Cyclic Interference Neutralization on the Full-Duplex Relay-Interference Channel

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***Abstract*—In this work, we investigate the *Interference Neutralization* scheme on a $2\times2\times2$ relay-interference channel w. r. t. the *Cyclic Interference Alignment* framework. We formulate sufficient and necessary conditions for this particular Interference Neutralization scheme to achieve the min-cut upper bound, i. e., two degrees of freedom, confirming the work on *Aligned Interference Neutralization* by Gou et al.**

Furthermore, we translate our proposed scheme to a generalized version of the asymptotic Aligned Interference Neutralization scheme based on spatial Interference Alignment.

I. Introduction

Efficiently mitigating interference has been a very challenging and long standing problem in multi-user communications. To approach this problem, the concept of *Interference Alignment* (IA) was introduced in [1]. The idea of IA is to fusion all the interference signals at undesired receivers, while keeping dedicated signals distinct and decodable. Then, approximately half of the capacity, the users would achieve if the dedicated channels were interference-free, can be achieved. An extensive overview on diverse IA techniques is provided in [2].

The particular *Cyclic IA* scheme is presented in [3] for the X-channel and the K-user interference channel. We will denote the underlying channel model as the *cyclic polynomial channel model*. It is strongly related to the *linear deterministic channel model* [4], and mainly serves as a conceptual model focussing on the effects of interference rather than on noise.

We consider a system of two transmitters, two parallel interjacent relays and two destinations, i. e., a $2\times2\times2$ *relay-interference channel* as depicted in Fig. 1. *Interference Neutralization* (IN) [5]–[8] is a novel approach to achieve the min-cut upper bounds on the approximate capacity. IN is a cooperative signalling scheme for both the sources and relays such that the interfering signals at undesired destinations are literally erased over the air. The effective communication from a source to a dedicated destination is entirely interference-free.

Moreover, the authors of [9] provide a more generalized IN scheme for a $K\times K\times K$ relay-interference channel and achieve the corresponding min-cut upper bound.

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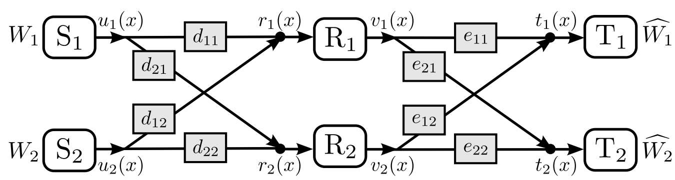

Fig. 1. The cyclic polynomial relay-interference channel with a discrete channel matrix $\boldsymbol{D} = (d_{ji})^{2\times2}$ between transmitters S_i and relays R_j and a discrete channel matrix $\boldsymbol{E} = (e_{ji})^{2\times2}$ between relays R_i and destinations T_i.

Contributions. In the present work, we propose an IN scheme based on the Cyclic IA framework for the full-duplex relay-interference channel. We formulate a set of *interference-neutralization conditions* and *no-signal-neutralization conditions* to ensure decodability of the dedicated signals at each destination. With these conditions fulfilled, we derive a communication scheme that achieves the min-cut upper bound of the approximate capacity. Furthermore, our proposed scheme for *Cyclic IN* is translated to the Gaussian channel model and shown to generalize the *Aligned IN* (AIN) scheme [7].

But in contrast to [7], we will neither translate the model to the *Rational Dimensions IA* nor to the *Asymmetric Complex Signalling IA* framework.

Organization. The system model for the full-duplex $2\times2\times2$ relay-interference channel is given in Section II-A. The Cyclic IN scheme is presented in II-B. The corresponding model for AIN is provided in III-A and our generalized representation of AIN in III-B. The upper bound is given in IV. We conclude this work in Section V.

Notation. The operator $\mathrm{diag}(a_1,\dots,a_n)$ specifies a diagonal matrix with the entries $a_1,\dots,a_n$ on the main diagonal and zero elsewhere. A univariate polynomial of degree $n-1$ in the indeterminate x is denoted by $p(x) = \sum_{k=0}^{n-1} p^{[k]} x^k$ with coefficients $p^{[k]}$. We define $\boldsymbol{X} := \mathrm{diag}(x^0, x^1, x^2, \dots, x^{n-1})$.

II. Cyclic Interference Neutralization

A. System Model - The Cyclic Polynomial Channel

The communication model of a cyclic polynomial channel refers to [3]. In the present case, each source S_i desires to communicate a message W_i to a dedicated destination T_i for $i = 1, 2$. There is no direct link between sources and destinations. The communication is performed over two hops by the aid of two full-duplex relays R_1 and R_2. The relays apply a causal relaying function to forward the signals of the previous first hop and concurrently receive the signals of next first hop.

We assume that a block of transmitted signals in one hop is limited to n dimensions per user[1]. Each dimension $k \in \{0, \dots, n-1\}$ is addressed by a corresponding offset x^k. The influence of the wireless channel is represented by a cyclic right-shift of the offsets over n dimensions. We describe the individual shifts per user-pair by channel matrices with discrete entries in $\mathcal{D} \coloneqq \{x^k \mid k \in \mathbb{N}\}$. The channel of the first hop is $\boldsymbol{D} = (d_{ji})_{1 \le i,j \le 2}$ and the channel of the second hop is $\boldsymbol{E} = (e_{ji})_{1 \le i,j \le 2}$ with $d_{ji}, e_{ji} \in \mathcal{D}$. These channel matrices are known to all users. Furthermore, we denote the offset exponents by δ_{ji}, $\eta_{ji} \in \mathbb{N}$, so that $d_{ji} = x^{\delta_{ji}}$ and $e_{ji} = x^{\eta_{ji}}$.

1) *First hop*: The sources S_i map their message W_i to the transmit polynomials $u_i(x)$. The received polynomials at relays R_j are:

$$r_j(x) = \textstyle\sum_{i=1}^{2} d_{ji} u_i(x) \mod (x^n - 1). \tag{1}$$

2) *Second hop*: The relays R_i map their received polynomials $r_i(x)$ to the forwarded polynomials $v_i(x)$. This mapping may involve a permutation of coefficients, a change of sign, and even discarding some specified coefficients of the received polynomials.

The received polynomials at destinations T_j yield:

$$t_j(x) = \textstyle\sum_{i=1}^{2} e_{ji} v_i(x) \mod (x^n - 1). \tag{2}$$

The destinations D_i can decode $\widehat{W}_i$, if the dedicated messages in $t_j(x)$ are received interference-free.

The given model does not consider the effects of additive noise at the relays. Our setup is also illustrated in Fig. 1.

In a more convenient vectorized notation, the transmission vector of the first hop is denoted by $\boldsymbol{u} = (u_1(x), u_2(x))$. and the received vector by $\boldsymbol{r} = (r_1(x), r_2(x))$. Accordingly, the transmission vector for the second hop is $\boldsymbol{v} = (v_1(x), v_2(x))$ and the received vector is $\boldsymbol{t} = (t_1(x), t_2(x))$. Then, the transfer functions of both hops can be expressed in a compact way:

$$\boldsymbol{r}^\mathsf{T} = \boldsymbol{D}\boldsymbol{u}^\mathsf{T} \mod (x^n - 1), \tag{3}$$
$$\boldsymbol{t}^\mathsf{T} = \boldsymbol{E}\boldsymbol{v}^\mathsf{T} \mod (x^n - 1), \tag{4}$$

where the modulo operation is taken component-wise.

To evaluate the achieved data rate, the *Degrees of Freedom* (DoF) serve as a metric for the polynomial channel. The metric is defined for the polynomial channel as the number M of messages conveyed interference-free per n dimensions [3]:

$$\mathrm{DoF} = \frac{M}{n}. \tag{5}$$

B. Cyclic Interference Neutralization Scheme

The main goal is to convey the maximal number of interference-free messages from each source to each dedicated destination. Instead of decoding single messages at both relays, only functions of superimposed messages are decoded. These superimposed messages are forwarded to the destinations using a proper relaying function. Let each message W_i be partitioned into a vector $\boldsymbol{w}_i$ of n submessages $W_i^{[k]}$:

$$\boldsymbol{w}_i = (W_i^{[0]}, W_i^{[1]}, \dots, W_i^{[n-1]}). \tag{6}$$

1) *First hop*: Each of the n submessages $W_i^{[k]}$ from source S_i is allocated to the corresponding dimension at offset x^k. The transmitted polynomial from source S_i yields:

$$u_i(x) = \textstyle\sum_{k=0}^{n-1} W_i^{[k]} x^k = \boldsymbol{X}\boldsymbol{w}_i^\mathsf{T}. \tag{7}$$

As given by (1), the relays R_j receive the following superposition of coefficients of two submessages per dimension:

$$r_j^{[k]} = W_1^{[k-\delta_{j1}]} + W_2^{[k-\delta_{j2}]}. \tag{8}$$

Let the superscript indices denoted in squared brackets, i.e., $[\cdot]$, be reduced modulo n for notational convenience.

2) *Second hop*: The two relays forward their previously received polynomials by $v_1(x) = x^{\gamma_1} r_1(x) \mod (x^n - 1)$ and $v_2(x) = -x^{\gamma_2} r_2(x) \mod (x^n - 1)$, using the offset parameters $\gamma_1, \gamma_2 \in \{0, \dots, n-1\}$, respectively.

$$\begin{aligned} t_1^{[k]} = \;& W_1^{[k-\delta_{11}-\gamma_1-\eta_{11}]} + W_2^{[k-\delta_{12}-\gamma_1-\eta_{11}]} \\ & - W_1^{[k-\delta_{21}-\gamma_2-\eta_{12}]} - W_2^{[k-\delta_{22}-\gamma_2-\eta_{12}]}, \end{aligned} \tag{9}$$
$$\begin{aligned} t_2^{[k]} = \;& W_1^{[k-\delta_{11}-\gamma_1-\eta_{21}]} + W_2^{[k-\delta_{12}-\gamma_1-\eta_{21}]} \\ & - W_1^{[k-\delta_{21}-\gamma_2-\eta_{22}]} - W_2^{[k-\delta_{22}-\gamma_2-\eta_{22}]}. \end{aligned} \tag{10}$$

At both destinations, the desired submessages are superimposed by interference. The idea of IN is to combine two identical inter-user interference signals with complementary signs within the same dimension k, such that their sum is zero. To suppress the inter-user interference at both destinations, these two *interference-neutralization conditions* must hold:

$$\delta_{12} + \gamma_1 + \eta_{11} \equiv \delta_{22} + \gamma_2 + \eta_{12} \pmod n, \tag{11}$$
$$\delta_{11} + \gamma_1 + \eta_{21} \equiv \delta_{21} + \gamma_2 + \eta_{22} \pmod n. \tag{12}$$

In other words, inter-user interference is aligned and *neutralized* over two hops.

On the other hand, we must also ensure that the desired signals remain intact and are not neutralized. Thence, the following *no-signal-neutralization conditions* must hold:

$$\delta_{11} + \gamma_1 + \eta_{11} \not\equiv \delta_{21} + \gamma_2 + \eta_{12} \pmod n, \tag{13}$$
$$\delta_{12} + \gamma_1 + \eta_{21} \not\equiv \delta_{22} + \gamma_2 + \eta_{22} \pmod n. \tag{14}$$

Let $\boldsymbol{\Gamma} = \mathrm{diag}(x^{\gamma_1}, -x^{\gamma_2})$. The above conditions (11) to (14) indicate that the matrix product $\boldsymbol{E\Gamma D}$ must be a diagonal matrix of full rank [8]. If these conditions are satisfied, the superposition of submessages in (9) and (10) is reduced to:

$$t_1^{[k]} = W_1^{[k-\delta_{11}-\gamma_1-\eta_{11}]} - W_1^{[k-\delta_{21}-\gamma_2-\eta_{12}]}, \tag{15}$$
$$t_2^{[k]} = W_2^{[k-\delta_{12}-\gamma_1-\eta_{21}]} - W_2^{[k-\delta_{22}-\gamma_2-\eta_{22}]}. \tag{16}$$

The superposition of desired submessages received at destination T_j as given by (15) and (16) is compactly expressed by:

$$t_j(x) = (\boldsymbol{X}\boldsymbol{C}_j)\boldsymbol{w}_j^\mathsf{T}, \tag{17}$$

[1] In contrast to [3], we do not interpret the n dimensions in terms of propagation delays here.

for the coefficient matrix $\boldsymbol{C}_j = (c_{j,lm})_{0\le l,m\le n-1}$ with:

$$c_{j,lm} = \begin{cases} 1 & \text{, if } m-l \equiv \delta_{1j}+\gamma_1+\eta_{j1} \pmod{n}, \\ -1 & \text{, if } m-l \equiv \delta_{2j}+\gamma_2+\eta_{j2} \pmod{n}, \\ 0 & \text{, else.} \end{cases} \quad (18)$$

The superimposed submessages of $\boldsymbol{w}_j$ are resolvable by a *linear decoding scheme* if $\det(\boldsymbol{C}_j) \ne 0$ holds.

Lemma 1. *Linear decoding at destination T_j can not resolve n desired submessages $W_j^{[k]}$ from the received polynomial $t_j(x)$ for the given Interference Neutralization scheme.*

Proof: $\boldsymbol{C}_j$ corresponds to an $n\times n$ circulant matrix $\widetilde{\boldsymbol{C}}_j$ as in [10] with entries $c_{j,lm} = \tilde{c}_{j,(m-l \bmod n)}$. Thus, we have n eigenvectors of $\boldsymbol{C}_j$, namely $\boldsymbol{v}_i = \frac{1}{\sqrt{n}}(1,\omega_i,\omega_i^2,\ldots,\omega_i^{n-1})^\mathsf{T}$ for $i=0,\ldots,n-1$ with the roots of unity $\omega_i = \exp(\frac{\jmath 2\pi i}{n})$ and the complex symbol $\jmath = \sqrt{-1}$. The n corresponding eigenvalues are $\lambda_i = \sum_{k=0}^{n-1}\tilde{c}_k\omega_i^k$. Let $\nu_{ji} = \delta_{ij}+\gamma_i+\eta_{ji} \pmod{n}$. Then, $\det(\boldsymbol{C}_j)$ yields from the multiplication of n eigenvalues:

$$\begin{aligned}\det(\boldsymbol{C}_j) &= \textstyle\prod_{j=0}^{n-1}\lambda_j = \prod_{j=0}^{n-1}(\omega_j^{\nu_{j1}}-\omega_j^{\nu_{j2}}) \\ &= (1^{\nu_{j1}}-1^{\nu_{j2}})\cdot \textstyle\prod_{j=1}^{n-1}(\omega_j^{\nu_{j1}}-\omega_j^{\nu_{j2}}) = 0. \end{aligned} \quad (19)$$

Thence, the messages in $\boldsymbol{w}_j$ can not be linearly resolved. ■

The conditions (11) to (14) are too strict for a total number of $2n$ submessages. To enable Cyclic IN with linear decoding, we propose an *asymptotic IN scheme* for $2n-1$ submessages: 1) *First Hop*: Let source S_1 transmit n submessages as in (7) and let S_2 transmit only $n-1$ submessages, discarding submessage $W_2^{[\tau]}$ for a parameter $\tau\in\{0,\ldots,n-1\}$:

$$u_1(x) = \textstyle\sum_{k=0}^{n-1} W_1^{[k]}x^k, \quad (20)$$

$$u_2(x) = \textstyle\sum_{k=0,k\ne\tau}^{n-1} W_2^{[k]}x^k. \quad (21)$$

Now, the $k=0,\ldots,n-1$ received dimensions at relays R_j are:

$$r_j^{[k]} = W_1^{[k-\delta_{j1}]} + W_2^{[k-\delta_{j2}]}, \quad (22)$$

$$r_j^{[k]} = W_1^{[k-\delta_{j1}]}, \text{ if } k\equiv\tau+\delta_{j2}. \quad (23)$$

2) *Second hop*: Relay R_1 forwards all n dimensions and R_2 forwards only $n-1$ of the n received dimensions. In particular, relay R_2 discards forwarding the dimension received at $k_2\equiv\tau+\delta_{22} \pmod{n}$. One γ_1,γ_2 is arbitrarily chosen and the other is computed by (11). The transmitted polynomials are:

$$v_1(x) = x^{\gamma_1}r_1(x) \bmod (x^n-1), \quad (24)$$

$$v_2(x) = -x^{\gamma_2}\textstyle\sum_{k=0,k\ne k_2}^{n-1} r_2^{[k]}x^k \bmod (x^n-1). \quad (25)$$

The received signals at D_1, D_2 correspond to (9), (10). The discarded messages for $\sigma_{ji} = \tau+\delta_{i2}+\gamma_i+\eta_{ji} \pmod{n}$ yield:

$$\begin{aligned} t_1^{[\sigma_{11}]} &= W_1^{[\sigma_{11}-\delta_{11}-\gamma_1-\eta_{11}]} - W_1^{[\sigma_{11}-\delta_{21}-\gamma_2-\eta_{12}]} \\ &\quad - W_2^{[\sigma_{11}-\delta_{22}-\gamma_2-\eta_{12}]}, \end{aligned} \quad (26)$$

$$t_1^{[\sigma_{12}]} = W_1^{[\sigma_{12}-\delta_{11}-\gamma_1-\eta_{11}]} + W_2^{[\sigma_{12}-\delta_{12}-\gamma_1-\eta_{11}]}, \quad (27)$$

$$\begin{aligned} t_2^{[\sigma_{21}]} &= W_1^{[\sigma_{21}-\delta_{11}-\gamma_1-\eta_{21}]} - W_1^{[\sigma_{21}-\delta_{21}-\gamma_2-\eta_{22}]} \\ &\quad - W_2^{[\sigma_{21}-\delta_{22}-\gamma_2-\eta_{22}]}, \end{aligned} \quad (28)$$

$$t_2^{[\sigma_{22}]} = W_1^{[\sigma_{22}-\delta_{11}-\gamma_1-\eta_{21}]} + W_2^{[\sigma_{22}-\delta_{12}-\gamma_1-\eta_{21}]}. \quad (29)$$

Theorem 2. *The asymptotic Interference Neutralization scheme achieves $\frac{2n-1}{n}$ DoF on the cyclic polynomial channel, if the interference-neutralization conditions* (11), (12) *and no-signal-neutralization conditions* (13), (14) *hold.*

Proof: For the given conditions, the received signals at T_1 and T_2 further simplify to (15), (16) and to these special cases:

$$t_1^{[\sigma_{1j}]} = W_1^{[\sigma_{1j}-\delta_{11}-\gamma_1-\eta_{11}]}, \quad (30)$$

$$t_2^{[\sigma_{21}]} = -W_2^{[\sigma_{21}-\delta_{22}-\gamma_2-\eta_{22}]}, \quad (31)$$

$$t_2^{[\sigma_{22}]} = W_1^{[\sigma_{22}-\delta_{11}-\gamma_1-\eta_{21}]} + W_2^{[\sigma_{22}-\delta_{12}-\gamma_1-\eta_{21}]}. \quad (32)$$

Note that $\sigma_{11}\equiv\sigma_{12} \pmod{n}$ holds here. Furthermore, the conditions (11) to (14) imply a proper choice of γ_1 and γ_2. At destination T_1, the coefficient matrix $\boldsymbol{C}_1$ has almost the same structure as in (18). The exception is an additional zero entry in $\boldsymbol{C}_1$ at row σ_{1j} and column $\sigma_{1j}-\delta_{21}-\gamma_2-\eta_{12}$ as given by (30). By Laplace's formula, we can recursively expand the determinant of $\boldsymbol{C}_1$ along the rows with only one non-zero entry, i. e., row σ_{1j} in the first iteration. The determinant yields $\det(\boldsymbol{C}_1) = 1$ and each submessage dedicated for T_1 is linear decodable.

Destination T_2 discards row σ_{22} and column τ in $\boldsymbol{C}_2$ since it only needs to decode the remaining $n-1$ submessages and neglects $W_2^{[\tau]}$. Furthermore, the interfering submessage $W_1^{[\sigma_{22}-\delta_{11}-\gamma_1-\eta_{21}]}$ in (32) is not neutralized anyway. Thus, we consider a reduced coefficient matrix $\widehat{\boldsymbol{C}}_2$ which is a corresponding $(n-1)\times(n-1)$ matrix of $\boldsymbol{C}_2$. $\widehat{\boldsymbol{C}}_2$ has a single row with only one non-zero entry at σ_{21} as given in (31). In analogy to $\boldsymbol{C}_1$, the determinant yields $\det(\widehat{\boldsymbol{C}}_2) = 1$ so that each submessage dedicated for T_2 is also linear decodable.

Altogether, a total number of $M = 2n-1$ submessages is conveyed interference-free over $n\ge 2$ dimensions using IN by Cyclic IA and linear decoding. The asymptotic IN scheme achieves $\lim_{n\to\infty}\frac{2n-1}{n} = 2$ DoF in the limit. ■

Valid parameters for $n\ge 2$ do exist, e. g., $\gamma_1=\gamma_2=0$, $\delta_{12}=\eta_{12}=1$, $\delta_{11}=\delta_{21}=\delta_{22}=\eta_{11}=\eta_{21}=\eta_{22}=0$ as in [7].

Corollary 3. *The conditions of Theorem 2 also imply that:*

(a) $\delta_{12}+\delta_{21}+\eta_{11}+\eta_{22} \equiv \delta_{11}+\delta_{22}+\eta_{12}+\eta_{21} \pmod{n}$, $\delta_{12}+\delta_{21}+\eta_{12}+\eta_{21} \not\equiv \delta_{11}+\delta_{22}+\eta_{11}+\eta_{22} \pmod{n}$,
(b) $\det(\boldsymbol{D}) \not\equiv 0 \bmod (x^n-1)$, $\det(\boldsymbol{E}) \not\equiv 0 \bmod (x^n-1)$,
(c) *and $n\ge 2$ dimensions must hold for Cyclic IN.*

Proof:

(a) The first condition, is obtained by substituting (11) and (12) w. r. t. γ_1 or γ_2. The same is done in (13) and (14) for the second condition respectively.
(b) Assuming $\det(\boldsymbol{D}) \equiv 0 \bmod (x^n-1)$, yields $\delta_{11}+\delta_{22}\equiv\delta_{12}+\delta_{21} \pmod{n}$. Inserting this into the first condition of (a), it follows $\det(\boldsymbol{E})\equiv 0 \bmod (x^n-1)$. Further inserting $\delta_{11}+\delta_{22}\equiv\delta_{12}+\delta_{21} \pmod{n}$ and $\eta_{11}+\eta_{22}\equiv\eta_{12}+\eta_{21} \pmod{n}$ into the second condition of (a) leads to a contradiction.
(c) By assuming $n=1$, $\det(\boldsymbol{D})\equiv\det(\boldsymbol{E})\equiv 0 \bmod (x^n-1)$ always holds. This also leads to a contradiction as shown in (b). ■

III. Aligned Interference Neutralization

A. System Model

In this section, we refer to the channel model that was introduced for AIN in [7] as depicted in Fig. 2.

1) *First hop*: The channel from source S_i to relay R_j is characterized by the channel cofficient $F_{ji} \in \mathbb{C}$. The relays R_j receive a superposition of the signals from the two sources S_i plus additive i. i. d. Gaussian noise $Z_{\mathrm{R}_j}(t) \sim \mathcal{SCN}(0,1)$:

$$Y_{\mathrm{R}_j}(t) = \textstyle\sum_{i=1}^{2} F_{ji}(t) X_{\mathrm{S}_i}(t) + Z_{\mathrm{R}_j}(t). \tag{33}$$

2) *Second hop*: The channel from relay R_i to destination T_j is characterized by the channel coefficient $G_{ji} \in \mathbb{C}$. The destinations T_j receive a superposition of the signals from the relays R_i plus additive i. i. d. Gaussian noise, $Z_{\mathrm{T}_j}(t) \sim \mathcal{SCN}(0,1)$:

$$Y_{\mathrm{T}_j}(t) = \textstyle\sum_{i=1}^{2} G_{ji}(t) X_{\mathrm{R}_i}(t) + Z_{\mathrm{T}_j}(t). \tag{34}$$

All sources, relays and destinations have single antennas. The channel coefficients are generic and assumed to be time-varying in each discrete time-slot t and furthermore they are bounded between a non-zero minimum and a finite maximum. For the sake of simplicity, we assume that the channel state information (CSI) is fully known to all sources, relays and destinations. Since the channel coefficients are generic, the matrices have full rank almost surely and are invertible accordingly.

An *n-symbol extension* over n timeslots as also utilized in [1], [7] provides diagonal channel matrices enabling spatial IA over time-varying channel coefficients:

$$\boldsymbol{F}_{ji}(t) = \mathrm{diag}(F_{ji}(nt+1), \ldots, F_{ji}(nt+n)), \tag{35}$$

$$\boldsymbol{G}_{ji}(t) = \mathrm{diag}(G_{ji}(nt+1), \ldots, G_{ji}(nt+n)). \tag{36}$$

From now on, time indices t are dropped for brevity. We obtain the following channel model as also depicted in Fig. 2:

$$\boldsymbol{Y}_{\mathrm{R}_j} = \textstyle\sum_{i=1}^{2} \boldsymbol{F}_{ji} \boldsymbol{X}_{\mathrm{S}_i} + \boldsymbol{Z}_{\mathrm{R}_j}, \tag{37}$$

$$\boldsymbol{Y}_{\mathrm{T}_j} = \textstyle\sum_{i=1}^{2} \boldsymbol{G}_{ji} \boldsymbol{X}_{\mathrm{R}_i} + \boldsymbol{Z}_{\mathrm{T}_j}. \tag{38}$$

$\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ are $n \times 1$ vectors, i. e., the n-symbol extensions of X, Y, Z. Sources and relays encode their messages into Gaussian codebooks of length n, with codeword symbols $w_{\mathrm{S}_i}^{[k]}$, $w_{\mathrm{R}_i}^{[k]}$, and use beamforming vectors $\boldsymbol{v}_{\mathrm{S}_i}^{[k]}$ and $\boldsymbol{v}_{\mathrm{R}_i}^{[k]}$ to transmit the codewords over the given channel. The transmitted signals[2] from the S_i and R_i are:

$$\boldsymbol{X}_{\mathrm{S}_i} = \textstyle\sum_{k=0}^{n-1} \boldsymbol{v}_{\mathrm{S}_i}^{[k]} w_{\mathrm{S}_i}^{[k]}, \tag{39}$$

$$\boldsymbol{X}_{\mathrm{R}_i} = \textstyle\sum_{k=0}^{n-1} \boldsymbol{v}_{\mathrm{R}_i}^{[k]} w_{\mathrm{R}_i}^{[k]}. \tag{40}$$

The average transmit power for each transmit vector is limited by P. The *Degrees of Freedom* (DoF) are an approximate metric to measure the maximal sum-rate. Here, they are

[2]Our notation of indices slightly differs from [7].

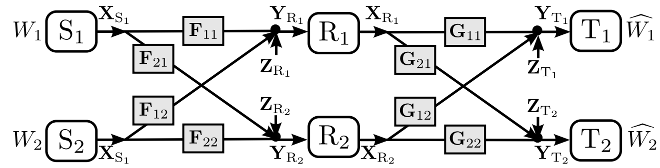

Fig. 2. The channel model of [7] for the relay-interference channel with diagonal channel matrices $\boldsymbol{F}_{ji}$ between transmitters S_i, relays R_j and diagonal channel matrices $\boldsymbol{G}_{ji}$ between relays R_i and destinations T_j for $i, j \in \{1,2\}$.

defined by the pre-log factor of the sum-capacity $C_\Sigma(P)$ for the high SNR regime [1], [7]:

$$\mathrm{DoF} = \lim_{P \to \infty} \frac{C_\Sigma(P)}{\log(P)}. \tag{41}$$

Thence, the influence of noise may be neglected at the relays and destinations.

B. Generalized Aligned Interference Neutralization Scheme

An explanatory toy example for AIN is given in [7, Sect. I-D] for a symbol extension of $n = 2$ time-slots. An asymptotic AIN scheme is given in [7, Sect. III-A] for general $n \geq 2$. Therein, the symbols $w_{\mathrm{S}_2}^{[n-1]}, w_{\mathrm{R}_2}^{[n-1]}$ are discarded and the beamforming vectors for $i = 0, \ldots, n-2$ are aligned by:

$$\boldsymbol{F}_{11} \boldsymbol{v}_{\mathrm{S}_1}^{[i+1]} = \boldsymbol{F}_{12} \boldsymbol{v}_{\mathrm{S}_2}^{[i]}, \tag{42}$$

$$\boldsymbol{F}_{21} \boldsymbol{v}_{\mathrm{S}_1}^{[i]} = \boldsymbol{F}_{22} \boldsymbol{v}_{\mathrm{S}_2}^{[i]}, \tag{43}$$

$$\boldsymbol{G}_{11} \boldsymbol{v}_{\mathrm{R}_1}^{[i+1]} = -\boldsymbol{G}_{12} \boldsymbol{v}_{\mathrm{R}_2}^{[i]}, \tag{44}$$

$$-\boldsymbol{G}_{21} \boldsymbol{v}_{\mathrm{R}_1}^{[i]} = \boldsymbol{G}_{22} \boldsymbol{v}_{\mathrm{R}_2}^{[i]}, \tag{45}$$

achieving $\frac{2n-1}{n}$ DoF on the relay-interference channel. We now show how this scheme yields a special case of Theorem 2.

1) *First hop*: Source S_1 sends n and S_2 sends $n-1$ symbols $w_{\mathrm{S}_j}^{[0]}, \ldots, w_{\mathrm{S}_j}^{[n-1]}$, along beamforming vectors $\boldsymbol{v}_{\mathrm{S}_j}^{[0]}, \ldots, \boldsymbol{v}_{\mathrm{S}_j}^{[n-1]}$, for $j = 1, 2$, discarding $w_{\mathrm{S}_2}^{[\tau]}$ and $\boldsymbol{v}_{\mathrm{S}_2}^{[\tau]}$, respectively. In order to imitate the separate dimensions of the polynomial model, the beamforming vectors align at the relays for $i = 0, \ldots, n-1$ as:

$$\boldsymbol{F}_{11} \boldsymbol{v}_{\mathrm{S}_1}^{[i-\delta_{11}]} = \boldsymbol{F}_{12} \boldsymbol{v}_{\mathrm{S}_2}^{[i-\delta_{12}]}, i \not\equiv \tau + \delta_{12} \pmod{n}, \tag{46}$$

$$\boldsymbol{F}_{21} \boldsymbol{v}_{\mathrm{S}_1}^{[i-\delta_{21}]} = \boldsymbol{F}_{22} \boldsymbol{v}_{\mathrm{S}_2}^{[i-\delta_{22}]}, i \not\equiv \tau + \delta_{22} \pmod{n}. \tag{47}$$

Let $\boldsymbol{F} = \boldsymbol{F}_{11}^{-1} \boldsymbol{F}_{12} \boldsymbol{F}_{22}^{-1} \boldsymbol{F}_{21}$, with B_m denoting the m-th diagonal entry of $\boldsymbol{F}$ and let $\Delta_{\boldsymbol{D}} \equiv \delta_{12} - \delta_{11} + \delta_{21} - \delta_{22} \pmod{n}$. The dependencies of the beamforming vectors are resolved w. r. t. $\boldsymbol{v}_{\mathrm{S}_1}^{[\tau_1]}$ with $\tau_1 = \tau + \delta_{12} - \delta_{11}$ for $i = 1, \ldots, n-1$ by:

$$\boldsymbol{v}_{\mathrm{S}_1}^{[\tau_1 + i\Delta_{\boldsymbol{D}}]} = \boldsymbol{F}^i \boldsymbol{v}_{\mathrm{S}_1}^{[\tau_1]}, \tag{48}$$

$$\boldsymbol{v}_{\mathrm{S}_2}^{[\tau_1 + i\Delta_{\boldsymbol{D}} - \delta_{12} + \delta_{11}]} = \boldsymbol{F}^{i-1} \boldsymbol{F}_{22}^{-1} \boldsymbol{F}_{21} \boldsymbol{v}_{\mathrm{S}_1}^{[\tau_1]}. \tag{49}$$

Since $\det(\boldsymbol{D}) \not\equiv 0 \bmod (x^n - 1)$ is assumed, $\Delta_{\boldsymbol{D}} \not\equiv 0 \pmod{n}$ holds. We choose $\boldsymbol{v}_{\mathrm{S}_1}^{[\tau_1]} = \mathbf{1}_n$. The remaining $n-1$ vectors from S_1 are computed by (48). To ensure that each vector from S_1 is allocated, n and $\Delta_{\boldsymbol{D}}$ must be coprime, i. e., $\gcd(n, \Delta_{\boldsymbol{D}}) = 1$. To show the linear independence of these beamforming vectors, we construct the following matrix:

$$\boldsymbol{B} = (\boldsymbol{v}_{\mathrm{S}_1}^{[\tau_1]}, \boldsymbol{v}_{\mathrm{S}_1}^{[\tau_1 + \Delta_{\boldsymbol{D}}]}, \ldots, \boldsymbol{v}_{\mathrm{S}_1}^{[\tau_1 + (n-1)\Delta_{\boldsymbol{D}}]}). \tag{50}$$

Then, $\boldsymbol{B}$ is a Vandermonde matrix as in [7]:

$$\boldsymbol{B} = \begin{pmatrix} 1 & B_0 & B_0^2 & \dots & B_0^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & B_{n-1} & B_{n-1}^2 & \dots & B_{n-1}^{n-1} \end{pmatrix}. \quad (51)$$

The determinant of such a Vandermonde matrix yields:

$$\det(\boldsymbol{B}) = \prod_{0 \le i < j \le n-1} (B_j - B_i) \neq 0, \quad (52)$$

since the B_m, for $m = 0, \dots, n-1$, are distinct almost surely. Thus, all beamforming vectors in $\boldsymbol{B}$ are linear independent. The beamforming vectors transmitted from S_2 are also linear independent by an analogous computation for (49).

For the given alignment scheme, $(\boldsymbol{F}_{jj}\boldsymbol{v}_{\mathrm{S}_1}^{[k]})^{-1}\boldsymbol{Y}_{\mathrm{R}_j}^{[k]}$ is computed for each dimension $k = 0, \dots, n-1$ so that a superposition of the codeword symbols $w_{\mathrm{S}_i}^{[k]}$ is received. The resulting symbols are comparably ordered as in (22), (23). Note that noise is negligible due to the assumption of high SNR.

2) *Second hop*: The relays amplify and forward their received signals from the previous first hop. Furthermore, the forwarded symbols are also index-shifted by the offsets γ_1, γ_2 given in (24), (25).

Relay R_1 sends n and R_2 sends a number of $n-1$ symbols $x_{\mathrm{R}_j}^{[0]}, \dots, x_{\mathrm{R}_j}^{[n-1]}$ along beamforming vectors $\boldsymbol{v}_{\mathrm{R}_j}^{[0]}, \dots, \boldsymbol{v}_{\mathrm{R}_j}^{[n-1]}$. Relay R_2 discards to forward $x_{\mathrm{R}_2}^{[\tau+\delta_{22}+\gamma_2]}$ along $\boldsymbol{v}_{\mathrm{R}_2}^{[\tau+\delta_{22}+\gamma_2]}$. The vectors align at T_1 and T_2 for $i = 0, \dots, n-1$ as:

$$\boldsymbol{G}_{11}\boldsymbol{v}_{\mathrm{R}_1}^{[i-\eta_{11}]} = -\boldsymbol{G}_{12}\boldsymbol{v}_{\mathrm{R}_2}^{[i-\eta_{12}]}, \quad (53)$$
$$i \not\equiv \tau + \delta_{22} + \gamma_2 + \eta_{12} \pmod{n},$$
$$\boldsymbol{G}_{21}\boldsymbol{v}_{\mathrm{R}_1}^{[i-\eta_{21}]} = -\boldsymbol{G}_{22}\boldsymbol{v}_{\mathrm{R}_2}^{[i-\eta_{22}]}, \quad (54)$$
$$i \not\equiv \tau + \delta_{22} + \gamma_2 + \eta_{22} \pmod{n}.$$

The dependencies of the beamforming vectors are resolved w. r. t. $\boldsymbol{v}_{\mathrm{R}_1}^{[\tau_2]} = \mathbf{1}_n$ with parameter $\tau_2 = \tau + \delta_{22} + \gamma_2 + \eta_{12} - \eta_{11}$ for $i = 1, \dots, n-1$ by:

$$\boldsymbol{v}_{\mathrm{R}_1}^{[\tau_2 + i\Delta_{\boldsymbol{E}}]} = \boldsymbol{G}^i \boldsymbol{v}_{\mathrm{R}_1}^{[\tau_2]}, \quad (55)$$
$$\boldsymbol{v}_{\mathrm{R}_2}^{[\tau_2 + i\Delta_{\boldsymbol{E}} - \eta_{12} + \eta_{11}]} = -\boldsymbol{G}^{i-1}\boldsymbol{G}_{22}^{-1}\boldsymbol{G}_{21}\boldsymbol{v}_{\mathrm{R}_1}^{[\tau_2]}. \quad (56)$$

The parameters are $\Delta_{\boldsymbol{E}} \equiv \eta_{12} - \eta_{11} + \eta_{21} - \eta_{22} \pmod{n}$ and $\boldsymbol{G} = \boldsymbol{G}_{11}^{-1}\boldsymbol{G}_{12}\boldsymbol{G}_{22}^{-1}\boldsymbol{G}_{21}$. Note that (55),(56) are analogous to (48),(49). Since $\det(\boldsymbol{E}) \not\equiv 0 \mod (x^n - 1)$ is assumed, $\Delta_{\boldsymbol{E}} \not\equiv 0 \pmod{n}$ holds. As in the first hop, n and $\Delta_{\boldsymbol{E}}$ must be coprime. Then, the linear independence of beamforming vectors is analogous to the scheme of the first hop.

The received signals $\boldsymbol{Y}_{\mathrm{D}_j}$ are filtered by $(\boldsymbol{G}_{jj}\boldsymbol{v}_{\mathrm{D}_j}^{[k]})^{-1}$ in each dimension k. The resulting received symbols are comparably ordered as in (9) and (10) with the special cases of (30) to (32). Thence, we can apply the Cyclic IA framework for IN of Section II on the transmitted symbols, and achieve $\frac{2n-1}{n}$ DoF by Theorem 2.

The linear scheme of [7] expressed by (42) to (45) can be translated to the cyclic polynomial channel model: We may use $n \geq 2$ dimensions, $\tau = n-1$, and the parameters $\gamma_1, \gamma_2, \delta_{ji}, \eta_{ji}$ given in the last paragraph of the proof for Theorem 2.

IV. Upper Bound

The capacity is limited by the min-cut upper bounds [7] which are valid for both given channel models. Thus, presuming that each message is received interference-free at its dedicated receiver, each user-pair would achieve the capacity of the corresponding point-to-point link.

Both channels given in Sections II-A and III-A are constrained to n dimensions. There is a maximum $M = 2n$ interference-free messages possible for n dimensions so that the maximal data rate is upper bounded by 2 DoF.

V. Conclusion

We study the concept of Interference Neutralization (IN) [5], [7] w. r. t. the Cyclic Interference Alignment scheme [3].

A set of *interference neutralization conditions* in (11), (12) and *no-signal-neutralization conditions* (13), (14) is defined to ensure that interfering messages are neutralized while dedicated signals are kept intact. We propose a Cyclic Interference Alignment scheme for IN that achieves $\frac{2n-1}{n}$ Degrees of Freedom for a symbol extension over n dimensions. The IN scheme is shown to asymptotically achieve the min-cut upper bound 2 DoF and confirms the results given in [7].

Our scheme is translated to the *Aligned Interference Neutralization* (AIN) framework of [7], which is based on spatial Interference Alignment as in [1]. A main insight is that the AIN scheme is generalized w. r. t. the alignment of beamforming vectors. Furthermore, we observe that AIN imposes additional constraints in comparison to the Cyclic IA framework in order to prevent loops in the dependencies of beamforming vectors.

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