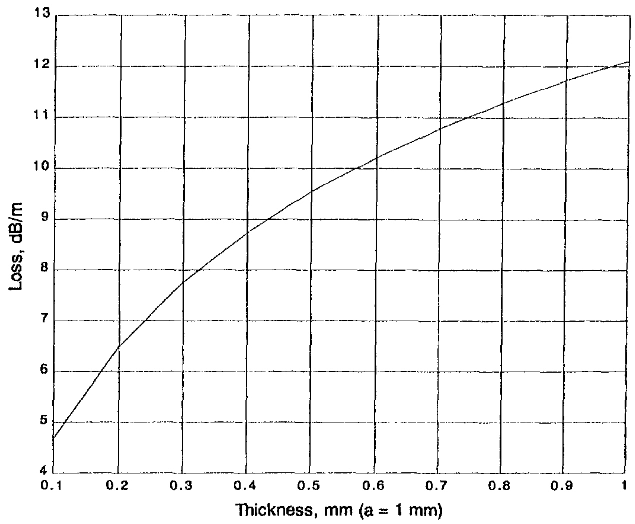

Figure 5. The attenuation of a coated conductor as a function of the coating thickness (3 GHz).

with $A = -(0.89b)^2(\varepsilon_r - 1)\beta_0^2$ and $B = \frac{\varepsilon_r b}{t}$. Now, a series of transformations of variables is performed:

1. Let $y = \ln x$, ($x = e^y$), which gives $e^{2y}(y - 1/B) = A/B$;

2. Let $z = y - 1/B$, which gives $2ze^{2(z+1/B)} = 2A/B$;

3. Let $u = 2z$, which gives $ue^u = 2Ae^{-2/B}/B$.

The solution for u is $W\left(2Ae^{-2/B}/B\right)$, and the attenuation constant is $\alpha_\rho^0 = \frac{\exp(u + 1/B)}{0.89b}$. Figure 5 shows the attenuation as a function of coating thickness for $\varepsilon_r = 2.56$ and $a = 0.001$ m at 300 MHz. As Balanis points out, increasing the dielectric thickness increases the attenuation constant, and binds the wave more tightly to the surface.

4. Summary

The solution to many EM problems can be formulated in terms of the Lambert W function. Several have been presented here. The function frequently occurs in the dispersion relations for transmission-line problems when the limiting cases of small radius or thickness are considered. Numerical algorithms of the function have been incorporated into the popular mathematical computation packages, which allows the efficient and rapid solution of problems that have traditionally been done graphically.

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