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# Buckling near a hole in an infinite plate under tension

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# BUCKLING NEAR A HOLE IN AN INFINITE PLATE UNDER TENSION

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# UNDER TENSION

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# ABSTRACT

A study is made of the circumstances under which there will be incipient buckling in an infinite plate containing a circular hole and subjected to uniform uniaxial tension at a great distance from the hole. This buckling is a consequence of the fact that near the hole and in line with the direction of the applied tensile loading, there are areas in which both principal membrane stresses are compressive. Values of critical tensile stress are given, together with expressions for the corresponding lateral deflection patterns. The results are in the form

$$S_{cr} = KE(t/a)^2$$

where E is Young's modulus of elasticity, t is plate thickness, a is hole radius, and K is a buckling coefficient having the following values for the first four modes: 7.65, 7.80, 11.7, 12.3. A comparison is made with some experimental determinations of Danis for plates of finite width.

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# NOMENCLATURE

| | |
|---|---|
| a | Hole radius |
| $A_{NM}$ | A set of unknown coefficients in a series describing plate deflection. |
| $\underline{A}$ | An unknown eigenvector |
| A | An integer index used in Appendix C |
| a(J,L) | A trigonometric product used as an intermediate function in Tables 5 and 6 |
| b | The half-width of an infinite strip in convenient units |
| b(J,L) | A trigonometric product used as an intermediate function in Tables 5 and 6 |
| $\underline{\underline{B}}$ | A coefficient matrix in Appendix D |
| B | An integer index in Appendix C |
| C | An integer index in Appendix C |
| d | Hole diameter |
| D | An integer index in Appendix C |
| D | Flexural rigidity of the plate. = $Eh^3/12\ (1 - \nu^2)$ |
| E | Young's Modulus |
| $E_1, E_2$, etc. | Terms in the membrane energy integral |
| f | Functions of I,J,k, and L used in Table 5 |
| g | Functions of I,J,K, and L used in Table 6 |
| $G_1, G_2$, etc. | Terms in the bending energy integral |
| h | Plate thickness in units of hole radii |
| $\underline{\underline{I}}$ | Identity matrix |
| I | A dummy integer index for the index N |
| I' | A single subscript representing the double subscript IJ |
| $\bar{I}$ | The maximum value of I' |
| J | A dummy integer index for the index M |

| | |
|---|---|
| $J'$ | A single subscript representing the double subscript KL |
| $\bar{J}$ | The maximum value of $J'$ |
| K | A dummy integer index for the index N |
| K | Buckling coefficient based on hole radius |
| L | A dummy integer index for the index M |
| M | An integer index |
| $M_r$ | Bending moment per unit length |
| $M_{rt}$ | Twisting moment per unit length |
| N | An integer index |
| $N_x, N_y$ | Membrane forces per unit length, $= \sigma_x h$, etc. |
| $P_{IJKL}$, $P_{I'J'}$ | A term in the quadruple series that expresses the bending energy integral |
| $\underline{\underline{Q}}$ | A matrix of the eigenvectors of $\underline{\underline{I}}$ |
| $Q_{IJKL}$, $Q_{I'J'}$ | A term in the quadruple series that expresses the membrane energy integral |
| r | Distance from the hole center in units of hole radii; also a coordinate of the cylindrical system |
| $R_{IJKL}$, $R_{I'J'}$ | A term in the quadruple series that expresses the membrane energy integral |
| $\underline{\underline{R}}$ | The matrix of elements $R_{IJKL}$ |
| $s_1, s_2, s_3$ | Functions of N, M and $\nu$ in Appendix A. |
| S | Externally applied stress |
| $S_{cr}$ | The value of S at which incipient buckling occurs |
| $t_1, t_2, t_3$ | Functions of N, M and $\nu$ in Appendix A. |
| t | Plate thickness |
| $\underline{\underline{T}}$ | The matrix of elements $T_{IJKL}$ |
| $T_{IJKL}$, $T_{I'J'}$ | A term in the quadruple series that expresses the bending energy integral |
| u | Component of displacement in the x direction |

| | |
|---|---|
| $U_1, U_2$ | Membrane energy in the plate |
| $U_3$ | Bending energy in the plate |
| $v$ | Component of displacement in the y direction |
| $W$ | Plate deflection in units of hole radii |
| $W_r, W_{rr}$, etc. | Derivatives of plate deflection with respect to the subscripts |
| $x^*$ | A compacted notation of $X_{NM}$ in Table 5 with I and J replacing N and M respectively |
| $x$ | A compacted notation for $X_{NM}$ in Table 5 with K and L replacing N and M respectively; also a coordinate in the Cartesian system |
| $X_{NM}$ | An unknown coefficient which is used to satisfy the boundary conditions for each term in N or M |
| $\underline{X}$ | An unknown eigenvector in Appendix D |
| $y^*$ | A compacted notation for $Y_{NM}$ in Table 5 with I and J replacing N and M respectively |
| $y$ | A compacted notation for $Y_{NM}$ in Table 5 with K and L replacing N and M respectively; also a coordinate of the Cartesian system; also a scalar parameter in an eigenvalue problem in Appendix D |
| $Y_{NM}$ | An unknown coefficient which is used to satisfy the boundary conditions for each term in N or M |
| $\alpha$ | The number of terms taken in N in the series for W |
| $\beta$ | The number of terms taken in M in the series for W |
| $\lambda$ | A scalar parameter in eigenvalue problems |
| $\sigma_x, \sigma_y, \sigma_r, \sigma_\theta$ | Normal stress |
| $\tau_{r\theta}, \tau_{xy}$ | Shear stress |
| $\theta$ | An angular coordinate in the cylindrical system |
| $\nu$ | Poisson's ratio ($\nu$ = .3 is used throughout) |

OTHER NOTATION

A double underline indicates a square matrix. ($\underline{\underline{A}}$)

A single underline indicates a column vector. ($\underline{A}$)

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# 1. INTRODUCTION

The problem of determining the stress situation near a hole in a plate under tension (see Figure 1) has a well known solution which is of great practical importance.

The original work on this problem was done by Kirsch (19), who solved the problem of the stress distribution in an infinite plate pierced with a round hole and subject to uniaxial tension at a great distance from the hole. Howland (17) treated a similar problem with an infinite strip of finite width. Savin (5) and Hogan (15) have compiled solutions produced by other authors for variations in hole geometry, as square, triangular, etc., for both finite and infinite plates. Lur'e (3) has solved the problem with a hole in a shell of uniform curvature.

An important result obtained from these solutions is that the stresses near the hole become much greater than those imposed upon the plate at a distance from the hole. This has given rise to the wide publication of many stress concentration factors which are important in design work and which relate the maximum stress to the nominal stress at a distance from the hole.

Another feature of the stress situation, which is not so widely recognized, is that certain areas of the plate are under uniaxial or biaxial compression,* although tension is imposed externally. These areas of compression, given the proper conditions of plate thickness

---

* These terms will be employed herein to mean that one or both principal stresses are compressive. Specifically, "uniaxial compression" means that one principal stress is compressive and one is tensile; "biaxial compression" means that both are compressive.

and external tensile load, will buckle, possibly causing failure of the plate. Houghton and Rothwell (16) have shown that in plates with holes having a very small gross curvature the stress concentration factor can increase appreciably. It is reasonable to assume then, that even though local distortions of a plate may be acceptable, such distortions could increase the stress concentration beyond that expected for a flat plate. Failure of the plate by yielding could then occur.

There is no known investigation of this buckling problem, other than an experimental investigation by Danis (14), to determine the critical stress for the buckling of a finite plate with a round hole.

This paper presents a theoretical investigation of the incipient buckling problem for an infinite plate containing a round hole. This problem is of interest in itself and for the practical reasons given in the preceding discussion. Results are presented on page 33.

## 2. PROBLEM DESCRIPTION

Consider a plate infinitely extended in two directions and pierced with a hole, as shown in Figure 1.

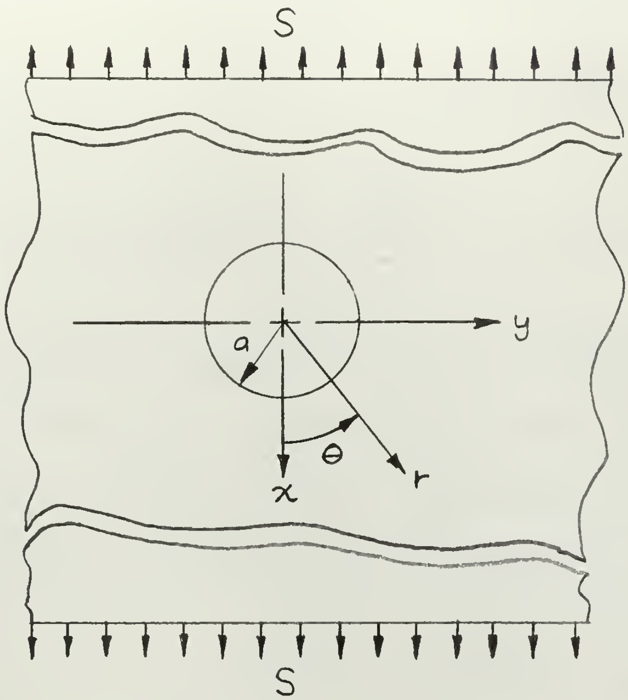


Figure 1

x and y are Cartesian coordinates, r and θ, cylindrical coordinates. The z axis, not shown, is perpendicular to the paper at the center of the hole, and extends in the positive direction toward the reader. Consider all quantities to be in units of hole radius, a, which essentially reduces them to dimensionless quantities, with hole radius equal to one.

The plate is placed in uniform tension, S psi, at infinity in the x direction. The solution of the stress distribution problem for this plate, as originally solved by Kirsch (19), and clearly presented by Timoshenko (7) is:

$$\sigma_r = \frac{S}{2}\left[\left(1 - \frac{1}{r^2}\right) + \left(1 + \frac{3}{r^4} - \frac{4}{r^2}\right) \cos 2\theta\right] \tag{1}$$

$$\sigma_\theta = \frac{S}{2}\left[\left(1 + \frac{1}{r^2}\right) - \left(1 + \frac{3}{r^4}\right) \cos 2\theta\right] \tag{2}$$

$$\tau_{r\theta} = \frac{S}{2}\left[1 - \frac{3}{r^4} + \frac{2}{r^2}\right] \sin 2\theta \tag{3}$$

This set of equations is henceforth referred to as the Kirsch solution.

It is observed that the maximum tensile stress is 3 times the stress applied at infinity, and occurs at r = 1, and $\theta = \pm\pi/2$. The maximum compressive stress, equal in magnitude to the applied tensile stress, occurs at r = 1, and $\theta = \pm\pi$. From this it can be seen that if buckling were to occur in the plate, it would be most apparent near the hole along the x axis.

A better understanding of the buckling problem is obtained by considering the minimum principal stresses (most negative, largest compressive). From the Kirsch solution, the minimum principal stresses are found and shown in Figures 2 and 3.

Figure 2 shows one quarter of the plate, with the hole centered at the top left. The stress distribution is symmetrical with respect to the x and y axes. This picture was prepared with consideration only to whether the stress situation at a particular point is biaxial tensile, biaxial compressive, or uniaxial compressive.

Figure 3 shows the same section of plate as Figure 2. This presentation maps the relative values of the algebraic minimum principal stress at each point.


FIG 2 — MAP OF THE PRINCIPAL STRESSES AROUND A HOLE

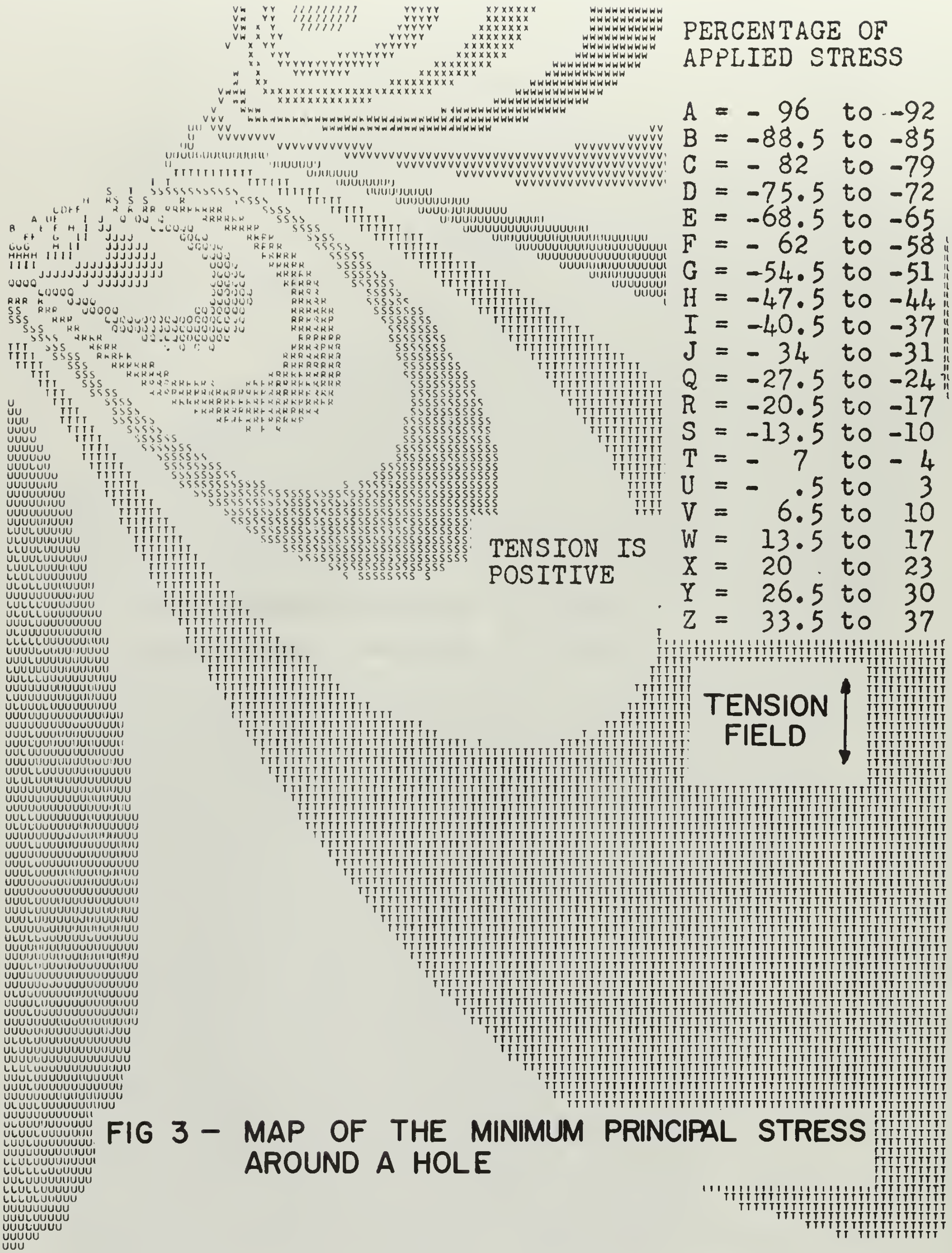


FIG 3 – MAP OF THE MINIMUM PRINCIPAL STRESS AROUND A HOLE

It can be seen from these figures that there is an area having biaxial compression. Accordingly, buckling is expected to take place for some sufficiently great value of applied load.

Numerous methods have been developed for solutions of plate buckling problems. In general, these methods involve either the satisfaction of the equilibrium equations or minimizing the strain energy of the plate (Energy Methods). A compilation of many solutions to plate buckling problems was made by Gerard and Becker (11). This is an excellent reference for both method and practical solutions.

One of the first energy methods was developed by Ritz (20), and independently, by Lord Rayleigh (4). This method is presented in a rigorous manner by Sokolnikoff (6). Budiansky and Hu (12) give a detailed example of the application of this method to a plate buckling problem.

The Rayleigh-Ritz method involves determining the total strain energy in the plate as a function of an assumed buckled configuration. Equilibrium, and the stable buckled deflection, $\bar{W}(x,y)$, exists when this strain energy, $U_t$, is minimized with respect to $W(x,y)$.

To do this, first, a family of functions, $\Phi$, are assumed, which are dependent upon parameters $A_k$, such that the buckled deflection, $W(x,y)$, can be represented by:

$$W(x,y) = \Phi(x, y, A_1, A_2, \cdots) \qquad (4)$$

where $\Phi$ is such that the boundary conditions on the plate are satisfied for any values of the $A_i$.

The functions $\Phi$ are substituted into the strain energy expression $U_t$ so that

$$U_t = \iint ftn\left[\Phi(x, y, A_1, A_2, \cdots)\right] dx\,dy \qquad (5)$$

where $U_t$ is the integrated total energy in the plate.

In order to minimize the energy function, the following set of equations is formed

$$\left.\begin{aligned} \frac{\partial U_t}{\partial A_1} &= 0 \\ \frac{\partial U_t}{\partial A_2} &= 0 \\ &\vdots \end{aligned}\right\} \qquad (6)$$

If this set of equations has solutions $\bar{A}_1, \bar{A}_2, \ldots,$ the function

$$\bar{W}(x,y) = \Phi(x, y, \bar{A}_1, \bar{A}_2, \cdots)$$

represents the buckled equilibrium condition, the boundary conditions of the problem being satisfied by $\Phi$ for any values of the $A_i$.

## 3. BOUNDARY CONDITIONS

For the problem of an infinite plate with a hole, two boundaries are of concern. At infinity, deflection, slope, and curvature are zero. Around the hole, a free edge condition exists.

The Kirsch solution is used, which already implies zero tractions around the hole, and uniaxial tension at infinity, so these conditions have already been satisfied.

For a free edge to exist around the hole means that there is no shear, twist, or moment applied at the edge of the hole. Following Timoshenko (8), and considering the cylindrical coordinate system established in Figure 1, the conditions for a free edge are

$$(Q_r)_{r=1} = 0 \qquad \text{(shear = 0)} \qquad (7)$$

$$(M_{rt})_{r=1} = 0 \qquad \text{(twist = 0)} \qquad (8)$$

$$(M_r)_{r=1} = 0 \qquad \text{(moment = 0)} \qquad (9)$$

Kirchoff (18) has shown that the twist condition can be reduced to a statically equivalent shear force. The original requirements of zero shear and zero twist can be reduced to one requirement such that the sum of the shear forces and the static equivalent of the twisting moments equals zero. The conditions then become

$$\left| Q_r - \frac{\partial M_{rt}}{\partial \theta} \cdot \frac{1}{r} \right|_{r=1} = 0 \qquad \text{(Kirchoff shear = 0)} \qquad (10)$$

$$\left| M_r \right|_{r=1} = 0 \qquad \text{(moment = 0)} \qquad (11)$$

where $Q_r$, $M_r$, and $M_{rt}$ are expressed as

$$Q_r = -D\frac{\partial}{\partial r}\left[ \frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial W}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 W}{\partial \theta^2} \right] \qquad (12)$$

$$M_r = -D\left[ \frac{\partial^2 W}{\partial r^2} + \nu\left( \frac{1}{r} \cdot \frac{\partial W}{\partial r} + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2} \right) \right] \qquad (13)$$

$$M_{rt} = D(1-\nu)\left[ \frac{1}{r} \cdot \frac{\partial^2 W}{\partial r \partial \theta} - \frac{1}{r^2} \cdot \frac{\partial W}{\partial \theta} \right] \qquad (14)$$

When the appropriate differentiations and substitions are made, the boundary condition equations become

$$\left[\frac{\partial^3 W}{\partial r^3} + \frac{1}{r}\cdot\frac{\partial^2 W}{\partial r^2} - \frac{1}{r^2}\cdot\frac{\partial W}{\partial r} + \frac{(2-\nu)}{r^2}\cdot\frac{\partial^3 W}{\partial r\,\partial\theta^2} - \frac{(3-\nu)}{r^3}\cdot\frac{\partial^2 W}{\partial\theta^2}\right]_{r=1} = 0 \qquad \text{(Kirchoff Shear = 0)} \qquad (15)$$

$$\left[\frac{\partial^2 W}{\partial r^2} + \frac{\nu}{r}\left(\frac{\partial W}{\partial r} + \frac{1}{r}\cdot\frac{\partial^2 W}{\partial\theta^2}\right)\right]_{r=1} = 0 \qquad \text{(Moment = 0)} \qquad (16)$$

$$\left[W = \frac{\partial W}{\partial r} = \frac{\partial^2 W}{\partial r^2}\right]_{r=\infty} = 0 \qquad (17)$$

As previously noted, each member of the family of functions must satisfy the boundary conditions in the Rayleigh-Ritz method. For a choice of $\Phi$ let

$$W(x,y) = \sum_{N=1}^{\infty}\sum_{M=0}^{\infty} r^{-N}\left[1 + \frac{X_{NM}}{r} + \frac{Y_{NM}}{r^2}\right]\left\{\begin{matrix} A_{NM}\cos M\theta \\ \text{or} \\ B_{NM}\sin M\theta \end{matrix}\right\} \qquad (18)$$

where $A_{NM}$ and $B_{NM}$ are arbitrary coefficients that are varied to minimize strain energy in the plate.

The boundary conditions of eq. 17 are satisfied by this series since at $r = \infty$, W and all of its derivatives are zero. Conditions of equations 15 and 16 are satisfied by substituting equation 18 into these two boundary condition equations (see Appendix A). Such a substitution results in two equations, 15 and 16, with two unknowns, $X_{NM}$ AND $Y_{NM}$, for each conbination of N and M. A unique value of $X_{NM}$ and $Y_{NM}$ is found which allows each term* of equation 18 to satisfy the free edge boundary conditions for any N and M chosen.

From Figures 1 and 3, it is seen that there is symmetry with respect to both the x and y axes. This means that when W is written as a Fourier series in θ (eq. 18), the series contains either cosine

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* A term in this sense means the full expression

$$r^{-N}\left[1 + \frac{X_{NM}}{r} + \frac{Y_{NM}}{r^2}\right]\cdot\left[\begin{matrix} A_{NM}\cos M\theta \\ \text{or} \\ B_{NM}\sin M\theta \end{matrix}\right]$$

terms or sine terms, but not both. Additionally, the requirement of symmetry around both the x and y axes means that only odd or even terms in M need be taken. Thus the Fourier portion of equation 18 reduces to:

$$A_{NM} \cos M\theta, \text{ with } M = 1,3,5,...$$

or

$$A_{NM} \cos M\theta, \text{ with } M = 0,2,4,...$$

or

$$B_{NM} \sin M\theta, \text{ with } M = 1,3,5,...$$

or

$$B_{NM} \sin M\theta, \text{ with } M = 2,4,6,...$$

This gives four separate expressions for W, one for each possible condition of symmetry. The full Fourier series could be taken, but if this is done, three quarters of the coefficients, $A_{NM}$ or $B_{NM}$ are zero and considerable computer storage space is wasted. It is shown later than each of these series represents a different buckling mode.

## 4. ENERGY IN THE PLATE

To this point, a family of functions $W(r,\theta)$ (equation 18) have been developed, which satisfy the boundary conditions independently of the coefficients, $A_{NM}$ or $B_{NM}$, and are adequate to represent possible shapes of the plate. Attention is now turned upon formulating an expression for the total strain energy, and then minimizing this expression to obtain the critical buckling stress.

The energy in the plate can be divided into two forms. First, there is the membrane energy which is the energy stored because the mid-plane is stretched. Second, there is the bending and twisting energy which results from the plate buckling out of its plane.

A basic assumption is that the plate deflection, $W(x,y)$, is infinitesimally small, and that the mid-plane strains may be represented by

$$\left.\begin{aligned} e_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 \\ e_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2 \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial W}{\partial x}\cdot\frac{\partial W}{\partial y} \end{aligned}\right\} \qquad (19)$$

Terms of higher order in W than these are neglected, since W is infinitesimal; those terms which are retained are essential for the calculation of bending energy.

In the above expressions $\frac{\partial u}{\partial x}$, etc., represent the stretching of the plate in its own plane. The terms $\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2$, etc., represent the difference between a length of flat plate, and the corresponding straight line length when the plate is bent (difference between arc and chord).

Stresses in the x and y directions are composed of

(a) Membrane stresses of finite magnitude, which come from the Kirsch Solution and are denoted by $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ below; and (b) Infinitesimal bending and twisting stresses which do not appear explicitly below, but which are involved in the bending and twisting energy.

The normal forces per unit length ($N_x$, $N_y$) can be determined from $N_x = \sigma_x h$, etc. The membrane energy in the plate is

$$U_{MEM} = \frac{1}{2} \iint \left( N_x e_x + N_y e_y + N_{xy} \gamma_{xy} \right) dx\, dy \qquad (20)$$

Substituting the strains (eqns. 19), and letting

$$U_{MEM} = U_1 + U_2 \qquad (21)$$

where

$$U_1 = \frac{h}{2} \iint \left[ \sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] dx\, dy \qquad (22)$$

and

$$U_2 = \frac{h}{4} \iint \left[ \sigma_x \left( \frac{\partial W}{\partial x} \right)^2 + \sigma_y \left( \frac{\partial W}{\partial y} \right)^2 + \tau_{xy} \cdot \frac{\partial W}{\partial x} \cdot \frac{\partial W}{\partial y} \right] dx\, dy \qquad (23)$$

an expression for $U_{MEM}$ is found in terms of the stresses, strains, and the deflection.

It can be seen that the membrane energy is composed of two terms. One, $U_1$, relates the work done by the stresses, $\sigma_x$, etc, through a distance $\frac{\partial u}{\partial x}$, etc. The other term, $U_2$, relates the work done by the stresses, $\sigma_x$, etc., through a distance $\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2$, etc., as the plate bends out of plane.

At this point, the assumption is made that the Kirsch solution is still valid, even if the plate has buckled infinitesimally. This means that $U_1$ is independent of the lateral deflection, W, and can be treated as a constant. This assumption is justified by considering that the problem is one of incipient buckling, not postbuckling. This assumption considerably simplifies the problem, because it uncouples the membrane stress distribution from the buckled deflection.

An expression for the bending and twisting energy, $U_3$, is obtained by considering the moments in the plate and the respective rotations at a point as functions of plate deflection. Timoshenko (9) shows this expression to be;

$$U_3 = \frac{D}{2}\iint\left\{\left[\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right]^2-(2-\nu)\left[\frac{\partial^2 W}{\partial x^2}\cdot\frac{\partial^2 W}{\partial y^2}-\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right]\right\}dx\,dy \qquad (24)$$

Adopting the notation $W_r = \frac{\partial W}{\partial r}$, $W_\theta = \frac{\partial W}{\partial \theta}$, etc., and transforming the previous energy equations to cylindrical coordinates, with the proper limits of integration for the problem of a hole in an infinite plate, equations 22, 23, and 24 become, respectively

$$U_1 = \text{CONSTANT} \qquad (25)$$

$$U_2 = \frac{h}{4}\int_0^{2\pi}\int_1^{\infty}\left(\sigma_r W_r^2 + \frac{\sigma_\theta W_\theta^2}{r^2} + \frac{2\tau_{r\theta} W_r W_\theta}{r}\right) r\,dr\,d\theta \qquad (26)$$

$$U_3 = \frac{D}{2}\int_0^{2\pi}\int_1^{\infty}\left\{\left[W_{rr}+\frac{W_r}{r}+\frac{W_{\theta\theta}}{r^2}\right]^2-(2-\nu)\left[W_{rr}\left(\frac{W_r}{r}+\frac{W_{\theta\theta}}{r^2}\right)-\left(\frac{W_\theta}{r^2}-\frac{W_{r\theta}}{r}\right)^2\right]\right\} r\,dr\,d\theta \qquad (27)$$

$\sigma_r$, $\sigma_\theta$ and $\tau_{r\theta}$ are obtained from the Kirsch solution (eqns. 1,2, and 3). W and its derivatives are obtained from the assumed series (eq 18).

## 5. SOLUTION OF THE PROBLEM

With the expression for total energy ($U_t = U_1 + U_2 + U_3$) formulated, the next step is to substitute the Kirsch solution, and the assumed series for deflection and its derivatives. Then this expression must be minimized with respect to each of the unknown coefficients, $A_{NM}$ or $B_{NM}$. The substitutions and integrations are shown in Appendices B, C, and D. It is sufficient for our purpose here to say that what arises from the integration of the expressions for $U_2$ and $U_3$, are two quadruple series, with indices I and K summed over the range of N, and J and L summed over the range of M. The quadruple series stems from the multiplication of two double sums. Equations 26 and 27 now become*

$$U_2 = \frac{Sh}{4} \sum_I \sum_J \sum_K \sum_L Q_{IJKL} \cdot A_{IJ} \cdot A_{KL} \qquad (28)$$

$$U_3 = \frac{D}{2} \sum_I \sum_J \sum_K \sum_L P_{IJKL} \cdot A_{IJ} \cdot A_{KL} \qquad (29)$$

where the flexural rigidity of the plate, D, equals $\frac{Eh^3}{12(1-\nu^2)}$. $Q_{IJKL}$ and $P_{IJKL}$ are different for each value of I, J, K, or L. S is the stress applied to the plate at infinity and is factored from the Kirsch solution after those expressions are substituted into the energy integrals. $A_{IJ}$ and $A_{KL}$ are the unknown coefficients in the series expression for deflection (eq. 18).

Following the Rayleigh-Ritz procedure, $U_t$ is differentiated with respect to each of the unknown coefficients, so that the

---------------

* The unknown coefficients for the cosine series, $A_{NM}$, only will be used henceforth, although these could be the sine series coefficients, $B_{NM}$, equally as well.

following set of simultaneous equations result.

$$0=\frac{\partial U_t}{\partial A_{11}}=\left[\sum_K\sum_L\left(\frac{Sh}{4}Q_{11KL}+\frac{D}{2}P_{11KL}\right)A_{KL}\right]+\left[\frac{Sh}{4}Q_{1111}+\frac{D}{2}P_{1111}\right]A_{11}$$

$$0=\frac{\partial U_t}{\partial A_{12}}=\left[\sum_K\sum_L\left(\frac{Sh}{4}Q_{12KL}+\frac{D}{2}P_{12KL}\right)A_{KL}\right]+\left[\frac{Sh}{4}Q_{1212}+\frac{D}{2}P_{1212}\right]A_{12} \qquad (30)$$

$$\vdots$$

The last term in each equation arises from the fact that $\frac{\partial}{\partial A_{IJ}}(A_{IJ}\cdot A_{IJ})=2A_{IJ}$.

As a practical matter, the equations arrived at in this investigation are too complicated to permit considering the use of a doubly infinite set of terms in the double series. Accordingly, it is henceforth presumed that truncated series are used and that there is but a finite number of unknown $A_{NM}$'s.

Simplification in notation is made at this point if the quadruple subscripts are replaced by a double subscript arrangement.

Let $\beta$ equal the number of terms taken in M, and $\alpha$ equal the number of terms taken in N. Define

$$I'=(I-1)\beta+J$$

$$J'=(K-1)\beta+L$$

Using these equations a unique pair of subscripts can be changed to a unique single subscript, so that $A_{IJ}=A_{I'}$, and $Q_{IJKL}=Q_{I'J'}$, etc.

For example, if 4 terms were taken in M, and we wished to convert $Q_{7492}$ to some doubly subscripted $Q_{I'J'}$, then I = 7, J = 4, K = 9, and L = 2; and

$$I'=(7-1)\,4+4, \qquad J'=(9-1)\,4+2$$

$$=28 \qquad\qquad =34$$

so that $Q_{7492}$ becomes $Q_{(28)(34)}$. Similarly

$$A_{11} \rightarrow A_1 \qquad Q_{1111} \rightarrow Q_{11}$$
$$A_{12} \rightarrow A_2 \qquad Q_{1211} \rightarrow Q_{21}$$
$$A_{13} \rightarrow A_3 \qquad Q_{1112} \rightarrow Q_{12}$$
etc. etc.

Also, let
$$\bar{J} = (\alpha - 1)\cdot\beta + \beta = \alpha\cdot\beta$$
where $\bar{J}$ is the largest value of I' and J' respectively.

Substituting
$$D = \frac{Eh^3}{12(1-\nu^2)}$$
and
$$R_{I'J'} = Q_{I'J'} \quad \text{and} \quad T_{I'J'} = P_{I'J'} \quad \text{if} \quad I' \neq J'$$
$$R_{I'J'} = 2\cdot Q_{I'J'} \quad \text{and} \quad T_{I'J'} = 2\cdot P_{I'J'} \quad \text{if} \quad I' = J'$$
equation 30 simplifies to

$$\left.\begin{aligned}
\frac{\partial U_t}{\partial A_1} &= 0 = \sum_{J'} \left[ \left( \frac{6S(1-\nu^2)}{Eh^2} \right) R_{1J'} + T_{1J'} \right] \\
\frac{\partial U_t}{\partial A_2} &= 0 = \sum_{J'} \left[ \left( \frac{6S(1-\nu^2)}{Eh^2} \right) R_{2J'} + T_{2J'} \right] \\
&\vdots \\
\frac{\partial U_t}{\partial A_{\bar{J}}} &= 0 = \sum_{J'} \left[ \left( \frac{6S(1-\nu^2)}{Eh^2} \right) R_{\bar{J}J'} + T_{\bar{J}J'} \right]
\end{aligned}\right\} \tag{31}$$

When the substitution
$$\lambda = \frac{6S(1-\nu^2)}{Eh^2} \tag{32}$$
is made, and the above set of equations expressed in matrix notation, the following matrix eigenvalue problem results
$$(\underline{\underline{R}}\lambda + \underline{\underline{T}})\underline{A} = 0 \tag{33}$$
where

$$\underline{\underline{R}} = \begin{bmatrix} R_{11} & R_{12} & \cdot & \cdot & \cdot & R_{1\bar{J}} \\ R_{21} & R_{22} & & & & R_{2\bar{J}} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ R_{\bar{J}1} & R_{\bar{J}2} & \cdot & \cdot & \cdot & R_{\bar{J}\bar{J}} \end{bmatrix} \tag{34}$$

$$\underline{\underline{T}} = \begin{bmatrix} T_{11} & T_{12} & \cdot & \cdot & \cdot & T_{1\bar{J}} \\ T_{21} & T_{22} & \cdot & \cdot & \cdot & T_{2\bar{J}} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ T_{\bar{J}1} & T_{\bar{J}2} & \cdot & \cdot & \cdot & T_{\bar{J}\bar{J}} \end{bmatrix} \tag{35}$$

and

$$\underline{A} = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_{\bar{J}} \end{bmatrix} \tag{36}$$

The matrix $\underline{\underline{T}}$ is positive definite, since it arose from the bending energy integral. The $\underline{\underline{R}}$ matrix arose from the membrane energy integral, which contains terms from the Kirsch solution. This matrix is not necessarily positive definite. Both matrices are symmetric. These properties are important to the method utilized in solving the matrix eigenvalue problem (see Appendix D).

The question of whether differentiating $U_t$ and setting these derivatives equal to zero, produces a minimum or a maximum, has not been discussed. Either situation can occur in the problem as it has been developed. Some of the eigenvalues obtained in a solution of equation 33 represent maxima which are not relevant to the plate buckling problem. The criterion for rejecting such answers is the set of equations

$$\frac{\partial^2 U_t}{\partial A_1^2} > 0$$

$$\left.\begin{array}{l} \dfrac{\partial^2 U_t}{\partial A_2^2} > 0 \\ \vdots \\ \dfrac{\partial^2 U_t}{\partial A_{\bar{J}}^2} > 0 \end{array}\right\} \tag{37}$$

If the set of equations 37 is satisfied, then $U_t$ is minimized.

When the set of equations 31 is again differentiated with respect to the appropriate coefficient, the set of equations 37 become

$$\left.\begin{array}{l} \dfrac{\partial^2 U_t}{\partial A_1^2} = \lambda R_{11} + T_{11} > 0 \\ \dfrac{\partial^2 U_t}{\partial A_2^2} = \lambda R_{22} + T_{22} > 0 \\ \vdots \qquad\qquad \vdots \\ \dfrac{\partial^2 U_t}{\partial A_{\bar{J}}^2} = \lambda R_{\bar{J}\bar{J}} + T_{\bar{J}\bar{J}} > 0 \end{array}\right\} \tag{38}$$

where $R_{I'I'}$ and $T_{I'I'}$ are the diagonal elements of the matrices $\underline{\underline{R}}$ and $\underline{\underline{T}}$, respectively. All of these terms are positive numbers, since they arise from the "squares" in the energy integrals (i.e., no cross product terms).

By rearranging equation 38 so that it becomes

$$\lambda > -\frac{T_{J'J'}}{R_{J'J'}}, \qquad 1 \leq J' \leq \bar{J} \tag{39}$$

It is seen that for some negative values of $\lambda$, and all positive values, the solution of equation 33 represents a minimization of energy.

We are interested in only positive values, so that for any answer of interest, a minimization is assured. It should be noted that the negative values of $\lambda$ that are admissible stem from buckling modes in the plate that correspond to an applied compressive loading. These modes arise from the reversal of sign in the Kirsch solution and the associated reversed stress distribution around the hole. Modes of this type should be expected from a theoretical standpoint.

When a nontrivial solution has been found for equation 33, giving an eigenvalue $\lambda$ and a corresponding eigenvector $\underline{A}$, then the buckling stress may be found by use of equation 32 and the buckled mode shape from equation 18.

## 6. RESULTS AND CONCLUSIONS

The results of this thesis may be presented compactly in the form of a formula for incipient buckling, derived from equation 37, viz.

$$S_{cr} = KE\left(\frac{t}{a}\right)^2 \tag{40}$$

where $S_{cr}$ is the value of S, the tension at infinity, which causes incipient buckling, t is plate thickness, a is the hole radius and E is Young's modulus of elasticity. Values of the buckling coefficient, K, are given in table 1.

Values of K were obtained from a digital computer solution (see Appendix F), for the particular case

$E = 30 \times 10^6$ psi (Young's Modulus)

$\nu = .3$ (Poisson's ratio)

h = .001 hole radii

Equation 40 permits prediction of $S_{cr}$ for other values of t/a and E. However, the effect of different values of $\nu$ cannot be studied easily since $\nu$ appears in the bending energy integral and in the boundary conditions. All results given are for $\nu = .3$, but the computer program can be modified by changing card listed as 0011 on page 84.

Four distinct mode shapes are apparent, as determined by different buckling coefficients appearing for each. A tabulation of K is shown in Table 1 below.

Table 1

Values of the Buckling Coefficient, K

| MODE | K |
|---|---|
| 1 | 7.65 |
| 2 | 7.80 |
| 3 | 11.7 |
| 4 | 12.3 |

Since the energy method involves the use of a series which cannot be made infinite in the present method, the actual buckling coefficients obtained will always be higher than the true value. This is because the deflection of the plate is only approximated, and the energy never quite minimized. A solution with less plate energy could be found, if more terms were taken in the series for deflection.

Different values of the buckling coefficient, K, actually calculated by use of the digital computer program are shown in Table 2, and are plotted against the number of terms taken in both the N and M parameters of the deflection series, equation 18. These plots are shown in Figures 4a through 4e. The projected values listed in Table 1 are taken graphically from these figures as the asymptotes which the curves for buckling coefficient appear to approach, when the number of terms in N and M increase. These projected values are considered to be the best evaluation of the buckling coefficient.

Figure 4f is taken from the experimental data of Danis (14). All of his points for a finite plate are plotted as K versus d/b,

Table 2

Values of the Buckling Coefficient, K, for Various Numbers of Terms in the Series

Mode 1

| | Terms in M | | | | | |
|---|---|---|---|---|---|---|
| Terms in N | 4 | 5 | 6 | 7 | 8 | 9 |
| 6 | | | | 11.258 | | |
| 8 | 9.9897 | 8.6493 | 8.4420 | 8.4173 | 8.4157 | 8.4153 |
| 10 | 9.1333 | 8.0673 | 7.8967 | 7.8763 | 7.8748 | 7.8746 |
| 12 | 8.9990 | 7.9743 | 7.8060 | 7.7857 | 7.7840 | |
| 14 | 8.9927 | 7.9683 | 7.7987 | 7.7777 | | |
| 16 | 8.9750 | 7.9537 | 7.7847 | | | |
| 18 | 8.9347 | 7.9237 | | | | |
| 20 | 8.8897 | 7.8913 | | | | |

Mode 2

| | Terms in M | | | | | |
|---|---|---|---|---|---|---|
| Terms in N | 4 | 5 | 6 | 7 | 8 | 9 |
| 8 | 9.1250 | 8.5877 | 8.5107 | 8.5033 | 8.5026 | 8.5026 |
| 10 | 8.5133 | 8.0630 | 7.9977 | 7.9913 | 7.9906 | 7.9906 |
| 12 | 8.4310 | 7.9860 | 7.9197 | 7.9130 | 7.9127 | |
| 14 | 8.4253 | 7.9773 | 7.9100 | 7.9030 | | |
| 16 | 8.4023 | 7.9577 | 7.8907 | | | |
| 18 | 8.3637 | 7.9257 | | | | |
| 20 | 8.3253 | 7.8940 | | | | |

Mode 3

| | Terms in M | | | | | |
|---|---|---|---|---|---|---|
| Terms in N | 4 | 5 | 6 | 7 | 8 | 9 |
| 8 | | 15.592 | 14.545 | 14.371 | 14.348 | 14.347 |
| 10 | | 13.327 | 12.616 | 12.501 | 12.487 | 12.486 |
| 12 | 16.288 | 12.679 | 12.046 | | 11.929 | |
| 14 | 15.959 | 12.516 | 11.897 | 11.793 | | |
| 16 | 15.910 | 12.493 | 11.872 | | | |
| 18 | 15.907 | 12.488 | | | | |
| 20 | 15.882 | 12.470 | | | | |

Mode 4

| | Terms in M | | | | | |
|---|---|---|---|---|---|---|
| Terms in N | 4 | 5 | 6 | 7 | 8 | 9 |
| 8 | | 16.418 | 15.765 | 15.660 | 15.647 | 15.646 |
| 10 | | 13.714 | 13.318 | 13.258 | 13.252 | 13.252 |
| 12 | 14.761 | 12.929 | 12.589 | | 12.533 | |
| 14 | 14.463 | 12.719 | 12.388 | 12.337 | | |
| 16 | 14.415 | 12.682 | 12.350 | | | |
| 18 | 14.412 | 12.677 | | | | |
| 20 | 14.392 | 12.662 | | | | |

the ratio of hole diameter to plate width. The buckling coefficients in Table 1, for modes 1 and 2, are plotted on the y axis (for zero d/b).

The values for modes 1 and 2 are in fair agreement with experimental data. No correlation between the theoretical prediction and the experimental observation is attempted for modes 3 and 4.

The buckled deflection of the plate for modes 1 through 4 is shown in Figures 5 through 8 respectively. It is seen in these figures that the four possible conditions of symmetry are satisfied by the four modes. This result is given in Table 3.

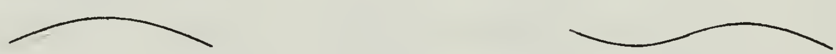
Table 3

Resultant Symmetry for the Various Mode Shapes

| MODE | SYMMETRY WITH RESPECT TO X AXIS | SYMMETRY WITH RESPECT TO Y AXIS |
|---|---|---|
| 1 | $f(y) = f(-y)$ | $f(x) = f(-x)$ |
| 2 | $f(y) = f(-y)$ | $f(x) = -f(-x)$ |
| 3 | $f(y) = -f(-y)$ | $f(x) = f(-x)$ |
| 4 | $f(y) = -f(-y)$ | $f(x) = -f(-x)$ |

In looking at Figure 4c, it is apparent that mode 2 is converging to a value of buckling coefficient that is somewhat higher than that for mode 1. This can be qualitatively justified by considering the symmetry of buckled shape, and the equivalent one dimensional problem.

Consider two beams, buckled by some unspecified loading, into shapes as shown in Figure 9.



a b

Figure 9

It is known from beam theory that the shape of Figure 9b has a higher critical buckling load than the shape of Figure 9a. This is exactly the relationship between modes 1 and 2 of the buckled plate, as can be seen in Figures 5 and 6, respectively.

There is a small area near the intersection of the hole edge and the y axis, which assumes cross sectional shapes similar to that in Figure 9a for mode 1, and in Figure 9b for mode 2. It is only in these small areas connecting the "top" of the plate to the "bottom", that this sort of bending occurs, so that the disparity between the buckling coefficients of the first and second plate modes is much smaller than that for the beam modes.

This same argument applies to the difference between modes 3 and 4, and to the gross differences between modes 3 and 4, and modes 1 and 2.

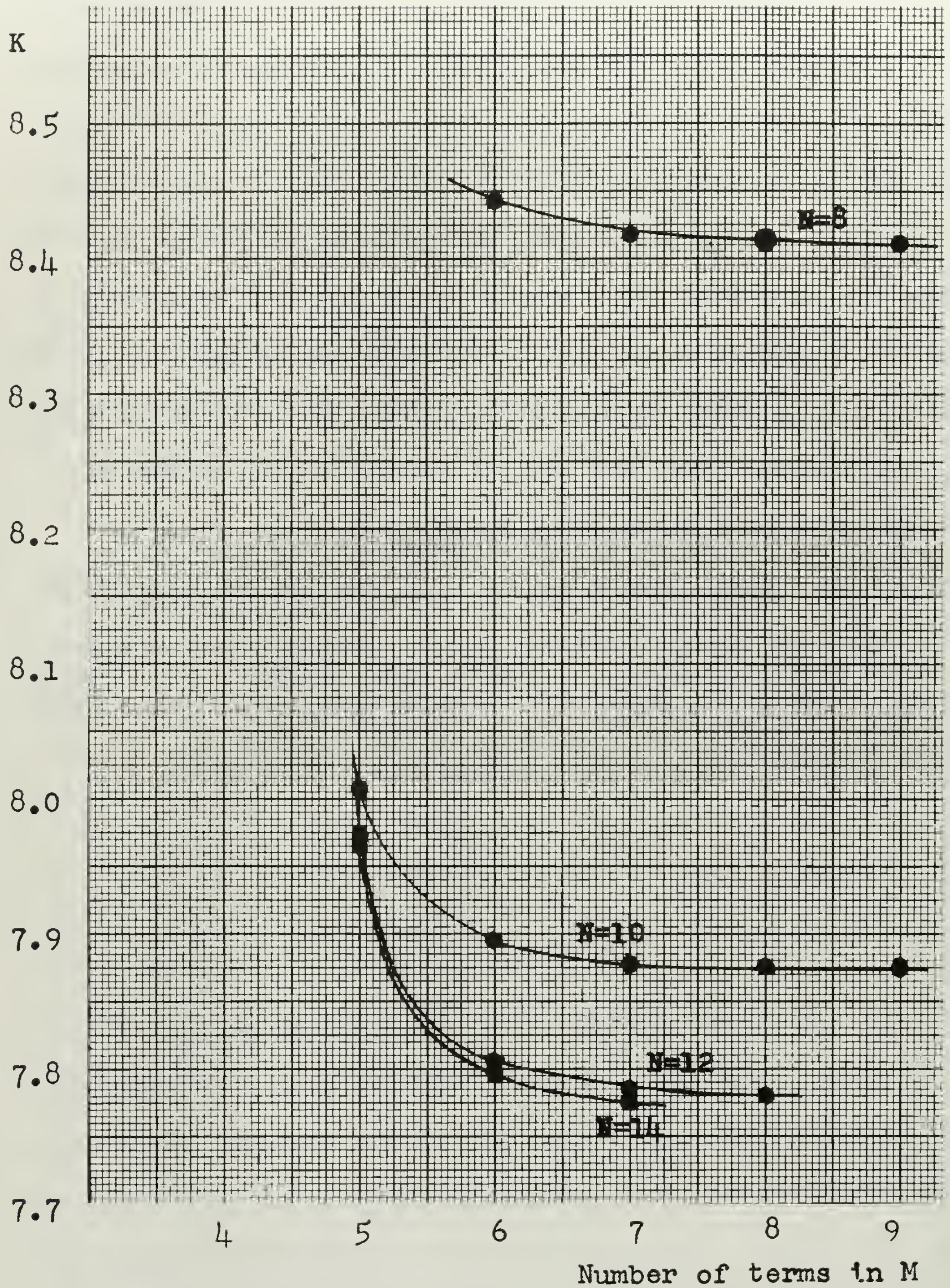



FIGURE 4a - Plot of the Buckling Coefficient K vs the number of terms taken in M for mode 1.

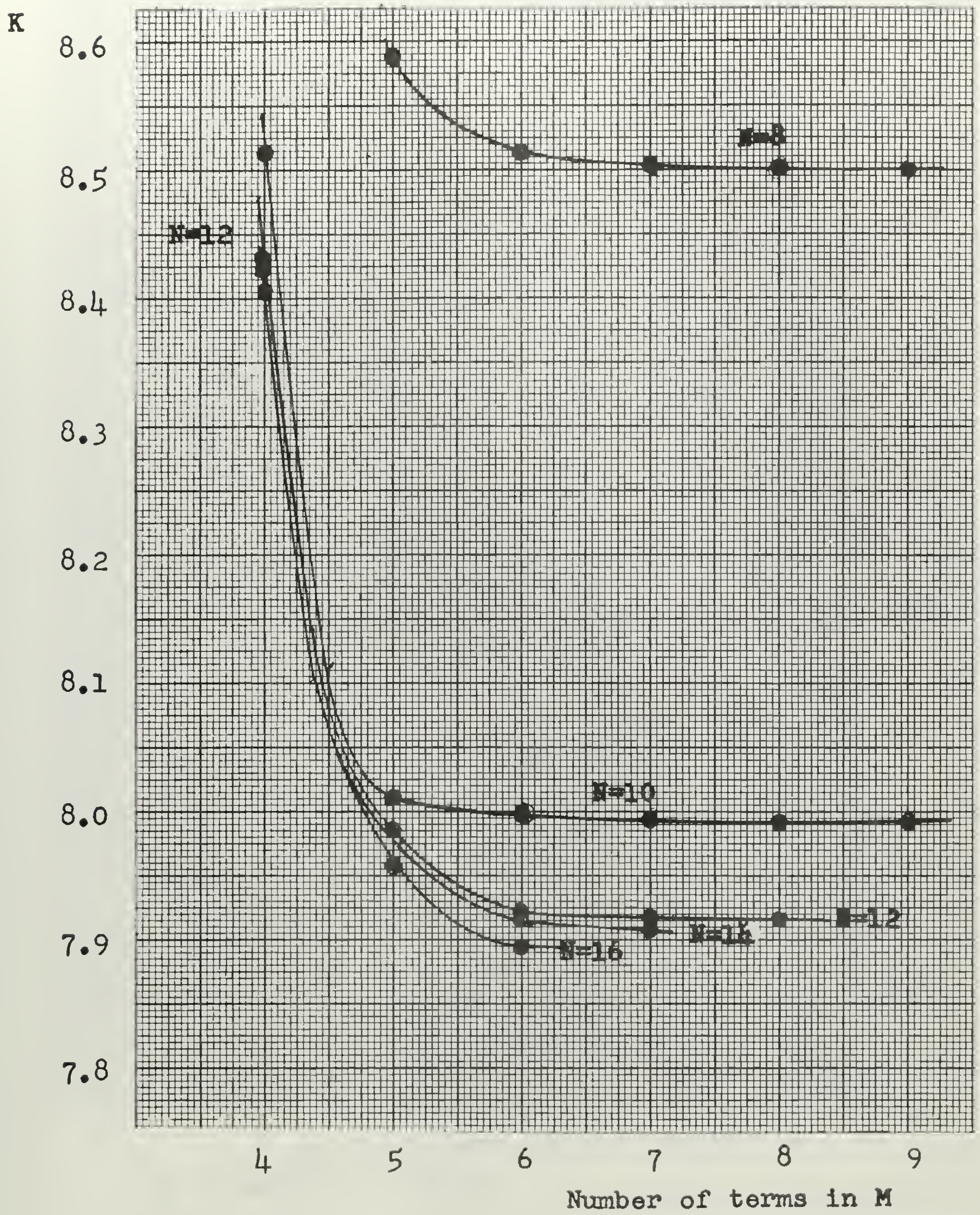



FIGURE 4b - Plot of the Buckling Coefficient K vs the number of terms taken in M, for mode 2.

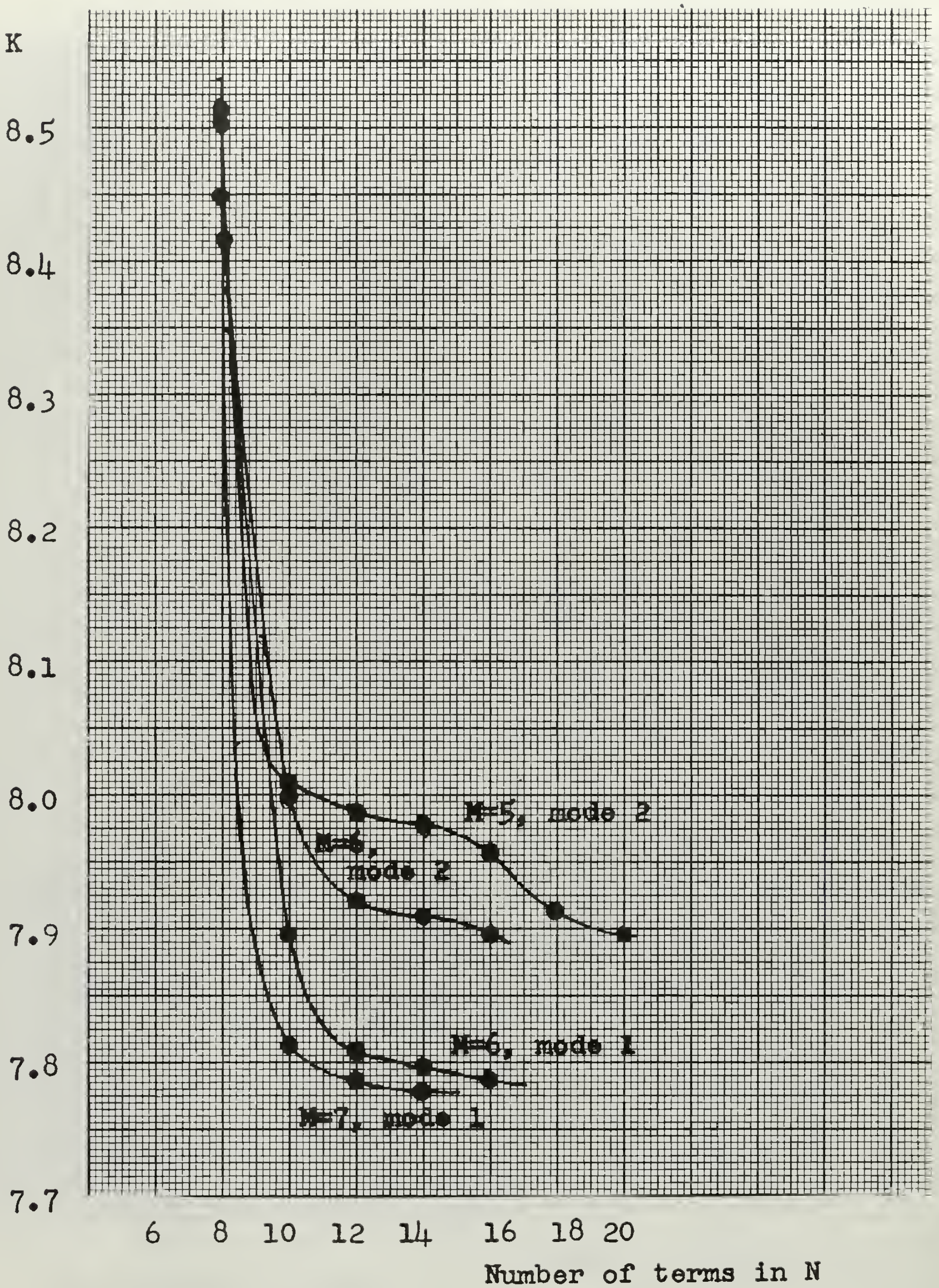



FIGURE 4c - Plot of the Buckling Coefficient K vs the number of terms taken in N, for modes 1 and 2

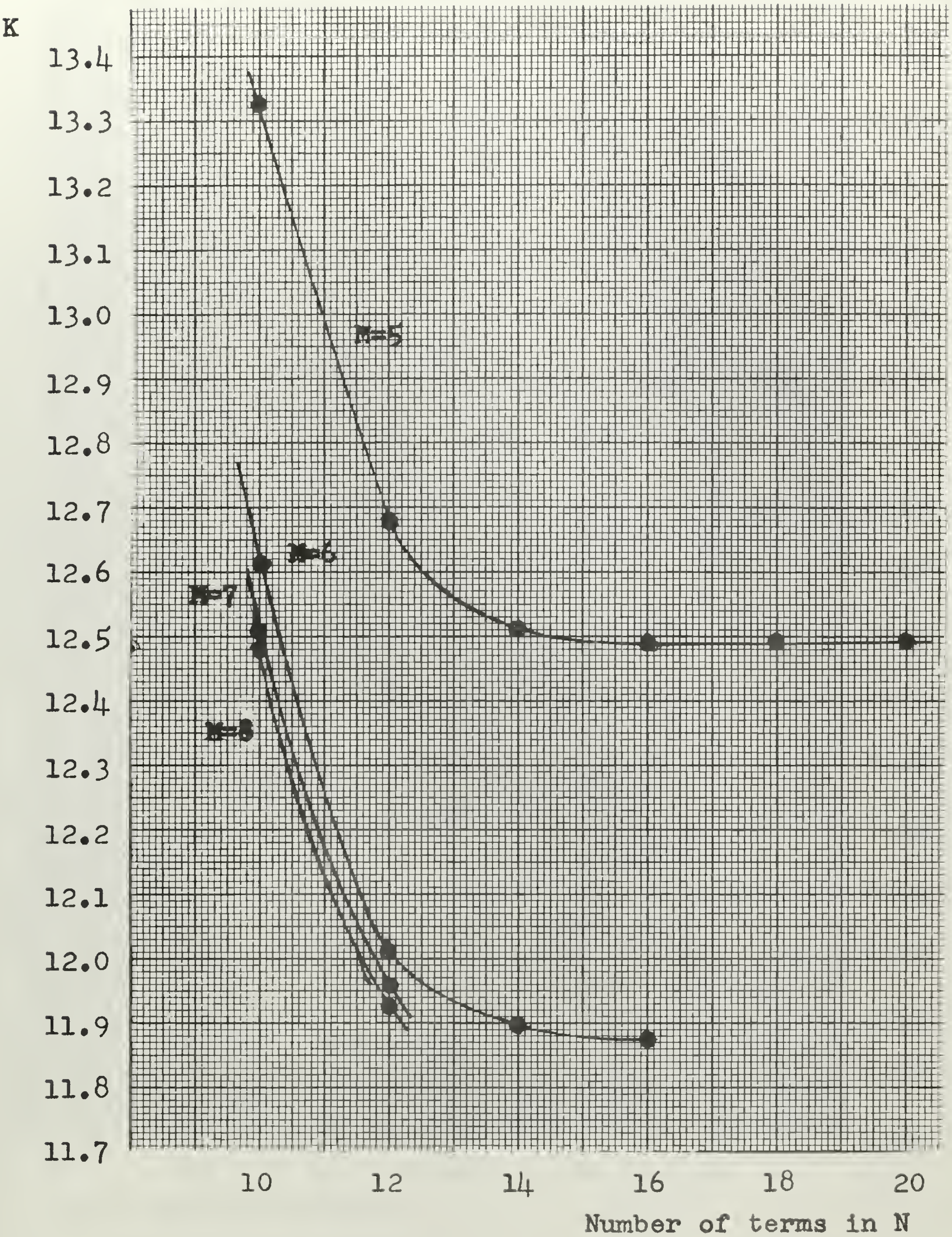



FIGURE 4d - Plot of the Buckling Coefficient K vs the number of terms taken in N for mode 3.

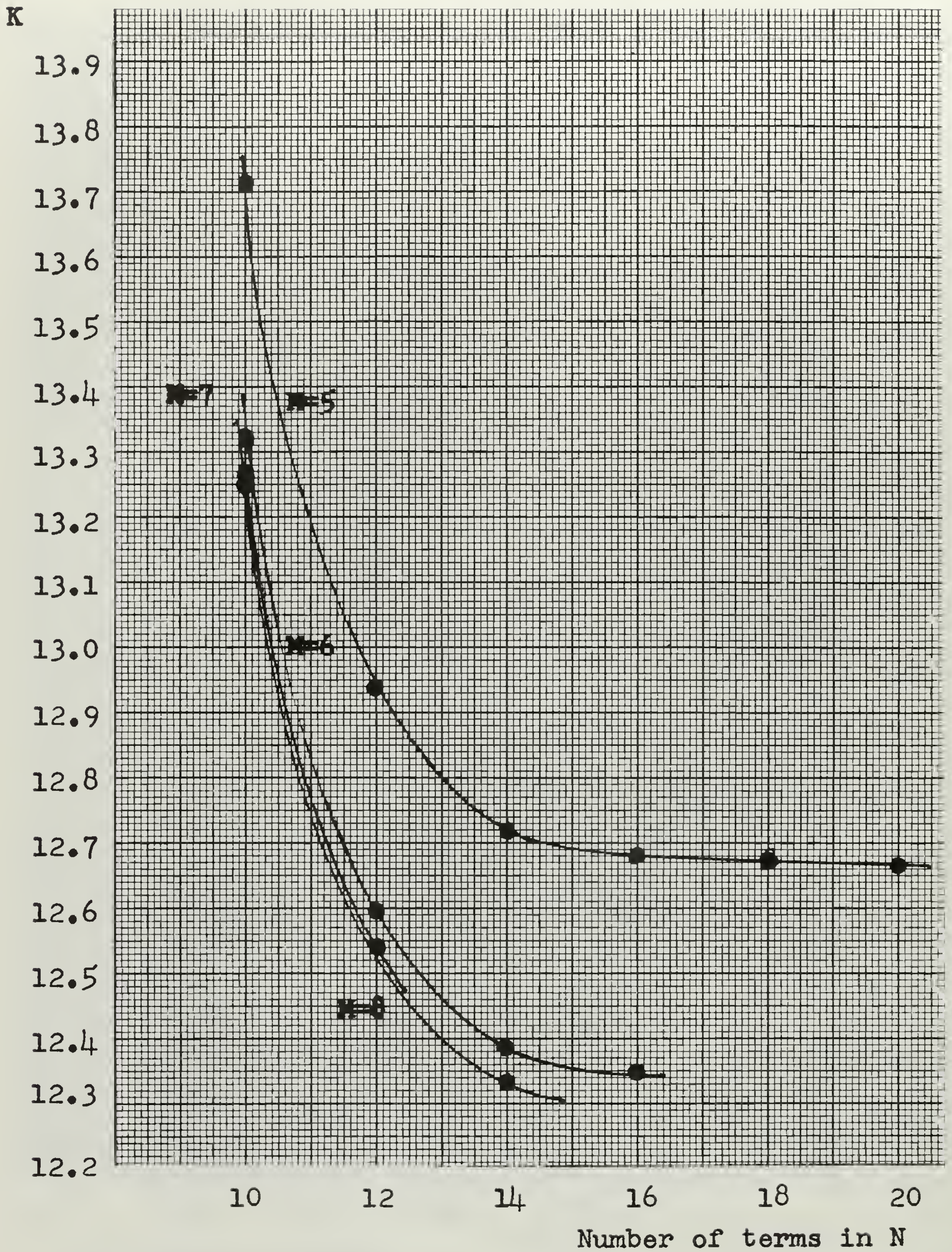
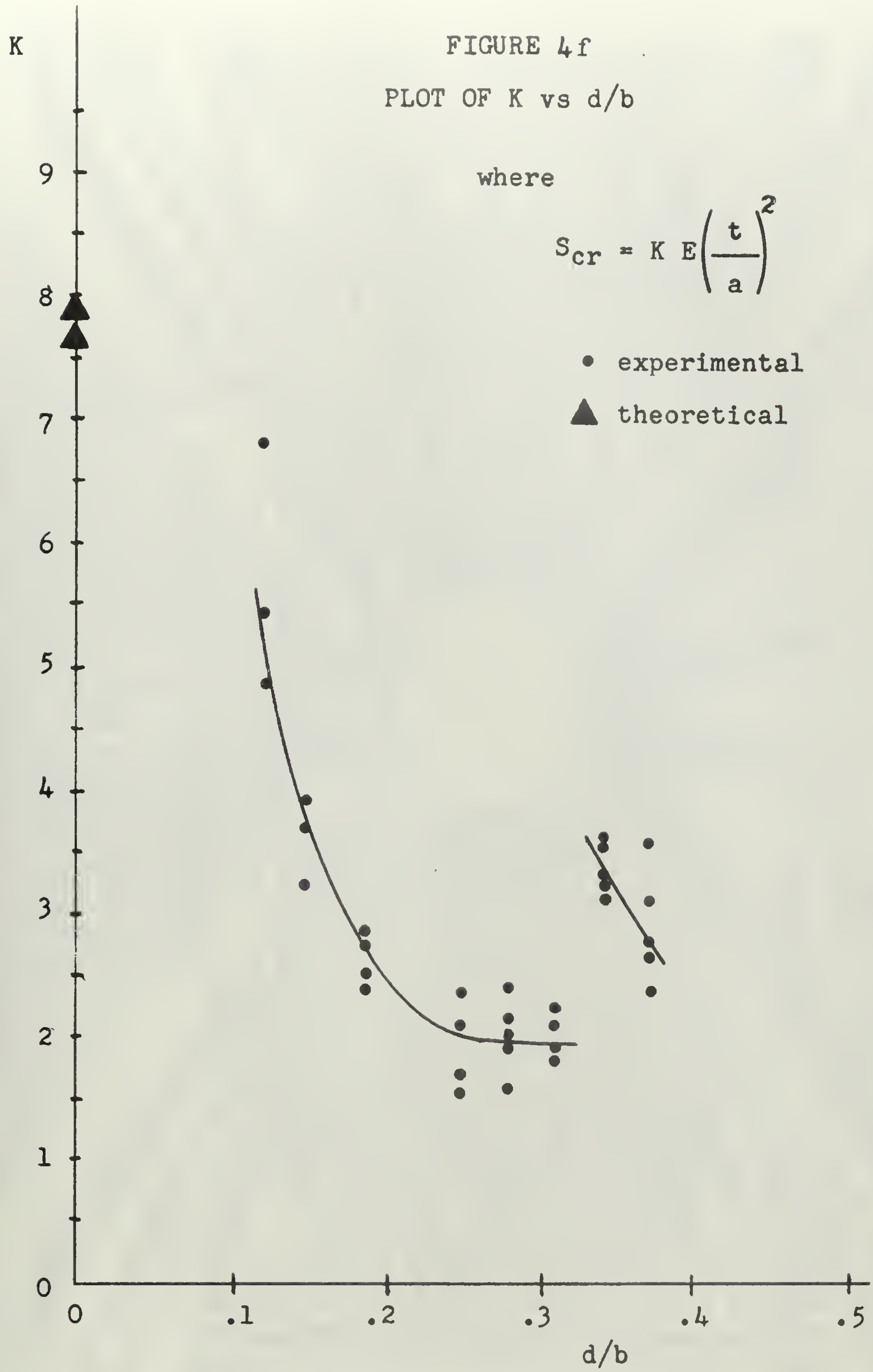



FIGURE 4e - Plot of the Buckling Coefficient K vs the number of terms taken in N, for mode 4.



K
FIGURE 4f
PLOT OF K vs d/b
where
$S_{cr} = K E \left(\frac{t}{a}\right)^2$
experimental
theoretical
9
8
7
6
5
4
3
2
1
0
0
.1
.2
.3
.4
.5
d/b

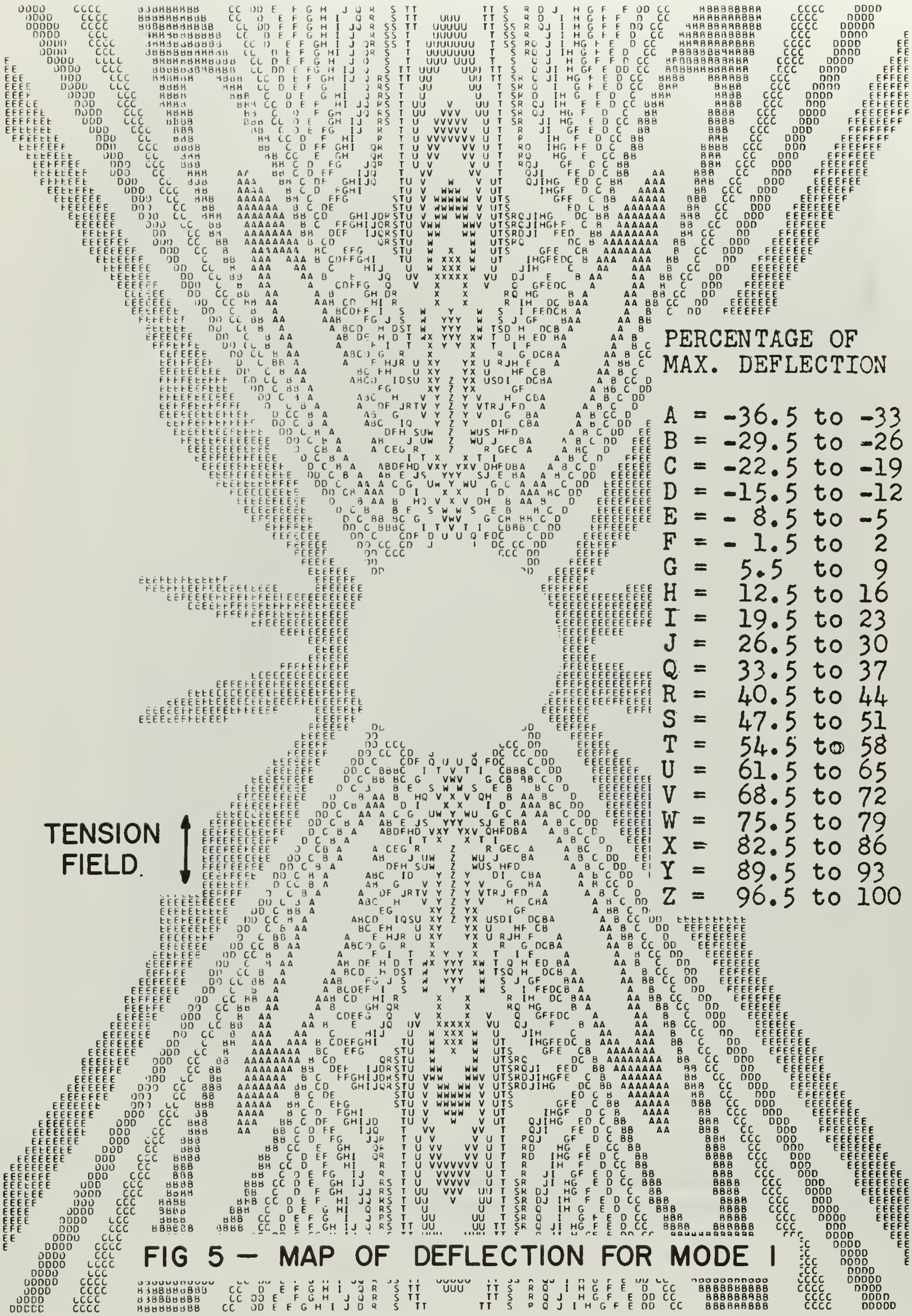



FIG 5 — MAP OF DEFLECTION FOR MODE I

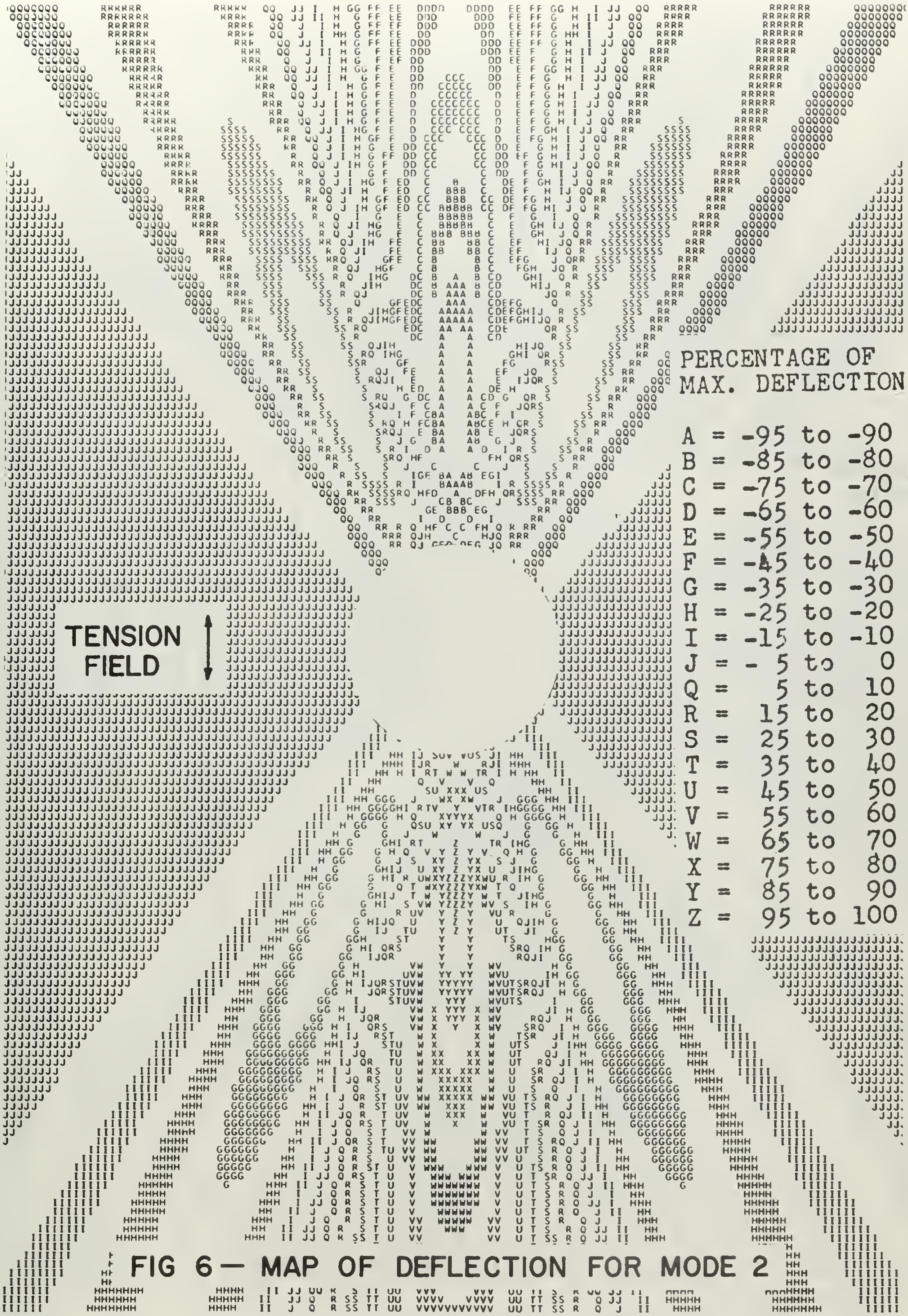



FIG 6 — MAP OF DEFLECTION FOR MODE 2

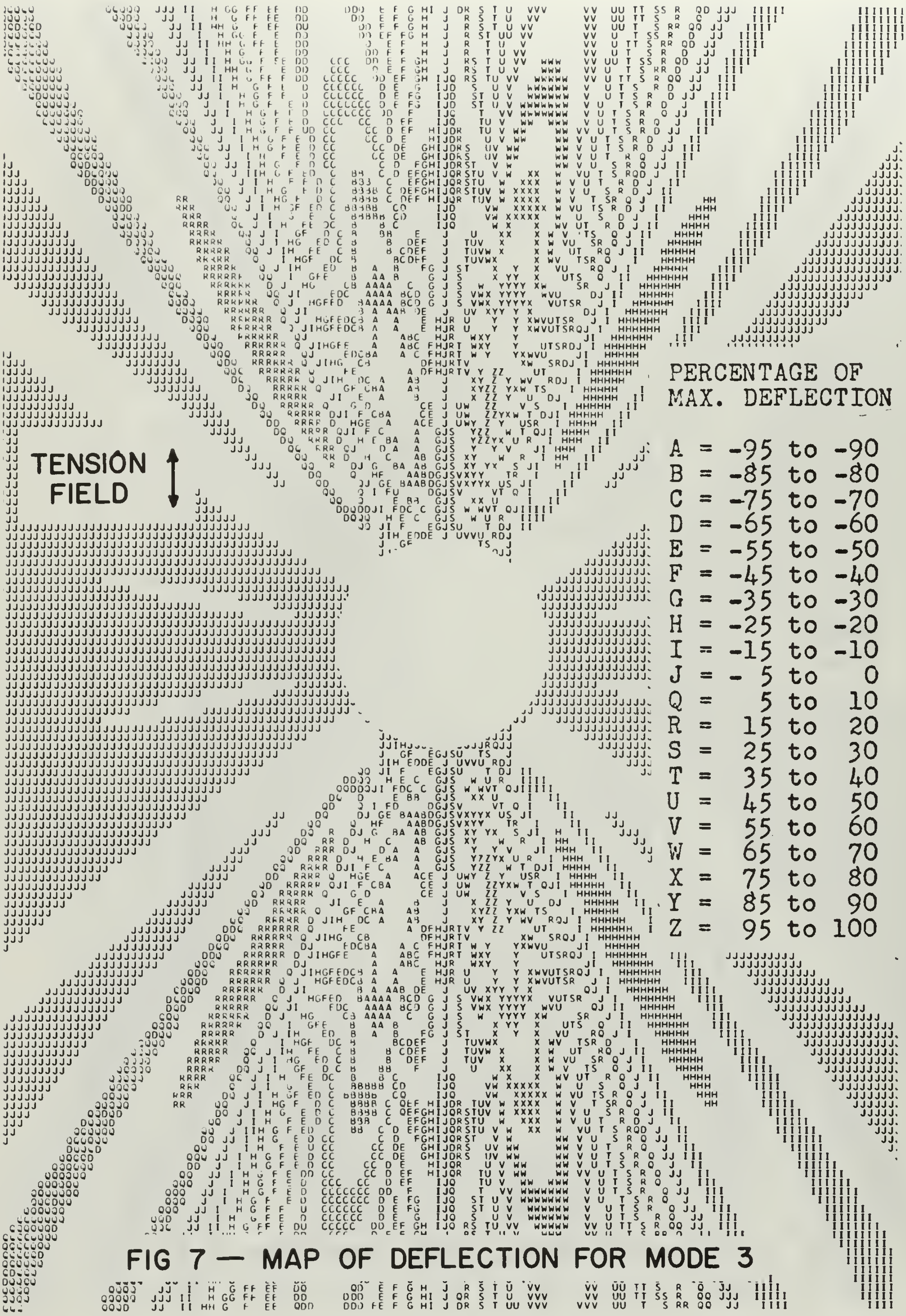



FIG 7 — MAP OF DEFLECTION FOR MODE 3

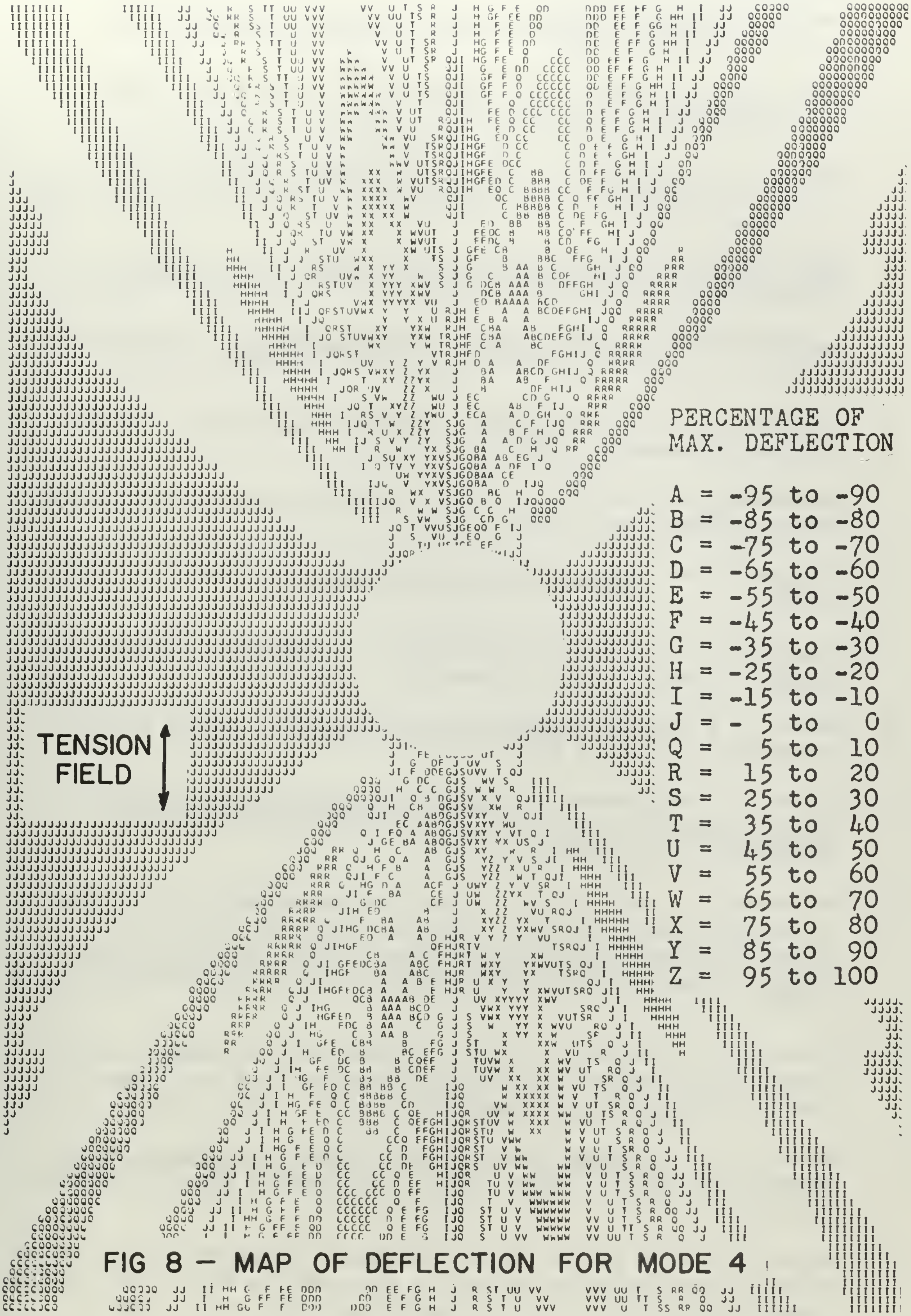



FIG 8 — MAP OF DEFLECTION FOR MODE 4

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## APPENDIX A

## SATISFACTION OF THE BOUNDARY CONDITIONS

The free edge boundary conditions for the hole are stated in equations 15 and 16 as

$$\left[ W_{rrr} + \frac{W_{rr}}{r} - \frac{W_r}{r^2} + \frac{(2-\nu)}{r^2} \cdot W_{r\theta\theta} - \frac{(3-\nu)}{r^3} \cdot W_{\theta\theta} \right]_{r=1} = 0 \qquad \text{(A-1)}$$

and

$$\left[ W_{rr} + \frac{\nu}{r}\left( W_r + \frac{W_{\theta\theta}}{r} \right) \right]_{r=1} = 0 \qquad \text{(A-2)}$$

The plate deflection, W, is taken as

$$W(r,\theta) = \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} r^{-N} \left[ 1 + \frac{X_{NM}}{r} + \frac{Y_{NM}}{r^2} \right] \begin{bmatrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{bmatrix} \qquad \text{(A-3)}$$

where the coefficients $A_{NM}$ and $B_{NM}$ are varied to minimize the plate energy, and the coefficients $X_{NM}$ and $Y_{NM}$ are chosen so that the boundary conditions expressed by equations A-1 and A-2 are satisfied.

The derivatives of W are tabulated in Table 4. When these derivatives are substituted into equations A-1 and A-2, and evaluated at r = 1, the following equationsresult

$$0 = \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \left( s_1 + s_2 X_{NM} + s_3 Y_{NM} \right) \begin{bmatrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{bmatrix} \qquad \text{(A-4)}$$

and

$$0 = \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \left( t_1 + t_2 X_{NM} + t_3 Y_{NM} \right) \begin{bmatrix} A_{NM} \cos M\theta \\ \text{or} \\ B_{NM} \sin M\theta \end{bmatrix} \qquad \text{(A-5)}$$

where

$$S_1 = N(N+1) - N\nu - M^2\nu \quad \text{(A-6)}$$

$$S_2 = (N+1)(N+2) - (N+1)\nu - M^2\nu \quad \text{(A-7)}$$

$$S_3 = (N+2)(N+3) - (N+2)\nu - M^2\nu \quad \text{(A-8)}$$

$$t_1 = -N(N+1)(N+2) + N(N+1) + N + (2-\nu)M^2 N + (3-\nu)M^2 \quad \text{(A-9)}$$

$$t_2 = -(N+1)(N+2)(N+3) + (N+1)(N+2) + (N+1) + (2-\nu)M^2(N+1) + (3-\nu)M^2 \quad \text{(A-10)}$$

$$t_3 = -(N+2)(N+3)(N+4) + (N+2)(N+3) + (N+2) + (2-\nu)M^2(N+2) + (3-\nu)M^2 \quad \text{(A-11)}$$

Equations A-4 and A-5 must hold for all values of θ, which can only be true if

$$S_1 + S_2 X_{NM} + S_3 Y_{NM} = 0 \quad \text{(A-12)}$$

$$t_1 + t_2 X_{NM} + t_3 Y_{NM} = 0 \quad \text{(A-13)}$$

for any value of N and M. Thus, a value of $X_{NM}$ and $Y_{NM}$ can be found for each value of N and M, such that the boundary conditions are satisfied.

By application of Cramer's Rule:

$$X_{NM} = \frac{\begin{vmatrix} -S_1 & S_3 \\ -t_1 & t_3 \end{vmatrix}}{\begin{vmatrix} S_2 & S_3 \\ t_2 & t_3 \end{vmatrix}} = \frac{-S_1 t_3 + S_3 t_1}{S_2 t_3 - S_3 t_2}$$

and

$$Y_{NM} = \frac{\begin{vmatrix} S_2 & -S_1 \\ t_2 & -t_1 \end{vmatrix}}{\begin{vmatrix} S_2 & S_3 \\ t_2 & t_3 \end{vmatrix}} = \frac{-S_2 t_1 + S_1 t_2}{S_2 t_3 - S_3 t_2}$$

The above results are incorporated into SUBROUTINE BOUND (see APPENDIX F) which generates values of $X_{NM}$ and $Y_{NM}$ for given values of N and M.

Table 4

**Tabulation of the Derivatives of W**

$$W = \sum_{N=1}^{\infty}\sum_{M=0}^{\infty}\left[r^{-N} + X_{NM}r^{-(N+1)} + Y_{NM}r^{-(N+2)}\right]\begin{Bmatrix} A_{NM}\cos M\theta \\ \text{or} \\ B_{NM}\sin M\theta \end{Bmatrix}$$

$$W_r = \sum_{N=1}^{\infty}\sum_{M=0}^{\infty}\left[-Nr^{-(N+1)} - (N+1)X_{NM}r^{-(N+2)} - (N+2)Y_{NM}r^{-(N+3)}\right]\begin{Bmatrix} A_{NM}\cos M\theta \\ \text{or} \\ B_{NM}\sin M\theta \end{Bmatrix}$$

$$W_{rr} = \sum_{N=1}^{\infty}\sum_{M=0}^{\infty}\left[N(N+1)r^{-(N+2)} + (N+1)(N+2)X_{NM}r^{-(N+3)} + (N+2)(N+3)Y_{NM}r^{-(N+4)}\right]\begin{Bmatrix} A_{NM}\cos M\theta \\ \text{or} \\ B_{NM}\sin M\theta \end{Bmatrix}$$

$$W_{rrr} = \sum_{N=1}^{\infty}\sum_{M=0}^{\infty}\left[-N(N+1)(N+2)r^{-(N+3)} - (N+1)(N+2)(N+3)X_{NM}r^{-(N+4)} - (N+2)(N+3)(N+4)Y_{NM}r^{-(N+5)}\right]\begin{Bmatrix} A_{NM}\cos M\theta \\ \text{or} \\ B_{NM}\sin M\theta \end{Bmatrix}$$

$$W_{r\theta} = \sum_{N=1}^{\infty}\sum_{M=0}^{\infty}\left[-Nr^{-(N+1)} - (N+1)X_{NM}r^{-(N+2)} - (N+2)Y_{NM}r^{-(N+3)}\right]M\begin{Bmatrix} -A_{NM}\sin M\theta \\ \text{or} \\ B_{NM}\cos M\theta \end{Bmatrix}$$

$$W_{r\theta\theta} = \sum_{N=1}^{\infty}\sum_{M=0}^{\infty}\left[Nr^{-(N+1)} + (N+1)X_{NM}r^{-(N+2)} + (N+2)Y_{NM}r^{-(N+3)}\right]M^2\begin{Bmatrix} A_{NM}\cos M\theta \\ \text{or} \\ B_{NM}\sin M\theta \end{Bmatrix}$$

$$W_{\theta\theta} = \sum_{N=1}^{\infty}\sum_{M=0}^{\infty}\left[-r^{-N} - X_{NM}r^{-(N+1)} - Y_{NM}r^{-(N+2)}\right]M^2\begin{Bmatrix} A_{NM}\cos M\theta \\ \text{or} \\ B_{NM}\sin M\theta \end{Bmatrix}$$

$$W_{\theta} = \sum_{N=1}^{\infty}\sum_{M=0}^{\infty}\left[r^{-N} + X_{NM}r^{-(N+1)} + Y_{NM}r^{-(N+2)}\right]M\begin{Bmatrix} -A_{NM}\sin M\theta \\ \text{or} \\ B_{NM}\cos M\theta \end{Bmatrix}$$

# APPENDIX B

## EVALUATION OF THE ENERGY INTEGRALS

The total energy in the plate, from equations 25, 26, and 27, is:

$$U_t = U_1 + U_2 + U_3$$

where

$U_1$ = constant

$$U_2 = \frac{h}{4}\int_0^{2\pi}\int_1^{\infty}\left[\sigma_r W_r^2 + \frac{\sigma_\theta W_\theta^2}{r^2} + \frac{2\tau_{r\theta} W_r W_\theta}{r}\right] r\,dr\,d\theta \tag{B-1}$$

$$U_3 = \frac{D}{2}\int_0^{2\pi}\int_1^{\infty}\left\{\left[W_{rr} + \frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2}\right]^2 - 2(1-\nu)\left[W_{rr}\left(\frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2}\right) - \left(\frac{W_\theta}{r^2} - \frac{W_{r\theta}}{r}\right)^2\right]\right\} r\,dr\,d\theta \tag{B-2}$$

Note that these integrals contain second degree terms in the derivatives of the deflection, W. Since W is represented by a series, care must be taken to insure that the series are properly multiplied.

The product of two double sums, both summed over the same range for the subscripts N and M, is a quadruple series, with dummy subscripts I and K summed over the range of N, and K and L, summed over the range of M.

For example;

$$\left(\sum_{N=1}^{\infty}\sum_{M=0}^{\infty}(1+r^{-N})\cos M\theta\right)^2 = \left(\sum_{I=1}^{\infty}\sum_{J=0}^{\infty}(1+r^{-I})\cos J\theta\right)\left(\sum_{K=1}^{\infty}\sum_{L=0}^{\infty}(1+r^{-K})\cos L\theta\right)$$

$$= \sum_{I=1}^{\infty}\sum_{J=0}^{\infty}\sum_{K=1}^{\infty}\sum_{L=0}^{\infty}(1+r^{-I})(1+r^{-K})\cos J\theta\,\cos L\theta$$

$$= \sum\sum\sum\sum(1+r^{-I}+r^{-K}+r^{-(I+K)})\cos J\theta\,\cos L\theta$$

It is seen that the cross product terms above do not combine directly into one term. One cross product term involves $r^{-I}$; the other involves $r^{-K}$. For this reason, cross product terms in the energy integrals B-1 and B-2 must be identified and separated.

As an aid in keeping the cross product terms separated, the notation is adopted that starred (*) functions are functions of I and J, and that non-starred functions are functions of K and L, in place of N and M, respectively. Equation B-1 may be written

$$U_2 = \frac{h}{2}\int_0^{2\pi}\int_1^{\infty}\Big[\sigma_r W_r^* W_r + \frac{\sigma_\theta W_\theta W_\theta^*}{r^2} + \frac{1}{r}\tau_{r\theta}\left(W_r^* W_\theta + W_r W_\theta^*\right)\Big]\, r\,dr\,d\theta \qquad \text{(B-3)}$$

and with the substitution of the Kirsch solution becomes

$$\begin{aligned} U_2 = \frac{Sh}{4}\int_0^{2\pi}\int_1^{\infty}\Big[&\left(1-\frac{1}{r^2}\right)W_r^* W_r + \left(1+\frac{3}{r^4}-\frac{4}{r^2}\right)W_r^* W_r \cos 2\theta \\ &+\left(1+\frac{1}{r^2}\right)\frac{W_\theta^* W_\theta}{r^2} - \left(1+\frac{3}{r^4}\right)\frac{W_\theta^* W_\theta}{r^2}\cos 2\theta \\ &-\left(1-\frac{3}{r^4}+\frac{2}{r^2}\right)\cdot\left(W_\theta^* W_r + W_\theta W_r^*\right)\frac{\sin 2\theta}{r}\Big]\, r\,dr\,d\theta \end{aligned} \qquad \text{(B-4)}$$

Equation B-2 is now

$$U_3 = \frac{D}{2}\int_0^{2\pi}\int_1^{\infty}\left(U_{3a} + U_{3b} + U_{3c}\right) r\,dr\,d\theta \qquad \text{(B-5)}$$

where

$$U_{3a} = \left(W_{rr} + \frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2}\right)\left(W_{rr}^* + \frac{W_r^*}{r} + \frac{W_{\theta\theta}^*}{r^2}\right) \qquad \text{(B-6)}$$

$$U_{3b} = (1-\nu)\left[W_{rr}^*\left(\frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2}\right) + W_{rr}\left(\frac{W_r^*}{r} + \frac{W_{\theta\theta}^*}{r^2}\right)\right] \qquad \text{(B-7)}$$

$$U_{3c} = 2(1-\nu)\left[\frac{W_{r\theta}}{r} - \frac{W_\theta}{r^2}\right]\left[\frac{W_{r\theta}^*}{r} - \frac{W_\theta^*}{r^2}\right] \qquad \text{(B-8)}$$

A listing of the expansion of the products of the derivatives necessary for the evaluation of $U_2$ and $U_3$ are shown in Tables 5 and 6, respectively. Further presentation of the energy integrals is in the notation delineated in those tables.

Equations B-4 and B-5 now become

$$U_2 = \frac{Sh}{4}\sum\sum\sum\sum\int_0^{2\pi}\Big[E_1 - E_2 + \left(E_1 + 3E_3 - 4E_2\right)\cos 2\theta + E_4 + E_5 \tag{B-9}$$

$$- \left(E_4 + 3E_6\right)\cos 2\theta - \left(E_{7a} + E_{7b} - 3E_{9a} - 3E_{9b} + 2E_{8a} + 2E_{8b}\right)\text{SIN } 2\theta\Big]\, d\theta$$

and

$$U_3 = \frac{D}{2}\sum\sum\sum\sum\int_0^{2\pi}\Big[G_1 + G_{2a} + G_{2b} + G_{3a} + G_{3b} + G_{5a} + G_{5b} \tag{B-10}$$

$$+ G_4 + G_6 + (1-\nu)\left(G_{2a} + G_{2b} + G_{3a} + G_{3b}\right)$$

$$+ 2(1-\nu)\left(G_7 + G_9 + G_{8a} + G_{8b}\right)\Big]\, d\theta$$

When the integration in $\theta$ is done, equations B-9 and B-10 reduce to

$$U_2 = \sum_{I=1}^{\infty}\sum_{J=0}^{\infty}\sum_{K=1}^{\infty}\sum_{L=0}^{\infty} Q_{IJKL}\begin{bmatrix} A_{IJ}\cdot A_{KL} \\ \text{or} \\ B_{IJ}\cdot B_{KL}\end{bmatrix} \tag{B-11}$$

and

$$U_3 = \sum_{I=1}^{\infty}\sum_{J=0}^{\infty}\sum_{K=1}^{\infty}\sum_{L=0}^{\infty} P_{IJKL}\begin{bmatrix} A_{IJ}\cdot A_{KL} \\ \text{or} \\ B_{IJ}\cdot B_{KL}\end{bmatrix} \tag{B-12}$$

Tables 5 and 6

Tabulation of the Integrals of the Radius Dependent Terms

NOTATION:

$G_1$, $G_2$,...$E_1$, $E_2$,.... etc. are functions of I, J, K, and L.

x means $X_{NM}$ as used previously, only as a function of K & L.

$x^*$ means $X_{NM}$ as used previously, only as a function of I and J.

y means $Y_{NM}$ as used previously, only as a function of K and L.

$y^*$ means $Y_{NM}$ as used previously, only as a function of I and J.

THE INTEGRALS ARE TABULATED ON THE FOLLOWING PAGES. THE FIRST NINE TERMS IN EACH COLUMN ARE SUMMED TO FORM THE WHOLE EXPRESSION. THE TERMS a(J,L) or b(J,L) SHOULD BE CHOSEN FOR THE UNKNOWN COEFFICIENTS $A_{NM}$ or $B_{NM}$ RESPECTIVELY.

Table 5

Tabulation of the Compacted Form of the Terms in the Membrane Energy Integral

$$E_1 = \int_0^\infty W_r W_r^* \, dr = f_1 \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$E_2 = \int_0^\infty \frac{W_r W_r^*}{r^2} \, dr = f_2 \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$E_3 = \int_0^\infty \frac{W_r W_r^*}{r^4} \, dr = f_3 \cdot \begin{bmatrix} a(J,L) \; A_{IJ} \; A_{KL} \\ \text{or} \\ b(J,L) \; B_{IJ} \; B_{KL} \end{bmatrix}$$

$$E_4 = \int_0^\infty \frac{W_\theta W_\theta^*}{r^2} \, dr = f_4 \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$E_5 = \int_0^\infty \frac{W_\theta W_\theta^*}{r^4} \, dr = f_5 \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$E_6 = \int_0^\infty \frac{W_\theta W_\theta^*}{r^6} \, dr = f_6 \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$E_{7a} = \int_0^\infty \frac{W_\theta^* W_r}{r} \, dr = f_{7a} \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$E_{7b} = \int_0^\infty \frac{W_\theta W_r^*}{r} \, dr = f_{7b} \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

Table 5 (continued)

$$E_{8a} = \int_0^\infty \frac{W_\theta^* W_r}{r^3}\, dr = f_{8a} \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$E_{8b} = \int_0^\infty \frac{W_\theta W_r^*}{r^3}\, dr = f_{8b} \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$E_{9a} = \int_0^\infty \frac{W_\theta^* W_r}{r^5}\, dr = f_{9a} \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$E_{9b} = \int_0^\infty \frac{W_\theta W_r^*}{r^5}\, dr = f_{9b} \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot A_{IJ} \cdot A_{KL} \end{bmatrix}$$

Table 5 (continued)

$$
\begin{aligned}
f_1 = & \frac{IK}{(I+K)} \\
& + \frac{I(K+1)x}{(I+K+1)} \\
& + \frac{I(K+2)y}{(I+K+2)} \\
& + \frac{(I+1)Kx^*}{(I+K+1)} \\
& + \frac{(I+1)(K+1)xx^*}{(I+K+2)} \\
& + \frac{(I+1)(K+2)yx^*}{(I+K+3)} \\
& + \frac{(I+2)Ky^*}{(I+K+2)} \\
& + \frac{(I+2)(K+1)y^*x}{(I+K+3)} \\
& + \frac{(I+2)(K+2)yy^*}{(I+K+4)}
\end{aligned}
$$

where

$a(J,L) = \cos J\theta \cos L\theta$

$b(J,L) = \sin J\theta \sin L\theta$

$$
\begin{aligned}
f_2 = & \frac{IK}{(I+K+2)} \\
& + \frac{I(K+1)x}{(I+K+3)} \\
& + \frac{I(K+2)y}{(I+K+4)} \\
& + \frac{(I+1)Kx^*}{(I+K+3)} \\
& + \frac{(I+1)(K+1)xx^*}{(I+K+4)} \\
& + \frac{(I+1)(K+2)yx^*}{(I+K+5)} \\
& + \frac{(I+2)Ky^*}{(I+K+4)} \\
& + \frac{(I+2)(K+1)xy^*}{(I+K+5)} \\
& + \frac{(I+2)(K+2)yy^*}{(I+K+6)}
\end{aligned}
$$

where

$a(J,L) = \cos J\theta \cos L\theta$

$b(J,L) = \sin J\theta \sin L\theta$

$$f_3 = \frac{IK}{(I+K+4)} + \frac{I(K+1)x}{(I+K+5)} + \frac{I(K+2)y}{(I+K+6)} + \frac{(I+1)Kx^*}{(I+K+5)} + \frac{(I+1)(K+1)xx^*}{(I+K+6)} + \frac{(I+1)(K+2)yx^*}{(I+K+7)} + \frac{(I+2)Ky^*}{(I+K+6)} + \frac{(I+2)(K+1)xy^*}{(I+K+7)} + \frac{(I+2)(K+2)yy^*}{(I+K+8)}$$

where

$a(J,L) = \cos J\theta \cos L\theta$

$b(J,L) = \sin J\theta \sin L\theta$

$$f_4 = \frac{JL}{(I+K)} + \frac{JLx^*}{(I+K+1)} + \frac{JLy^*}{(I+K+2)} + \frac{JLx}{(I+K+1)} + \frac{JLxx^*}{(I+K+2)} + \frac{JLxy^*}{(I+K+3)} + \frac{JLy}{(I+K+2)} + \frac{JLyx^*}{(I+K+3)} + \frac{JLyy^*}{(I+K+4)}$$

where

$a(J,L) = \sin J\theta \sin L\theta$

$b(J,L) = \cos J\theta \cos L\theta$

Table 5 (continued)

$$f_5 = \frac{JL}{(I+K+2)} + \frac{JLx^*}{(I+K+3)} + \frac{JLy^*}{(I+K+4)} + \frac{JLx}{(I+K+3)} + \frac{JLxx^*}{(I+K+4)} + \frac{JLxy^*}{(I+K+5)} + \frac{JLy}{(I+K+4)} + \frac{JLyx^*}{(I+K+5)} + \frac{JLyy^*}{(I+K+6)}$$

where

$a(J,L) = \sin J\theta \sin L\theta$

$b(J,L) = \cos J\theta \cos L\theta$

$$f_6 = \frac{JL}{(I+K+4)} + \frac{JLx^*}{(I+K+5)} + \frac{JLy^*}{(I+K+6)} + \frac{JLx}{(I+K+5)} + \frac{JLxx^*}{(I+K+6)} + \frac{JLxy^*}{(I+K+7)} + \frac{JLy}{(I+K+6)} + \frac{JLyx^*}{(I+K+7)} + \frac{JLyy^*}{(I+K+8)}$$

where

$a(J,L) = \sin J\theta \sin L\theta$

$b(J,L) = \cos J\theta \cos L\theta$

$$
\begin{aligned}
f_{7a} = & - \frac{JK}{(I+K)} \\
& - \frac{J(K+1)x}{(I+K+1)} \\
& - \frac{J(K+2)y}{(I+K+2)} \\
& - \frac{JKx^{*}}{(I+K+1)} \\
& - \frac{J(K+1)xx^{*}}{(I+K+2)} \\
& - \frac{J(K+2)yx^{*}}{(I+K+3)} \\
& - \frac{JKy^{*}}{(I+K+2)} \\
& - \frac{J(K+1)xy^{*}}{(I+K+3)} \\
& - \frac{J(K+2)yy^{*}}{(I+K+4)}
\end{aligned}
$$

where

$a(J,L) = -\sin J\theta \cos L\theta$

$b(J,L) = \sin L\theta \cos J\theta$

$$
\begin{aligned}
f_{7b} = & - \frac{LI}{(I+K)} \\
& - \frac{L(I+1)x^{*}}{(I+K+1)} \\
& - \frac{L(I+2)y^{*}}{(I+K+2)} \\
& - \frac{LIx}{(I+K+1)} \\
& - \frac{L(I+1)xx^{*}}{(I+K+2)} \\
& - \frac{L(I+2)xy^{*}}{(I+K+3)} \\
& - \frac{LIy}{(I+K+2)} \\
& - \frac{L(I+1)yx^{*}}{(I+K+3)} \\
& - \frac{L(I+2)yy^{*}}{(I+K+4)}
\end{aligned}
$$

where

$a(J,L) = -\sin L\theta \cos J\theta$

$b(J,L) = \sin J\theta \cos L\theta$

Table 5 (continued)

$$f_{8a} = -\frac{JK}{(I+K+2)} - \frac{J(K+1)x}{(I+K+3)} - \frac{J(K+2)y}{(I+K+4)} - \frac{JKx^*}{(I+K+3)} - \frac{J(K+1)xx^*}{(I+K+4)} - \frac{J(K+2)yx^*}{(I+K+5)} - \frac{JKy^*}{(I+K+4)} - \frac{J(K+1)xy^*}{(I+K+5)} - \frac{J(K+2)yy^*}{(I+K+6)}$$

where

$a(J,L) = -\sin J\theta \cos L\theta$

$b(J,L) = \sin L\theta \cos J\theta$

$$f_{8b} = -\frac{LI}{(I+K+2)} - \frac{L(I+1)x^*}{(I+K+3)} - \frac{L(I+2)y^*}{(I+K+4)} - \frac{LIx}{(I+K+3)} - \frac{L(I+1)xx^*}{(I+K+4)} - \frac{L(I+2)xy^*}{(I+K+5)} - \frac{LIy}{(I+K+4)} - \frac{L(I+1)yx^*}{(I+K+5)} - \frac{L(I+2)yy^*}{(I+K+6)}$$

where

$a(J,L) = -\sin L\theta \cos J\theta$

$b(J,L) = \sin J\theta \cos L\theta$

Table 5 (continued)

$$f_{9a} = -\frac{JK}{(I+K+4)} - \frac{J(K+1)x}{(I+K+5)} - \frac{J(K+2)y}{(I+K+6)} - \frac{JKx^*}{(I+K+5)} - \frac{J(K+1)xx^*}{(I+K+6)} - \frac{J(K+2)yx^*}{(I+K+7)} - \frac{JKy^*}{(I+K+6)} - \frac{J(K+1)xy^*}{(I+K+7)} - \frac{J(K+2)yy^*}{(I+K+8)}$$

where

$a(J,L) = -\sin J\theta \cos L\theta$

$b(J,L) = \sin L\theta \cos J\theta$

$$f_{9b} = -\frac{LI}{(I+K+4)} - \frac{L(I+1)x^*}{(I+K+5)} - \frac{L(I+2)y^*}{(I+K+6)} - \frac{LIx}{(I+K+5)} - \frac{L(I+1)xx^*}{(I+K+6)} - \frac{L(I+2)xy^*}{(I+K+7)} - \frac{LIy}{(I+K+6)} - \frac{L(I+1)yx^*}{(I+K+7)} - \frac{L(I+2)yy^*}{(I+K+8)}$$

where

$a(J,L) = -\sin L\theta \cos J\theta$

$b(J,L) = \sin J\theta \cos L\theta$

Table 6

Tabulation of the Compacted Form of the Terms in the Bending Energy Integral

$$G_1 = \int_0^\infty W_{rr} W_{rr}^* \, dr = g_1 \begin{bmatrix} a(J,L)\cdot A_{IJ}\cdot A_{KL} \\ \text{or} \\ b(J,L)\cdot B_{IJ}\cdot B_{KL} \end{bmatrix}$$

$$G_{2a} = \int_0^\infty \frac{W_r^* W_{rr}}{r} \, dr = g_{2a} \begin{bmatrix} a(J,L)\cdot A_{IJ}\cdot A_{KL} \\ \text{or} \\ b(J,L)\cdot B_{IJ}\cdot B_{KL} \end{bmatrix}$$

$$G_{2b} = \int_0^\infty \frac{W_r W_{rr}^*}{r} \, dr = g_{2b} \begin{bmatrix} a(J,L)\cdot A_{IJ}\cdot A_{KL} \\ \text{or} \\ b(J,L)\cdot B_{IJ}\cdot B_{KL} \end{bmatrix}$$

$$G_{3a} = \int_0^\infty \frac{W_{rr}^* W_{\theta\theta}}{r^2} \, dr = g_{3a} \begin{bmatrix} a(J,L)\cdot A_{IJ}\cdot A_{KL} \\ \text{or} \\ b(J,L)\cdot B_{IJ}\cdot B_{KL} \end{bmatrix}$$

$$G_{3b} = \int_0^\infty \frac{W_{rr} W_{\theta\theta}^*}{r^2} \, dr = g_{3b} \begin{bmatrix} a(J,L)\cdot A_{IJ}\cdot A_{KL} \\ \text{or} \\ b(J,L)\cdot B_{IJ}\cdot B_{KL} \end{bmatrix}$$

$$G_4 = \int_0^\infty \frac{W_r W_r^*}{r^2} \, dr = g_4 \begin{bmatrix} a(J,L)\cdot A_{IJ}\cdot A_{KL} \\ \text{or} \\ b(J,L)\cdot B_{IJ}\cdot B_{KL} \end{bmatrix}$$

$$G_{5a} = \int_0^\infty \frac{W_{\theta\theta}^* W_r}{r^3} \, dr = g_{5a} \begin{bmatrix} a(J,L)\cdot A_{IJ}\cdot A_{KL} \\ \text{or} \\ b(J,L)\cdot B_{IJ}\cdot B_{KL} \end{bmatrix}$$

$$G_{5b} = \int_0^\infty \frac{W_{\theta\theta} W_r^*}{r^3} \, dr = g_{5b} \begin{bmatrix} a(J,L)\cdot A_{IJ}\cdot A_{KL} \\ \text{or} \\ b(J,L)\cdot B_{IJ}\cdot B_{KL} \end{bmatrix}$$

Table 6 (continued)

$$G_6 = \int_0^\infty \frac{W_{\theta\theta} W^*_{\theta\theta}}{r^4}\, dr = g_6 \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$G_7 = \int_0^\infty \frac{W_{r\theta} W^*_{r\theta}}{r^2}\, dr = g_7 \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$G_{8a} = \int_0^\infty \frac{W^*_{\theta} W_{r\theta}}{r^3}\, dr = g_{8a} \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$G_{8b} = \int_0^\infty \frac{W_{\theta} W^*_{r\theta}}{r^3}\, dr = g_{8b} \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

$$G_9 = \int_0^\infty \frac{W_{\theta} W^*_{\theta}}{r^4}\, dr = g_9 \cdot \begin{bmatrix} a(J,L) \cdot A_{IJ} \cdot A_{KL} \\ \text{or} \\ b(J,L) \cdot B_{IJ} \cdot B_{KL} \end{bmatrix}$$

Table 6 (continued)

$$g_1 = \frac{IK(I+1)(K+1)}{(I+K+2)}$$

$$+ \frac{(I+1)(K+1)(I+2)(K+2)xx^*}{(I+K+4)}$$

$$+ \frac{(I+2)(K+2)(I+3)(K+3)yy^*}{(I+K+6)}$$

$$+ \frac{K(K+1)(I+1)(I+2)x^*}{(I+K+3)}$$

$$+ \frac{I(I+1)(K+1)(K+2)x}{(I+K+3)}$$

$$+ \frac{K(K+1)(I+2)(I+3)y^*}{(I+K+4)}$$

$$+ \frac{I(I+1)(K+2)(K+3)y}{(I+K+4)}$$

$$+ \frac{(I+1)(I+2)(K+2)(K+3)yx^*}{(I+K+5)}$$

$$+ \frac{(K+1)(K+2)(I+2)(I+3)xy^*}{(I+K+5)}$$

where

$a(J,L) = \cos J\theta \cos L\theta$

$b(J,L) = \sin J\theta \sin L\theta$

$$g_{2a} = -\frac{IK(K+1)}{(I+K+2)} - \frac{I(K+1)(K+2)x}{(I+K+3)} - \frac{I(K+2)(K+3)y}{(I+K+4)} - \frac{(I+1)(K+1)Kx^*}{(I+K+3)} - \frac{(I+1)(K+1)(K+2)xx^*}{(I+K+4)} - \frac{(I+1)(K+2)(K+3)yx^*}{(I+K+5)} - \frac{(I+2)(K+1)Ky^*}{(I+K+4)} - \frac{(I+2)(K+1)(K+2)xy^*}{(I+K+5)} - \frac{(I+2)(K+2)(K+3)yy^*}{(I+K+6)}$$

where

$a(J,L) = \cos J\theta \cos L\theta$

$b(J,L) = \sin J\theta \sin L\theta$

$$g_{2b} = -\frac{IK(I+1)}{(I+K+2)} - \frac{K(I+1)(I+2)x^*}{(I+K+3)} - \frac{K(I+2)(I+3)y^*}{(I+K+4)} - \frac{(K+1)I(I+1)x}{(I+K+3)} - \frac{(K+1)(I+1)(I+2)xx^*}{(I+K+4)} - \frac{(K+1)(I+2)(I+3)xy^*}{(I+K+4)} - \frac{(K+2)I(I+1)y}{(I+K+4)} - \frac{(K+2)(I+1)(I+2)yx^*}{(I+K+5)} - \frac{(K+2)(I+2)(I+3)yy^*}{(I+K+6)}$$

where

$a(J,L) = \cos J\theta \cos L\theta$

$b(J,L) = \sin J\theta \sin L\theta$

Table 6 (continued)

$$g_{3a} = - \frac{I(I+1)L^2}{(I+K+2)} - \frac{I(I+1)L^2 x}{(I+K+3)} - \frac{I(I+1)L^2 y}{(I+K+4)} - \frac{(I+1)(I+2)L^2 x^*}{(I+K+3)} - \frac{(I+1)(I+2)L^2 xx^*}{(I+K+4)} - \frac{(I+1)(I+2)L^2 yx^*}{(I+K+5)} - \frac{(I+2)(I+3)L^2 y^*}{(I+K+4)} - \frac{(I+2)(I+3)L^2 xy^*}{(I+K+5)} - \frac{(I+2)(I+3)L^2 yy^*}{(I+K+6)}$$

where

$a(J,L) = \cos J\theta \cos L\theta$

$b(J,L) = \sin J\theta \sin L\theta$

$$g_{3b} = - \frac{K(K+1)J^2}{(I+K+2)} - \frac{K(K+1)J^2 x^*}{(I+K+3)} - \frac{K(K+1)J^2 y^*}{(I+K+4)} - \frac{(K+1)(K+2)J^2 x}{(I+K+3)} - \frac{(K+1)(K+2)J^2 xx^*}{(I+K+4)} - \frac{(K+1)(K+2)J^2 xy^*}{(I+K+5)} - \frac{(K+2)(K+3)J^2 y}{(I+K+4)} - \frac{(K+2)(K+3)J^2 yx^*}{(I+K+5)} - \frac{(K+2)(K+3)J^2 yy^*}{(I+K+6)}$$

where

$a(J,L) = \cos J\theta \cos L\theta$

$b(J,L) = \sin J\theta \sin L\theta$

$$g_4 = \frac{IK}{(I+K+2)} + \frac{K(I+1)x^*}{(I+K+3)} + \frac{K(I+2)y^*}{(I+K+4)} + \frac{(K+1)Ix}{(I+K+3)} + \frac{(K+1)(I+1)xx^*}{(I+K+4)} + \frac{(K+1)(I+2)xy^*}{(I+K+5)} + \frac{(K+2)Iy}{(I+K+4)} + \frac{(K+2)(I+1)yx^*}{(I+K+5)} + \frac{(K+2)(I+2)yy^*}{(I+K+6)}$$

where

$a(J,L) = \cos J\theta \cos L\theta$

$b(J,L) = \sin J\theta \sin L\theta$

$$g_{5a} = \frac{J^2L}{(I+K+2)} + \frac{J^2(K+1)x}{(I+K+3)} + \frac{J^2(K+2)y}{(I+K+4)} + \frac{J^2Kx^*}{(I+K+3)} + \frac{J^2(K+1)xx^*}{(I+K+4)} + \frac{J^2(K+2)yx^*}{(I+K+5)} + \frac{J^2Ky^*}{(I+K+4)} + \frac{J^2(K+1)xy^*}{(I+K+5)} + \frac{J^2(K+2)yy^*}{(I+K+6)}$$

where

$a(J,L) = \cos J\theta \cos L\theta$

$b(J,L) = \sin J\theta \sin L\theta$

Table 6 (continued)

$$g_{5b} = \frac{L^2 I}{(I+K+2)} + \frac{L^2(I+1)x^*}{(I+K+3)} + \frac{L^2(I+2)y^*}{(I+K+4)} + \frac{L^2 Ix}{(I+K+3)} + \frac{L^2(I+1)xx^*}{(I+K+4)} + \frac{L^2(I+2)xy^*}{(I+K+5)} + \frac{L^2 Iy}{(I+K+4)} + \frac{L^2(I+1)yx^*}{(I+K+5)} + \frac{L^2(I+2)yy^*}{(I+K+6)}$$

where

$a(J,L) = \cos J\theta \cos L\theta$

$b(J,L) = \sin J\theta \sin L\theta$

$$g_6 = \frac{J^2L^2}{(I+K+2)} + \frac{J^2L^2x^*}{(I+K+3)} + \frac{J^2L^2y^*}{(I+K+4)} + \frac{J^2L^2x^*}{(I+K+3)} + \frac{J^2L^2xx^*}{(I+K+4)} + \frac{J^2L^2xy^*}{(I+K+5)} + \frac{J^2L^2y}{(I+K+4)} + \frac{J^2L^2yx^*}{(I+K+5)} + \frac{J^2L^2yy^*}{(I+K+6)}$$

where

$a(J,L) = \cos J\theta \cos L\theta$

$b(J,L) = \sin J\theta \sin L\theta$

$$g_7 = \frac{IJKL}{(I+K+2)} + \frac{IJ(K+1)Lx}{(I+K+3)} + \frac{IJ(K+2)Ly}{(I+K+4)} + \frac{(I+1)JKLx^*}{(I+K+3)} + \frac{(I+1)J(K+1)Lxx^*}{(I+K+4)} + \frac{(I+1)J(K+2)Lyx^*}{(I+K+5)} + \frac{(I+2)JKLy^*}{(I+K+4)} + \frac{(I+2)J(K+1)Lxy^*}{(I+K+5)} + \frac{(I+2)J(K+2)Lyy^*}{(I+K+6)}$$

where

$a(J,L) = \sin J\theta \sin L\theta$

$b(J,L) = \cos J\theta \cos L\theta$

$$g_{8a} = -\frac{JK}{(I+K+2)} - \frac{J(K+1)x}{(I+K+3)} - \frac{J(K+2)y}{(I+K+4)} - \frac{JKx^*}{(I+K+3)} - \frac{J(K+1)xx^*}{(I+K+4)} - \frac{J(K+2)yx^*}{(I+K+5)} - \frac{JKy^*}{(I+K+4)} - \frac{J(K+1)xy^*}{(I+K+5)} - \frac{J(K+2)yy^*}{(I+K+6)}$$

where

$a(J,L) = \sin J\theta \sin L\theta$

$b(J,L) = \cos J\theta \cos L\theta$

Table 6 (continued)

$$g_{8a} = -\frac{LI}{(I+K+2)} - \frac{L(I+1)x^*}{(I+K+3)} - \frac{L(I+2)y^*}{(I+K+4)} - \frac{LIx}{(I+K+3)} - \frac{L(I+1)xx^*}{(I+K+3)} - \frac{L(I+2)xy^*}{(I+K+4)} - \frac{LIy}{(I+K+4)} - \frac{L(I+1)yx^*}{(I+K+5)} - \frac{L(I+2)yy^*}{(I+K+6)}$$

where

$a(J,L) = \sin J\theta \sin L\theta$

$b(J,L) = \cos J\theta \cos L\theta$

$$g_9 = \frac{JL}{(I+K+2)} + \frac{JLx^*}{(I+K+3)} + \frac{JLy^*}{(I+K+4)} + \frac{JLx}{(I+K+3)} + \frac{JLxx^*}{(I+K+4)} + \frac{JLxy^*}{(I+K+5)} + \frac{JLy}{(I+K+4)} + \frac{JLyx^*}{(I+K+5)} + \frac{JLyy^*}{(I+K+6)}$$

where

$a(J,L) = \sin J\theta \sin L\theta$

$b(J,L) = \cos J\theta \cos L\theta$

## APPENDIX C

## INTEGRATION OF SINE-COSINE TRIPLE PRODUCTS

When evaluating the energy integrals for the plate, it is necessary to evaluate the following two integrals

$$I_c = \int_0^{2\pi} \left| \cos A\theta \ \cos B\theta \ \cos C\theta \right| d\theta \tag{C-1}$$

and

$$I_s = \int_0^{2\pi} \left| \sin A\theta \ \sin B\theta \ \cos C\theta \right| d\theta \tag{C-2}$$

where A, B, and C are integers.

The above integrals can best be evaluated by expanding the integrands using the identities

$$\sin x \ \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right] \tag{C-3}$$

$$\cos x \ \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right] \tag{C-4}$$

$$\cos x \ \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right] \tag{C-5}$$

$$\sin x \ \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] \tag{C-6}$$

The object is to reduce the integrals to the form

$$\int_0^{2\pi} \sin D\theta \ d\theta \qquad \text{or} \qquad \int_0^{2\pi} \cos D\theta \ d\theta \tag{C-7}$$

where D is an integer sum or difference of A, B, and C. Note that the sine integral in C-7 is always zero and that the cosine integral is nonzero only if D is zero.

Expanding equation C-2, using identity C-3, we get

$$I_s = \frac{1}{2}\int_0^{2\pi} \sin A\theta \left[\sin(B+C)\theta + \sin(B-C)\theta\right] d\theta \tag{C-8}$$

Continuing the process with identity C-6

$$I_s = \frac{1}{4}\int_0^{2\pi} \left[\cos(A-B-C)\theta - \cos(A+B+C)\theta + \cos(A-B+C)\theta - \cos(A+B-C)\theta\right] d\theta \qquad \text{(C-9)}$$

$I_s$ is now expressed as the sum of four integrals

$$I_s = I_1 + I_2 + I_3 + I_4 \qquad \text{(C-10)}$$

where

$$I_1 = \begin{cases} \pi/2 & \text{if } (A-B-C) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$I_2 = \begin{cases} -\pi/2 & \text{if } (A+B+C) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$I_3 = \begin{cases} \pi/2 & \text{if } (A-B+C) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$I_4 = \begin{cases} -\pi/2 & \text{if } (A+B-C) = 0 \\ 0 & \text{otherwise} \end{cases}$$

A similar expansion for $I_c$ (equation C-1) can be made. The results are the same as above, except that the signs of $I_2$ and $I_4$ are positive.

APPENDIX D

SOLUTION OF THE EIGENVALUE PROBLEM $(\underline{\underline{R}}x + \underline{\underline{T}}) \underline{A} = 0$

The canonical form of the matrix eigenvalue problem is

$$(\underline{\underline{I}}\lambda - \underline{\underline{B}})\underline{X} = 0 \qquad \text{(D-1)}$$

where $\underline{\underline{I}}$ is the identity matrix, $\underline{X}$ an unknown eigenvector, $\lambda$ is an unknown scalar parameter (eigenvalue), and $\underline{\underline{B}}$ a known coefficient matrix.

Equation D-1 has a solution if determinant of

$$(\underline{\underline{I}}\lambda - \underline{\underline{B}}) = 0 \qquad \text{(D-2)}$$

If this determinant is expanded, a polynomial equation in $\lambda$ results. However, if the determinant is of order greater than three or four, it is generally inefficient to obtain this polynomial equation and then solve it.

There are alternative procedures for solution of the eigenvalue problem. A method which gives extremely accurate eigenvectors and eigenvalues for symmetric matrices is one named for Jacobi, which is presented by Fox (1). Brown and White (13) give a comparison of the accuracy of the Jacobi method with that of other methods available. The necessity for extremely accurate eigenvectors is shown later.

Consider the eigenvalue problem (equation 33)

$$(\underline{\underline{R}}\lambda + \underline{\underline{T}})\underline{A} = 0 \qquad \text{(D-3)}$$

where both $\underline{\underline{R}}$ and $\underline{\underline{T}}$ are symmetric matrices and $\underline{\underline{T}}$ is positive definite. Dividing equation D-3 by $\lambda$ , and making the substitution

$$y = -\frac{1}{\lambda} \qquad \text{(D-4)}$$

gives

$$(\underline{\underline{T}}y - \underline{\underline{R}})\underline{A} = 0 \qquad \text{(D-5)}$$

Any symmetric matrix with real elements can be reduced to a product of the eigenvector and eigenvalue matrices, so that

$$\underline{\underline{T}} = \underline{\underline{Q}}\,\underline{\underline{D}}\,\underline{\underline{Q}}^T \tag{D-6}$$

where $\underline{\underline{D}}$ is a diagonal matrix of the eigenvalues ($d_i$) of $\underline{\underline{T}}$, and $\underline{\underline{Q}}$ is a matrix of normalized eigenvectors ($q_i$), obtained from a solution of the problem

$$(\underline{\underline{I}}d_i - \underline{\underline{T}})\underline{q}_i = 0 \tag{D-7}$$

Note that $\underline{\underline{Q}}^T = \underline{\underline{Q}}^{-1}$, since the matrix $\underline{\underline{T}}$ is symmetric and has orthogonal eigenvectors.

Equation D-5 may now be written

$$(\underline{\underline{Q}}\,\underline{\underline{D}}\,\underline{\underline{Q}}^T y - \underline{\underline{R}})\underline{A} = 0 \tag{D-8}$$

Premultiplying by $\underline{\underline{Q}}^T$, we get

$$(\underline{\underline{D}}\,\underline{\underline{Q}}^T y - \underline{\underline{Q}}^T\underline{\underline{R}})\underline{A} = 0 \tag{D-9}$$

Factoring a Q out of $\underline{A} = \underline{\underline{I}}\underline{A} = \underline{\underline{Q}}\,\underline{\underline{Q}}^T\underline{A}$, and letting

$$\underline{Y} = \underline{\underline{Q}}^T\underline{A} \tag{D-10}$$

the problem reduces to

$$(\underline{\underline{D}}y - \underline{\underline{Q}}^T\underline{\underline{R}}\,\underline{\underline{Q}})\underline{Y} = 0 \tag{D-11}$$

Since $\underline{\underline{T}}$ is positive definite, $\underline{\underline{D}}$ contains all positive terms. By again premultiplying, this time by $\underline{\underline{D}}^{-\frac{1}{2}}$, factoring $\underline{\underline{D}}^{-\frac{1}{2}}$, and making the substitution

$$\underline{Z} = \underline{\underline{D}}^{1/2}\underline{Y} \tag{D-12}$$

equation D-11 reduces to

$$(\underline{\underline{I}}y - \underline{\underline{D}}^{-1/2}\underline{\underline{Q}}^T\underline{\underline{R}}\,\underline{\underline{Q}}\,\underline{\underline{D}}^{-1/2})\underline{Z} = 0 \tag{D-13}$$

and becomes

$$(\underline{\underline{I}}y - \underline{\underline{B}})\underline{Z} = 0 \qquad \text{(D-14)}$$

which is of canonical form.

Note that $\underline{\underline{D}}^{\frac{1}{2}}$ and $\underline{\underline{D}}^{-\frac{1}{2}}$ are easily obtained by taking the square roots, and the square root reciprocals, respectively, of the diagonal terms. Also note that $\underline{\underline{B}}$ is symmetric. The problem is now well suited for solution by use of the Jacobi method. It is seen that inaccuracies in obtaining eigenvectors $\underline{\underline{Q}}$ would cause compounded errors in the final solution for $\underline{A}$ and $\lambda$.

The problem is returned to original form with the substitutions

$$\lambda = -\frac{1}{y} \qquad \text{(D-15)}$$

and

$$\underline{A} = \underline{\underline{Q}}\,\underline{\underline{D}}^{-1/2}\,\underline{Z} \qquad \text{(D-16)}$$

APPENDIX E

SUGGESTIONS FOR EXTENSION OF THE PROBLEM

There are two obvious directions for extension of this problem of a thin plate with a circular hole. The buckling problem for a strip of finite width and the postbuckling problem for both finite and infinite plates remain to be treated analytically.

The postbuckling problem is the more difficult of the two because of the coupling between the stress distribution and the buckled deflection. Johns (2) outlines the postbuckling situation for plates and suggests several methods of solution. Solution of the postbuckling problem would be important because of the revised stress concentration factors that would be obtained; cf. Houghton and Rothwell (16).

Extension of this problem to a finite strip, of width 2b, would require the following:

a. Using the Howland solution (17) instead of that due to Kirsch, to obtain the stress distribution in the plate.

b. Satisfying additional boundary conditions for two "free" straight edges.

Integration of the energy terms for the finite strip is between the limits $r = a$ and $r = \dfrac{b}{\sin\theta}$, so that the integration with respect to $\theta$ becomes more complicated than for the infinite plate.

This integration is over the interval 0 to $2\pi$. Wylie (10) presents a method for the evaluation of real trigonometric integrals over this interval. The method involves substitution of appropriate complex variables for the sine and cosine terms, and then evaluation of the residues inside a unit circle.

There are two possible methods for satisfaction of the additional boundary conditions of the finite strip. Introduction of additional unknowns, similar to $X_{NM}$ and $Y_{NM}$ in equation A-3, into the assumed series for deflection could be made. These unknowns could be determined by simultaneous solution of all the boundary condition equations. Another method, the Lagrangian multiplier method, introduces dummy variables so that additional restraining equations, those expressing the boundary conditions, can be solved simultaneously with the energy expressions. Budiansky and Hu (12) present a clear example of the use of this method in minimizing the plate energy, while satisfying plate boundary conditions.

## APPENDIX F

## COMPUTER PROGRAM LISTINGS

The following pages contain a listing of the computer programs used in the solution of the problem. All programs are written in FORTRAN IV for use on an IBM System 360 computer.

The control card listing on the first page is effective as of September 1967. Users of this program should check current procedures.

The maximum terms to be taken in N and M, and the plate thickness read into the computer by data card (see comment cards regarding input requirements). Young's modulus, E, and Poisson's ratio are established by cards serial 0010 and 0011 in the main program.

Printed output from the computer is a column matrix EIVU which is an array of all of the values of $S_{cr}$ in psi for the parameters of plate thickness, Poisson's ratio, Young's modulus, terms in N, and terms in M. Matrix EIVU is followed by a map of the buckled plate deflection for the minimum value of tensile critical buckling stress.

```
//OBJECT JOB '400','PELLETT',MSGLEVEL=1
// EXEC FORTCLG
//FORT    EXEC  PGM=IEYFORT                                    00000010
//SYSLIN DD     DSNAME=SYS1.SYSUT4,DISP=OLD                    00000020
//SYSPRINT DD   SYSOUT=A                                       00000030
//SYSPUNCH DD   UNIT=SYSCP                                     00000040
//FORT.SYSIN  DD  *
```

The above card listing is the initial control card sequence needed for the IBM System 360 as of September, 1967. Users of this program should insure that the above cards are valid before using them. Only the unnumbered cards need to be supplied by the user.

```
      IMPLICIT REAL*8(A-H,O-Z)
C     THIS IS THE MAIN PROGRAM FOR DETERMINING THE CRITICAL BUCKLING
C     STRESS FOR AN INFINITE PLATE UNDER TENSION.
C
C     SEE -  PELLETT, C. A. 'BUCKLING OF AN INFINITE PLATE WITH A HOLE
C     UNDER TENSION, NPGS MASTERS THESIS, 1967
C
C     THIS PROGRAM  FINDS THE ENERGY STORED IN AN INFINITE PLATE WITH
C      A HOLE,  AND MINIMZES THIS ENERGY WITH RESPECT TO AN UNKNOWN
C     PARAMETER.  CRITICAL BUCKLING STRESS IS PRINTED IN THE MATRIX
C     EIVU.  THE UNKNOWN COEFFICIENTS ARE IN MATRIX A.  BUCKLED MODE
C     SHAPES FOR THE PLATE ARE PRINTED.
C
C     INPUT REQUIREMENT-  DATA CARDS AS SPECIFIED BY FORMATS 991 $ 930
C     OUTPUT -  PRINTED OUTPUT OF EIVU, A, AND BUCKLED MODE SHAPES
      COMMON/PASS/MCOUNT(100),JOY,N,M,INIT,THICK
      COMMON/PAST/JROW,JCOL
      COMMON/BANE/XX(100),YY(100)
      COMMON/CHUNK/A(80,80),B(80,80),EIVR(80,80)
      COMMON/MUCK/EIVU(100),TEMPA(100)
      COMMON/TAME/START
        COMMON/NUMBER/US1,US2,US3,US4,US5,US6,US7A,US7B,US8A,US8B,US9A,
     1US9B,UB1,UB2A,UB2B,UB3A,UB3B,UB4,UB5A,UB5B,UB7,UB8A,UB8B,UB9,UB6,
     2 UBD,USD
      START=ITIME(C)*.C1
      E=30.D+06
      VNU=.3D0
C     READ THE NUMBER OF DATA CARDS THAT ARE TO FOLLOW
      READ 991,NCARD
  991 FORMAT(I2)
      DO 844 JARK=1,NCARD
C     READ  THE MAXIMUM NUMBER OF TERMS IN N,M,AND THE PLATE THICKNESS
      READ(5,930) N,M,THICK
      RIGID=E*THICK**3/(12.*(1.-VNU**2))
  930 FORMAT(2I3,D8.7)
      INIT=N*M
C     GENERATE ODD OR EVEN M'S,WITH SIN OR COS TERMS
      DO 844 JOY=1,3,2
      DO 364 ITLE=1,M
      GO TO (365,366,367,368),JOY
  365 MCOUNT(ITLE)=2*ITLE-1
      GO TO 364
  366 MCOUNT(ITLE)=2*ITLE-2
      GO TO 364
  367 MCOUNT(ITLE)=2*ITLE-1
      GO TO 364
  368 MCOUNT(ITLE)=2*ITLE
  364 CONTINUE
```

```
      DO 363 NOD=1,INIT
      DO 363 NUT=1,INIT
      B(NUT,NOD)=0.DO
  363 A(NUT,NOD)=0.DO
      NEWPAG =50
      PIE=3.14159265358979323 8
      DO 30 I=1,N
      DO 30 J=1,M
      JCOL=(I-1)*M+J
   30 CALL BOUND(I,MCOUNT(J),XX(JCOL),YY(JCOL))
      DO 800 I=1,N
      DO 800 JAA=1,M
      DO 800 K=1,N
      DO 800 LAA=1,M
      J=MCOUNT(JAA)
      L=MCOUNT(LAA)
C     CONVERT QUADRUPLE SUBSCRIPTS TO DOUBLE SUBSCRIPTS.
      JCOL=(K-1)*M+LAA
      JROW=(I-1)*M+JAA
C     INITIALIZE THE VALUE OF THE TRIGONOMETRIC INTEGRALS
C     NOTATION IS OF THE FORM CACBCD.  THIS MEANSTHE INTEGRAL OF
C     COSINE (A*THETA) COSINE(B*THETA) COSINE (D*THETA) OVER THE
C     INTERVAL ZERO TO TWO PI
      CLCJ=0.DO
      SLSJ=0.DO
      SLSJC2=0.DO
      CLCJC2=0.DO
      SJS2CL=0.DO
      SLS2CJ=0.DO
C     FIND THE VALUE OF THE TRIGONOMETRIC INTEGRALS FOR DIFFERENT I,J,L,K
      IF(L-J)4,3,4
    3 CALL CCC(CLCJ,L,J,0)
      CALL SSC (SLSJ,L,J,0)
    4 IF((L-J-2)*(L+J-2)*(L-J+2)*(L+J+2))5,6,5
    6 CALL CCC (CLCJC2,L,J,2)
      CALL SSC (SLSJC2,L,J,2)
      CALL SSC (SJS2CL,J,2,L)
      CALL SSC (SLS2CJ,L,2,J)
C     IF ALL OF THE TRIGONOMETRIC INTEGRAL ARE ZERO, SET THE ENERGY
C     INTEGRAL EQUAL TO ZERO FOR THESE PARTICULAR VALUES OF I,J,K,L.
    5 IF(CLCJ**2+SLSJ**2+SLSJC2**2+CLCJC2**2+SLS2CJ**2+SJS2CL**2)7,800,7
    7 CONTINUE
C     FORM THE TERMS IN THE ENERGY INTEGRALS.
      CALL TERMS (I,J,K,L)
C     SUM THE TERMS IN THE ENERGY INTEGRALS
      UBTI=UB1+UB2A+UB3B+UB4+UB3A+UB2B+UB5B+UB5A+UB6
      UBTII=(1.DO-VNU)*(UB2B+UB3A+UB2A+UB3B)
      UBTIII=2.DO*(1.DO-VNU)*(UB7-UB8A-UB8B+UB9)
```

```
0068            UBTA=((UBTI-UBTII)*CLCJ+(UBTIII)*SLSJ)/UBD
0069            UBTB=((UBTI-UBTII)*SLSJ+(UBTIII)*CLCJ)/UBD
0070            USRTA= (US7A-3.*US9A+2.*US8A)*SJS2CL+(US7B-3.*US9B+2.*US8B)*SLS2CJ
0071            USRTB=-(US7A-3.*US9A+2.*US8A)*SLS2CJ-(US7B-3.*US9B+2.*US8B)*SJS2CL
0072            USRA=(US1-US2)*CLCJ+(US1+3.*US3-4.*US2)*CLCJC2
0073            USRB=(US1-US2)*SLSJ+(US1+3.*US3-4.*US2)*SLSJC2
0074            USTB=(US4+US5)*CLCJ-(US4+3.*US6)*CLCJC2
0075            USTA=(US4+US5)*SLSJ-(US4+3.*US6)*SLSJC2
0076            USA=(USRA+USTA+USRTA)/USD
0077            USB=(USRB+USTB+USRTB)/USD
          C     TEST TO SEE IF SINE OR COSINE TERMS OR PRESENT  AND SUM ACCORDINGLY
0078            IF (JOY-2) 10,10,9
          C     THE A MATRIX REPRESENTS THE STRETCH ENERGY INTEGRAL
          C     THE B MATRIX REPRESENTS THE BENDING ENERGY INTEGRAL
0079         10 A(JROW,JCOL)=USA*THICK/2.D0
0080            B(JROW,JCOL)=UBTA*RIGID
0081            GO TO 8
0082          9 A(JROW,JCOL)=USB*THICK/2.D0
0083            B(JROW,JCOL)=UBTB*RIGID
          C      DOUBLE THE DIAGONAL TERMS OF THE A AND B  MATRICES
0084          8 IF(JCOL-JROW)12,13,12
0085         13 A(JROW,JCOL)=2.*A(JROW,JCOL)
0086            B(JROW,JCOL)=2.*B(JROW,JCOL)
0087        990 FORMAT (1H1,9HSYMMETRY=,2X,I3,3X,2HN=,2X,I3,3X,2HM=,2X,I3,3X,6HTER
               1MS=,2X,I3,3X,10HTHICKNESS=,G12.6/)
0088         12 CONTINUE
          C IF A 'GO TO 800' CARD FOLLOWS, MATRIX ELEMENT PRINT IS SUPPRESSE
0089            GO TO 800
0090        887 IF(NEWPAG -50)23,25,25
0091         25 WRITE (6,990) JOY,N,M,INIT,THICK
0092            WRITE(6,910)
0093        910 FORMAT(1H ,13H I   J   K   L   ,5X,4HCLCJ,7X,4HSLSJ,7X,6HSLSJC2,5X,6HC
               1LCJC2,5X,6HSLS2CJ,5X,6HSJS2CL,5X,7HSTRETCH,9X,4HBEND,//)
0094            NEWPAG = 0
0095         23 WRITE(6,920)I,J,K,L,CLCJ,SLSJ,SLSJC2,CLCJC2,SLS2CJ,SJS2CL,A(JROW,J
               1COL),B(JROW,JCOL)
0096        920 FORMAT (1X,4(I2,1X),2X,6(G10.3,1X),2X,2G15.8)
0097            NEWPAG =NEWPAG +1
0098        800 CONTINUE
0099            WRITE(6,990) JOY,N,M,INIT,THICK
          C     SOLVE THE EIGENVALUE PROBLEM AX+B=0
0100            CALL JUGGLE (INIT)
0101            DO 225 I=1,INIT
0102            DO 225 J=1,INIT
0103        225 EIVR(I,J)=A(I,J)
0104            WRITE(6,990)JOY,N,M,INIT,THICK
0105            BETWEN=ITIME(0)*.01
0106            TIME=BETWEN-START
```

```
0107          WRITE(6,995) TIME
0108          NULL=0
0109          DO 220 I=1,INIT
      C       DETERMINE WHICH EIGENVALUE S ARE POSITIVE
0110          IF (EIVU(I))220,220,221
0111      221 NULL=NULL+1
0112          TEMPA(NULL)=EIVU(I)
0113          DO 222 J=1,INIT
0114      222 EIV3(J,NULL)=EIVR(J,I)
0115      220 CONTINUE
0116          IF (NULL.EG.0) GC TC 844
0117          NULL=1
0118          DC 223 NO=1,NULL
      C        PRINT THE MODE SHAPES FOR THE POSITIVE EIGENVALUES.
0119          CALL PICTUR (NC,50)
0120      223 CONTINUE
0121          BETWEN=ITIME(C)*.01
0122          TIME=BETWEN-START
0123          WRITE(6,995) TIME
0124      844 CONTINUE
0125      995 FORMAT(1X,'ELAPSED TIME =  ',G13.5,'SECONDS')
0126          END
```

```
      SUBROUTINE JUGGLE (LINK)
C     THIS SUBROUTINE CONVERTS A MATRIX EIGENVALUE OF THE FORM (AX+B)Y=0
C     TO THE FORM (IX+C)Z=0,  WHERE A,  B, C , ARE COEFFICIENT MATRICES,
C     X IS THE EIGENVALUE MATRIX, AND Y $ Z ARE THE EIGENVECTOR MATRICES
C     SEE APPENDIX G. OF 'BUCKLING NEAR A HOLE IN AN INFINITE PLATE UNDER
C     TENSION' BY C. A. PELLETT, NPGS MASTER'S THESIS.
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/MUCK/EIVU(100),TEMPA(100)
      COMMON/CHUNK/A(80,80),B(80,80),EIVR(80,80)
      DO 1 I=1,LINK
      DO 1 J=1,LINK
      EIVR(I,J)=A(I,J)
      A(I,J)=B(I,J)
    1 B(I,J)=EIVR(I,J)
      CALL JACVAT (LINK)
      DO 2 J=1,LINK
      DO 2 I=1,LINK
      SUM=0.D0
      DO 3 JOKE=1,LINK
    3 SUM=EIVR(JOKE,I)*B(JOKE,J)+SUM
    2 A(I,J)=SUM
      DO 4 J=1,LINK
      DO 4 I=1,LINK
      SUM=0.D0
      DO 5 JOKE=1,LINK
    5 SUM=A(I,JOKE)*EIVR(JOKE,J)+SUM
    4 B(I,J)=SUM
      DO 6 I=1,LINK
      TEMPA(I)=DSQRT(EIVU(I))
    6 EIVU(I)=1.D0/DSQRT(EIVU(I))
      DO 7 I=1,LINK
      DO 7 J=1,LINK
      A(I,J)=B(I,J)*EIVU(I)*EIVU(J)
    7 B(I,J)=EIVR(I,J)
      CALL JACVAT(LINK)
      DO 8 I=1,LINK
      DO 8 J=1,LINK
      SUM=0.D0
      DO 9 JOKE=1,LINK
    9 SUM=B(JOKE,I)*EIVR(JOKE,J)+SUM
    8 A(I,J)=SUM*TEMPA(I)
      DO 10 I=1,LINK
   10 EIVU(I)=-1.D0/EIVU(I)
      CALL PRINN(4,LINK,0)
      RETURN
      END
```

```
      SUBROUTINE CENTR
      COMMON/CHUNK/A(150,130),DUMM(6100),DUM2(12800)
      N=150
      M=130
C THIS SUBROUTINE WILL GIVE A MAP OF THE NUMBERS CONTAINED IN THE ARRAY A
C THREE PARAMETERS MUST BE FURNISHED TO THE SUBROUTINE
C              A IS THE NAME OF THE ARRAY TO BE MAPPED
C              N IS THE NUMER OF LINES IN T&E MAP
C              M IS THE WIDTH OF THE LINE, THE MAXIMUM IS 115
      DIMENSION LINE(130),ISYMB(41),BV(21)
      DATA ISYMB/1H0,1H ,1HA,1H ,1HB,1H ,1HC,1H ,1HD,1H ,1HE
     1,1H ,1HF,1H ,1HG,1H ,1HH,1H ,1HI,1H ,1HJ,1H ,1HQ,1H ,1HR,1H ,1HS,
     21H ,1HT,1H ,1HU,1H ,1HV,1H ,1HW,1H ,1HX,1H ,1HY,1H ,1HZ/
      DATA IBLK/1H /
      AMIN=1.E+70
       DO 6 I=1,N
      DO 5 J=1,M
    5 AMIN=AMIN1(AMIN,A(I,J))
    6 CONTINUE
      DO 8 I=1,N
      DO 7 J=1,M
    7 A(I,J)=A(I,J)-AMIN
    8 CONTINUE
      AMAX=-1.E+70
      DO 11 I=1,N
       DO 10 J=1,M
   10 AMAX=AMAX1(AMAX,A(I,J))
   11 CONTINUE
      SCF=40./AMAX
      DO 21 I=1,N
      DO 20 J=1,M
   20 A(I,J)=SCF*A(I,J)
   21 CONTINUE
      DO 30 I=1,21
      AI=DFLOAT(I-1)
   30 BV(I)=(2.*AI/SCF)+AMIN
      WRITE(6,1050)
      WRITE(6,1300) BV(1),BV(1),BV(21)
      WRITE(6,1100) BV(2),BV(3),BV(4),BV(5),BV(6)
      WRITE(6,1200) BV(7),BV(8),BV(9),BV(10),BV(11)
      WRITE(6,1210) BV(12),BV(13),BV(14),BV(15),BV(16)
      WRITE(6,1220) BV(17),BV(18),BV(19),BV(20),BV(21)
      WRITE(6,1050)
      M1=M+1
      DO 50 K=1,N
      DO 40 J=1,M
      AJ=A(K,J)+1.50000
```

```
      $AJ=AJ
      JJ=IFIX($AJ)
   40 LINE(J)=ISYMP(JJ)
      IF(M1-115) 41,41,46
   41 CONTINUE
      DO 45 I=M1,130
   45 LINE(I)=IBLK
   46 CONTINUE
      WRITE(6,2000) LINE
   50 CONTINUE
      DO 61 I=1,N
      DO 60 J=1,M
   60 A(I,J)=(A(I,J)/SCF)+AMIN
   61 CONTINUE
      WRITE(6,2500) NMAP
 2500 FORMAT(//,5X,4H MAP,I3)
 1050 FORMAT(//)
 1100 FORMAT( 5X,'A=',G13.5,3X,'B=',G13.5,3X,'C=',G13.5,3X,'D=',G13.5,
     13X,'E=',G13.5)
 1200 FORMAT( 5X,'F=',G13.5,3X,'G=',G13.5,3X,'H=',G13.5,3X,'I=',G13.5,
     13X,'J=',G13.5)
 1210 FORMAT( 5X,'Q=',G13.5,3X,'R=',G13.5,3X,'S=',G13.5,3X,'T=',G13.5,
     13X,'U=',G13.5)
 1220 FORMAT( 5X,'V=',G13.5,3X,'W=',G13.5,3X,'X=',G13.5,3X,'Y=',G13.5,
     13X,'Z=',G13.5)
 1300 FORMAT(5X,3H D=,G15.6,5X,7H RANGE(,G15.6,5X,G15.6,2H ))
 2000 FORMAT(1X,130A1)
      END
```

```
C      THIS SUBROUTINE FORMS THE DEFLECTION  MATRIX OF REPRESENTATIVE
C      POINTS ON THE PLATE FOR USE IN SUBROUTINE CONTR
       SUBROUTINE PICTUR (NO,IRAD)
       IMPLICIT REAL*8 (A-H,O-Z)
       REAL*4 DATA,DUMM
       COMMON/BANE/ X(100),Y(100)
       COMMON/TAME/START
       COMMON/CHUNK/DATA(150,130),DUMM(6100),A(80,80)
       COMMON/MUCK/EIVU(100),TEMPA(100)
       COMMON/PASS/MCOUNT(100),JOY,N,M,INIT,THICK
       RADIUS=IRAD
       WRITE (6,990) JOY,N,M,INIT,THICK
       WRITE(6,90) TEMPA(NO)
       DO 125 NID=1,N
       DO 125 MID=1,M
       JROW=(NID-1)*M+MID
       EIVU(JROW)=A(JROW,NO)
       WRITE(6,91)NID,MCOUNT(MID),A(JROW,NO)
   125 CONTINUE
       DO 210 IY=1,130
       WY=IY
       WY=WY-65.D0
       DO 210 IX=1,150
       EX=IX/.8D0
       OVARE=RADIUS/(DSQRT(EX*EX+WY*WY))
       WWW=0.D0
       IF(1.D0-OVARE)210,126,126
   126 CONTINUE
       THETA=DATAN2(WY,EX)
       PARE=OVARE*OVARE
       DO 213 NID=1,N
       PARE=PARE*OVARE
       N1=NID+1
       N2=NID+2
       N3=NID+3
       DO 213 MID=1,M
       JROW=(NID-1)*M+MID
       IF (JOY-2)214,214,215
   214 WWW=WWW+(PARE*(NID*N1+OVARE*N2*(X(JROW)*N1+Y(JROW)*N3*OVARE)))*
      1DCOS(MCOUNT(MID)*THETA)
       GO TO 213
   215 WWW=WWW+(PARE*(NID*N1+OVARE*N2*(X(JROW)*N1+Y(JROW)*N3*OVARE)))*
      1DSIN(MCOUNT(MID)*THETA)
   213 CONTINUE
   210 DATA(IX,IY)=WWW
       WRITE (6,990) JOY,N,M,INIT,THICK
       WRITE(6,92)
```

```
   92 FORMAT(1X,'MAP OF THE RELATIVE CURVATURE IN THE RADIAL DIRECTION')
      CALL CONTR
   90 FORMAT(1X,'CRITICAL BUCKLING STRESS = ',G12.5,' PSI'//1X,'COEFFICI
     1ENTS OF THE SERIES'//)
   91 FORMAT(1X,'N= 'I2,'  M=',I2,'     A(N,M) = ',G16.6)
  990 FORMAT (1H1,9HSYMMETRY=,2X,I3,3X,2HN=,2X,I3,3X,2HM=,2X,I3,3X,6HTER
     1MS=,2X,I3,3X,10HTHICKNESS=,G12.6/)
      RETURN
  444 FORMAT(1X,'ELAPSED TIME =',G13.5, '   SECONDS')
      END
```

```
      SUBROUTINE BCUND(N,M,X,Y)
      IMPLICIT REAL*8 (A-H,O-Z)
C     THIS SUBROUTINE FINDS THE VALUES OF X AND Y FOR A GIVEN N AND M
C     SO THAT THE BOUNDARY CONDITIONS WILL BE SATISFIED.
      VNU=.3D0
      OMEN1=N*(N+1)-N*VNU-M**2*VNU
      OMEN2=(N+1)*(N+2)-(N+1)*VNU-M**2*VNU
      OMEN3=(N+2)*(N+3)-(N+2)*VNU-M**2*VNU
      SHE1=-N*(N+1)*(N+2)+N*(N+1)+N+(2-VNU)*M**2*N+(3-VNU)*M**2
      SHE2=-(N+1)*(N+2)*(N+3)+(N+1)*(N+2)+(N+1)+(2-VNU)*M**2*(N+1)+(3-VN
     1U)*M**2
      SHE3=-(N+2)*(N+3)*(N+4)+(N+2)*(N+3)+(N+2)+(2-VNU)*M**2*(N+2)+(3-VN
     1U)*M**2
      X=(-OMEN1*SHE3+OMEN3*SHE1)/(OMEN2*SHE3-OMEN3*SHE2)
      Y=(-OMEN2*SHE1+OMEN1*SHE2)/(OMEN2*SHE3-OMEN3*SHE2)
      RETURN
      END
```

```
C          THIS SUBROUTINE FINDS THE VALUE OF THE INTEGRAL
C
C          SINF(IXX*THETA) SINF(IYY*THETA) COSINF(IZZ*THETA)
C
C         AS THETA RANGES FROM ZERO TO TWO PI
          SUBROUTINE SSC (VALUE,IXX,IYY,IZZ)
          IMPLICIT REAL*8 (A-H,O-Z)
          B1=0.D0
          B2=0.0D0
          B3=0.0D0
          B4=0.0D0
          VALUE=0.0D0
          IF (IXX*IYY)10,9,10
       10 IF (IXX-IYY-IZZ)1,2,1
        2 B1=.5D0
        1 IF(IXX+IYY+IZZ)3,4,3
        4 B2=-.5D0
        3 IF(IXX-IYY+IZZ)5,6,5
        6 B3=.5D0
        5 IF(IXX+IYY-IZZ)7,8,7
        8 B4=-.5D0
        7 VALUE=(B1+B2+B3+B4)*3.14159265358979323 8
        9 RETURN
          END
```

```
      SUBROUTINE CCC(VALUE,IXX,IYY,IZZ)
C     THIS SUBROUTINE FINDS THE VALUE OF THE INTEGRAL OF
C
C     COSINE (IXX*THETA) COSINE (IYY*THETA) COSINE (IZZ*THETA)
C
C     AS THETA RANGES FROM ZERO TO TWO PI
C
      IMPLICIT REAL*8 (A-H,O-Z)
      B1=0.D0
      B2=0.0D0
      B3=0.0D0
      B4=0.0D0
      IF(IXX-IYY-IZZ)1,2,1
    2 B1=.5D0
    1 IF(IXX+IYY+IZZ)3,4,3
    4 B2= .5D0
    3 IF(IXX-IYY+IZZ)5,6,5
    6 B3=.5D0
    5 IF(IXX+IYY-IZZ)7,8,7
    8 B4= .5D0
    7 VALUE=(B1+B2+B3+B4)*3.14159265358979323 8
      RETURN
      END
```

```
      SUBROUTINE JACVAT (N)
C      THIS SUBROUTINE SOLVES THE MATRIX EIGENVALUE PROBLEM (IX-A)Y=0.
C     WHERE X IS THE EIGENVALUE MATRIX IN EIVU, Y IS THE EIGENVECTOR MATRIX
C     IN EIVR, AND A IS THE COEFFICIENT MATRIX IN A
       IMPLICIT REAL*8(A-H,O-Z)
      COMMON/CHUNK/A(80,80),B(80,80),EIVR(80,80)
      COMMON/MUCK/EIVU(100),TEMPA(100)
      NOYES=1
      IF(N-1)20,23,21
   20 WRITE (6,22) N
   22 FORMAT(4H N= ,I3,68H IS TOO SMALL.  LIMIT IS 1.  RETURN TO CALLING
     1 ROUTINE FROM JACVAT. )
      RETURN
   23 WRITE (6,24) A(1,1)
   24 FORMAT (24H IN JACVAT , MATRIX A = ,D14.6)
      RETURN
   21 IF(N-160)1,1,3
    3 WRITE(6,2) N
    2 FORMAT(3H N=,I5,70H IS TOO LARGE.  LIMIT IS 160.  RETURN TO CALLI-
     1NG ROUTINE FROM JACVAT. )
      RETURN
    1 IF(NOYES)99,102,99
C
C        SET THE MATRIX EIVR EQUAL TO THE  IDENTITY MATRIX
C
   99 CONTINUE
      DO 101 J=1,N
      DO 100 I=1,N
  100 EIVR(I,J)=0.0D0
  101 EIVR(J,J)=1.0D0
C
C        CHECK SYMMETRY OF THE GIVEN MATRIX A.  IF TWO SYMMETRIC ELEMENTS ARE
C        NOT EQUAL, REPLACE EACH OF THE UNEQUAL ELEMENTS WITH THE MEAN OF THE
C        TWO ELEMENTS, PRINT OUT A MESSAGE, AND CONTINUE WITH THE NEW MATRIX.
C
  102 ATOP=0.0D0
      DO 112 J=1,N
      DO 111 I=1,J
      IF(A(I,J)-A(J,I))90,103,90
C  90 WRITE(6,106) N,N
  106 FORMAT(14H IN JACVAT (A(,I3,1H,,I3,3H)),)
      WRITE (6,108) I,J,J,I
  108 FORMAT(3H A(,I3,1H,,I3,10H)  AND  A(,I3,1H,,I3,54H) WERE UNEQUAL,
     1SO THEY WERE REPLACED WITH THEIR MEAN )
   90 CONTINUE
      A(I,J)=.5*(A(I,J)+A(J,I))
      A(J,I)=A(I,J)
```

```
  103 CONTINUE
C
C         FIND THE ABSOLUTELY LARGEST ELEMENT OF A
C
      IF(ATOP-DABS(A(I,J)))104,111,111
  104 ATOP=DABS(A(I,J))
  111 CONTINUE
  112 EIVU(J)=A(J,J)
      IF(ATOP)109,109,113
  109 WRITE (6,110)
  110 FORMAT(26H IN JACVAT, MATRIX A  = 0 )
      RETURN
C
C         CALCULATE THE STOPPING CRITERION -- DSTOP
C
  113 AVGF=DFLOAT(N*(N-1))*.55
      D=0.0D0
      DO 114 JJ=2,N
      DO 114 II=2,JJ
      S=A(II-1,JJ)/ATOP
  114 D=S*S+D
      DSTOP=(1.D-12)*D
C
C         CALCULATE THE THRESHOLD, THRSH
C
      THRSH = DSQRT(D/AVGF)*ATOP
C
C         START A SWEEP
C
  115 IFLAG=0
      DO 130 JCOL=2,N
      JCOL1=JCOL-1
      DO 130 IROW=1,JCOL1
      AIJ=A(IROW,JCOL)
C
C         COMPARE THE OFF-DIAGONAL ELEMENT WITH THRSH
C
      IF(DABS(AIJ)-THRSH)130,130,117
  117 AII=A(IROW,IROW)
      AJJ=A(JCOL,JCOL)
      S=AJJ-AII
C
C         CHECK TO SEE IF THE CHOSEN ROTATION IS LESS THAN THE ROUNDING ERROR.
C         IF SO , THEN DO NOT ROTATE.
C
      IF(DABS(AIJ)-1.D-17*DABS(S))130,130,118
  118 IFLAG=1
C
```

```
C          IF THE ROTATION IS VERY CLOSE TO 45 DEGREES, SET SIN AND COS
C          TO 1/(ROOT 2).
C
           IF(1.D-10*DABS(AIJ)-DABS(S))116,119,119
      119  S=DSQRT(.5DO)
           C=S
           GO TO 120
C
C          CALCULATION OF SIN AND COS FOR ROTATION THAT IS NOT VERY CLOSE
C          TO 45 DEGREES
C
      116  T=AIJ/S
           S=0.25/DSQRT(0.25+T*T)
C
C          COS = C ,  SIN= S
C
           C=DSQRT(0.5+S)
           S=2.*T*S/C
C
C          CALCULATION OF THE NEW ELEMENTS OF MATRIX A
C
      120  DO 121 I=1,IROW
           T=A(I,IROW)
           U=A(I,JCOL)
           A(I,IROW)=C*T-S*U
      121  A(I,JCOL)=S*T+C*U
           I2=IROW+2
           IF(I2-JCOL)127,127,123
      127  CONTINUE
           DO 122 I=I2,JCOL
           T=A(I-1,JCOL)
           U=A(IROW,I-1)
           A(I-1,JCOL)=S*U+C*T
      122  A(IROW,I-1)=C*U-S*T
      123  A(JCOL,JCOL)=S*AIJ+C*AJJ
           A(IROW,IROW)=C*A(IROW,IROW)-S*(C*AIJ-S*AJJ)
           DO 124 J=JCOL,N
           T=A(IROW,J)
           U=A(JCOL,J)
           A(IROW,J)=C*T-S*U
      124  A(JCOL,J)=S*T+C*U
C
C          ROTATION COMPLETED.
C          SEE IF EIGENVECTORS ARE WANTED BY USER
C
           IF(NOYES)131,126,131
      131  CONTINUE
           DO 125 I=1,N
```

```
      T=EIVR(I,IROW)
      EIVR(I,IROW)=C*T-EIVR(I,JCOL)*S
  125 EIVR(I,JCOL)=S*T+EIVR(I,JCOL)*C
C
C        CALCULATE THE NEW NORM D AND COMPARE WITH DSTOP
C
  126 CONTINUE
      S=AIJ/ATOP
      D=D-S*S
      IF(D-DSTOP)1260,129,129
C
C        RECALCULATE DSTOP AND THRSH TO DISCARD ROUNDING ERRORS
C
 1260 D=0.0D0
      DO 128 JJ=2,N
      DO 128 II=2,JJ
      S=A(II-1,JJ)/ATOP
  128 D=S*S+D
      DSTOP=(1.D-12)*D
  129 THRSH=DSQRT(D/AVGF)*ATOP
  130 CONTINUE
      IF(IFLAG)115,134,115
C
C        ENDING ROUTINE -- RESET MATRIX A AND PLACE EIGENVALUES IN EIVU
C
  134 T=A(1,1)
      A(1,1)=EIVU(1)
      EIVU(1)=T
      DO 132 J=2,N
      T=A(J,J)
      A(J,J)=EIVU(J)
      EIVU(J)=T
      DO 132 I=2,J
  132 A(I-1,J)=A(J,I-1)
  133 RETURN
      END
```

```
0001         SUBROUTINE TERMS(I,J,K,L)
       C     THIS SUBROUTINE FINDS THE INTEGRATED VALUE OF THE TERMS OF THE
       C     STRETCH AND BENDING INTEGRALS FOR PLATE ENERGY
0002         IMPLICIT REAL*8(A-H,O-Z)
0003         COMMON/PAST/JROW,JCOL
0004         COMMON/BANE/XX(100),YY(100)
0005           COMMON/NUMBER/US1,US2,US3,US4,US5,US6,US7A,US7B,US8A,US8B,US9A,
            1US9B,UB1,UB2A,UB2B,UB3A,UB3B,UB4,UB5A,UB5B,UB7,UB8A,UB8B,UB9,UB6,
            2 UB0,USD
0006         DIMENSION BUB(9)
0007         X=XX(JCOL)
0008         XSTAR=XX(JROW)
0009         Y=YY(JCOL)
0010         YSTAR=YY(JROW)
0011         US1=0.D0
0012         US2=0.D0
0013         US3=0.D0
0014         US4=0.D0
0015         US5=0.D0
0016         US6=0.D0
0017         US7A=0.D0
0018         US7B=0.D0
0019         US8A=0.D0
0020         US8B=0.D0
0021         US9A=0.D0
0022         US9B=0.D0
0023         UB1=0.D0
0024         UB2A=0.D0
0025         UB2B=0.D0
0026         UB3A=0.D0
0027         UB3B=0.D0
0028         UB4 =0.D0
0029         UB5A=0.D0
0030         UB5B=0.D0
0031         UB6 =0.D0
0032         UB7 =0.D0
0033         UB8A=0.D0
0034         UB8B=0.D0
0035         UB9 =0.D0
0036         TOR=I+K
0037         TOR1=I+K+1
0038         TOR2=I+K+2
0039         TOR3=I+K+3
0040         TOR4=I+K+4
0041         TOR5=I+K+5
0042         TOR6=I+K+6
0043         TOR7=I+K+7
```

```
      TOR8=I+K+8
      DENOM=TOR1*TCR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
      DENOM1=TOR*TCR2*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
      DENOM2=TOR*TCR1*TOR3*TOR4*TOR5*TOR6*TOR7*TOR8
      DENOM3=TCR*TCR1*TOR2*TOR4*TOR5*TOR6*TOR7*TOR8
      DENOM4=TOR*TCR1*TOR2*TOR3*TOR5*TOR6*TOR7*TOR8
      DENOM5=TCR*TCR1*TOR2*TOR3*TOR4*TOR6*TOR7*TOR8
      DENOM6=TCR*TCR1*TOR2*TOR3*TOR4*TOR5*TOR7*TOR8
      DENOM7=TCR*TCR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR8
      DENOM8=TCR*TCR1*TOR2*TOR3*TOR4*TOR5*TOR6*TOR7
      USD=TOR*DENOM
      UBD=TOR*DENCM
      BUB(1)=I*K*(I+1)*(K+1)*DENOM2
      BUB(2)=(I+1)*(K+1)*(I+2)*(K+2)*X*XSTAR*DENOM4
      BUB(3)=(I+2)*(K+2)*(I+3)*(K+3)*Y*YSTAR*DENOM6
      BUB(4)=K*(K+1)*(I+1)*(I+2)*XSTAR*DENOM3
      BUB(5)=I*(I+1)*(K+1)*(K+2)*X*DENOM3
      BUB(6)=K*(K+1)*(I+2)*(I+3)*YSTAR*DENOM4
      BUB(7)=(I+1)*(I+2)*(K+2)*(K+3)*XSTAR*Y*DENOM5
      BUB(8)=I*(I+1)*(K+2)*(K+3)*Y*DENOM4
      BUB(9)=(K+1)*(K+2)*(I+2)*(I+3)*X*YSTAR*DENOM5
      UB1=0.
      DO1 KK=1,9
    1 UB1=BUB(KK)+UB1
      BUB(1)=-I*K*(K+1)*DENOM2
      BUB(2)=-I*(K+1)*(K+2)*X*DENOM3
      BUB(3)=-I*(K+2)*(K+3)*Y*DENOM4
      BUB(4)=-(I+1)* K*(K+1)*XSTAR*DENOM3
      BUB(5)=-(I+1)*(K+1)*(K+2)*XSTAR*X*DENOM4
      BUB(6)=-(I+1)*(K+2)*(K+3)*XSTAR*Y*DENOM5
      BUB(7)=-(I+2)* K*(K+1)*YSTAR*DENOM4
      BUB(8)=-(I+2)*(K+1)*(K+2)*YSTAR*X*DENOM5
      BUB(9)=-(I+2)*(K+2)*(K+3)*YSTAR*Y*DENOM6
      DO 2 KK=1,9
    2 UB2A=UB2A+BUB(KK)
      BUB(1)=-K*I*(I+1)*DENOM2
      BUB(2)=-K*(I+1)*(I+2)*XSTAR*DENOM3
      BUB(3)=-K*(I+2)*(I+3)*YSTAR*DENOM4
      BUB(4)=-(K+1)*I*(I+1)*X*DENOM3
      BUB(5)=-(K+1)*(I+1)*(I+2)*X*XSTAR*DENOM4
      BUB(6)=-(K+1)*(I+2)*(I+3)*X*YSTAR*DENOM5
      BUB(7)=-(K+2)* I*(I+1)*Y*DENOM4
      BUB(8)=-(K+2)*(I+1)*(I+2)*Y*XSTAR*DENOM5
      BUB(9)=-(K+2)*(I+2)*(I+3)*Y*YSTAR*DENOM6
      DO 3 KK=1,9
    3 UB2B=UB2B+BUB(KK)
      BUB(1)=-I*(I+1)*L**2*DENOM2
      BUB(2)=-I*(I+1)*L**2*X*DENCM3
```

```
      BUB(3)=-I*(I+1)*L**2*Y*DENOM4
      BUB(4)=-(I+1)*(I+2)*XSTAR*L**2*DENOM3
      BUB(5)=-(I+1)*(I+2)*XSTAR*X*L**2*DENOM4
      BUB(6)=-(I+1)*(I+2)*XSTAR*Y*L**2*DENOM5
      BUB(7)=-(I+2)*(I+3)*YSTAR*L**2*DENOM4
      BUB(8)=-(I+2)*(I+3)*YSTAR*L**2*X*DENOM5
      BUB(9)=-(I+2)*(I+3)*YSTAR*L**2*Y*DENOM6
      DO 4 KK=1,9
    4 UB3A=UB3A+BUB(KK)
      BUB(1)=-K*(K+1)*J**2*DENOM2
      BUB(2)=-K*(K+1)*J**2*XSTAR*DENOM3
      BUB(3)=-K*(K+1)*J**2*YSTAR*DENOM4
      BUB(4)=-(K+1)*(K+2)*J**2*X*DENOM3
      BUB(5)=-(K+1)*(K+2)*J**2*X*XSTAR*DENOM4
      BUB(6)=-(K+1)*(K+2)*J**2*X*YSTAR*DENOM5
      BUB(7)=-(K+2)*(K+3)*Y*J**2*DENOM4
      BUB(8)=-(K+2)*(K+3)*Y*J**2*XSTAR*DENOM5
      BUB(9)=-(K+2)*(K+3)*Y*J**2*YSTAR*DENOM6
      DO 5 KK=1,9
    5 UB3B=UB3B+BUB(KK)
      BUB(1)=I*K*DENOM2
      BUB(2)=K*(I+1)*XSTAR*DENOM3
      BUB(3)=K*(I+2)*YSTAR*DENOM4
      BUB(4)=(K+1)*I*X*DENCM3
      BUB(5)=(K+1)*(I+1)*X*XSTAR*DENOM4
      BUB(6)=(K+1)*(I+2)*X*YSTAR*DENOM5
      BUB(7)=(K+2)*(I)*Y*DENOM4
      BUB(8)=(K+2)*(I+1)*XSTAR*Y*DENOM5
      BUB(9)=(K+2)*(I+2)*Y*YSTAR*DENOM6
      DO 6 KK=1,9
    6 UB4=UB4+BUB(KK)
      BUB(1)=J**2*K*DENOM2
      BUB(2)=J**2*(K+1)*X*DENOM3
      BUB(3)=J**2*(K+2)*Y*DENOM4
      BUB(4)=J**2*K*XSTAR*DENOM3
      BUB(5)=J**2*(K+1)*XSTAR*X*DENOM4
      BUB(6)=J**2*(K+2)*XSTAR*Y*DENOM5
      BUB(7)=J**2*K*YSTAR*DENOM4
      BUB(8)=J**2*(K+1)*YSTAR*X*DENOM5
      BUB(9)=J**2*(K+2)*YSTAR*Y*DENOM6
      DO 7 KK=1,9
    7 UB5A=UB5A+BUB(KK)
      BUB(1)=L**2*I*DENCM2
      BUB(2)=L**2*(I+1)*XSTAR*DENOM3
      BUB(3)=L**2*(I+2)*YSTAR*DENOM4
      BUB(4)=L**2*I*X*DENCM3
      BUB(5)=L**2*(I+1)*XSTAR*X*DENOM4
      BUB(6)=L**2*(I+2)*X*YSTAR*DENOM5
```

```
      BUB(7)=L**2*I*Y*DENOM4
      BUB(8)=L**2*(I+1)*XSTAR*Y*DENOM5
      BUB(9)=L**2*(I+2)*YSTAR*Y*DENOM6
      DO 8 KK=1,9
    8 UB5B=UB5B+BUB(KK)
      BUB(1)=J**2*L**2*DENOM2
      BUB(2)=J**2*L**2*XSTAR*DENOM3
      BUB(3)=J**2*L**2*YSTAR*DENOM4
      BUB(4)=J**2*L**2*X*DENOM3
      BUB(5)=J**2*L**2*X*XSTAR*DENOM4
      BUB(6)=J**2*L**2*X*YSTAR*DENOM5
      BUB(7)=J**2*L**2*Y*DENOM4
      BUB(8)=J**2*L**2*Y*XSTAR*DENOM5
      BUB(9)=J**2*L**2*Y*YSTAR*DENOM6
      DO 9 KK=1,9
    9 UB6=UB6+BUB(KK)
      BUB(1)=I*J*K*L*DENOM2
      BUB(2)=I*J*(K+1)*L*X*DENOM3
      BUB(3)=I*J*(K+2)*L*Y*DENOM4
      BUB(4)=(I+1)*J*K*L*XSTAR*DENOM3
      BUB(5)=(I+1)*J*(K+1)*L*XSTAR*X*DENOM4
      BUB(6)=(I+1)*J*(K+2)*L*XSTAR*Y*DENOM5
      BUB(7)=(I+2)*J*K*L*YSTAR*DENOM4
      BUB(8)=(I+2)*J*(K+1)*L*YSTAR*X*DENOM5
      BUB(9)=(I+2)*J*(K+2)*L*YSTAR*Y*DENOM6
      DO 10 KK=1,9
   10 UB7=UB7+BUB(KK)
      BUB(1)=-J*K*DENOM2
      BUB(2)=-J*(K+1)*X*DENOM3
      BUB(3)=-J*(K+2)*Y*DENOM4
      BUB(4)=-J*K*XSTAR*DENOM3
      BUB(5)=-J*(K+1)*XSTAR*X*DENOM4
      BUB(6)=-J*(K+2)*XSTAR*Y*DENOM5
      BUB(7)=-J*K*YSTAR*DENOM4
      BUB(8)=-J*(K+1)*YSTAR*X*DENOM5
      BUB(9)=-J*(K+2)*YSTAR*Y*DENOM6
      DO 11 KK=1,9
   11 UB8A=UB8A+BUB(KK)
      BUB(1)=-L*I*DENOM2
      BUB(2)=-L*(I+1)*XSTAR*DENOM3
      BUB(3)=-L*(I+2)*YSTAR*DENOM4
      BUB(4)=-L*I*X*DENOM3
      BUB(5)=-L*(I+1)*X*XSTAR*DENOM4
      BUB(6)=-L*(I+2)*X*YSTAR*DENOM5
      BUB(7)=-L*I*Y*DENOM4
      BUB(8)=-L*(I+1)*Y*XSTAR*DENOM5
      BUB(9)=-L*(I+2)*Y*YSTAR*DENOM6
      DO 12 KK=1,9
```

```
12 UB8B=UB8B+BUB(KK)
   BUB(1)=J*L*DENCM2
   BUB(2)=J*L*XSTAR*DENCM3
   BUB(3)=J*L*YSTAR*DENCM4
   BUB(4)=J*L*X*DENCM3
   BUB(5)=J*L*X*XSTAR*DENOM4
   BUB(6)=J*L*X*YSTAR*DENOM5
   BUB(7)=J*L*Y*DENCM4
   BUB(8)=J*L*Y*XSTAR*DENOM5
   BUB(9)=J*L*Y*YSTAR*DENOM6
   DO 13 KK=1,9
13 UB9=UB9+BUB(KK)
   BUB(1)=I*K*DENCM
   BUB(2)=I*(K+1)*X*DENCM1
   BUB(3)=I*(K+2)*Y*DENCM2
   BUB(4)=(I+1)*K*XSTAR*DENOM1
   BUB(5)=(I+1)*(K+1)*X*XSTAR*DENOM2
   BUB(6)=(I+1)*(K+2)*Y*XSTAR*DENOM3
   BUB(7)=(I+2)*K*YSTAR*DENOM2
   BUB(8)=(I+2)*(K+1)*YSTAR*X*DENOM3
   BUB(9)=(I+2)*(K+2)*YSTAR*Y*DENOM4
   DO 14 KK=1,9
14 US1=US1+BUB(KK)
   BUB(1)=I*K*DENOM2
   BUB(2)=I*(K+1)*X*DENCM3
   BUB(3)=I*(K+2)*Y*DENCM4
   BUB(4)=(I+1)*K*XSTAR*DENOM3
   BUB(5)=(I+1)*(K+1)*X*XSTAR*DENOM4
   BUB(6)=(I+1)*(K+2)*Y*XSTAR*DENOM5
   BUB(7)=(I+2)*K*YSTAR*DENOM4
   BUB(8)=(I+2)*(K+1)*YSTAR*X*DENOM5
   BUB(9)=(I+2)*(K+2)*YSTAR*Y*DENOM6
   DO 15 KK=1,9
15 US2=US2+BUB(KK)
   BUB(1)=I*K*DENCM4
   BUB(2)=I*(K+1)*X*DENCM5
   BUB(3)=I*(K+2)*Y*DENOM6
   BUB(4)=(I+1)*K*XSTAR*DENOM5
   BUB(5)=(I+1)*(K+1)*X*XSTAR*DENOM6
   BUB(6)=(I+1)*(K+2)*Y*XSTAR*DENOM7
   BUB(7)=(I+2)*K*YSTAR*DENOM6
   BUB(8)=(I+2)*(K+1)*YSTAR*X*DENOM7
   BUB(9)=(I+2)*(K+2)*YSTAR*Y*DENOM8
   DC 16 KK=1,9
16 US3=US3+BUB(KK)
   BUB(1)=J*L*DENOM
   BUB(2)=J*L*XSTAR*DENOM1
   BUB(3)=J*L*YSTAR*DENCM2
```

```
      BUB(4)=J*L*X*DENCM1
      BUB(5)=J*L*X*XSTAR*DENOM2
      BUB(6)=J*L*X*YSTAR*DENOM3
      BUB(7)=J*L*Y*DENCM2
      BUB(8)=J*L*Y*XSTAR*DENOM3
      BUB(9)=J*L*Y*YSTAR*DENOM4
      DO 17 KK=1,9
   17 US4=US4+BUB(KK)
      BUB(1)=J*L*DENCM2
      BUB(2)=J*L*XSTAR*DENOM3
      BUB(3)=J*L*YSTAR*DENCM4
      BUB(4)=J*L*X*DENCM3
      BUB(5)=J*L*X*XSTAR*DENOM4
      BUB(6)=J*L*X*YSTAR*DENOM5
      BUB(7)=J*L*Y*DENCM4
      BUB(8)=J*L*Y*XSTAR*DENOM5
      BUB(9)=J*L*Y*YSTAR*DENOM6
      DO 18 KK=1,9
   18 US5=US5+BUB(KK)
      BUB(1)=J*L*DENOM4
      BUB(2)=J*L*XSTAR*DENCM5
      BUB(3)=J*L*YSTAR*DENOM6
      BUB(4)=J*L*X*DENCM5
      BUB(5)=J*L*X*XSTAR*DENOM6
      BUB(6)=J*L*X*YSTAR*DENOM7
      BUB(7)=J*L*Y*DENCM6
      BUB(8)=J*L*Y*XSTAR*DENOM7
      BUB(9)=J*L*Y*YSTAR*DENOM8
      DO 19 KK=1,9
   19 US6=US6+BUB(KK)
      BUB(1)=-J*K*DENOM
      BUB(2)=-J*(K+1)*X*DENCM1
      BUB(3)=-J*(K+2)*Y*DENOM2
      BUB(4)=-J*K*XSTAR*DENOM1
      BUB(5)=-J*(K+1)*XSTAR*X*DENOM2
      BUB(6)=-J*(K+2)*XSTAR*Y*DENOM3
      BUB(7)=-J*K*YSTAR*DENOM2
      BUB(8)=-J*(K+1)*YSTAR*X*DENOM3
      BUB(9)=-J*(K+2)*YSTAR*Y*DENOM4
      DO 20 KK=1,9
   20 US7A=US7A+BUB(KK)
      BUB(1)=-L*I*DENOM
      BUB(2)=-L*(I+1)*XSTAR*DENOM1
      BUB(3)=-L*(I+2)*YSTAR*DENOM2
      BUB(4)=-L*I*X*DENOM1
      BUB(5)=-L*(I+1)*XSTAR*X*DENOM2
      BUB(6)=-L*(I+2)*X*YSTAR*DENOM3
      BUB(7)=-L*I*Y*DENOM2
```

```
      BUB(8)=-L*(I+1)*Y*XSTAR*DENOM3
      BUB(9)=-L*(I+2)*YSTAR*Y*DENOM4
      DO 21 KK=1,9
   21 US7B=US7B+BUB(KK)
      BUB(1)=-J*K*DENOM2
      BUB(2)=-J*(K+1)*X*DENOM3
      BUB(3)=-J*(K+2)*Y*DENOM4
      BUB(4)=-J*K*XSTAR*DENOM3
      BUB(5)=-J*(K+1)*XSTAR*X*DENOM4
      BUB(6)=-J*(K+2)*XSTAR*Y*DENOM5
      BUB(7)=-J*K*YSTAR*DENOM4
      BUB(8)=-J*(K+1)*YSTAR*X*DENOM5
      BUB(9)=-J*(K+2)*YSTAR*Y*DENOM6
      DO 22 KK=1,9
   22 US8A=US8A+BUB(KK)
      BUB(1)=-L*I*DENOM2
      BUB(2)=-L*(I+1)*XSTAR*DENOM3
      BUB(3)=-L*(I+2)*YSTAR*DENOM4
      BUB(4)=-L*I*X*DENOM3
      BUB(5)=-L*(I+1)*XSTAR*X*DENOM4
      BUB(6)=-L*(I+2)*X*YSTAR*DENOM5
      BUB(7)=-L*I*Y*DENOM4
      BUB(8)=-L*(I+1)*Y*XSTAR*DENOM5
      BUB(9)=-L*(I+2)*YSTAR*Y*DENOM6
      DO 23 KK=1,9
   23 US8B=US8B+BUB(KK)
      BUB(1)=-J*K*DENOM4
      BUB(2)=-J*(K+1)*X*DENOM5
      BUB(3)=-J*(K+2)*Y*DENOM6
      BUB(4)=-J*K*XSTAR*DENOM5
      BUB(5)=-J*(K+1)*XSTAR*X*DENOM6
      BUB(6)=-J*(K+2)*XSTAR*Y*DENOM7
      BUB(7)=-J*K*YSTAR*DENOM6
      BUB(8)=-J*(K+1)*YSTAR*X*DENOM7
      BUB(9)=-J*(K+2)*YSTAR*Y*DENOM8
      DO 24 KK=1,9
   24 US9A=US9A+BUB(KK)
      BUB(1)=-L*I*DENOM4
      BUB(2)=-L*(I+1)*XSTAR*DENOM5
      BUB(3)=-L*(I+2)*YSTAR*DENOM6
      BUB(4)=-L*I*X*DENOM5
      BUB(5)=-L*(I+1)*XSTAR*X*DENOM6
      BUB(6)=-L*(I+2)*X*YSTAR*DENOM7
      BUB(7)=-L*I*Y*DENOM6
      BUB(8)=-L*(I+1)*Y*XSTAR*DENOM7
      BUB(9)=-L*(I+2)*YSTAR*Y*DENOM8
      DO 25 KK=1,9
   25 US9B=US9B+BUB(KK)
```

```
      RETURN
      END
```

```
      SUBROUTINE PRINN (N,M,NPAGE)
C     THIS SUBROUTINE PRINTS THE MATRIX A,B,OR EIVR, DEPENDING ON
C     WHETHER N IN THE CALL ARGUMENT IS 1,2, OR 3.  IF NPAGE EQUALS ZERO
C     NC PAGE HEACING WILL BE PRODUCED.  IF NPAGE EQUALS 1 A NEW PAGE
C     WILL BE STARTEC.  THIS SUBROUTINE FINDS UTILITY IN DEBUGGING
C     WHEN MATRIX PRINTOUT IS NEEDED.
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSICN B(1,1)
      COMMON/MUCK/EIVU(100),TEMPA(100)
      COMMON/CHUNK/A(8C,80),B(80,80),EIVR(80,80)
      IF (NPAGE.EQ.1) WRITE (6,99)
   99 FORMAT ('1')
      GC TO (1,2,3,14,15),N
    1 WRITE(6,91)
   91 FORMAT(' ','MATRIX IN A'//)
      GO TO 4
    2 WRITE (6,92)
   92 FORMAT(' ','MATRIX IN B'//)
      GC TO 4
    3 WRITE (6,93)
   93 FORMAT(' ','MATRIX IN EIVR'//)
      GO TO 4
   14 WRITE(6,84)
   84 FORMAT(' ','MATRIX IN EIVU'//)
   18 WRITE (6,88)((J,EIVU(J)),J=1,M)
      GO TO 100
   15 WRITE (6,85)
   85 FORMAT(' ','MATRIX IN TEMPA'//)
   19 WRITE (6,88)((J,TEMPA(J)),J=1,M)
      GO TO 100
    4 DO 8 I=1,M
      GC TO (5,6,7),N
    5 WRITE (6,94)(A(I,J),J=1,M)
      WRITE(6,95)
      GC TO 8
    6 WRITE (6,94)(B(I,J),J=1,M)
      WRITE(6,95)
      GC TO 8
    7 WRITE (6,94)(EIVR(I,J),J=1,M)
      WRITE(6,95)
    8 CCNTINUE
  100 CCNTINUE
      RETURN
   88 FORMAT(1X,'EIG(',I2,')= ',G15.7)
   94 FORMAT (1X,10G12.5)
```

```
      95 FORMAT(1X)
         END
```

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**13. ABSTRACT**


A study is made of the circumstances under which there will be incipient buckling in an infinite plate containing a circular hole and subjected to uniform uniaxial tension at a great distance from the hole. This buckling is a consequence of the fact that near the hole and in line with the direction of the applied tensile loading, there are areas in which both principal membrane stresses are compressive. Values of critical tensile stress are given, together with expressions for the corresponding lateral deflection patterns. The results are in the form

$$S_{cr} = KE(t/a)^2$$

where E is Young's modulus of elasticity, t is plate thickness, a is a hole radius, and K is a buckling coefficient having the following values for the first four modes: 7.65, 7.80, 11.7, 12.3. A comparison is made with some experimental determinations of Danis for plates of finite width.


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| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
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| Plate Buckling | | | | | | |
| Energy Method | | | | | | |
| Rayleigh Ritz Method | | | | | | |
| Infinite Plate | | | | | | |
| Stress Concentration | | | | | | |
| Plate with a Hole | | | | | | |
| Eigenvalue Problem | | | | | | |