

SOPHIE GERMAIN:
A MATHEMATICAL BIOGRAPHY

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This thesis is a biography of Sophie Germain, a French mathematician who lived from 1776 to 1831. In addition to a traditional biographical account of her life, it discusses her work, focusing on her mathematical accomplishments in both number theory and the theory of elasticity. It also mentions some of her philosophical ideas.

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INTRODUCTION

What is the life of a mathematician? Is it a collection of isolated theorems and proofs? A list of awards won and articles published? A series of amusing personal anecdotes? An ideal biography would include all of these. However, in the recent past the biographies of many female mathematicians have concentrated on a woman's struggles to succeed in such a male-dominated subject. While these travails are extremely important sociologically, a biography which concentrates on these social and political aspects of a female mathematician's life tends to completely ignore her actual mathematical work and achievements, or, at best, only gives them a cursory look.

I intend to remedy this situation in the case of one female mathematician. Sophie Germain was an important person in the history of mathematics, not just an important woman. Her work in number theory has been the foundation for the works of countless mathematicians. She was the first to make a bold step into the theory of elasticity, and inspired others to venture into this relatively unexplored realm. Without her work, the mathematical world would have suffered.

Even with this goal in mind, one cannot completely ignore her gender. The fact that she was a woman did greatly affect her life, as she was denied access to resources that could have allowed her

mathematical abilities to develop even farther. Her parents tried to discourage her from her studies, she was never given a tutor, and she could not attend an institute of higher learning. Even after her abilities were recognized by the academic world, her gender kept her in the position of an outsider, a lone genius who could not significantly interact with the people who should be her peers.

While her status as an outsider allowed her to achieve greatness in the area of number theory, it was extremely detrimental when she tried to work in the area of applied mathematics. The mathematical theory of elasticity was just beginning to develop when Germain began her work; as others became interested in the subject, they left Germain behind. She could not keep up with the latest developments because she did not have access to the ever-growing set of knowledge, or even the benefit of professional academic conversation.

From the time she was thirteen years old, Sophie Germain wanted to be a mathematician. Most biographies have focused on her womanhood. By studying her actual work as well as her struggles, we honor her in the way she probably would have wished: as a mathematician.

IN THE BEGINNING

On April 1, 1776, in a house on the Rue St. Denis in Paris, Marie-Sophie Germain was born. Her parents, Ambrose-François and Marie-Madeleine Germain, were moderately wealthy members of the bourgeois. Sophie Germain was financially supported by them throughout her life, as she did not marry and was unable to secure a professional academic position due to her sex. They were prosperous enough so that Sophie did not have to worry about a means of support, even during times of shifting politics. Ambrose-François was somewhat active in the political events that led up to the French Revolution, and served as a deputy to the States-General as a representative of the Third Estate. He also helped to transform that body into the Constituent Assembly.¹ In some of the speeches that he gave in this position, he fought against “the bankers and all the men who call themselves businessmen” and also stated that he “always professed publicly to regard speculation as a crime of the state”.² He also proposed a project concerning the Caisse d’Escompte,

¹Marilyn B. Ogilvie, Women in Science: Antiquity Through the Nineteenth Century: A Biographical Dictionary with Annotated Bibliography (Cambridge: MIT Press, 1986), p. 90.

² Both quotes from Hippolyte Stupuy, “Notice sur Sophie Germain” in Oeuvres philosophiques de Sophie Germain (Paris: Paul Ritti, 1879), p. 9. The first quote is “les banquiers et tous ces messieurs qu’on appelle faiseurs d’affaires,” from a speech given to the Constituent Assembly October 8, 1790. He had “toujours fait profession publique de regarder l’agiotage comme un crime d’Etat” from a speech May 5, 1791.

the national bank of France at this time.³ He describes himself as a merchant in one of his speeches, and it is known that he dealt in silk. Later in his life, he became one of the directors of the Bank of France.⁴

Marie-Madeleine came from a wealthy family, the Gruguelus. Little else is known of her. Also in the family was an older sister, named Marie-Madeleine after their mother. Her son Armand-Jacques Lherbette was Sophie Germain's literary executor. Germain also had a younger sister, Angélique-Ambroise. Angélique married René-Claude Geoffroy, a doctor, around 1816. The entire Germain family moved into the Geoffroy's town house after this, thus improving their living conditions from modest to grand.⁵

Sophie Germain was educated at home. Luckily, her father had an extensive library so she was able to read about subjects that normally she, as a female, would not have had access to. In 1789, when Germain was thirteen years old, she came to this library to find something to divert her mind from the Revolution going on practically outside her door. She began reading Histoire de Mathématiques by Jean-Étienne Montucla and came to the accomplishments of Archimedes. Montucla stated that Archimedes was so involved in mathematics that he "would forget food and

³Stupuy, p. 9.

⁴Charles Coulston Gillispie, editor, Dictionary of Scientific Biography (New York: Scribner, [1970-1980]), p. 375.

⁵Louis L. Bucciarelli and Nancy Dworsky, Sophie Germain: An Essay in the History of the Theory of Elasticity (Dordrecht, Holland: D. Dreidel Publishing Co., 1980), p. 9. The house still exists at 4 Rue du Braque in Paris.

drink. His servants would have to remember them for him and would almost have to force him to satisfy these human needs.”⁶ Even more dramatic is the account of his death. During the siege of Syracuse, Archimedes was so engrossed in a geometry problem he was working out in the sand that he failed to notice the approach of a Roman soldier. He was so absorbed that he did not answer the questions of the soldier, and was subsequently slain. Germain was fascinated at the idea that mathematics was so engaging that it could wipe away all other cares. At this moment, she decided upon the direction she wanted her life to take: she wanted to be a mathematician.

Germain’s family was by no means supportive of this decision. As Germain’s passion for mathematics grew, she devoured every book her father’s library had on the subject. Her parents grew concerned for her mental and physical health, as at the time it was common knowledge that girls who were too studious turned wild, and, as evidenced by the popular play Les Femmes Savantes by Molière, could not truly become intellectuals anyway. As Germain was from the bourgeois class, this studying seemed even more useless as she could not converse with educated aristocratic women in the salon circles. When she refused to stop her quest for knowledge, her parents kept her from studying during the day. During the night, in order to force her to sleep, they denied her heat and light for her bedroom and confiscated her clothes. Germain pretended to follow their authoritarian rulings, but after her parents were in bed for the

⁶Bucciarelli and Dworsky, p. 10. Translation from Montucla, Histoire des mathématiques, 2 vol. (Paris: C.A. Jombert, 1758).

night, she would wrap herself in quilts, light the candle stubs she had hidden away, and work at her books the entire night. One morning, they woke to find her asleep at her desk, her slate covered with calculations and her ink frozen in the ink horn. Upon this discovery, her parents decided to relent in their opposition, but without giving her any encouragement.

Although she was still alone and without a tutor, Germain was finally able to pursue her love in peace. She began with Étienne Bezout's standard mathematics text, Traité d'Arithmétique, a book commonly found on the bookshelves of educated people of the time, and then moved on to much more difficult material.⁷ After reading a text on differentiation by Jacques Antoine-Joseph Cousin, Le Calcul Différential, she fell in love with the relatively new science of calculus.⁸ She taught herself Latin in order to read the works of Isaac Newton and Leonhard Euler, as these were the next logical step in her study of this subject.⁹ In 1794, just as she was beginning to exhaust the resources of her father's library, the École centrale des travaux publics, later called the École polytechnique, opened in Paris. It seemed to be the perfect opportunity for a budding young mathematician (Germain was eighteen at this time), but the school, blindly following the dictates of social custom, did not admit females. However, Germain had become a serious student, and still had the

⁷Amy Dahan Dalmédico, "Sophie Germain" in Scientific American, vol. 265, n.6, December 1991, p. 117.

⁸Stupuy, p. 14

⁹She also taught herself Greek.

determination that had previously allowed her to conquer her parent's objections to her studies. She was able to obtain copies of the lecture notes of various courses, including a chemistry course taught by Antoine-François Fourcroy and, more importantly to her career in mathematics, a course in analysis taught by Joseph-Louis Lagrange.¹⁰ At the end of the course students were to hand in written observations to the professor. Germain wanted to submit a paper, but could not do so under her real name. There was a student at the school by the name of Antoine-August LeBlanc, who had grown up in Paris as well. It is unknown exactly how Germain was acquainted with him, especially as she was rather reclusive socially. Using his name as a pseudonym, she sent her paper on analysis to Lagrange. Lagrange was impressed by its originality and publicly praised the paper. He searched for its author, and found that his brilliant Monsieur LeBlanc was in reality a Mademoiselle Germain. He immediately became her sponsor and mathematical mentor, and provided support for years to come.

The discovery of such a mathematical talent in a young woman apparently was something of a sensation in the intellectual circles in Paris. Several superior scientists and mathematicians made Germain's acquaintance. Many of these exchanges were made through correspondence, doubtless because of the difficulty in arranging a socially proper meeting with a young, unmarried woman. Although these savants took a definite interest in Germain's talent and clearly wanted to aid in her education, their help was not a

¹⁰Stupuy, p. 19.

substitute for an ordered, more conventional education from a university. The problems that they discussed were interesting but somewhat random. A letter from Gaspard Monge discusses the equilibrium of a lever, where a finite weight located at a finite distance from the fulcrum can be moved by an infinitely small weight located at an infinite distance from the fulcrum. Others discussed mathematical paradoxes, which were interesting but somewhat isolated examples and did not lead to further study. Cousin requested a meeting with Germain and offered his resources for her use.

While the vast majority of intellectuals were supportive of Germain's talent, she had a minor feud with Joseph-Jérôme Lalande, the famous astronomer. He visited her in 1797, paying his respects much like any of the other savants who visited her during this time period. However, this meeting turned into an incident in which Lalande grossly underestimated her mathematical abilities and insulted her. According to Lalande's note of apology for the incident, they had been discussing Pierre-Simon Laplace's Système du Monde. Lalande suggested that she could not possibly understand this work without first reading his own book on astronomy. Astronomie des Dames, however, was a short text written for "the education of women," a greatly simplified course that was clearly far too primary for someone of Germain's standing in the academic world. She was greatly angered at this suggestion and took it as a professional insult. Despite his letter of apology, Germain never forgave Lalande for lumping her in the category of the intended audience for his book. Some time later, an invitation for a dinner from Alexander Tessier

shows that their dispute was widely known. Tessier encourages her to come by tempting her with a description of the delicious meal, a promise of safe transportation (a social necessity as Germain was an unmarried woman), and the fact that many people of importance would be there, except for Lalande as she had “not yet reconciled with him.”¹¹

Germain was well known not only in the scientific circles of the time, but to other intellectuals as well. In 1802, a Greek scholar, d’Ansse de Villoison, wrote two poems in which Germain was one of the main figures. However, Germain was not pleased with this poetic praise of her talents. At her request, Villoison destroyed the Greek poem, but the Latin one, a birthday poem for Lalande, was already in the process of being published. An English translation of the section involving Germain shows that the poem was quite flattering of her talents.

Ariadne, by whom skilled Germain’s visage is already envied,
Sees and dislikes what she sees, yielding her crown.
“What new Epigone enters the starry realm?”
She cries. “Most boldly she tries to enter
Our house, Gods stop her flight:
While you can, rein in this Icarian girl;
For her burning efforts will conquer giants.
This ambitious woman already wanders in LaPlace’s realm!
And drinks the airy fires with greedy gulps!”¹²

According to the letters of apology that Villoison wrote to her and her mother, this displeasure with the poetry was linked with

¹¹Letter from Alex.-H. Tessier, 1741-1837. “Point de M***, puisque vous ne vous êtes pas encore raccommodee avec lui.” Published in Oeuvres Philosophiques de Sophie Germain, p. 289.

¹²Bucciarelli and Dworsky, p. 15.

Germain's social timidity: the poems had "wounded [her] excessive modesty."¹³ It is also likely that she did not care to have her actions published in such a manner that would detract from the seriousness of her work. Germain probably resented having her name associated with Lalande as well, especially since the poem seems to refer to the reason for their discord. Whatever her reasons, they were not fully understood by Villoison, and he comments on this in a post-script to his letter of apology.

...if you are the only young woman who possesses such a superior knowledge of mathematics, you are also the only who has known and feared the danger of a Greek poem.¹⁴

In 1798, Adrien Marie Legendre published his work Essai sur la Théorie des Nombres. Germain studied it diligently, and began a correspondence with Legendre, submitting some of her own work that had stemmed from her observations of his text. Years later, he used some of these proofs in the second edition of his book and mentions her most famous theorem in his monograph "Sur le Théorème de Fermat."

She also studied Gauss' Disquisitiones Arithmeticae, published in 1802, and wrote her first letter to him November 21, 1804. She once again assumes the name M. LeBlanc, fearing that Gauss would not take her letter seriously if he knew she were a woman. She even goes so far as to have Gauss send his replies to a member of the First

¹³Letter from D'Ansse de Villoison, July 12, 1802. "...cette pièce qui a pu blesser l'excessive modestie de mademoiselle votre fille." O.P., p. 294.

¹⁴Same, in a letter dated July 14, 1802. "...si vous êtes la seule demoiselle qui possède si supérieurement les mathématiques, vous êtes aussi la seule qui ait connu et redouté le danger d'un poème grec." O.P., p. 295.

Class, M. Silvestre de Sacy (after her true identity is revealed, Gauss sends her letters to her father's address).¹⁵ In her letter, she demonstrates a generalization of one of his equations, and states that one of his methods could also be applied to a special case. She also discusses a proof of Fermat's Last Theorem for $n = p-1$, where p is a prime of the form $8k + 7$. She calls herself an "amateur enthusiast." Gauss responded enthusiastically to her letter and they began a rather extensive correspondence.

Gauss was extremely impressed with this young mathematician, so much so that he even praised her in letters to others. He wrote to Heinrich Wilhelm Matthias Olbers, a German physician and astronomer, on December 7, 1804, that

Recently I had the pleasure to receive a letter from LeBlanc, a young geometer in Paris, who made himself enthusiastically familiar with higher mathematics and showed how deeply he penetrated into my *Disqu. Arithm...*¹⁶

He also entrusted her with business matters. He discussed the difficulty of having one's work published, and when he had problems with a publisher in Paris who had not paid him any royalties, Germain researched the matter for him.¹⁷

¹⁵The First Class was the "official center for scientific exchange" in Paris at the time, according to Bucciarelli and Dworsky. It met weekly to hear papers, set up competitions for difficult problems, and review scientific progress throughout the world.

¹⁶W.K. Buhler, *Gauss: A Biographical Study* (New York: Springer-Verlag, 1981), p. 53. At this time, "geometer" simply means a pure mathematician.

¹⁷Letter from Gauss, June 16, 1806. According to Gray, the publisher had declared bankruptcy, so Gauss never received the money owed him.

Gauss did not discover the true identity of his correspondent until 1807. Napoleon's troops were in the process of invading Prussia, including the area near Gauss' home. Germain remembered the fate of Archimedes at the hands of Roman soldiers, and feared for Gauss' safety. She was acquainted with General Pernety, the man in charge of besieging Breslau, as he was a friend of her father.¹⁸ She asked him to send an emissary to Gauss' home to check that he was safe and healthy. General Pernety sent an officer named Chantel to his house, but upon encountering Gauss and his wife, a great deal of confusion ensued.¹⁹ Officer Chantel knew only that he was offering assistance in the name of a Mademoiselle Germain in Paris, but the only woman Gauss believed he knew in Paris was Madame Lalande. Gauss thanked the officer, but did not solve the mystery until Germain cleared up the situation herself. She writes a letter confessing that

...I am not as completely unknown to you as you might believe, but that fearing the ridicule attached to a female scientist, I have previously taken the name of M. LeBlanc in communicating to you...I hope that the information that I have today confided to you will not deprive me of the honor you have accorded me under a borrowed name...²⁰

Gauss was understandably surprised upon finding out that she was female, but there was no way he could devalue her mathematical abilities that he had already witnessed. He wrote to her, saying

¹⁸Marie-Louise Dubreil-Jacotin, "Figures de Mathématiciennes" in Les grands courants de la pensée mathématique (Paris: Albert Blanchard, 1962), p. 261.

¹⁹Gauss' wife was Johanna Elisabeth Rosina Ostoff.

²⁰Bucciarelli and Dworsky, pp. 24-25.

But how to describe to you my admiration and astonishment at seeing my esteemed correspondent M. LeBlanc metamorphose himself into this illustrious personage who gives such a brilliant example of what I would find difficult to believe. A taste for the abstract sciences in general and above all the mysteries of numbers is excessively rare...when a person of the sex which, according to our customs and prejudices, must encounter infinitely more difficulties than men to familiarize herself with these thorny researches, succeeds nevertheless in surmounting these obstacles and penetrating the most obscure parts of them, then without doubt she must have the noblest courage, quite extraordinary talents and a superior genius. Indeed nothing could prove to me in so flattering and less equivocal manner that the attractions of this science, which has enriched my life with so many joys, are not chimerical, as the predilection with which you have honored it.²¹

Clearly this letter goes beyond a courteous thank-you for her concern. He also read the mathematics she had enclosed in her previous letter and comments on a proposition that she had suggested.²² It becomes even more clear that this is not merely polite flattery when we read another letter he wrote to Olvers, on March 24, 1807.

Recently, I was greatly surprised on account of my *Disq. Arithm.* Did I not repeatedly write you of a correspondent in Paris, one M. LeBlanc, who had perfectly understood all my investigations? You will certainly be surprised as I was when you hear that LeBlanc is the assumed name of a young woman, Sophie Germain.²³

Unfortunately, quite soon after the discovery of her true identity, their correspondence came to a halt. Gauss had received a position

²¹E. T. Bell, Men of Mathematics (New York: Simon and Schuster, 1937), p. 262.

²²See my pages 55-57 for further discussion.

²³Buhler, p. 53.

as professor of astronomy in Göttingen, and thus had reached a professional position which allowed him to concentrate more fully on his work and to publish without much difficulty. He usually took several months to respond to her letters, and with his new position his time became even more limited. He essentially became too busy to take the time to correspond with Germain. His last letter to her is flattering and warm, but he does not discuss the mathematical proofs she had sent him. He writes

Pardon me that this time I have not extended myself any farther on the beautiful demonstration of my mathematical theorems. I admire the sagacity with which you have been able to arrive at them in so little time...Always be happy, my dear friend...and continue from time to time to renew the sweet assurance that I can count myself among the number of your friends, a title of which I will always be proud.²⁴

Germain continued to write to him, but he never again responded. They had at least one more instance of communication through Jean-Baptiste Delambre, Perpetual Secretary of the First Class. In 1810, Gauss was presented a medal worth 500 francs for his astronomical work. Gauss did not want to accept any money from France for political reasons, so instead wanted that the money be used to buy a pendulum clock for his wife. He requests that Delambre ask Germain

²⁴Letter from Gauss, January 19, 1808. "Vous me pardonnerez que cette fois je ne puis m'étendre davantage sur la belle démonstration de mes théorèmes arithmétiques. J'admire la sagacité avec laquelle vous avez pu en si peu de temps y parvenir...Soyez toujours aussi heureuse, ma chère amie, que vos rares qualités d'esprit et de coeur le méritent, et continuez de temps en temps de me renouveler la douce assurance que je puis me compte parmi le nombre de vos amis, titre duquel je serai toujours orgueilleux." *Q.P.*, pp. 320-321.

to choose it for him. Germain agreed, and the clock was used in Gauss' home until he died.²⁵

²⁵Bucciarelli and Dworsky states that "we cannot know" how Germain responded to the request, but Dunnington, a biographer of Gauss, puts this ending on the story.

ELASTICITY

In the fall of 1808, a German scholar named Ernst Florens Freidrich Chladni came to Paris to demonstrate a simple, beautiful, and astonishing experiment to the members of the First Class of the Institute. He would take a glass plate, sprinkle it with fine sand, hold it with two fingers on opposite sides of the plate, and draw a bow across the edge. If done correctly, a pure tone would be emitted. The sand would move to the sections of the plate that were not vibrating, the nodes, and form symmetrical shapes. The patterns could be preserved by transferring the sand to a wet sheet of paper. A pattern could be reproduced if the conditions were duplicated, and if the conditions were varied by changing the number of supports, the shape of the plate, or where and how hard the bow was drawn, different patterns would appear.

This phenomenon had never been seen before, and thus, naturally, there were no explanations for why certain parts of the plate moved and others remained at rest. The Emperor Napoleon, who was interested in advancing France's scientific achievements as well as its military ones, supported a system of prizes designed to promote new scholarship. Chladni performed the experiment for

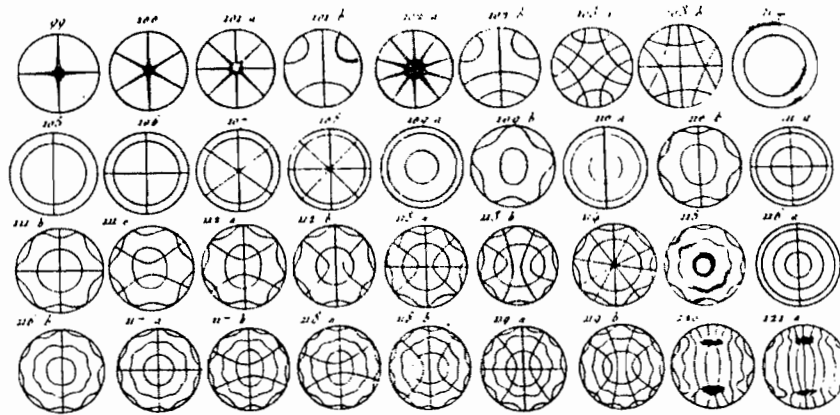


Figure 1. Diagrams of Chladni patterns produced on round plates.¹

Napoleon, and apparently he was rather impressed. These vibrations seemed to be an ideal subject for such a prize, and in April of 1809 the contest was announced:

His Majesty the Emperor and King...being struck by the impact that the discovery of a rigorous theory explaining all phenomena rendered sensible by these experiments would have on the progress of physics and analysis, desires that the Class make this the subject of a prize...The Class has thus proposed, for the subject of the prize, the development of a mathematical theory of the vibration of elastic surfaces, and a comparison of this theory with experiments.²

The deadline for entries was set for October 1, 1811, and if no entry was deemed acceptable, the prize would not be awarded.

The excitement of the experiments and the resulting *prix extraordinaire* caught the attention of Germain. Later, she writes

¹An illustration in the 1809 edition of Chladni's Traité d'Acoustique. From Dalmédico, p. 119.

²Bucciarelli and Dworsky, p.35. The announcement was published in the appendix to Chladni's Traité d'acoustique, pp. 353-357.

As soon as I learned of the first experiments of M. Chladni, it seemed to me that analysis could determine the laws by which they were governed. But I happened to learn from a great geometer [Lagrange]...that this question contained the difficulties that I had not suspected. I ceased to think about it.

During M. Chladni's visit to Paris, viewing his experiments excited my curiosity anew. I studied Euler's memoir on the linear case, certainly not with the intention to compete for the prize that the Institute had proposed, but only with the desire to appreciate the difficulties that the terms of the program brought to my mind.³

Many other mathematicians were put off by Lagrange's remark as well. While it is unknown what his exact comment was, it is conjectured that he pointed out that a solution to the plate problem would involve considering two spatial dimensions, a situation that analysis of the time did not encompass.⁴ It is likely that Germain, with her limited and spotty education, did not fully recognize the difficulties that would arise from this "new" calculus, so was not as daunted by Lagrange's warning as other mathematicians. However, at this time, she did not intend to find a solution but merely to understand the question. She corresponded extensively with Legendre, using his knowledge to help in her study of Euler's work in the one dimensional, linear case.

³Sophie Germain, Recherches sur la Théorie des Surfaces Élastiques (Paris: Mme. Ve. Courcier, 1826), p. v. "Aussitôt que les premières expériences de M. Chladni me furent connues, il me parut que l'analyse pouvait déterminer les lois auxquelles elles sont assujetties. Mais j'eus occasion d'apprendre d'un grand géomètre...que cette question contenait des difficultés que je n'avais pas même soupçonnées. Je cessai d'y penser. A l'époque du séjour à Paris de M. Chladni, la vue de ses expériences excita de nouveau ma curiosité. J'étudiai le Mémoire d'Euler sur le cas linéaire, non pas certainement dans l'intention de concourir au prix extraordinaire que l'Institut proposa alors, mais avec l'unique désir d'apprécier les difficultés dont les termes mêmes du programme me renouvelaient l'idée."

⁴Bucciarelli and Dworsky, p. 41.

Euler considered the forces acting on a horizontal beam that was experiencing small displacements in the vertical direction Q , but none in the horizontal P (see Figure 2). He was concerned with the motions that moved each point along the beam in a simple harmonic motion, and sought solutions that would describe the movement of the beam as a whole. These solutions were described by the following equation:

$$y(s,t) = \sin(\zeta + t\sqrt{2g/k}) \{ \alpha e^{s/f} + \beta e^{-s/f} + \gamma \sin(s/f) + \delta \cos(s/f) \}$$

where s is the position of the beam, t is the time, $\alpha, \beta, \gamma, \delta$ are parameters that fix the node shapes, and f is related to the frequency of vibration $\sqrt{2g/k}$, by

$$1/f^2 = \text{constant} \cdot \sqrt{2g/k}.$$

In order to find the possible values of f and establish the four parameters, one needs to know the conditions at the ends of the beam. The case in which Germain was most interested was the one in which both ends of the beam, E and F , are restrained by hinges to prohibit movement, and there is an additional hinge or stylus at some point along the beam which prohibits displacement but not rotation. If the beam is of length a , the hinge is located at a point $\delta \cdot a$ along the beam. Euler provided solutions for the case $\delta = 1/2$, when the stylus is at the midpoint of the beam, and Germain attempted to solve for any rational δ .

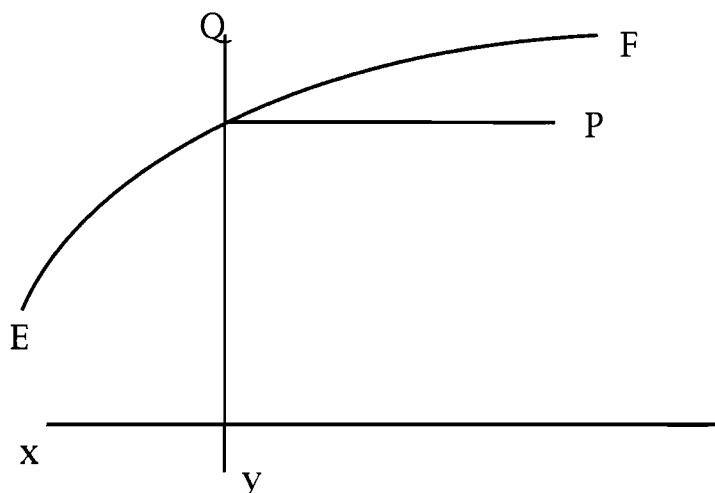


Figure 2. Beam ends are E and F; forces are P and Q.⁵

Germain corresponded with Legendre extensively about this problem. As she grew to understand it more and more, she came to believe that the plate problem could be solved by a method analogous to this special case. Euler suggested that a force of elasticity at a point along the beam is proportional to the curvature of the beam at that point. Germain suggested that for the plate problem, the force of elasticity is proportional to the sum of the major curvatures at that point. Now completely captivated by the problems of elasticity, she worked for the next eight months to complete a paper to submit to the contest.

Selected to be on the judging panel were Legendre, Laplace, Lagrange, Silvestre-François Lacroix, and Étienne Malus. All entries were to be secret. In order to identify each entry, the author was to write a quotation or saying on the first page of his or her memoir, and attach a sealed envelope that contained that quotation and the

⁵Adapted from an illustration in Bucciarelli and Dworsky, p. 42.

author's name. This envelope was only to be opened if the essay won the prize.⁶ However, Germain was the only person to submit an entry to this contest; all others had been frightened away by Lagrange's warning. Germain asked Legendre if her paper had been received; obviously he figured out that she was the author of the single entry. He writes

Your memoir is not lost; it is the only one that has been received concerning the problem of the vibration of surfaces...I have said nothing, I advise you, as well, to keep silent until a definite judgment is made.⁷

Clearly the rules governing secrecy were taken with a grain of salt. While Legendre did not publicize his knowledge, he did not disqualify himself as a judge either. It is extremely unlikely that he believed that her memoir was useless and thus his knowledge did no harm, as his letters evidence that he spent a considerable amount of time helping her in her endeavors. This reveals instead a degree of friendship between the two that went beyond a purely professional, intellectual relationship. Also, this bending of the rules may have been common; it is only because of Germain's status as a woman that this letter exists. Men could see each other easily and make such a communication orally, leaving no incriminating evidence behind.

⁶Germain's first entry used the following quotation from Newton as identification: "Effectuum naturalium ejusdem generis eadem sunt causae."

⁷Letter from Legendre, October 22, 1811. "Votre mémoire n'est pas perdu; il est le seul qu'on ait reçu sur la question des vibrations des surfaces...Je n'ai rien dit, je vous conseille également de garder le silence jusqu'au jugement définitif." Q.P., p. 334.

Her memoir did not win the prize. She used as her basis an equation that was analogous to Euler's equation for the vibrating beam:

$$\int dzdy \int Pds + \int dzdx \int Qds - 2 \int dx dy \int Rds = V((1/r) + (1/r')).$$

This was to represent equilibrium for a point (x, y, z) of the plate, where P , Q , and R are the external forces acting in the x , y , and z direction respectively. On the right hand side, the constant V refers to the elasticity of the material of the plate. As r and r' are the two principal radii of curvature of the deformed plate, the term $(1/r) + (1/r')$ represents the mean curvature. This mean curvature of the surface was proportional to the movement of the plate. Using the above equation, she differentiated four times with respect to x and y and, assuming that in time the behavior of the plate would be harmonic so that the equation is not dependent on time, came up with the following:

$$z(x,y) = \frac{f^6}{2} \left(\frac{d^6 z}{dx^4 dy^2} + \frac{d^6 z}{dy^4 dx^2} \right)$$

Unfortunately, this derivation is wrong. Germain's lack of expertise in the realm of analysis caused her to commit some computational errors. Legendre writes:

Your principal equation is not correct, even assuming the hypothesis that the elasticity at each point can be represented by $(1/r) + (1/r')$...Your error seems to arise from the manner with which you tried to deduce the equation of a vibrating surface from the equation of a simple lamina; you became confused with the double integrals.⁸

⁸Letter from Legendre, December 4, 1811. "...votre équation principale n'est pas exacte, même admettant l'hypothèse que l'élasticité en chaque point peut être représenté par $(1/r) + (1/r')$...La source de votre erreur paraît être dans la manière dont vous avez cru pouvoir déduire l'équation de la surface vibrante

However faulty her basic mathematical skills, the judges did not immediately dismiss her entry. Lagrange took her hypothesis and applied the variational method from his own Mécanique Analytique, a book Germain had not mastered. Using these derivations, and assuming that z , the amplitude of vibration, is small, he found the equation

$$\frac{d^2z}{dt^2} + k^2 \left(\frac{d^4z}{dx^4} + 2 \frac{d^4z}{dx^2dy^2} + \frac{d^4z}{dy^4} \right) = 0.$$

where k is a constant, t is time, and x and y represent points on the surface. This equation, which Legendre reported to Germain in a letter, is correct. Bucciarelli and Dworsky claim it is the same equation used today as the basis for analyzing elastic plates after it is supplemented with the appropriate boundary conditions, but Skudrzyk tempers this statement by referring to it as part of “classic plate theory,” pointing out that it is adequate only for lower frequencies, when the wave length is greater than five times the thickness of the plate.^{9,10}

As there had been only one entry for the contest, and it was not judged to be sufficient enough to win the prize, the Class decided to extend the contest. New entries would be received until October of 1813, so Germain had almost two years to improve her work. She believed in her hypothesis, but needed to exhibit the correct

de l'équation d'une simple lame; c'est dans les doubles intégrales que vous vous êtes égarée.” Q.P., p. 337.

⁹Bucciarelli and Dworsky, p. 55.

¹⁰Eugen Skudrzyk, Simple and Complex Vibratory Systems (USA: Pennsylvania State University Press, 1968), pp. 488, 500. There does exist a more complex and precise modern theory that will approximate the behavior of a plate for higher frequencies.

derivation for the equation found by Lagrange, and be more exact in her justification of this equation based upon physical evidence. Germain did not fully understand the method by which Lagrange had derived this equation from her hypothesis, and, judging from her later entries and lack of correspondence, did not receive any help from Lagrange himself.

Even though Lagrange died in April of 1813, so he was not even to be one of the judges of her new work, Germain began to fear that his opinion about the difficulty of the problem would affect the other judges. She thought that perhaps the judges would not recognize the correct equation since they doubted that one could be found. She states in a letter written October, 1813 that

Without doubt, the problem has been abandoned only because this grand geometer judged it difficult. Possibly this same prejudgment will mean a condemnation of my work without a reflective examination...[T]he notion that the problem is difficult...might prevent one from devoting any effort to the examination of a memoir condemned once before...¹¹

She submitted her memoir on September 21, 1813. Once again it was the only submission. The judges this time were Laplace, Lacroix, Legendre, Lazare Carnot, and Siméon-Denis Poisson, who had just recently been elected to the First Class.

In the first section of her hundred-page memoir, she attempts to derive the plate equation stated above. Her starting point was as in her first paper, and she eventually obtained the correct equation by the end, but the method of her analysis in between these two

¹¹Germain, letter to unknown recipient, October 1813. From Bucciarelli and Dworsky, p. 61.

points was full of errors. After this, she then worked on establishing appropriate boundary conditions in order to obtain more particular solutions. These were based upon Euler's examination of the vibrating beam, and were applicable to the plate problem.¹² Finally, she compared Chladni's results with the predictions of her equations. Her results predicted the nodal lines and frequency ratios of these lines in both square and rectangular plates. The judges, while understandably not satisfied by her derivation of the equation, were impressed with this correspondence between mathematical theory and real-life experimentation. Despite the mistakes in her analysis, the equation was recognized as correct and her memoir was rewarded with an honorable mention. However, as there was still no correct derivation for the equation, the contest was again renewed, with the new entry deadline being October 1815.

It is at this point that academic politics began to rear its ugly head. On August 1, 1814, during a session of the First Class, Poisson began presenting a memoir on the subject of vibrating elastic surfaces. This was highly improper. Not only were members of the First Class not to compete for prizes that they themselves had established, but Poisson was actually a judge for a prize of this same subject! Legendre interrupted the reading to object, and a committee was formed to discuss the matter, but this committee was never again mentioned and Poisson was allowed to continue his reading.

¹²Bucciarelli and Dworsky, p. 63.

Not surprisingly, the equation Poisson derives in his paper is exactly the same as the equation Germain had derived in her own memoir. When this equation was first presented by Lagrange, it was merely the outcome of her hypothesis using the correct mathematics. It was of little merit until Germain had demonstrated in her second memoir that it corresponded to Chladni's experiments in several cases. Germain's work had not been made public as it had not won the prize but only received an honorable mention; Poisson used his privileged position as a judge to obtain this equation. As he believed that the equation was correct, he set about deriving the equation rigorously, a feat Germain had not accomplished.

This paper presented by Poisson also revealed another aspect of academic politics: the paradigm. The scientific elite of this time believed wholeheartedly in the corpuscular hypothesis, or "molecular mentality." This belief system thought of matter not as a continuum, but as a collection of discrete particles. A theory of elasticity should, therefore, deal with the displacement of these particles and the forces in between them. This is quite different than the modern view of elasticity, which involves a continuous piece of matter through which stress and strain are distributed. In order to explain how elasticity fits into this conceptualization, Laplace writes in 1809 that

In order to determine the equilibrium and motion of a naturally straight, elastic lamina, bent along some arbitrary curve, it has been assumed that at each point its spring is inversely proportional to the radius of curvature. But this law

is only secondary and derives from the attractive and repulsive action between molecules.¹³

Laplace was Poisson's main patron. Poisson based his work in elasticity upon this conceptual scheme, believing that he was working with basic truths and not a particular view of the world.

Poisson's analysis is based on an abstract plane and the relationship of molecules to one another in that plane. When the surface was stretched or bent in some way, the distances between the molecules would change, and these changes produced a force which would return the surface to its original shape. His analysis created the following equation:

$$n^2 \epsilon^2 \left(\frac{1+q^2}{k} \frac{d^2P}{dx^2} + \frac{1+p^2}{k} \frac{d^2P}{dy^2} - p \frac{PdP}{dx} - q \frac{PdP}{dy} + \frac{kP}{2} (P^2 - 4Q) \right) \\ = Z - pX - qY - kP\pi.$$

Here, $P = (1/r) + (1/r')$ and $Q = 1/rr'$, functions of the principal radii of curvature. If Poisson had not known the equation he was setting out to prove, there is no apparent way he could have arrived the correct equation through the process of linearization that he used. Bucciarelli and Dworsky state that this is a "frightening equation, fraught with nonlinearities."¹⁴ Basically, Poisson took the correct equation, made it conform to the hypothesis he believed to be right, and worked backwards so everything seemed to be correct.

While the actual equation involves the bending of the plate, this molecular model does not take into account the redistribution of molecules during the bending process, the fact that the molecules

¹³Bucciarelli and Dworsky, p. 71.

¹⁴Bucciarelli and Dworsky, p. 74.

would stretch apart at the outer surface of the curve and come closer together at the inner surface. Also, it would be close to impossible to find boundary conditions using this model. In his memoir, Poisson postponed determining the boundary conditions. Germain later writes that

It seems to me...that in admitting the existence of repulsive forces [between molecules] one will be led to suppose that the plane is infinite. Moreover, the able geometer [Poisson], against whose principles I regretfully combat, has not concealed the difficulties that are presented in the study of the conditions of the extremes...I have waited a long time for the author to publish the determination of the question here; I had desired, in the interest of the question, that he develop all the consequences of the hypothesis that he had adopted.¹⁵

Germain senses that the problem lies in Poisson's hypothesis, but as he does not discuss the matter further, she cannot use this to prove his work definitively wrong. She is only able to assert that her hypothesis may be the better of the two as hers does not present such a difficulty.

After the presentation of this memoir to the First Class, Poisson had it published in the Bulletin des Sciences, par la Société Philomatique de Paris, a journal for which he worked as the mathematical editor. An extract was also published in Correspondance l'École Polytechnique under the guise of being "very

¹⁵Germain, Recherches, pp. 9-10. "Il m'a paru...qu'en admettant l'existence des forces répulsives, on serait mené à supposer infinie la surface plane. Au reste, l'habile géometre dont je combats à regret les principes, n'a pas dissimulé les difficultés que présenterait la recherche des conditions des extrémités...J'ai long-temps attendu que l'auteur publiât la détermination dont il s'agit ici; j'aurais désiré, dans l'intérêt de la question, qu'il développât lui-même toutes les conséquences de l'hypothèse qu'il a adoptée."

useful to those young geometers who compete for the prize.”¹⁶ However, Poisson’s derivation was completely adequate for his fellow molecular thinkers. There were no “young geometers” working on the prize other than Germain, and she was thirty-nine at this time. Even if she were to enter, Poisson would most likely be one of the judges, and as her hypothesis was entirely different from his, it would be questionable whether her paper would be given the attention it deserved.

However, there was still the question of propriety. Poisson should not have presented his solution to a contest that he was ineligible to enter, and he probably should not have used parts of Germain’s work without a thorough acknowledgment of her efforts. It is postulated that an oral agreement was reached between Germain, Legendre (who had objected to Poisson’s first reading), Poisson, and any other relevant party, that the contest would be continued and the prize would be awarded to Germain if her memoir was at all worthy.¹⁷ Germain would then be able to finish her memoir without the fear of hostile judgment. Whatever the case, we know the following: the prize was continued, Germain submitted the only entry, and she was awarded the prize.

In this third paper, Germain looks briefly at Poisson’s work. The fact that they both presented the same equation apparently increased her belief in her own work. She writes

¹⁶Bucciarelli and Dworsky, p. 75.

¹⁷Bucciarelli, p. 79.

I may have entirely renounced [my] research...if I hadn't learned...that the equation obtained by a different hypothesis than the one I had proposed gave the same result as mine. In effect, I see each day some new reasons to regard my hypothesis as incontestable...¹⁸

Lagrange had derived the equation from her hypothesis, and now the equation was generally accepted as being correct. This seemed to validate the hypothesis from which the equation originally came.

She also attempted to extend her work. Rather than limit her equation to planar surfaces, she studied initially curved surfaces as well. In order to do this, she needed to remove some of the ambiguity from concepts in her basic hypothesis, a task she needed to take on anyway. Previously, she had considered the geometry of four points on the surface in order to justify her hypothesis that elastic force is proportional to the change in curvature, a definition that was unsatisfactory. This time, she stated that the elastic force is proportional to the difference between the undeformed and deformed surface. Since shape is determined by curvature, elastic force is proportional to difference in curvature. In the case of the beam, this curvature can only be represented by one radius of curvature, but in the case of a surface there are many possible choices of curvatures through any given point. In order to express this curvature in a specific, concrete form, she associated the curvature with two particular perpendicular planes, one containing

¹⁸Germain, third entry. "J'aurais même entièrement renoncé aux recherches...si je n'avais appris,...que l'équation obtenue dans une hypothèse différente de celle que j'avais proposée, résulterais également de cette dernière hypothèse. En effet je voyais chaque jour de nouvelles raisons de regarder mon hypothèse comme incontestable..." From Bucciarelli and Dworsky, p. 79.

the maximum curvature, the other containing the minimum. On each of these planes, the curve can be approximated by a circle tangent to the curve. The curvature is then equal to the sum of the inverses of the radii of the circles. Once these two curvatures are known, all other curves can be obtained from them.

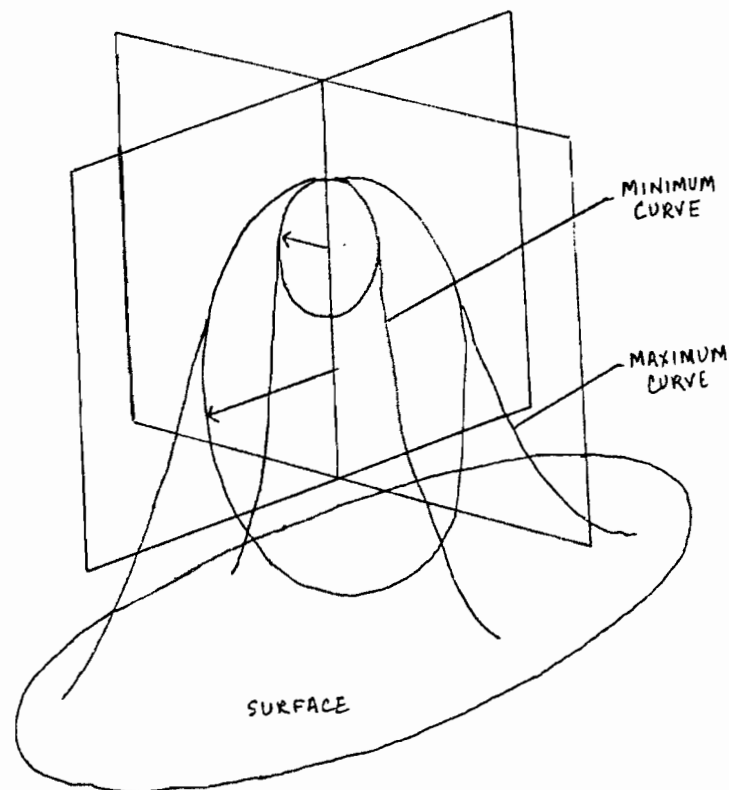


Figure 3. Approximating the curve of a surface.¹⁹

By setting the two principal curvatures of the undeformed, “natural” surface equal to R and R' , and the two of the deformed, “elastic”

¹⁹Based on an illustration in Dalmédico, p. 120.

surface equal to r and r' , the elastic force is proportional to the following:

$$\left(\frac{1}{r} + \frac{1}{r'}\right) - \left(\frac{1}{R} + \frac{1}{R'}\right)$$

the difference between the curvature of the elastic surface and that of the natural surface.²⁰

She also describes some experiments she had been trying with these initially curved surfaces to show that her generalized equation was as accurate as the plate equation. Just as Chladni had made the node patterns visible on flat plates, she wanted to exhibit the patterns of cylindrical surfaces. Chladni's technique did not translate well to curved surfaces, and the results of these experiments were only partially successful.

It is on the basis of these new experiments that she was awarded the prize, as her demonstration of the equation was still incorrect. Her demonstration still suffered from a lack of rigor, but she fulfilled the second part of the contest, which was to show that her equation could predict the nodal lines, and her new work on cylindrical surfaces was impressive if not entirely successful. The judges this time were Poisson, Laplace, Legendre, Louis Poincot, and Jean-Baptiste Biot. The announcement of the winner included the fact that

The differential equation given by the author is correct although it has not resulted from the demonstration. Yet the manner in which the particular integrals satisfying it have been discussed, the comparison made with the results observed by M. Chladni and finally the new experiments attempted on

²⁰“Natural” and “elastic” are the terms Germain uses in her work.

plane and curved surfaces in order to test the indications of the analysis appear to merit the award of the prize...²¹

Germain was the first woman ever to have won an award of this importance from the Institute. There was a good deal of public interest in this, mostly due to the novelty of a successful female scientist, but Germain did not attend the prize ceremony.

As the announcement of the prize stated that there were still some problems with the memoir, Germain desired to know what the difficulties were. She did not believe that her mistake was in the way in which the equation was deduced from the hypothesis, where indeed it was, but rather that the judges did not believe her hypothesis to be sufficiently justified. She wrote a letter to Poisson after the prize had been awarded in an attempt to engage him in a dialogue on this subject. In this letter, she gives the chain of reasoning that led to her hypothesis; it explains why she uses the sum of the principal curvatures in her work and gives a valuable insight into how she came to construct her theory.

Whatever the nature of the forces considered, they are proportional to the effect they produce or tend to produce.

The forces of elasticity tend to destroy the differences between the natural shape of the bodies endowed with this force and the shape that those same bodies are forced to take by an external cause.

The forces of elasticity acting in any elastic body are therefore measured by the difference in the natural shape of the body and the shape that an external force would cause it to take.

The effect produced by a force is implicitly or explicitly the sum of the effects produced by that same force: explicitly,

²¹Bucciarelli and Dworsky, p. 82. From a transcript of the Public Session of the Institut de France First Class, January 8, 1816.

if one successively considers all the diverse effects without taking into account their interdependence; implicitly, if the connection existing between these same effects permits them to be considered as a single thing.

The effect of the forces of elasticity that act on a surface is to destroy the difference between the natural curvature of the surface and the curvature that the same surface is forced to take through the action of an exterior cause. But the question of curvature of a surface cannot be answered simply: it is composed of the group of questions relative to the curvature of curves resulting from sectioning the same surface in all directions and under every possible inclination.

The sum of the differences between the curvatures of the curves formed by the various sections of the surface, considered before and after the action of the exterior force, is therefore explicitly the measure of the forces of elasticity acting on this surface.

There exists between the curvatures of the curves formed by the various sections of the surface a relationship such that it is permissible to express their sum by that of the principal sections only.

The effect of the forces of elasticity is then implicitly expressed by the sum of the differences between the principal curvatures of the surface, considered before and after the action of the external cause.²²

²²Germain, letter to Poisson, January, 1816. "Quelles que soient les forces que l'on considère, elles sont proportionnelles à l'effet qu'elles produisent ou tendent à produire. Les forces d'élasticité tendent à détruire la différence entre la forme naturelle des corps qui en sont douées et la forme que les mêmes corps ont été forcés de prendre par l'action d'une cause extérieure. Les forces d'élasticité qui agissent sur un corps élastique quelconque, ont donc pour mesure la différence, entre la forme naturelle de ce corps et la forme qu'une cause extérieure la force de prendre. L'effet produit par une force est explicitement ou implicitement l'ensemble des effets produits par le même forcé. Explicitement si on considère successivement tous les divers effets sans exprimer qu'ils dépendent les uns des autres; implicitement, si la liaison qui existe entre les mêmes effets permet de les considérer comme un fait unique. L'effet des forces d'élasticité qui agissent sur une surface est de détruire la différence entre la courbure naturelle de la surface et la courbure que la même surface a été forcée de prendre par l'action d'une cause extérieure. Mais la question sur la courbure d'une surface n'est pas susceptible d'une réponse simple; elle se compose de l'ensemble des questions relatives à la courbure des courbes résultantes de sections de la même surface faites dans toutes les directions et sous toutes les inclinations possibles. L'ensemble des différences entre les courbures des courbes résultantes des diverses sections de la surface, considérées avant et après l'action de la cause extérieure, est

There is no fault in this line of reasoning, as Poisson admits politely but somewhat grudgingly in his reply. He does not answer her questions, but instead simply sends a brief note:

The reproach the commission made concerns not so much the hypothesis as the manner in which you applied the calculus to the hypothesis. The result to which these calculations have led you do not agree with mine except in the single case wherein the surface extends itself infinitely little from a plane, be it in a state of equilibrium or of movement. My memoir will be printed shortly and I am considering offering you a copy, as soon as the printing is finished.

Permit then, Mademoiselle, that we adjourn this discussion until the time when you will have been able to compare my results with yours.²³

It is clear that he does not feel her work merits a professional discussion. There is no evidence that he ever discussed the problem further with her. Five years later, in another memoir, she again makes an effort to engage Poisson in some sort of discussion:

One can easily understand with what repugnance I had decided to contradict the principles of an author whose talents inspire in me the highest esteem. If he does not disdain to respond to

donc explicitement la mesure des forces d'élasticité, qui agissent sur cette surface. Il existe entre les courbures des courbes formées par les diverses sections de la surface une liaison telle qu'il est permis d'exprimer leur somme par celle des seules sections principales. L'effet des forces d'élasticité est donc implicitement exprimé par la somme des seules différences entre les courbures principales de la surface, considérées avant et après l'action de la cause extérieure." *Q.P.*, pp. 344-346.

²³Letter from Poisson, January 15, 1816. "Le reproche que la commission lui a fait porte moins sur l'hypothèse dont vous êtes partie que sur la manière dont vous avez appliqué le calcul à cette hypothèse. Le résultat auquel ce calcul vous a conduit ne s'accorde avec le mien que dans le seul cas où la surface s'écarte infiniment peu d'un plan, soit dans l'état d'équilibre, soit dans l'état de mouvement. On imprime succinctement mon mémoire, et je me propose de vous en offrir un exemplaire, aussitôt que l'impression sera achevée. Permettez donc, mademoiselle, que nous ajournions la discussion à l'époque où vous aurez pu comparer mes résultats aux vôtres." *Q.P.*, pp. 347-348.

my objections, I will be eager to retract the errors that he points out.²⁴

Despite her flattery and continued desire to discuss their mutual theories, she never received the requested criticism.

Up to this time, none of Germain's work in elasticity had been published. Legendre had suggested that she publish her second memoir, and after her third memoir had won the prize she again considered this option. Without publication of her theory, the only view that would be preserved is that of Poisson. While his work still had the stamp of authority, Germain did not believe that hers was without true merit. She writes

...there still remain, between the doctrine of this savant author [Poisson] and my own, some differences too essential for me to not need to refer the choice to the mathematicians.

I tried in vain to renounce the hypothesis that I had adopted; it resisted all of the objections with which I attempted to fight it.²⁵

She does not believe in Poisson's hypothesis, but she must defer to his authority in this field. She is continually aware of this fact in her writing. If the other hypothesis had come from an obscure author, she says

...I would have limited myself to expose the question as I conceived of it. Far from this, the geometer with whom I have

²⁴Germain, *Recherches*, p. x. "On concevra aisément avec quelle répugnance j'ai dû me décider à contredire les principes d'un auteur dont les talens m'inspirent la plus haute estime. S'il ne dédaigne pas de répondre à mes objections, je m'empresserai de rétracter les erreurs qu'il aura signalées."

²⁵Germain, *Recherches*, p. ix. "...il reste encore, entre la doctrine de ce savant auteur et la mienne, des différences trop essentielles pour que je ne croie pas devoir en déférer le choix aux géomètres. Je tentais vainement de renoncer à l'hypothèse que j'avais adoptée; elle résistait à toutes les objections pas lesquelles j'essayais de la combattre."

the misfortune to be in disagreement has a right to this confidence, as the authority attached to his name halts my own judgment. I believe I would have hidden from the reader the strongest objection one could make against my hypothesis if I did not confess that it differed entirely from that of the savant author.²⁶

Indeed, her hypothesis was better than Poisson's by any modern standards and the principles behind her work were very solid; it was only her troubles with analysis and the variational technique of Lagrange that kept her from true genius. Unfortunately, no one was willing to give her instruction in these areas, or even tell her the severity of her problems. As a result, her memoir is filled with mathematical errors.

Germain published her work, Recherches sur la Théorie des Surfaces Elastiques, in July 1821 at her own expense. It was not endorsed by the Academy, but she sent copies to several members, and it was added to the library of the Academy. She received letters of congratulations and praise from Legendre, Delambre, Augustin-Louis Cauchy, and Claude Navier.

The memoir begins with a statement of her basic hypothesis, that the force of elasticity on a point on an initially curved surface is proportional to $\left(\frac{1}{r} + \frac{1}{r'}\right) - \left(\frac{1}{R} + \frac{1}{R'}\right)$. She states that Poisson's memoir (apparently he did actually send her a copy as promised) uses the quantity $\frac{1}{r} - \frac{1}{r'}$ as proportional to the force, rather than adding

²⁶Germain, Recherches, p. ix. "S'il s'agissait d'un auteur obscur, je me bornerais à exposer la question telle que je la conçois. Loin de là, le géomètre dont j'ai le malheur de ne pas partager l'opinion, a un tel droit propre jugement. Je croirais donc avoir caché au lecteur la plus forte objection que l'on puisse faire contre mon hypothèse, si je ne lui avouais pas qu'elle diffère entièrement de celle de ce savant auteur."

these two numbers as Germain does. She argues that this essentially makes no difference, as the two are proportional. This result is correct, but her general reasoning is not an adequate explanation.²⁷

She then discusses the problem of boundary conditions, asserting that her hypothesis is superior to Poisson's because it avoids any difficulty with this particular subject. She has a concrete example of why her hypothesis works better. The problem is as follows: take an elastic plate such as Chladni used, perform the basic experiment, then remove the portion of the plate between one of the nodal lines and the nearest edge. Replace this section with a non-elastic material of the same weight. Perform the experiment again, and the tone emitted and the nodal lines are the same as before the substitution took place. The only difference is that the intensity of the sound diminishes and the nodal lines become somewhat wider. Germain says that this is easy to explain under her hypothesis:

...for as each of the material points that compose the plaque are endowed with a force of their own, in virtue of which they tend to resume their natural situation, it suffices that these points remain submitted to the same exterior conditions in order for them to continue to move in the same manner. The relative position of the material points is conserved...²⁸

Each point on the elastic portion of the plaque moves the same way as before; there is no "molecular force" which would cause a change

²⁷Isaac Todhunter and Karl Pearson, A History of the Theory of Elasticity and of the Strength of Materials (Cambridge, England: Cambridge University Press, 1886), p. 150.

²⁸Germain, Recherches, p. 11. "...car chacun des points matériels qui composent la plaque, étant doué d'une force *propre*, en vertu de laquelle ils tendent à reprendre leur situation naturelle, il suffit que ces points restent soumis aux mêmes conditions extérieures, pour qu'ils continuent à se mouvoir de la même manière. La position relative des points matériels conserved.."

when a portion of the plaque is removed. The differences in intensity of tone and width of lines are due to the weakening of the movement of vibration.

Germain also spends some time discussing the problem of an elastic ring. The equation here is an extension of her equation for the vibrating plate. In her attempt to integrate it and make some numerical deductions, her lack of training undermines her work. On page 37 of her 96-page memoir, she makes a mistake in determining the constants in one of her equations; Todhunter says

It is not too much to say that the whole rest of the work is ruined by these mistakes, as almost every formula will have to be corrected...The lady does not appear to have paid that attention to the Calculus of Variations which might have been expected from the pupil and friend of its great inventor Lagrange.²⁹

This rather harsh comment exemplifies why Germain has not yet received the respect she deserves. She was as much a pupil of Lagrange as a pupil of Euler, who was dead by the time she was four. She taught herself from books and correspondence. Neither Lagrange nor anyone else ever filled the role of teacher for her; she was left to struggle on her own with no coach, only sideline cheers. Her lack of a solid background in analysis caused her to commit errors that allow others to disregard her work. Granted, there is no denying that these errors exist, but the reason for these errors is usually either misunderstood or ignored.

After her published paper, Germain did not give up on the subject of elasticity. She attempted to extend her research, and

²⁹Todhunter and Pearson, p. 156.

submitted a paper to the Academy in 1824. Once again, her work was filled with errors, and the Academy basically ignored it. The commission set up to read it included Poisson, Laplace, and Gaspard Riche de Prony. They did not report its errors to her, or give her any sort of critique. Poisson read it, then gave it to Prony, who did not even bother to return her paper. Mémoire sur l'emploi de l'épaisseur dans la théorie des surfaces élastiques was discovered in his estate after his death and was published in 1880.³⁰ In 1826 she submitted yet another memoir to the Academy. However, she published it first, then sent it for them to review.³¹ Cauchy was designated to go over her work and make a verbal report. There is no evidence of what he said in this report, or even if he gave her a critique. It is likely that they refrained from giving her the criticism she needed simply because she was a woman and not a “professional” mathematician; they probably felt that they were being polite.

In any case, Germain was definitely out of touch with the subject of elasticity by this time. The mathematical community had become interested in this subject, and Germain did not have access to others' memoirs, sessions at the Academy, or even regular professional conversation. Her isolation as much as her lack of a solid background in analysis kept her from achieving anything more in this area. The quest for a theory of elasticity was begun by

³⁰This is one year after the publication of Oeuvres Philosophiques, which contained Germain's philosophical works and Stupuy's biography of Germain.

³¹This was 21 pages in length and was published under the title Remarques sur la nature, les bornes et l'étendue de la question des surfaces élastiques, et l'équation générale de ces surfaces.

Germain, but as soon as her work was shown to be at least somewhat successful, the men of science latched onto it and pushed her away.

Poisson was not the only other party interested in elasticity. Joseph Fourier wrote a short essay on a solution technique for the plate equation in 1818. Navier was more intrigued by the subject; he desired to study the practical problem of a floor slab, supported at the edges and loaded with weight. In order to do this, he needed to establish boundary conditions as clearly a floor would not be infinitely large. His process, using the methods of Lagrange's Analytic Méchanique, succeeded in deriving Lagrange's plate equation and a set of boundary conditions. His grasp of analysis allowed him to succeed where Germain had failed. He presented this memoir to the Academy in August, 1820, and yet another memoir nine months later. Cauchy, one of the readers for his memoir, was interested in the subject too, and delayed the review of Navier's memoir so he could pursue his own investigations on the subject. He wrote his own article on the subject, presented it to the Academy in 1822, and had an abstract published in 1823. In this work, he made a definite advance towards the modern view of elasticity, defining stress, strain, and deriving equations in relation to these two concepts.³² Poisson wrote a massive work in 1828, still working within the molecular model. In the abstract published in the Annales de Chimie, he basically cites only ancient achievements in his historical introduction and ignores all recent work other than his own. His failure to mention Navier's work prompted a string of

³²Bucciarelli and Dworsky, p. 102.

angry correspondence between the two, all published in the journal.³³

Unlike Poisson, who obviously had a problem with proper acknowledgments, others mentioned and even praised Germain's work in their writings. In the abstract of his memoir in 1820, Navier writes:

The research that was awarded the prize was founded on an ingenious hypothesis, namely, that flexure gave birth, at each point of an elastic plate, to a force proportional to the sum of the inverse values of the two radii of principle curvature. Mademoiselle Germain gave the differential equation of equilibrium and movement of an elastic plane and some integrals of these equations...³⁴

Even though Germain's work is recognized in the introductions to these writings, today it is rare to find her work mentioned in textbooks. Often they cite only Poisson, Navier, and Cauchy as being responsible for the emerging theory of elasticity at this time. But if Germain had not blazed the trail, it is unlikely that their works would have developed at the time they did, if at all.

³³These are contained in Annales de Chimie, 1828-1829, vol 37-39.

³⁴Bucciarelli and Dworsky, p. 104.

NUMBER THEORY

Although her work in elasticity was not altogether successful, Germain's work in number theory was and still is very important. Her proofs were admired by Gauss, and Legendre published some of her results in his book Théorie des nombres. Her most famous theorem, which is called "Sophie Germain's Theorem," is included in many number theory textbooks published today. Germain's achievements in this area are not spoiled by a lack of accuracy; thus we can discuss them in a more rigorous, mathematical tone.

Notation

Is Read As

$a \mid b$

a divides b

$a \nmid b$

a does not divide b

$a \equiv b \pmod{m}$

a is congruent to b modulo m

$a \not\equiv b \pmod{m}$

a is not congruent to b modulo m

$\gcd(a, b) = m$

the greatest common denominator of a and b is m

$n!$

n factorial

Congruence

Congruence is a commonly used concept in basic number theory. It was introduced in the early nineteenth century by Gauss as a new

language to use when dealing with integers. Thus we begin with his definition as given on the first page of Disquisitiones Arithmeticae.

If a number a divides the difference of the numbers b and c , b and c are said to be congruent relative to a ; if not, b and c are noncongruent. The number a is called the modulus. If the numbers b and c are congruent, each of them is called a residue of the other. If they are noncongruent they are called nonresidues.¹

In order to fully understand this concept, it is necessary to first have a precise definition of divisibility.

Let a and b be integers, with $a \neq 0$. Then a divides b if there is an integer c such that $b = ac$. If a divides b , then we write $a \mid b$.

The symbol “ \equiv ” is read as “is congruent to.” Using this notation, we give another definition of congruence.

Let m , a , and b be integers with $m > 1$. Then $a \equiv b \pmod{m}$ if $m \mid (a-b)$.

For example, $7 \equiv 2 \pmod{5}$, as $5 \mid (7 - 2)$. If a is divisible by m , then $a \equiv 0 \pmod{m}$, as $m \mid (a - 0)$. A slightly different way of looking at congruences is that if $a \equiv b \pmod{m}$, a and b have the same remainder when divided by m . In the above example of $7 \equiv 2 \pmod{5}$, we see that 7 divided by 5 is 1 remainder 2, and 2 divided by 5 is 0 remainder 2.

Some important properties of congruences are as follows:

- A. Reflexive property. If a is an integer, then $a \equiv a \pmod{m}$.
- B. Symmetric property. If a and b are integers so that $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$.

¹Carl Friedrich Gauss, Disquisitiones Arithmeticae (Leipzig: G. Fleischer, 1801), p. 1. Translated by Arthur A. Clarke, S.J.

C. Transitive property. If a , b , and c are integers with $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.

D. If $a \equiv b \pmod{m}$ and c is any integer, then $a + c \equiv b + c \pmod{m}$, and $ac \equiv bc \pmod{m}$.

E. If $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$.

The only one of these which may be difficult to understand is Property E. The proof of this uses the Fundamental Theorem of Arithmetic, the statement that every positive integer greater than one can be written uniquely, up to the order of the factors, as the product of primes.

Proof of E.

If $ac \equiv bc \pmod{m}$, then $m \mid (ac - bc) = c(a - b)$. But as c and m have no common factors greater than one, the Fundamental Theorem of Arithmetic shows that all of the primes in the prime factorization of m must be contained in $(a - b)$. Thus $m \mid (a - b)$, so $a \equiv b \pmod{m}$. •

The following fact is used in the proof of Sophie Germain's theorem. It also relies on the Fundamental Theorem of Arithmetic.

Theorem A

Let r and s be relatively prime integers. If rs is an n^{th} power, then r and s must both be n^{th} powers.

Proof

First, assume that r and s are relatively prime and that $rs = t^n$. We can assume that $r > 1$ and $s > 1$, as if either were equal to one then

the theorem would obviously be true. The prime factorizations of r , s , and t can be shown as follows:

$$r = p_1^{a_1} p_2^{a_2} \dots p_u^{a_u},$$

$$s = p_{u+1}^{a_{u+1}} p_{u+2}^{a_{u+2}} \dots p_v^{a_v},$$

and

$$t = q_1^{b_1} q_2^{b_2} \dots q_k^{b_k}.$$

Since r and s are relatively prime, the primes occurring in their factorizations are distinct. Since $rs=t^n$,

$$p_1^{a_1} p_2^{a_2} \dots p_u^{a_u} p_{u+1}^{a_{u+1}} p_{u+2}^{a_{u+2}} \dots p_v^{a_v} = q_1^{nb_1} q_2^{nb_2} \dots q_k^{nb_k}.$$

By the Fundamental Theorem of Arithmetic, the primes occurring on the two sides of the equation are the same. Therefore, $v = k$, and after reindexing the primes q_i , we may assume that $p_i = q_i$ for all i . Their exponents must match, so then $a_i = nb_j$. Thus every exponent a_i is an n^{th} power, and so a_i/n is an integer. We may then see that $r=g^n$ and $s=h^n$, where g and h are the integers

$$g = p_1^{a_1/n} p_2^{a_2/n} \dots p_u^{a_u/n}$$

and

$$h = p_{u+1}^{a_{u+1}/n} p_{u+2}^{a_{u+2}/n} \dots p_v^{a_v/n}.$$

Hence r and s are both n^{th} powers. •

The theorem for which Sophie Germain is most famous concerns Fermat's Last "Theorem" (hereafter referred to as FLT), the statement made by Pierre de Fermat that $x^p + y^p = z^p$ is impossible in positive integers where $p > 2$. This is usually divided into two cases. Case 1 of FLT is the statement that $x^p + y^p = z^p$ is impossible in integers that are not divisible by p . Case 2 is the same, except that p divides one of x , y , or z . It is not necessary to look at cases where

two or three of x , y , or z are divisible by p . If two of the three are, then the third must be as well; if all three are divisible by p , the p can be factored out of the equation without changing its correctness. Thus, without loss of generality, it can be assumed that x , y , and z are pairwise relatively prime, as in a counterexample any common factor could be divided out without changing the result. In order to prove FLT, it is sufficient to prove it for $p = 4$ and all prime exponents $p \geq 3$, as all possible exponents can be constructed from these.

The theorem that Germain proved in 1823 is as follows:

If p is an odd prime such that $2p + 1$ is also a prime, then Case 1 of FLT holds for p .

Germain undoubtedly became interested in finding a proof for FLT because the Academy had established a contest for this in 1816 and again in 1818. At this time, there were only proofs for the cases of $n = 4$, which was proved by Fermat himself, and $n = 3$, which was proved by Euler. Legendre was interested in the problem as well, and was in the process of creating a proof for the case $n = 5$ around this time. It is quite possible that Germain began the outline of her proof in this case as well, where $p = 5$ and $2p + 1 = 11$, and expanded it from there.

Today, due to the expansions of this theorem to include primes p such that one of the following is a prime: $2p + 1$, $4p + 1$, $8p + 1$, and other combinations, the theorem is usually stated in the following manner. After the proof of this more popular version, I will show how p and $2p + 1$ satisfy its requirements, and then discuss some of the more recent generalizations.

Sophie Germain's Theorem

Let p be an odd prime. If there is an auxiliary prime q with the properties that

1. $x^p + y^p + z^p \equiv 0 \pmod{q}$ implies $x \equiv 0$ or $y \equiv 0$ or $z \equiv 0 \pmod{q}$, and
 2. $a^p \equiv p \pmod{q}$ is impossible for any integer a ,
- then Case I of Fermat's Last Theorem is true for p .

Proof

FLT can be reformulated as the statement that $x^p + y^p + z^p = 0$ is impossible in nonzero integers since p is odd. This is possible since $x^p + y^p = z^p$ is the same as $x^p + y^p - z^p = x^p + y^p + (-z)^p = 0$. Suppose, contrary to Case I of FLT, that p and q satisfy the conditions of the theorem and that x, y, z are integers, none divisible by p , such that $x^p + y^p + z^p = 0$. These assumptions will lead to a contradiction.

Assume that x, y , and z are pairwise relatively prime (this causes no loss of generality). We start with the equation

$$(-x)^p = y^p + z^p = (y + z)(y^{p-1} - y^{p-2}z + y^{p-3}z^2 - \dots + z^{p-1}).$$

This shows that $(y + z)$ and $(y^{p-1} - y^{p-2}z + y^{p-3}z^2 - \dots + z^{p-1})$ are relatively prime, as if n were a prime which divided them both,

$$y + z \equiv 0 \pmod{n}$$

and

$$(y^{p-1} - y^{p-2}z + y^{p-3}z^2 - \dots + z^{p-1}) \equiv 0 \pmod{n}, \quad [1]$$

so then

$$y \equiv -z \pmod{n}. \quad [2]$$

By combining [1] and [2],

$$(y^{p-1} - y^{p-2}(-y) + y^{p-3}(-y)^2 - \dots + (-y)^{p-1}) \equiv py^{p-1} \equiv 0 \pmod{n}.$$

This implies that either $p \equiv 0 \pmod{n}$ or $y \equiv 0 \pmod{n}$. The first cannot be true, since p and n are both primes and this statement would say that $p = n$. This would be contradictory to the assumption

that none of x , y , or z is divisible by p as if $p \mid y^p + z^p$, then $p \mid (-x)^p$ so $p \mid x$. Thus the second statement should be true. But if $y \equiv 0 \pmod{n}$, then n would divide both y and $(y + z)$, but y and z have no common factors. As neither of these can be true, there is no prime factor which divides both $(y + z)$ and $(y^{p-1} - y^{p-2}z + y^{p-3}z^2 - \dots + z^{p-1})$.

Since the factors are relatively prime, they are both p^{th} powers by Theorem A above. The equations $(-y)^p = (x^p + z^p)$ and $(-z)^p = (x^p + y^p)$ can be factored the same way. From this, it follows that there must be integers $a, b, c, \alpha, \beta,$ and γ such that

$$\begin{array}{lll} y + z = a^p & y^{p-1} - y^{p-2}z + y^{p-3}z^2 - \dots + z^{p-1} = \alpha^p & x = -a\alpha \\ z + x = b^p & z^{p-1} - z^{p-2}x + z^{p-3}x^2 - \dots + x^{p-1} = \beta^p & y = -b\beta \\ x + y = c^p & x^{p-1} - x^{p-2}y + x^{p-3}y^2 - \dots + y^{p-1} = \gamma^p & z = -c\gamma. \end{array}$$

Now consider arithmetic modulo q . Since $x^p + y^p + z^p \equiv 0 \pmod{q}$, the first condition on q in the theorem implies that $x, y,$ or z must be zero mod q . Assume, without loss of generality, that $x \equiv 0 \pmod{q}$. Then

$$2x = x + x = b^p + c^p + -(z + y) = b^p + c^p + (-a)^p \equiv 0 \pmod{q}$$

and, again by the first condition on q , it follows that $a, b,$ or c must be zero mod q . If b or c is $0 \pmod{q}$, then

$$y = -b\beta \equiv 0 \pmod{q}$$

or

$$z = -c\gamma \equiv 0 \pmod{q}.$$

This, together with the fact that $x \equiv 0 \pmod{q}$, implies that at least two of $x, y,$ and z are divisible by q , which contradicts the assumption that $x, y,$ and z are pairwise relatively prime. Therefore,

as neither b nor c is congruent to $0 \pmod{q}$, $a \equiv 0 \pmod{q}$. Then, since $y + z = a^p$, this implies that

$$y \equiv -z \pmod{q}.$$

So

$$\alpha^p = y^{p-1} - y^{p-2}z + y^{p-3}z^2 - \dots + z^{p-1} \equiv py^{p-1} \pmod{q}$$

as before and, since $x \equiv 0 \pmod{q}$,

$$\gamma^p = x^{p-1} - x^{p-2}y + x^{p-3}y^2 - \dots + y^{p-1} \equiv y^{p-1} \pmod{q}.$$

Putting these together gives

$$\alpha^p \equiv p\gamma^p \pmod{q}. \quad [3]$$

Since γ is not congruent to $0 \pmod{q}$, there is an integer g such that

$$\gamma g \equiv 1 \pmod{q},$$

as every element not congruent to zero must have a multiplicative inverse mod q . We can thus insert a factor of $(\gamma g)^p$ on the left side of [3] without changing the result, so

$$(\alpha\gamma g)^p \equiv p\gamma^p \pmod{q}.$$

By canceling the factor of γ^p , we reach

$$(\alpha g)^p \equiv p \pmod{q},$$

which is contrary to the second assumption on q . Thus, by this final contradiction, Sophie Germain's theorem is proved. •

Now it remains to show that p and $q = (2p + 1)$ satisfy the hypotheses of the theorem. In order to do this, we must first discuss two other concepts: Fermat's Little Theorem and the Legendre symbol.

Fermat's Little Theorem

If p is prime and a is a positive integer with $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.

Proof

Consider the integers $a, 2a, \dots, (p-1)a$. None of these $p-1$ integers is divisible by p , because if $p \mid aj$, then $p \mid j$, as we know $\gcd(a, p) = 1$ and $p \nmid a$, so we can use Property E of congruences. But as j is a number between 1 and $p-1$, it cannot possibly be divisible by p . Thus none of these is divisible by p . Also, no two of these integers are congruent mod p . If we assume that $ja \equiv ka \pmod{p}$ for some j and k such that $1 \leq j < k \leq (p-1)$, then, again from Property E, we have $j \equiv k$. But as j and k are different positive integers, both less than p , this is impossible.

Since the integers $a, 2a, \dots, (p-1)a$ are a set of $p-1$ integers with no two congruent mod p , and all incongruent to zero mod p , we know that each ia is congruent to one of the integers $1, 2, \dots, (p-1)$, although we do not know which. Even so, a result of this is that the product of the integers $a, 2a, \dots, (p-1)a$ is congruent mod p to the product of $1, 2, \dots, (p-1)$. Written out, this is

$$a \cdot 2a \cdots (p-1)a \equiv 1 \cdot 2 \cdots (p-1) \pmod{p}.$$

Hence,

$$a^{p-1} (p-1)! \equiv (p-1)! \pmod{p}.$$

Since $\gcd((p-1)!, p) = 1$, we can cancel $(p-1)!$ and reach the equation

$$a^{p-1} \equiv 1 \pmod{p}. \quad \bullet$$

In order to understand the Legendre symbol, a notation developed by Legendre, first we must discuss quadratic residues and nonresidues. We have the following definition.

Definition

If m is a positive integer, we say that the integer a is a quadratic residue of m if $\gcd(a, m) = 1$ and the congruence $x^2 \equiv a \pmod{m}$ has a solution. If this congruence has no solution, then a is a quadratic nonresidue of m .

Using this, we can thus define the Legendre symbol.

Definition

Let p be an odd prime and a an integer not divisible by p . The Legendre symbol $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue of } p \end{cases}.$$

The following criterion is used to demonstrate properties of the Legendre symbol. It is usually used to decide whether an integer is a quadratic residue of a prime number. We will use it in a different manner.

Euler's Criterion

Let p be an odd prime and let a be a positive integer not divisible by p . Then

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}.$$

Proof

First, consider the case when $\left(\frac{a}{p}\right) = 1$. Then the congruence

$x^2 \equiv a \pmod{p}$ has a solution, say $x = x_0$. By using Fermat's Little Theorem, we know

$$a^{(p-1)/2} = (x_0^2)^{(p-1)/2} = x_0^{p-1} \equiv 1 \pmod{p}.$$

Thus we know that $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}$ when $\left(\frac{a}{p}\right) = 1$.

Now look at the case when $\left(\frac{a}{p}\right) = -1$. This means that the congruence $x^2 \equiv a \pmod{p}$ has no solutions. For each integer i such that $1 \leq i \leq p - 1$, there is a unique integer j with $1 \leq j \leq p - 1$, such that $ij \equiv a \pmod{p}$.² Since $x^2 \equiv a \pmod{p}$ has no solutions, we know that $i \neq j$. We can thus group the integers $1, 2, \dots, (p - 1)$ into pairs, each with a product congruent to $a \pmod{p}$. Since there are $(p-1)/2$ of these pairs, multiplying them together gives

$$(p - 1)! \equiv a^{(p-1)/2} \pmod{p}.$$

According to Wilson's Theorem, $(p - 1)! \equiv -1 \pmod{p}$.³ Thus

$$-1 \equiv a^{(p-1)/2} \pmod{p}.$$

So as $\left(\frac{a}{p}\right) = -1$ in this case, and we once again have

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}. \quad \bullet$$

Using the above information, we can show that Germain's p and $q = (2p + 1)$ satisfy the requirements of the more general theorem. For the first condition, suppose that $x^p + y^p + z^p \equiv 0 \pmod{q}$ and q does not divide $x, y,$ or z . Since $p = (q-1)/2$, Fermat's Little Theorem implies that

²Theorem from Kenneth H. Rosen, Elementary Number Theory and Its Applications (USA: Addison-Wesley Publishing Company, 1993), p. 20: Let $a, b,$ and m be integers with $m > 0$ and $\gcd(a, m) = d$. If $d \nmid b$, then $ax \equiv b \pmod{m}$ has no solutions. If $d \mid b$, then $ax \equiv b \pmod{m}$ has exactly d incongruent solutions mod m . In this case, $d = 1$ as i and j are relatively prime to p . Thus, there is a unique solution.

³The first proof of this theorem was given by Lagrange, but it is named for John Wilson, who conjectured this result but did not prove it.

$$x^p \equiv \pm 1 \pmod{q},$$

$$y^p \equiv \pm 1 \pmod{q},$$

$$z^p \equiv \pm 1 \pmod{q}.$$

Thus, $0 = x^p + y^p + z^p \equiv \pm 1 \pm 1 \pm 1 \pmod{q}$, which is clearly impossible; one of x , y , or z must be divisible by q and thus congruent to $0 \pmod{q}$. For the second condition, if $p \equiv a^p \pmod{q}$, computing the Legendre symbol yields

$$\pm 1 = \left(\frac{a}{q}\right) \equiv a^{(q-1)/2} = a^{(2p+1-1)/2} = a^p \equiv p \pmod{q}$$

so $p \equiv \pm 1 \pmod{q}$, and this too is impossible. •

In 1823, Germain shared this theorem with Legendre, who presented it to the Institut de France for her. She had found auxiliary primes q for all primes $p < 100$, except for 2, of course. Legendre extended the theorem to include cases where the auxiliary prime q was equal to $4p + 1$, $8p + 1$, $10p + 1$, $14p + 1$, or $16p + 1$. He also proved that Germain's theorem could not use an auxiliary prime $q = (mn + 1)$ if m was divisible by 3.⁴ For example, $q = 12p + 1$ does not work. Using this information, auxiliary primes q were found for all primes $p < 197$, thus proving Case 1 for all of these primes. This happened before a proof existed for $p = 5$, and clearly showed that Case 2 was the place to focus attention.

⁴Leonard Eugene Dickson, History of the Theory of Numbers (New York: G. E. Stechert, 1934), p. 735.

In 1908, Leonard Eugene Dickson used Germain's generalized theorem to prove Case 1 for all primes $n < 7000$. J. Barkley Rosser used it to prove Case 1 for all primes $n < 41,000,000$ in 1940.

Germain's theorem has been expanded by more recent mathematicians as well. In 1940, M. Krasner proved the theorem:

Assume p is an odd prime, and h is an integer such that

1. $q = 2hp + 1$ is a prime,
2. 3 doesn't divide h ,
3. $3^{h/2} < 2hp + 1$,
4. $2^{2h} \not\equiv 1 \pmod{q}$.

Then Case 1 holds for p .

And in 1951, P. Dénes proved:

Let p be an odd prime, h an integer less than or equal to 55 and not divisible by 3. If $q = 2hp + 1$ is a prime, then Case 1 holds for p .

While the proofs of these theorems involve more complex concepts than those used to prove Germain's theorem, it is clear that the end results are very similar to the conclusion that Germain reached.⁵

Another theorem appears at first to be different from Germain's, but the result is still the same. E. Wendt proved in 1894 the following:

Wendt's Theorem

Let $p \neq 2$ and let $q = 2hp + 1$ ($h \leq 1$) be primes. If $q \nmid W_{2h}$ and $p^{2h} \not\equiv 1 \pmod{q}$, then the first case of Fermat's Theorem holds for the exponent p .

This sounds rather reasonable until one discovers that W_n is the determinant of the $n \times n$ matrix:

⁵Paulo Ribenboim, 13 Lectures on Fermat's Last Theorem (New York: Springer-Verlag, 1979), p. 57. Both Krasner's and Denes' theorems are mentioned here.

$$\det \begin{bmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-1} \\ \binom{n}{n-1} & 1 & \binom{n}{1} & \cdots & \binom{n}{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & 1 \end{bmatrix}$$

Obviously, this would be a rather unwieldy calculation for a large n . It is easy to see why so many textbooks refer to Germain's theorem as elegant and clever.

Germain's work extends beyond this very important theorem. Her work is also used by others as a basis for their own work. For example, in 1909 A. Fleck proved this theorem:

Assume p is an odd prime and x , y , and z are nonzero pairwise relatively prime integers satisfying the equation $x^p + y^p + z^p = 0$. If p does not divide x , then $x^{p-1} \equiv 1 \pmod{p^3}$.

He needed the following result by Germain to do so.

Theorem B

Assume that p is an odd prime and x , y , and z are nonzero pairwise relatively prime integers satisfying the equation $x^p + y^p + z^p = 0$. If p does not divide x , y , or z , then $\alpha \equiv 1 \pmod{p^2}$, $\beta \equiv 1 \pmod{p^2}$, and $\gamma \equiv 1 \pmod{p^2}$, where α , β , and γ are as in the proof of Sophie Germain's Theorem.

Proof⁶

For this proof, we need to remember that

⁶Ribenboim, 13 Lectures, p. 58. Follows the proof given by Pérez-Cacho in 1958.

$$\begin{array}{lll}
y + z = a^p & y^{p-1} - y^{p-2}z + y^{p-3}z^2 - \dots + z^{p-1} = \alpha^p & x = -a\alpha \\
z + x = b^p & z^{p-1} - z^{p-2}x + z^{p-3}x^2 - \dots + x^{p-1} = \beta^p & y = -b\beta \\
x + y = c^p & x^{p-1} - x^{p-2}y + x^{p-3}y^2 - \dots + y^{p-1} = \gamma^p & z = -c\gamma.
\end{array}$$

These equations relied only on the fact that x , y , and z were relatively prime and satisfied $x^p + y^p + z^p = 0$; we do not need an auxiliary prime q . To prove this theorem, we must only prove that $\alpha \equiv 1 \pmod{p^2}$; the rest follows by symmetry. It is enough to show that if q is any prime dividing α , then $q \equiv 1 \pmod{p^2}$.

So assume that q is a prime such that $q \mid \alpha$. It follows that $q \mid x$, but $q \nmid yz$. As we proved earlier that $(y + z)$ and $(y^{p-1} - y^{p-2}z + y^{p-3}z^2 - \dots + z^{p-1})$ are relatively prime, the $\gcd(a, \alpha) = 1$, so then we have that $q \nmid (y + z)$. Since $q \mid x$, then $x \equiv 0 \pmod{q}$ so we see that the equation

$$z^{p-1} - z^{p-2}x + z^{p-3}x^2 - \dots + x^{p-1} = \beta^p$$

becomes

$$z^{p-1} \equiv \beta^p \pmod{q}$$

and for the same reason we also have

$$y^{p-1} \equiv \gamma^p \pmod{q}.$$

Since $q \mid -(x^p)$, $q \mid y^p + z^p$. Thus,

$$0 \equiv y^p + z^p = y(y^{p-1}) + z(z^{p-1}) \equiv y\gamma^p + z\beta^p \pmod{q}$$

so $-y\gamma^p \equiv z\beta^p \pmod{q}$.

As $q \mid y^p + z^p$ and $q \nmid y + z$, then $q \equiv 1 \pmod{p}$.⁷ So $p \mid q - 1$, hence $(q-1)/p$ is an integer. We then raise each side of $-y\gamma^p \equiv z\beta^p \pmod{q}$ to the power of $(q-1)/p$. Thus

$$(-y)^{(q-1)/p} (\gamma^{(q-1)/p})^p \equiv z^{(q-1)/p} (\beta^{(q-1)/p})^p \pmod{q}. \quad [4]$$

⁷Ribenboim, 13 Lectures, p. 52. If $p \mid a^n + b^n$ but $p \nmid a^m + b^m$ for every proper divisor m of n , then $p \equiv 1 \pmod{n}$.

But by Fermat's Little Theorem, γ^{q-1} and α^{q-1} are both congruent to 1 (mod q), and so [4] yields that

$$(-y)^{(q-1)/p} \equiv z^{(q-1)/p} \pmod{q}.$$

However, if y' is such that $y'y \equiv -1 \pmod{q}$, then from $z^p \equiv -y^p \pmod{q}$, we can insert a factor of $(y')^p$ with the result that $(zy')^p \equiv 1 \pmod{q}$. However, if $(zy')^n \equiv 1 \pmod{q}$ for an integer n , then n is divisible by the order of $zy' \pmod{q}$.⁸ Since p is prime and thus only divisible by 1 and p , the order of zy' must be either 1 or p . As $z \not\equiv -y \pmod{q}$, the order of zy' cannot be 1. Thus, the multiplicative order of $zy' \pmod{q}$ is equal to p .

Also, since $z^{(q-1)/p} \equiv (-y)^{(q-1)/p} \pmod{q}$, inserting a factor of $(y')^{(q-1)/p}$ on each side yields that $(zy')^{(q-1)/p} \equiv 1 \pmod{q}$. Thus, as we know that the order of zy' is p , we know that p divides $(q-1)/p$. But this means that $p^2 \mid q-1$, that is, that $q \equiv 1 \pmod{p^2}$, as it was required to show. •

Germain communicated many of her discoveries in number theory to Gauss. Many of the problems she worked on were based on discussions in Gauss' Disquisitiones Arithmeticae. Gauss valued her results, as evidenced by the following letter he wrote to Olbers.

...the two test theorems (for what primes 2 is a cubic or a biquadratic residue), which I also communicated to [Lagrange] some time ago, he considers "among the most beautiful things and

⁸The order of an element is the least power an element must be raised to so it is equal to the identity, 1; to put it another way, if g is the element, the order is the smallest positive integer n such that $g^n \equiv 1$. Also, if $g^m \equiv 1$, then the order of g divides m .

among the most difficult to prove.” But Sophie Germain has sent me the proofs of these...⁹

These theorems concern finding odd primes p such that one or both of the congruences $x^3 \equiv 2 \pmod{p}$ and $x^4 \equiv 2 \pmod{p}$ are solvable. He writes to Germain that her new proof “...was very fine, although it seems to be isolated and cannot be applied to other numbers.”¹⁰ The following chart helps to give more solid understanding of why this problem may be interesting.

x	x ³	x ³ mod p, where p is					x ⁴	x ⁴ mod p, where p is				
		3	5	7	11	13		3	5	7	11	13
1	1	1	1	1	1	1	1	1	1	1	1	1
2	8	2	3	1	8	8	16	1	1	2	5	3
3	27	0	2	6	5	1	81	0	1	4	4	3
4	64	1	4	1	9	12	256	1	1	4	3	9
5	125	2	0	6	4	8	625	1	0	2	9	1
6	216	0	1	6	7	8	1296	0	1	1	9	9
7	343	1	3	0	2	5	2401	1	1	0	3	9
8	512	2	2	1	6	5	4096	1	1	1	4	1
9	729	0	4	1	3	1	6561	0	1	2	5	9
10	1000	1	0	6	10	12	10000	1	0	4	1	3
11	1331	2	1	1	0	5	14641	1	1	4	0	3
12	1728	0	3	6	1	12	20736	0	1	2	1	1
13	2197	1	2	6	8	0	28561	1	1	1	5	0

Looking down the columns reveals a rather interesting fact. All columns have a cyclic pattern, and some primes never have a 2 in their column.

⁹Bell, p. 262. Letter dated July 21, 1807.

¹⁰Letter from Gauss, June 16, 1806. “...votre nouvelle démonstration pour les nombres premiers, dont 2 est résidue ou nonrésidue, m’a extrêmement plu; elle est très fine, quoiqu’elle semble être isolée et ne pouvoir s’appliquer à d’autres nombres.” *Q.P.*, p. 303.

Since Gauss states that her proofs do not seem applicable to other numbers, the search for an historically accurate proof is difficult. In most modern number theory books there exists a very general theorem and proof for similar problems. The theorem is the following:

Let m be a positive integer with a primitive root. If k is a positive integer and a is an integer relatively prime to m , then the congruence $x^k \equiv a \pmod{m}$ has a solution if and only if $a^{\phi(m)/d} \equiv 1 \pmod{m}$, where $d = \gcd(k, \phi(m))$.^{11,12}

It is easy to see how a proof for $x^3 \equiv 2 \pmod{p}$ would use this theorem. Since $m = p$, a prime, then a primitive root exists, and $\phi(p) = p - 1$. Also, set $k = 3$, $a = 2$, and $d = \gcd(p - 1, 3)$. Thus we would know that a prime p has 2 as a cubic residue if $2^{p-1/d} \equiv 1 \pmod{p}$, a simple calculation. However, we know that Germain's proofs did not seem applicable to other numbers, so clearly they were not of a form very similar to this. Otherwise, surely either she or Gauss would have noticed that they could be expanded.

Bits and pieces of some of Germain's other discoveries appear in her correspondence. For example, a letter from Euler to Goldbach mentions the problem of factoring $p^4 + 4q^4$; Germain found that one could factor this into $p^2 \pm 2pq + 2q^2$.¹³ Another time, upon reading a memoir that Lagrange had written, she encountered the term

¹¹Rosen, p. 301.

¹²The Euler phi function, $\phi(n)$, is defined as the number of positive integers less than n that are relatively prime to n . For any prime p , $\phi(p) = p-1$. An integer has a primitive root if there exists a number n with order $\phi(n)$. All prime numbers have a primitive root.

¹³Dickson, p. 382.

$s^{10} - 11(s^8 - 4s^6r^2 + 7s^4r^4 - 5s^2r^6 + r^8)$. She writes to Gauss that she “saw with astonishment” that Lagrange had not reduced the term to the much simpler $t^2 - 11u^2$, although she does not reveal the substitution scheme in the body of her letter.¹⁴

Germain also completed a proof for a very special case of Fermat’s Last Theorem. In a letter to Gauss, she proved it held for $n = p - 1$, where p is a prime of the form $8k + 7$.¹⁵ However, this proof was contained only in the mathematical papers she sent along with her letter and has never been published.¹⁶

In the area of number theory, Germain’s name has been preserved in a different manner as well. E. Dubouis defined a “sophien” of a prime n to be a prime p of the form $(kn + 1)$, where n is such that $x^n \equiv y^n + 1 \pmod{p}$ is impossible in integers relatively prime to p . Pepin proved that 3 has a finite number of sophiens, but it is not known that any prime n has a finite number of sophiens.¹⁷

¹⁴Germain, letter to Gauss November 21, 1804.

¹⁵Ibid.

¹⁶Mary W. Gray, “Sophie Germain” in Women of Mathematics: A Bibliographic Sourcebook (Westport, Connecticut: Greenwood Press, Inc., 1987), p. 51.

¹⁷Gray, p. 51, states that Dubouis proved that this is true, but neither Dickson’s History of the Theory of Numbers or Ribenboim’s Book of Prime Number Records mentions this and I doubt its validity.

PHILOSOPHY

Although mathematics was Germain's first love, she was interested in a variety of other subjects as well. As previously mentioned, she obtained lecture notes from Fourcroy's chemistry course as well as Lagrange's analysis course. She read poetry and was interested in music. Also, she wrote two philosophical works. The first, Pensées Diverses, is a collection of short thoughts on different subjects, such as the nature of mathematicians and scientists. The other, Considérations générales sur l'état des sciences et des lettres aux différentes époques de leur culture, is a more unified, scholarly work. In it, she traces the history of human intellectual development in order to discuss the nature of society and the connections between science and art. What follows is an overview of the ideas expressed in Considérations. While some of her propositions are definitely debatable, I will not discuss the relative merits of her ideas, but merely present them.

Germain begins by discussing the similarities between artistic works and scientific ones. While it is undeniable that the impression produced by an artistic presentation is different than that produced by the study of a mathematical text, there are still underlying rules which both science and art must follow in order to be thought great or beautiful. Genius and eloquence are pleasing to us because they

reveal important relations between subjects that we had not previously seen. It is to this unexpected order that we respond.¹ People recognize easily that literature has style and eloquence, but the language of mathematics has this as well. The choice of characters corresponds to the choice of words, the choice of formulas to the choice of phrases. Just as in literature, all mathematical authors do not write with the same degree of perfection; those who are knowledgeable about mathematics find a charm in good writing. Good writers use their innate sense of style in order to write mathematical texts with finesse.²

Thus, although calculus and poetry seem on the surface to be quite unlike, they have strong similarities between them. They are both inspired by a sense of order and of proportion, and employ style to present their message in a pleasing manner. While their superficial differences tend to suggest that there is a real separation between them, the spirit which created them is the same.³ Science and art are inspired by the search for universal truth. All of our efforts in these subjects are directed towards order, simplicity, and unity of conception.⁴

After this introduction, Germain discusses the beginnings of human intellectual activity. At first, literature/storytelling and art

¹Sophie Germain, "Considérations sur l'état des sciences et lettres aux différentes époques de leur culture" in Oeuvres philosophiques, p. 100.

²Ibid. pp. 106-107.

³Ibid. p. 108.

⁴Ibid. p. 110.

were nothing more than exact copies of actual events. As they developed, the “man of genius” (l’homme de génie) could use his imagination and the stories he had heard from others and combine them into new tales. However, in order to create a successful story, this person must have an abstract notion of order. Without unity of action, unity of interest, and clarity of exposition, his story would fail.⁵ Once he achieved a sense of order, the problem became how to classify the world around him. Governing all of this was a strong sense of analogy. First came the sense of individuality; using this, he personified other beings, both inanimate and intellectual. There was thus a profound sentiment of a common bond between all beings in this first epoch of intellectual culture.⁶ At this time, there was no separation between science and art. There was the need to explain events, but these explanations were as poetic as any literature. The marvels of nature united the two.

Using this sense of science, humans began to see that acts of nature seemed to have an order and a succession that seemed to work toward a determined goal. Man could not conceive of any of this happening without supposing that this action must be caused by some sort of intelligent being. As he could not see anyone, this being must be invisible. Thus he imagined gods, demi-gods, and spirits, each having human traits such as passion, affection, varied interests, and dislikes. They were made in his own image, only immaterial. Since man considered his own existence to be the same type as all

⁵Ibid. p. 112.

⁶Ibid. p. 112.

other beings, he then searched for something comparable within himself. If the spirits exist and have knowledge and will, yet are immaterial, there must exist within humans something immaterial as well, since humans also have knowledge and will. Thus the soul was created.⁷

Germain asserts that this outline of what happened shows the origin of many of the ideas which were produced after this first era of human history. Literature preserved the fictions that were once regarded as truths; the physical sciences collected the observations which the fictions had explained; philosophy took its systems from the sciences; and religions found the elements of their beliefs.⁸ However, human knowledge marched on. The observation that celestial bodies followed a set of unchanging laws posed a problem. People's wills are varied and constantly changing; as concepts of divine beings were based upon this, their wills were similar. In order to have a set of immutable laws, there must be a single being which governs the universe, a being whose will is unchanging. Thus, God must exist.⁹

As people have a beginning, by analogy the universe must also have a beginning; as God created the universe, he must have existed before the universe. This concept pushes the limit of analogies, as suddenly man is faced with the concept of the infinite. He names this concept the eternal. As God has no beginning, by symmetry he

⁷Ibid. pp. 114-115.

⁸Ibid. p. 115.

⁹Ibid. p. 116.

has no end. The physical human body clearly had a beginning and an end, but humans had also appropriated spirituality in the form of the soul. This immaterial soul becomes our own means of everlasting life. Different religions have interpreted this in different manners, but there is still a prevailing idea of human immortality.

Germain also explains the manner in which people came to believe that all existence relates directly to that of humankind. Man knew facts about the universe, and tried to establish relations between them. He did this by a method of cause and effect, but he did not know all of the facts, and there was still the question of why things happened in the first place. Due to his own self-love, man believed himself to be the model of all other beings, to be essentially the center of the universe. Thus, everything happens for him. The sun and moon are there in order to light his villages. Animals and plants are there to feed him. In searching for a sense of order and unity, he conceived of imaginary relations between things. At first this was merely an error of judgment, but his own egotism sanctioned this error, and religions consecrated it.¹⁰

Two examples of this egotism are the false sciences of alchemy and astrology. Alchemy taught that the human body was the epitome of the universe, and named substances according to the organs which they resembled. It also searched for unity in the form of a universal solvent. Astrology taught that the stars influenced each and every individual. Germain shows how egotistical this belief is:

¹⁰Ibid. p. 123.

Man, persuaded of his own importance, believed himself to be menaced by the apparition of comets. The giants of the earth, cherishing through egotism their fellow man, could not see any event more remarkable than their own death. Also, they did not doubt that their death would be announced by these vagabond stars, who certainly would not have taken the pain to visit the earth if they were not directed to warn the habitants of such a grand misfortune.¹¹

Both alchemy and astrology had been renounced by scientists, but they give an important and easy example of the self-centeredness of humankind. There is still the habit to judge nature by what is understandable in relation to humankind; propositions are affirmed and denied based on whether we can conceive of their existence.¹²

However, innate within us is a model of truth. For example, if when first studying a circle we are distracted by sines and cosines, we would demand why such things take place. Since we are looking at only part of the subject, we are not content. Once we step back and look at the whole equation of the circle, our curiosity is satisfied, as we have defined the essence of a circle and see it in its true state.¹³ We would sense the truth in this, and our sense of unity and of order would be fulfilled.

Why then, if humankind is endowed with a sense of innate truth, have we committed so many errors of judgment? Germain attributes

¹¹Ibid. p. 125. "L'homme persuadé de son importance, se croyait menacé par l'apparition des comètes. Les grands de la terre, renchérissant sur l'amour-propre de leurs semblables ne voyaient par d'événements plus remarquables que leur propre mort. Aussi ne doutaient-ils pas qu'elle ne fut annoncée par ces astres vagabonds, qui bien certainement n'auraient pas pris la peine de visiter la terre s'ils n'eussent été chargés d'avertir les habitants d'un aussi grand malheur."

¹²Ibid. p. 125.

¹³Ibid. p. 135.

this to a lack of absolutism in our intellectual pursuits; at the base of this problem is the nature of language.¹⁴ Language was invented for common communication, for talking about things that were either present or perfectly known. As humankind developed, however, this same language was used to express and discuss abstract ideas. The difficulty arises when new words are invented to discuss these new concepts; they cannot be precisely defined. Technical expressions are interpreted in a thousand different ways, following the opinions of people searching for a support for their own beliefs. Two people can say the same thing and yet still have vastly different opinions.¹⁵

The only type of language which avoided such confusion was mathematics. The first people to study this subject looked at simple geometric figures and general properties of numbers. It was impossible to attribute any properties to these objects that they did not have. Signs were then created to express these properties exactly, and thus mathematics offered to the human spirit the realization of truth.¹⁶

In other subjects, such as scientific, religious, and political, there was not this kind of precision. There were thousands of different doctrines and hypotheses, all contradicting one another and disguising the spirit under which the original ideas were created. This began to change when Descartes essentially reconstructed the universe and created a new epoch of reason. He reunited algebra

¹⁴Ibid. p. 141.

¹⁵Ibid. pp. 141-142.

¹⁶Ibid. p. 143.

and geometry and gave the language of mathematics a new use. Newton was able to use the language of mathematics to describe and measure the movement of celestial bodies. The order and unity of the universe came closer to our reach. Mathematics was reunited with physical science. Subjects such as mechanics and hydrodynamics, whose theory and practice were known before this time, suddenly became measurable by calculus.¹⁷ The study of mathematics became much more widespread as people discovered it could express the laws of the heavens. Less than a century before, algebra had seemed to be a barbarous and indecipherable language; suddenly it could explain all sorts of diverse happenings.

Unfortunately, philosophy still could not be expressed in as an exact a manner as mathematics and science. The philosophers observed all that was happening to unite areas of mathematics and physical science, and thus had to find a way to explain this connection between the two. Rather than studying the causes of phenomena, as they had previous to this time, they began to consider the phenomena themselves. The question changed from “why?” to “how?” and “how much?”¹⁸ They made positive observations and renounced the satisfaction of explaining them, confident that liaisons between their observations would be found at a later date.

This was the beginning of the epoch of true knowledge of nature.¹⁹ Scientific progress established relations between events

¹⁷Ibid. pp. 146-147.

¹⁸Ibid. p. 150-151.

¹⁹Ibid. p. 151.

that were previously thought to be isolated; we began to envision these events as merely different parts of the same existence.²⁰

Germain explains this further:

The more one reflects, the more one acknowledges that necessity governs the world. At each new progress of science, that which seemed contingent is recognized as being necessary. Multiple relations are established between the branches that we had thought to be separate; we observe laws where we had thought there were only accidental events. We approach more and more the unity of being...²¹

This unity of being is the truth, and is inseparable from our existence. Notions of the good and the beautiful are derived from our innate sense of truth.

Germain clearly has a deep love and respect for mathematics; she returns to the subject again. She asserts that one day it may be possible to express moral, political, and metaphysical questions in the language of calculations. The special nature of the question would be represented by a constant, and the propositions related to each subject would be functions. This proposition is supported by an example. In calculus, phenomena generally have a tendency to be regular, and this is expressed in their formulas. Terms which express irregularity tend to disappear after a short time. Similar to this, in the area of morality the effects of fraud, lies, and injustices last for only a short time, while truth and justice tend to triumph

²⁰Ibid. p. 160.

²¹Ibid. p. 164. “Plus on réfléchit, plus aussi on reconnaît que la nécessité gouverne le monde. À chaque progrès nouveau des sciences, ce qui passait pour contingent est reconnu comme étant nécessaire. Il s’établit à nos yeux des liaisons multipliées entre des branches qu’on avait cru séparées; on observe des lois là où on n’avait encore vu que des faits accidentels. Nous approchons de plus en plus de l’unité d’être...”

over obstacles which oppose them. In politics, when looking at events that act on the system, one distinguishes between those which are accidental and soon cease, and those which are well known and must predominate. In the matter of taste, the fashion of the moment tends to disappear quickly. In any case, actions which disturb the natural order tend to come to nothing.²²

She also gives an analogy between mechanics and politics concerning the equilibrium of a situation. In mechanics, the equilibrium between many forces comes when the forces on one side are equivalent to the forces they oppose. If an exterior cause acts upon the system, the equilibrium reestablishes itself by the means of oscillations, whose amplitude diminishes quickly. However, this new equilibrium is different than the old. In society, a state of repose is maintained if there is an equilibrium of political forces. The natural tendency of people is to remain in this state of tranquillity. But if the government doesn't observe changes in the social climate, they can maintain this tranquillity only as long as no event agitates the spirits of the people. It is impossible to maintain the equilibrium for any length of time if they do not keep the center of gravity, the opinion of the people, at their base. The society will seem to move as one force against the government; a revolution results and the state of equilibrium is changed.²³

Music is another point of discussion. Just like mathematics, it too is a language, employing sounds, phrases, periods, and rules. Yet it is

²²Ibid. pp. 174-175.

²³Ibid. pp. 175-179.

a purely metaphysical language. It can only express emotions, but it is extremely powerful.²⁴

The work ends by restating the connection between all the seemingly different aspects of science and the humanities. We see them as being different only because we have “created” the universe according to our own wills rather than seeing how it really is. The analogies that can be drawn between science and art reflect the unity of the universe.

Considérations was praised by Auguste Comte, the founder of positive philosophy. His views are similar to many of her ideas. In his Course on the Positive Philosophy, first published in 1830, he searches for a set of laws which governed human history. He breaks history into three stages: the theological stage, in which man invents gods in order to explain the world around him and eventually moves towards the idea of a single god as more order is posed on the community; the metaphysical stage, in which intellect deifies itself, human reason becomes supreme, and the principle of authority is challenged and replaced with notions of equality; and the final positive stage, in which there is a true certainty of belief, providing a basis for reorganization of society, rationalism, and moral regeneration. He also believed in a unity of the sciences, that “the various sciences are branches from a single trunk; and thereby giving a character of unity to the variety of special studies that are

²⁴Ibid. p. 212.

now scattered abroad in a fatal dispersion.”²⁵ It is easy to see the similarities between this general description of Comte’s work and Germain’s own theories.

The similarities are so great that Stupuy asserts that perhaps Germain is the true founder of sociology, as

she did not distinguish between the logical processes which are owned by each category of knowledge; while asserting the organic similarity of the aesthetic and the scientific genius, she did not indicate the different destinations of art and science, and her work is devoid of all metaphysics.²⁶

Sociology, or social physics as Comte referred to it, supposedly completes the body of philosophy. Through the study of history in a scientific manner, its purpose is to show the unity between all of the sciences, a purpose Germain clearly worked toward.

Germain never published her philosophical works. Pensées has the same personal feel as journal writings, and Considerations was likely never finished, judging from the relatively non-conclusive ending. Her writings were compiled and printed by her nephew, Armand-Jacques Lherbette, in 1833, in honor of her memory. In 1879 they were republished in Oeuvres philosophiques de Sophie Germain, which contained these works, some correspondence, and a biography by Hippolyte Stupuy.

²⁵Frank N. Magill, editor, Masterpieces of World Philosophy in Summary Form (New York: Harper and Row, 1961), pp. 590-591.

²⁶Stupuy, p. 69. “...elle ne distingue pas entre les procédés logique celui qui est propre à chaque catégorie de la connaissance; elle n’indique pas, tout en constatant la similitude organique du génie esthétique et du génie scientifique, la destination différente de l’art et de la science, et son oeuvre n’est pas exempte de toute métaphysique.”

TOWARDS THE END

Despite the fact that she had won a great honor with her memoir in elasticity and her continuing professional contact with members of the First Class such as Legendre, Germain still had difficulties gaining entrance to public meetings of the First Class. At one time she requested a ticket from Delambre, the Permanent Secretary of the First Class until 1822. He responded with a letter discussing the difficulties of getting tickets to such an event, as

A number of these tickets are reserved for grand functionaries and celebrated foreigners, so that at each meeting of our Academy I have at most ten such tickets. I make it a rule to distribute them to those of my fellow members who want them for their wives.¹

Her professional standing had little sway here. However, Germain's friend and colleague Fourier was elected to be the next Permanent Secretary in November of 1822. By the end of May 1823, he sent her the following official letter.

I have the honor of informing you that every time you wish to attend the public meetings of the Institute you will be admitted to one of the reserved seats in the center of the hall. The Academy of Sciences wishes to demonstrate, by this distinction, all the interest that you mathematical works

¹Bucciarelli and Dworsky, p. 90. Letter from Delambre, July 25, 1821.

inspire, especially the scientific research that it has crowned through the award to you of one of its annual, grand prizes.²

Finally Germain was accorded a position that she well deserved. She was the first woman to attend the sessions who was not the wife of a member of the First Class.

At the age of 53, Sophie Germain contracted breast cancer.³ She continued her work in mathematics and elasticity and completed retrospective papers on both number theory and on the curvature of surfaces, as well composing her philosophical work Considérations. After two years of painful suffering, she died on June 27, 1831. In 1837, honorary doctorates were conferred to several persons at the centenary celebration of the University of Göttingen. Gauss regretted that she was not alive to receive one; he said that Germain “proved to the world that even a woman can accomplish something worthwhile in the most rigorous and abstract of the sciences and for that reason would have well deserved an honorary degree.”⁴ They would have met in person for the first time at the ceremony.

²Letter from Fourier, May 30, 1823. “J’ai l’honneur de vous prévenir que toutes les fois que vous vous proposerez d’assister aux séances publiques de l’Institut, vous y serez admise dans l’une des places réservés au centre de la salle. L’Académie des sciences désire témoigner par cette distinction tout l’intérêt que lui inspirent vos ouvrages mathématiques et spécialement les savantes recherches qu’elle a couronnées en vous décernant un de ses grands prix annuels.” Q.P., p. 363. In addition to this official notice, Fourier also sent a personal letter of congratulations.

³One of her biographers, Coolidge, states that she died of tuberculosis; I have not found any other corroboration of this statement and sincerely doubt its validity. Stupuy states only that she died of cancer, but Bucciarelli and Dworsky, Dahlmédico, Gray, and Perl all specifically mention breast cancer.

⁴Guy Waldo Dunnington, Carl Frederick Gauss. Titan of Science: A Study of His Life and Work (New York: Exposition Press, 1955), p. 68.

Germain was buried in the Cimetière du Père Lachaise. The inscription on her headstone reads

ICI REPOSE
DEMOISELLE
MARIE-SOPHIE GERMAIN
NÉE A PARIS
LE 1^{ER} AVRIL 1776
DÉCÉDÉE EN LA DITE VILLE
LE 27 JUIN 1831

On her death certificate, she is listed not as a mathematician, but as a “rentière,” a person of private means. The house in which she died, at 13 rue de Savoie in Paris, is now designated an historical landmark and has a commemorative plaque. She has two other monuments in Paris, the École Sophie Germain and the Rue Sophie Germain.

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