Job Market Signaling and Employer Learning*

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Abstract

We consider a signaling model where the receiver is able to update his belief about the sender's type after the signaling stage. We introduce Bayesian learning in a variety of environments ranging from simple two-period to continuous time models with stochastic production. Signaling equilibria present two major departures from those obtained in models without learning. First, new mixed-strategy equilibria involving multiple pooling are possible. Second, pooling equilibria can survive the Intuitive Criterion when learning is fast enough.

1 Introduction

Signaling models are concerned with situations where an agent is able to send messages about information that he could not otherwise credibly reveal. Private information is valuable to the extent that it helps to predict the outcome of the transaction between the sender and the receiver. When the relationship involves repeated interactions, observing each outcome allows the receiver to update his belief. He is therefore able to gather knowledge about the sender after the signaling stage, a possibility that is usually excluded in signaling games. We argue that this omission is not inconsequential by embedding Bayesian learning in a standard signaling model and establishing that: (i) qualitatively new equilibria emerge; and (ii) forward induction (as captured by the Intuitive Criterion) loses part of its predictive power.

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These features arise because expectations about future payoffs do not solely depend on the equilibrium belief, as is commonly assumed in the literature, but also on the sender's type. Even when agents are indistinguishable right after the signaling stage, their expectations vary with their productivity because they know that it will be identified over time. The standard premise that senders are equally rewarded in pooling equilibria is therefore violated.

For the sake of concreteness, this general mechanism and its ramifications are illustrated in a job-market signaling model along the lines of Spence's (1973) seminal work. We consider an economy where workers are better informed about their ability than prospective employers. In order to signal their capacity, talented individuals have an incentive to invest in education. Alternatively, they may decide to save on educational costs and trust that their actual productivity will be revealed by performances on-the-job. Spence's model does not take into account this countervailing incentive because it assumes that all information is collected prior to labor market entry. Most of the ensuing theoretical literature follows Spence's approach and has ignored employer learning.[1] Econometricians, on the other hand, have devised ingenious tests for unobservable characteristics in order to estimate the efficiency of signal extraction in labor markets. Empirical evidence documented in Lange (2007) shows that employers are not only able to elicit information about workers' abilities but that the speed at which they do so is quite fast, with nearly 95% of the statistically significant information being collected after solely 3 years.[2]

This finding suggests that it is important to assess whether the outcomes of the signaling game are indeed unaffected by the learning process. We show in this paper that it is not the case. Firstly, qualitatively new types of equilibria may arise where more than one common level of education is acquired by both types of workers with positive probability (multiple pooling). Secondly, the standard

[1] The paper by Gibbons and Katz (1991) is a notable exception. They do not focus on the game theoretical analysis but on the implications of asymmetric information for layoff decisions. Although not directly concerned with learning, the paper by Feltovich et al. (2002) also allows correlated information to be revealed after the signaling stage. They consider a set-up with three types and show that high types may pool with low types at the lowest level of education. More recently, Voorneveld and Weibull (2011) have studied markets for lemons in which buyers receive a private noisy signal of the product's quality; whereas Daley and Green (2009) have analyzed a signaling model with grades using the D1 criterion. We discuss some of Daley and Green's findings in Section 4.

[2] Lange (2007) is among the latest contributions to a strand of literature measuring the speed of employer learning. Earlier works are discussed in Section 3. Available measures do not distinguish between symmetric and asymmetric learning. This is why Pinkston (2009) can use an approach similar to that of Lange (2007) to estimate a model of asymmetric learning in which all firms receive a private signal but the current employer always accumulate more information about the worker. Hence, empirical models of employer learning do not test job market signaling. Yet, as explained by Lange (2007), the estimated speed of learning can be used to place an upper bound on the contribution of signaling to the gains of schooling.

refinement argument embodied in the Intuitive Criterion does not bite when employer learning is sufficiently fast.

The second observation is particularly relevant from a game-theoretic point of view. The multiplicity of equilibria in Spence's model has been a motivation for the vast literature on refinement concepts. For signaling games with only two types, the Intuitive Criterion of Cho and Kreps (1987) is the most commonly used refinement because it excludes all but one (separating) equilibrium. We prove that this does not always hold true when employers are able to update their beliefs. The key difference between the two environments is that learning yields higher asset values for talented individuals even when pooling is the equilibrium outcome. The gap increases with the efficiency of the updating process as low and high types become less and more optimistic, respectively. The stronger the correlation between a worker's ability and his observable performance, the more attractive it is for high types to reveal their ability on-the-job instead of paying the educational costs.[3] This is why high types find it optimal to pool with low types when learning is fast.

In order to illustrate the generality of this mechanism, we set-up our model in the most parsimonious fashion. Our framework embeds Spence's original game into a framework where firms receive additional, post-education signals on the worker's productivity; including dynamic settings with Bayesian learning on the side of firms. Workers of different abilities can acquire education before entering the labor market. We assume that their abilities are either high or low. The two types case enables us to concentrate on the conceptual differences between our model and basic signaling games because the Intuitive Criterion delivers a clear prediction that can be used as a benchmark. By contrast, allowing for more than two types would distract us from our main focus since we would have to consider more elaborate and diverse refinement concepts.

For similar reasons, we do not investigate all the potential interactions induced by the learning process but instead specify some generic correlations between signaling and workers' expected incomes. Then we show that these properties are fulfilled by the simplest model of passive learning where the worker's only signal is education and firms update their beliefs using Bayes' rule, so that types are *gradually* revealed over time. Under this interpretation, our set-up bears similarities to Jovanovic's (1979) matching model with the crucial difference that uncertainty is not match-specific but worker-specific. Hence it bridges the gap between the theoretic literature on signaling games and the labor market literature. This suggests that, even though our description of the labor market is admittedly stylized, the general properties laid out in Section 2 are likely to hold true in most models featuring

[3] On the other hand, the incentive for low types to send misleading signals, and thus to acquire education, increases with the speed of employer learning. Haberlmaz (2006) proposes a partial equilibrium model which underlies the ambiguity of the relationship between the value of job market signaling and the speed of employer learning.

signaling and learning.

To sum up, we do not attempt to characterize all the implications of signal extraction because it is our conviction that such a task is beyond the scope of any paper. We also do not share the objective of the "refinement program" by aiming at uniqueness results under powerful refinement concepts. We view our work as a parsimonious step towards making signaling models more realistic in a direction whose importance has been widely documented in the labor economics literature. It is our hope that our previously unforeseen findings will pave the way for a more systematic program of research.

The paper is organized as follows. Section 2 lays out the model's set-up. To underline the generality of the results, we initially adopt a reduced form approach. We define workers' value functions and specify, using intuitive arguments, which key properties they should fulfill in order to capture the learning process. To fix ideas, we provide a first, simple example called the Reports Model. In Section 3, we analyze the equilibria in pure and mixed strategies of our model. Here we show the existence of qualitatively new mixed-strategy equilibria ("multiple pooling") and outline some of their implications for empirical research. In Section 4, we discuss the conditions under which the Intuitive Criterion can refine the set of equilibria and explain why they are not met when employer learning is efficient enough. Then, Section 5 illustrates how the key properties of the workers' value functions can be derived from first principles, describing the signal extraction problem in examples with both discrete and continuous time settings. Section 6 concludes. Proofs are relegated to the Appendix.

2 Signaling and Employer Learning

Workers differ in their innate abilities. They can be of different *types* which determine their productivity. Nature initially selects types according to pre-specified probabilities. The main ingredient of the model is information asymmetry: Workers know their abilities whereas employers must infer them.

The game is as follows. In a first step, the worker chooses an education level. In a second step, the industry offers a starting wage based on beliefs derived from the education signal. In Spence's static framework, the game ends as the worker enters the labor market. In our set-up, in contrast, a third step is added, where the relationship develops, with the industry being able to extract information from noisy realizations of the worker's productivity. Rather than proposing a particular dynamic model, we adopt a reduced-form approach to employer learning and delay its microfoundation to Section 5.

I. Education decision: We restrict our attention to cases where workers (senders) are of only two types, $i = h$ (high) or $i = l$ (low). Nature assigns a productivity $p \in \{p_l, p_h\}$ to the worker with $p_h > p_l$. High-types account for a share $\mu_0 < 1$ of the population. For simplicity, we assume that workers are infinitely lived and that they discount the future at the common rate r. Before entering the labor market, workers choose their education level $e \in [0, +\infty)$. To isolate the effect of signaling, we do not allow education to increase productivity. Its only use is to signal abilities which are initially unobserved by the industry (receiver).

Let the function $c : \mathbb{R}^{++} \times \mathbb{R}^{+} \to \mathbb{R}^{+}$ specify the cost of acquiring education. That is, $c(p, e)$ is the cost that a worker with innate productivity p has to pay in order to acquire education level e.[4] The cost function is twice differentiable with $c_e(p, e) > 0$ and $c_{ee}(p, e) \geq 0$, hence strictly increasing and convex in the level of education. As commonly assumed in the literature, we also let $c_p(p, e) < 0$ and $c_{pe}(p, e) < 0$, so that total and marginal costs of education are strictly decreasing in ability. The last requirement ensures that low types have steeper indifference curves than high types because it implies the submodularity condition

$$c(p_h, e'') - c(p_h, e') < c(p_l, e'') - c(p_l, e') \quad \text{whenever } e'' > e' ,$$

which is commonly referred to as the *Single Crossing Property.*

II. Wage setting: As in Spence's model, a worker is paid his expected productivity. This wage setting rule is justified as a proxy for a competitive labor market or a finite number of firms engaged in Bertrand competition for the services of the worker. Under the first possibility, one can make sense of the condition by assuming a single player in place of the industry, with payoffs given by a quadratic loss function $-(w - p)^2$. In any Perfect Bayesian Equilibrium, optimal behavior of this player will lead to a wage offer equal to the expected productivity.[5] Under the second possibility, where Bertrand competition among multiple firms is explicitly introduced, the equilibrium concept must be refined to ensure that all firms share the same beliefs about the worker.[6] Let $\mu(e)$, which we abbreviate by μ when no confusion may arise, denote the probability that the industry attaches to the worker being of the high type given education signal e. The initial wage is given by $w(\mu) = (1 - \mu)p_l + \mu p_h$. We can therefore use the equilibrium belief of the industry to denote its response to a particular signal.

III. Expected income: The key departure from Spence's model is the third step where we specify the sender's expected income. We adopt in this section a reduced-form approach: Given the actual

[4] Even though workers will only have productivity levels in $\{p_l, p_h\}$, it is convenient to define the cost function for all potential productivities.

[5] Even though the resulting game is properly specified, there is no economic interpretation for the payoff function $-(w - p)^2$ but rather for the result of the optimization problem.

[6] This is the approach adopted by Mas-Colell *et al* (1995) in Section 13.C of their textbook. Notice that *common beliefs* follow from the concept of sequential equilibrium but not of Perfect Bayesian Equilibrium. However, sequential equilibrium is only defined for finite games.

productivity p and the employer's prior $\mu = \mu(e)$, expected earnings are given by a *value function*[7] $v(p, \mu)$ which captures the impact that employer learning has on future income.[8]

For the purposes of enabling comparative statics, we explicitly introduce two additional parameters. The first one is the common discount rate r of workers whose interpretation depends on the particular model behind the value function. We adopt the convention that the forward-discounted value is given by w/r when the agent receives a constant wage w, as in a continuous-time model with $r > 0$. The second parameter, $s > 0$, measures the informativeness or precision of additional, post-education signals on worker's productivity.[9] Thus we will write $v(p, \mu|r, s)$ whenever we wish to discuss the effect of those two parameters.

Let us now specify the properties that should be imposed on value functions and the rationales behind them. As employment histories unfold, employers observe the cumulative outputs produced by workers and use this information to revise their priors. Consider how the updating process affects the expectations of low types. For every $\mu \in (0, 1)$, they are offered an initial wage $w(\mu)$ that is above their actual productivity p_l. On average, realized outputs will induce the industry to lower its belief and thus wages. Their expected income $v(p_l, \mu)$ is therefore smaller than the forward discounted value of the starting wage w/r, but larger than the value p_l/r that they would have obtained if the industry had known their type with certainty. A symmetric argument holds for high types because their initial wage is lower than their actual productivity. On the other hand, when industry's beliefs collapse to certainty,[10] that is $\mu \in \{0, 1\}$, further information will be ignored and the initial wage will never be altered. Since we want to encompass Spence's model as a particular case, we first introduce a weak implication of this argument:

[7] From a purely game-theoretic point of view, the value function needs to be interpreted as an equilibrium device, in the sense that, given a specific microfoundation (i.e. a specification of the part of the game which the function summarizes), the computation of the outcome delivered by the function must rely on the explicit use of a uniquely defined equilibrium concept. As we will show in Section 5, alternative value functions can be obtained from different models.

[8] The value function depends on the industry's prior rather than the initial wage because, as commented before, we use the belief of the industry to denote its response to the education signal. This relies on the *equilibrium* requirement that beliefs uniquely determine wages. Strictly speaking, out-of-equilibrium behavior for the industry might involve wages which are not consistent with beliefs. This possibility, however, is inconsequential for the analysis because of the game structure. Alternatively, one can simply consider that Spence's modeling device, which yields a wage of $w(\mu)$ is replaced by $v(p, \mu)$ in our model.

[9] A microfoundation for s is derived in the continuous time model of Section 5. It shows that s can equivalently be interpreted as the speed of employer learning.

[10] Observe that, since we are describing the game and not an equilibrium outcome, we do not assume that beliefs are correct and thus define the value function of the low (high) type when $\mu = 1$ ($\mu = 0$).

P0. For all $\mu \in (0,1)$,

$$\frac{p_h}{r} > v(p_h, \mu) \geq \frac{w(\mu)}{r} \geq v(p_l, \mu) > \frac{p_l}{r}.$$

Further, when $\mu \in \{0,1\}$, $v(p_h, \mu) = v(p_l, \mu) = w(\mu)/r$.

Because of learning, we will actually expect a stronger variant of this property to hold:

P1. For all $\mu \in (0,1)$,

$$\frac{p_h}{r} > v(p_h, \mu) > \frac{w(\mu)}{r} > v(p_l, \mu) > \frac{p_l}{r}.$$

Further, when $\mu \in \{0,1\}$, $v(p_h, \mu) = v(p_l, \mu) = w(\mu)/r$.

Property P1 ensures that firms are able to update their beliefs towards the realization of the randomly assigned productivity. We will show in Section 5 that this is a general consequence of Bayesian learning.

Compare now two industries whose correlated signals have different precision. Workers whose productivity is overestimated prefer being employed in the industry where signal extraction is slow. Conversely, workers whose productivity is underestimated would rather choose the industry where actual abilities are quickly recognized. The following property captures this intuition.

P2. For all $\mu \in (0,1)$, we have that $\partial v(p_h, \mu|r,s)/\partial s > 0$ and $\partial v(p_l, \mu|r,s)/\partial s < 0$, hence $v(p_h, \mu|r,s)$ is strictly increasing in s whereas $v(p_l, \mu|r,s)$ is strictly decreasing in s.

We will show in Section 5 how to derive s from primitive parameters describing the production process and prove that its impact on $v(\cdot)$ is as postulated in P2.

Finally, consider limit cases where precision is very low or very high. As s goes to zero, employers have no possibility to update their initial beliefs: Wages are never revised and the value functions converge to their original specification in Spence's model. Conversely, when s goes to infinity, types are immediately recognized and signaling becomes redundant.

P3. For all $\mu \in (0,1)$, $\lim_{s \to 0} v(p, \mu|r,s) = w(\mu)/r$ and $\lim_{s \to \infty} v(p, \mu|r,s) = p/r$.

Definition 1. *A* ***value function*** *is a mapping assigning a lifetime income $v(p, \mu)$ to each belief $\mu \in [0,1]$ and productivity $p \in \{p_l, p_h\}$. It is assumed to be strictly increasing and twice differentiable in μ, and to fulfill property P0.*

A value function is said to exhibit ***weak learning*** *if it fulfills the more demanding property P1. It exhibits* ***strong learning*** *if it fulfills properties P1, P2, and P3.*

V. Reports model: Spence's problem is encompassed in our framework when the value function $v(p, \mu) = w(\mu)/r$, with the interpretation that the starting wage is final and no further adjustments (of wages or beliefs) are possible. This function fulfills the basic conditions in Definition 1. Alternatively, it can be interpreted as a limit case of a value function with strong learning when s goes to 0. The stronger properties listed in Definition 1, which capture the basic intuition on employer learning, can be shown to follow from Bayesian learning in a setting where employers receive an additional, informative but imprecise signal on workers' productivity.

In order to clarify the exposition, we relegate the dynamic microfoundations of the model to Section 5. Those include explicit learning based on actual production in multi-period or continuous-time settings. Here we present an illustrative model which allows for considerably simpler computations, and which we call the *Reports Model*.

In this particular model, we assume that updating occurs only once over the labor market career of a given worker. The employer has access to a detection technology which delivers a signal about the agent's type. We call this signal a "report" in order to avoid confusion with the education decision.[11] In this model, the parameter s is literally the informativeness of a post-education signal on the worker's productivity.

The report is extracted as soon as the agent starts working.[12] There are two possible reports, G(ood) and B(ad). If the productivity of the worker is high, a good report is delivered with probability $d(s) > 1/2$ for $s \in (0, +\infty)$, and a bad report is delivered with the complementary probability. Similarly, when the worker's productivity is low, the likelihoods of a good or bad report are $1-d(s) < \frac{1}{2}$ and $d(s)$, respectively. The function $d(s)$ is strictly increasing in s with the following lower and upper bounds: $\lim_{s\to 0} d(s) = 1/2$ and $\lim_{s\to\infty} d(s) = 1$. Since d is essentially a change of variable, we will simply write $d = d(s)$.

Firms use reports to update their beliefs and then pay workers their expected productivity over an infinite horizon.[13] Thus the expected lifetime income of an employee with ability i is simply

$$v(p, \mu|r, s) = \frac{1}{r}E[w(\mu')|p, \mu, s] \ , \tag{1}$$

where μ' denotes the updated belief of the industry and $E[\cdot|p, \mu, s]$ is the expectation operator conditional on the industry's prior, worker's type and precision of the correlated information s.

[11] That is, actual production by the worker does not influence learning. A possible justification is that output cannot be traced back to each individual. However, we insist that this is merely a "toy model". The reader is referred to Section 5 for more realistic microfoundations.

[12] One may think of the report as resulting from the worker's performance during his job interview, a trial period, etc.

[13] These timing conventions are introduced for consistency with the dynamic models discussed in Section 5 below but are inessential to the analysis. For the Reports Model, nothing changes if one merely sets $r = 1$.

As the updated belief μ' is formed according to Bayes' rule, we have

$$\mu'(\mu, G) = \frac{\mu d}{\mu d + (1-\mu)(1-d)} \quad \text{and} \quad \mu'(\mu, B) = \frac{\mu(1-d)}{\mu(1-d) + (1-\mu)d} \; .$$

Wages $w(\mu, S) = w(\mu'(\mu, S))$ therefore depends on the signal $S \in \{B, G\}$ in the following way

$$w(\mu, G) = \frac{\mu d p_h + (1-\mu)(1-d)p_l}{\mu d + (1-\mu)(1-d)} \quad \text{and} \quad w(\mu, B) = \frac{\mu(1-d)p_h + (1-\mu)d p_l}{\mu(1-d) + (1-\mu)d} \; .$$

Weighting these two payoffs with their respective probabilities yields the expected wage in the second period

$$E[w(\mu')|p_l, \mu] = (1-d)w(\mu, G) + dw(\mu, B) \quad \text{and} \quad E[w(\mu')|p_h, \mu] = dw(\mu, G) + (1-d)w(\mu, B) \; .$$

Straightforward but cumbersome computations show that $v(p, \mu|r, s)$ fulfills all the requirements for *strong learning* listed in Definition 1.

An alternative, slightly more involved model would be obtained if firms cannot immediately update their beliefs because reports are delivered e.g. at the end of a first period of production. In such a "two-period Reports Model", the expected lifetime income of an employee with ability i is given by[14]

$$v(p, \mu|r, s) = \frac{1}{1+r}\left(w(\mu) + \frac{E[w(\mu')|p, \mu, s]}{r}\right) \; . \tag{2}$$

The Value function now displays weak and not strong learning. Indeed, one can verify that $v(p, \mu|r, s)$ now violates property P3 because $\lim_{s\to\infty} v(p, \mu|r, s) \neq p/r$. This is because employer learning matters solely for earnings after the non-negligible first period. Whether the model generates weak or strong learning crucially depends on the timing of the report. We note here that, in a model with infinitely many production periods (e.g. production in continuous time), the initial beliefs and wages become negligible in the limit, and so strong learning is restored (see Section 5). The contrast between the one- and two-period version of the Reports Model, however, shows that this is not due to the difference between finite and infinite horizon, but to whether or not beliefs before the first additional signal have a negligible impact on lifetime income.

VI. Equilibrium concept: The equilibrium concept is just Perfect Bayesian Equilibrium for the game as specified before.[15] Combining the value and cost functions yields the payoff function for workers: $u(e, \mu|p) \triangleq v(p, \mu) - c(p, e)$. As discussed in subsection II, industry's payoffs are inconsequential as long as the specification leads to wage offers equal to the expected productivity given the beliefs.

[14] The scale factor $1/(1+r)$ is immaterial to the analysis and has been introduced for consistency across specifications, since it ensures that receiving the wage w forever yields a value of w/r.

[15] There is some confusion in the older game-theoretic literature with respect to the Perfect Bayesian Equilibrium (PBE) concept. For games with the structure of signaling games, PBEs coincide with "Weak PBEs" which are defined as

We will consider equilibria in pure and mixed strategies ("hybrid equilibria"). The very formalization of both Spence's model and ours is such that the industry never has an incentive to randomize. An equilibrium in mixed strategies can be defined as follows.

A *signaling equilibrium* is made out of a pair of probability distributions $(q(\cdot|p))_{p\in\{p_l,p_h\}}$ on $\mathbb{R}^+$ describing the education levels chosen by both types, and a belief system $(\mu(e))_{e\in\mathbb{R}^+}$ describing the priors of the industry given any possible signal, that satisfies the following properties:

- *Sequential rationality for the workers*: For $p \in \{p_l, p_h\}$, if e^* belongs to the support of $q(\cdot|p)$, then $e^* \in \arg\ \max_{e\in\mathbb{R}^+} u(e, \mu(e)|p)$.
- *(Weak) consistency of beliefs*: The industry's initial beliefs $\mu(e)$ are consistent with Bayes' rule for any educational attainment e in the support of either $q(\cdot|p_h)$ or $q(\cdot|p_l)$.
- *Sequential rationality for the industry*: Given any education level e, the industry offers an initial wage equal to $w(\mu(e))$.

In the pure strategy case, the support of the distributions $q(\cdot|p)$ is a singleton, implying that a *signaling equilibrium in pure strategies* can be described by a belief system as before, and a strategy profile (e_l, e_h) describing the deterministic education levels chosen by both types.[16]

3 Equilibrium Analysis

Having provided alternative microfoundations for the assumptions laid out in Section 2, we proceed to characterize the signaling equilibria of the game. We start with pure strategy equilibria, which are fully described by the beliefs $\mu(e)$ and the education levels selected by both types, e_l and e_h. If $e_l \neq e_h$, one speaks of a *separating* equilibrium, while the equilibrium is called *pooling* if $e_l = e_h$. In the second

pairs of beliefs and strategy profiles such that actions at any information set maximize payoffs given the beliefs (sequential rationality) and beliefs are consistent with strategies through Bayes' rule along the equilibrium path (weak consistency). This is the equilibrium concept that the Intuitive Criterion aims to refine. Unfortunately, some of the earlier literature called PBEs "sequential" in reference to the sequential rationality requirement. The concept of sequential equilibrium (Kreps and Wilson 1982), however, is defined for finite games only and hence does not apply in our framework.

[16] For the mixed-strategy case, we require workers to randomize among optimal education levels only. This eliminates from the onset technical difficulties associated with the inclusion of suboptimal strategies with zero probability in the support of the equilibrium strategies. The typical examples we have in mind at this point involve randomization over finitely many education levels. For such equilibria, belief consistency amounts to the assertion that, if there exists a $p \in \{p_l, p_h\}$ such that $q(e|p) > 0$, then $\mu(e) = \mu_0 q(e|p_h)/[\mu_0 q(e|p_h) + (1-\mu_0)q(e|p_l)]$. A priori, however, an equilibrium strategy might prescribe a randomization over an infinite set.

subsection, we characterize mixed-strategy equilibria and then discuss their empirical implications in the third subsection.

3.1 Pure-strategy equilibria

Separating equilibria are such that abilities are perfectly revealed: Depending on the education signal, the beliefs of the industry on the equilibrium path are either $\mu = 0$ or $\mu = 1$. It follows that each type receives an initial wage equal to his productivity and P1 implies that $v(p_i, 1) = p_i/r$ for $i \in \{l, h\}$. In other words, employer learning does not affect separating equilibria, and the proof of the following Proposition follows from standard arguments (we hence omit it).

Proposition 1. *Consider any value function. An education profile* (e_l, e_h) *with* $e_l \neq e_h$ *can be sustained as a (separating) signaling equilibrium if and only if*

(i) $e_l = 0$, *and*

(ii) $e_h \in [\underline{e_h}, \overline{e_h}]$ *where* $\overline{e_h} > \underline{e_h} > 0$ *and these two education levels are uniquely defined by*

$$c(p_l, \underline{e_h}) - c(p_l, 0) = \frac{1}{r}(p_h - p_l) = c(p_h, \overline{e_h}) - c(p_h, 0) .$$

Accordingly, the set of separating equilibria does not depend on the value function.

In pooling equilibria, both types select the same education level e_p and are therefore offered the same initial wage $w(\mu_0)$. Contrary to the model without learning, Bayesian updating leads to different lifetime incomes, $v(p_h, \mu_0) > v(p_l, \mu_0)$. As shown in the following proposition, this implies that the set of pooling equilibria shrinks as information precision increases.

Proposition 2. *Consider any value function. A common education level* e_p *can be sustained as a pooling equilibrium if and only if* $e_p \in [0, \overline{e_p}]$ *where* $\overline{e_p} > 0$ *is uniquely defined by the equation*

$$c(p_l, \overline{e_p}) - c(p_l, 0) = v(p_l, \mu_0) - \frac{p_l}{r} .$$

The upper bound fulfills $\overline{e_p} < \underline{e_h}$, *where* $\underline{e_h}$ *is as defined in Proposition 1. Further, under strong learning,* $\overline{e_p}$ *is strictly decreasing in the precision* s *and* $\lim_{s\to\infty} \overline{e_p} = 0$.

The set of pooling equilibria is given by a subset of the one in Spence's model. Learning lowers the upper-bound $\overline{e_p}$ because the incentives for low types to mimic high types decrease as the ability of firms to detect them improve. Under strong learning and as informativeness s goes to infinity, firms ignore education levels in favor of information collected after the signaling stage. There is no reason to spend resources on signaling which explains why the set of pooling equilibria collapses to the minimum level of education.

3.2 Mixed-strategy equilibria

As usual in signaling games, the model admits a plethora of mixed-strategy equilibria. Those are often referred to as *partial pooling* or *hybrid* equilibria. The following proposition shows that learning supports mixed-strategy equilibria which are qualitatively different from those possible in Spence's model.

Proposition 3. *Consider any value function. For any mixed-strategy signaling equilibrium, there exists a set of education levels $E_p \subseteq [0, \underline{e_h}]$, where $\underline{e_h}$ is as defined in Proposition 1, such that the following properties hold:*

(i) The support of the low types' strategy consists of E_p and at most a further education level, $e_l = 0$, such that $e_l < e$ for all $e \in E_p$.

(ii) The support of the high types' strategy consists of E_p and at most a further education level e_h, such that $e_h > e$ for all $e \in E_p$.

(iii) The industry's beliefs μ are strictly increasing on E_p.

In Spence's model, E_p contains at most one education level, but this is not true for general value functions.[17]

Equilibria with $E_p = \emptyset$ are the separating equilibria of Proposition 1. The pooling equilibria of Proposition 2 are such that $E_p = \{e_p\}$ where no additional education levels are chosen. In Spence's model, the set E_p (if nonempty) always consist of a unique education level, but this is not true in general for other value functions. Correlated information qualitatively enriches the set of outcomes: *multiple pooling* may arise, i.e. equilibria with more than one education level chosen by both types.

Although the set of mixed-strategy equilibria might be large and capture complex phenomena as multiple pooling, such equilibria are far from being arbitrary. In particular, the monotonicity property (iii) shows that there can be no counterintuitive correlation. Education remains informative in the sense that a higher (equilibrium) education level is always associated with a higher probability that the worker is of the high type.

[17] Proposition 3 can be refined assuming that, for all $\mu \in (0,1)$, $\partial^2 v(p_h,\mu)/\partial\mu^2 < \partial^2 v(p_l,\mu)/\partial\mu^2$, a property that can easily be established in the reports and two period models (and is also illustrated in Figure 4 in Section 5). Then, $P1$ implies that there exists a unique belief $\mu^* \in (0,1)$ such that $\partial v(p_l,\mu^*)/\partial\mu \lesseqgtr \partial v(p_h,\mu^*)/\partial\mu$ if and only if $\mu \lesseqgtr \mu^*$. It is then straightforward to show that, under weak or strong learning, $\mu(e) > \mu^*$ for all $e \in E_p$ except possibly the minimum education level in E_p (if a minimum exists).

Before looking at multiple pooling equilibria in more detail, let us briefly explain why they may arise. Consider first the model without correlated information. The argument for the impossibility of multiple pooling is as follows. If a given type chooses two different levels of education with positive probability, then he must be indifferent between those two education levels, implying that the difference in expected income must equal that in educational costs. Without employer learning, each level of education results in identical incomes for both types. By the single crossing property, however, differences in educational costs across signals are higher for low types. Hence both types cannot randomize over the same set.

The contradiction disappears with employer learning because, even when priors are identical, lifetime earnings differ across types. Then the difference in the increase in education costs required by the single-crossing property can be compensated by different gains in lifetime earnings across types.

We remark that the set of equilibria is also enriched if one moves away from the classical signaling setting by relaxing the single crossing property.[18] Then, the contradiction mentioned above does not follow because differences in educational costs across signals can be identical for both types. Araujo and Moreira (2010) establish that a violation of the single crossing property may lead to continuous pooling in adverse selection problems. It would be interesting to apply their methodology to signaling models in order to distinguish the impact of the signaling technology from that of the learning process.

Multiple pooling equilibria with linear costs. We now present an extended example to better understand multiple pooling equilibria. We consider the Reports Model with linear cost functions $c(p,e) = g(p)e$, where $g'(p) < 0$. Suppose that a mixed-strategy equilibrium exists and let $e_1, e_2 \in E_p$ with $e_1 < e_2$. By definition, both types have to be indifferent between e_1 and e_2, i.e.

$$v\left(p_i, \mu\left(e_1\right)\right) - g(p_i)e_1 = v\left(p_i, \mu\left(e_2\right)\right) - g(p_i)e_2 \ ,$$

for $i \in \{l, h\}$. Combining both equations, one obtains

$$v\left(p_h, \mu\left(e_2\right)\right) - v\left(p_h, \mu\left(e_1\right)\right) = \frac{g(p_h)}{g(p_l)}\left[v\left(p_l, \mu\left(e_2\right)\right) - v\left(p_l, \mu\left(e_1\right)\right)\right].$$

Thus if we define the function

$$h\left(\mu\right) \triangleq v\left(p_h, \mu\right) - \frac{g(p_h)}{g(p_l)} v\left(p_l, \mu\right) \ , \tag{3}$$

a *necessary* condition for the existence of mixed-strategy equilibria is that there exist $\mu_1, \mu_2 \in [0,1]$, with $\mu_1 \neq \mu_2$, such that

$$h\left(\mu_1\right) = h\left(\mu_2\right) \ . \tag{4}$$

[18] That is, allowing $c_{pe}(p,e)$ to be positive over some region of the parameter space.

We will use the function $h(\mu)$ for an informal discussion here, relegating the exact characterization of multiple pooling equilibria to the Appendix. Figure 1 depicts the function $h(\mu)$ for a particular set of parameters. It can be shown (see the proof of Proposition 4 in the Appendix) that h is strictly concave, i.e. the qualitative features of the figure hold in general. It follows that condition (4) can only hold true for a pair of beliefs, establishing that the set E_p contains at most two signals. Furthermore, $h(\mu)$ features an interior maximum. Thus, as illustrated by the horizontal dashed lines, beliefs that are close enough to the maximum can be matched so as to meet condition (4).

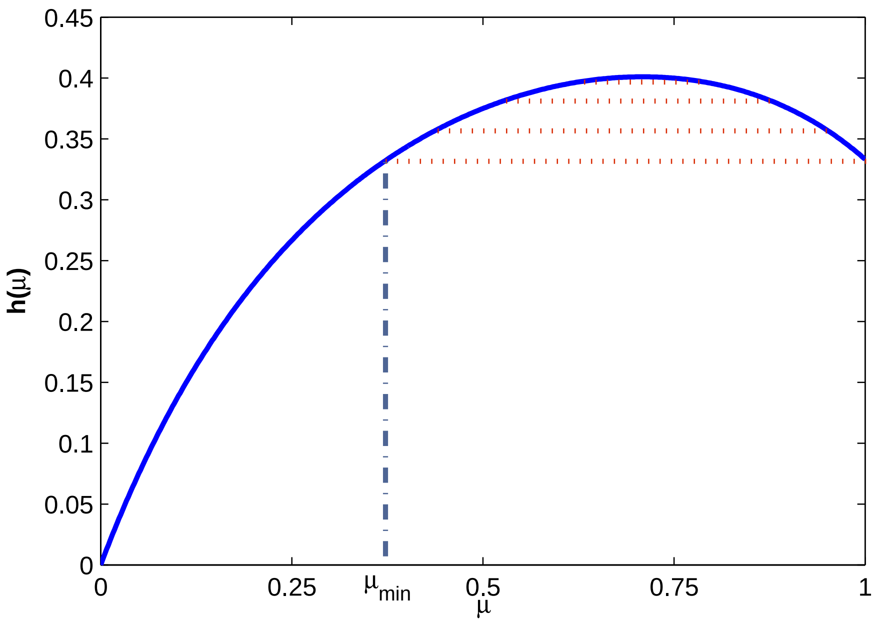

Figure 1: Beliefs supporting mixed-strategy equilibria. Parameters: $p_l = 0$, $p_h = 1$, $g(p_l) = 0.15$, $g(p_h) = 0.1$, $d = 3/4$, $r = 1$.

The set of PBE crucially depends on the share μ_0 of high types in the population. Let $\mu_{\min} < 1$ be such that $h(\mu_{\min}) = h(1)$ as in Figure 1. If $\mu_0 > \mu_{\min}$, one can always pick a pair of beliefs $\{\mu_1, \mu_2\}$ such that $\mu_0 \in (\mu_1, \mu_2)$ and $h(\mu_1) = h(\mu_2)$. Then it is straightforward to construct a "bipolar" equilibrium where neither type separates by sending a third education level. The randomization strategies $q(e|p)$ supporting the equilibria are uniquely pinned down by weak consistency of beliefs, because $q(e_1|\, p_l)$ and $q(e_1|\, p_h)$ leave us with two linear equations and two unknowns. The participation constraint of the low types[19] delivers an upper bound for the actual value of e_1 (while e_2 is uniquely determined

[19] As usual in signaling problems, the high types' constraint is redundant due to the single crossing property.

from e_1 by indifference), resulting in a set of possible multiple pooling equilibria for the selected μ_1, μ_2.

Consider now cases where $\mu_0 < \mu_{\min}$. Then it must be true that $\mu_0 < \mu_1 < \mu_2$, which can be satisfied in equilibrium if and only if low types randomize on E_p and the additional separating signal $e_0 = 0$. In other words, "bipolar" equilibria cannot anymore be sequentially rational. Consistent randomization strategies can nevertheless be constructed for appropriate values of $q\left(0|\, p_l\right)$. In fact, the construction is then easier, since one has a further degree of freedom. Hence, when $\mu_0 < \mu_{\min}$, there exists a continuum of randomization strategies associated to any equilibrium beliefs. The education levels, by contrast, are uniquely determined. Given that low types also separate by acquiring zero education, their participation constraint $u(e_1, \mu_1|p_l) \geq p_l/r$ will bind, pinning down the value of e_1.

This heuristic reasoning suggests that the set of mixed-strategy equilibria can be thoroughly characterized. The following proposition describes the actual features of such equilibria and provides an explicit, necessary and sufficient condition for the existence of multiple pooling equilibria. Intuitively, this condition requires correlated reports to be precise enough so the model sufficiently differs from Spence's.

Proposition 4. *Consider the Reports Model with linear costs, $c(p,e) = g(p)e$ and $0 < g(p_h) < g(p_l)$. Let μ_0 be the share of high types in the population. Multiple pooling equilibria exist if and only if*

$$d > \frac{1}{2}\left[1 + \sqrt{\frac{g(p_l) - g(p_h)}{g(p_l) + 3g(p_h)}}\right]. \tag{5}$$

Further,

- *(a) in any multiple pooling equilibrium, both types randomize on a common set containing exactly two signals, $E_p = \{e_1, e_2\}$ with $0 < e_1 < e_2$. Moreover, high types randomize exclusively among those two signals, i.e. there is no additional separating signal for high types;*
- *(b) there always exist multiple pooling equilibria where low types randomize on E_p and an additional separating signal $e_0 = 0$; and*
- *(c) there exists a $\mu_{\min}$ such that "bipolar" equilibria, where the low types randomize exclusively among the two signals in E_p, may arise if (and only if) $\mu_0 > \mu_{\min}$.*

As expected, pooling over more than one signal is ruled out when there is no correlated information ($d = 1/2$). Conversely, reducing the gap in educational costs between both types enlarges the parameter set over which mixed-strategies are sequentially rational. This is because differences in the slope of the value functions have to offset those in educational costs. As the latter is reduced, less requirements are imposed on the value functions. In the limit case where the two cost functions converge, $g(p_l) \to g(p_h)$, the necessary and sufficient condition (5) is always fulfilled.

3.3 Empirical Implications of Multiple Pooling

The previous sections were devoted to the theoretical analysis of the game. We now turn our attention to its empirical content and illustrate how it helps to rationalize the statistical discrimination documented in, e.g. Farber and Gibbons (1996) or Altonji and Pierret (2001). Both papers show that firms update their initial beliefs about the ability of their employees. The identifying assumption is that econometricians have access to a correlate of abilities that is not available to employers.[20] The impact of this correlate on wages increases with labor market experience, indicating that firms do learn over time. For learning to occur on-the-job, relevant information must be hidden after the signaling stage. In other words, some education levels have to be chosen by both types.

In pure pooling equilibria, educational choices do not convey any information. It is as if the signaling stage never occurred so that labor market outcomes are indistinguishable from those described in Jovanovic's (1979) model when skills are transferable across jobs and not match-specific. The updating process, and consequently wage dynamics, are independent of the educational choice. Instead, they depend solely on the informativeness of the production technology.

The multiple pooling equilibria described in Subsection 3.2 restore a connection between labor market and educational outcomes. According to Property (iii) in Proposition 3, expected earnings increase with educational attainments. The returns to education for each type are exogenously pinned down by their cost functions. In the data, however, one can only observe the pool of workers that have reached a given level of education. Observable returns are therefore affected by a composition effect.[21] It is most easily characterized in the Reports model with linear costs. By definition, expected returns are given by $E[v|e] - E[v|0]$, where

$$E[v|e] = \mu(e)v(p_h, \mu(e)) + (1 - \mu(e))v(p_l, \mu(e)) \ ,$$

i.e. $E[\cdot|e]$ is the expectation operator conditional on education, which captures the information available to econometricians. Consider situations where precision is high enough for equation (5) to hold. Then workers may randomize over three education levels and the indifference requirement for low types implies that $v(p_l, \mu(e)) - p_l/r = g(p_l)e$. Observable returns are therefore equivalent to

$$\frac{E[v|e] - E[v|0]}{e} = g(p_l) + \mu(e)\frac{v(p_h, \mu(e)) - v(p_l, \mu(e))}{e} \ .$$

The second term captures the composition effect, showing that it depends on two mechanisms. First, high types account for a larger share of workers as education, and consequently μ, increase. This

[20] In practice, most papers use the Armed Forces Qualification Test scores of workers.

[21] We thank an anonymous referee for bringing this mechanism to our attention.

raises the education premium because high types have higher expected earnings than low types. On the other hand, the gap in earnings decreases with education because high types also have lower marginal costs.[22] Whether observable returns are positively correlated with education depends on which of these two channels dominate. In most configurations, however, the second channel turns out to be stronger so that the expression above decreases in e.[23] This negative composition effect might help to explain empirical evidence about the slim premium commanded by PhDs over master's degrees.[24]

Mixed-strategy equilibria also imply that wage dynamics will differ across education levels. The less precise the initial belief, the more weight is assigned to information revealed on-the-job. This implies that wages are the most volatile when low and high types are equally likely to chose a given level of education, i.e. when the industry's belief is equal to 1/2. Whether this maximum is approximated at the upper or lower end of the educational choices depends on the exogenous parameters of the model as well as on the workers' randomization strategies.[25]

Human capital accumulation does not generate such correlation between educational attainments and wage volatility. Identifying it in the data could therefore be used as a test of job market signaling. We are not aware of papers investigating this question, probably because empirical studies usually rely on Informational equilibria rather than Perfect Bayesian equilibria.[26] Given that Informational equilibria are such that types perfectly separate, one needs to introduce other sources of uncertainty in order to generate employer learning. But since there is no reason to believe that the additional noise is correlated with the education level, Informational equilibria deliver learning speeds that are indistinguishable across educational groups in a given industry. As a consequence, the empirical literature has focused on sorting across industries with presumably different speeds of employer learning.[27]

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[22]This immediately follows from the indifference requirement across signals since $v(p_h, \mu(e_2)) - v(p_l, \mu(e_2)) - [v(p_h, \mu(e_1)) - v(p_l, \mu(e_1))] = [g(p_h) - g(p_l)](e_2 - e_1) < 0$.

[23]A straightforward computation shows that the composition effect is negative when $g(p_l)e/\mu(e) = (v(p_l, \mu(e)) - v(p_l, 0))/\mu(e)$ increases in e. It is therefore sufficient that the average slope of the value function for low types increases when equilibrium beliefs are raised from $\mu(e_1)$ to $\mu(e_2)$.

[24]Recent evidence on the premia across education levels can be found in Casey (2009).

[25]These features are easily established in the continuous-time model of Section 5. It can deliver wage volatilities that are increasing, decreasing or even inverted U-shaped in the level of education. It may also exhibit separation, and thus constant wages, at the highest and lowest qualification. The only general restriction is that U-shaped segments can be excluded because beliefs are increasing in education.

[26]See Riley (1979a, 1979b) for a thorough treatment of Informational equilibria in the context of signaling problems. As discussed above, econometricians have used this concept to study the interaction between the speed of learning and signaling, e.g. Farber and Gibbons (1996), Altonji and Pierret (2001), and more recently Kaymak (2006).

[27]We investigate this question in the working paper version of this article where we extend our model by assuming that workers are imperfectly aware of their productivity. We find that returns to ability and signal precision may have

4 The Intuitive Criterion

Even without introducing correlated information, signaling games typically have a large set of equilibria. In order to narrow it, a number of refinement concepts have been developed. The Intuitive Criterion (Cho and Kreps 1987), however, remains a milestone in the analysis of signaling games. When there are only two types, it is well known that the Intuitive Criterion confers a predictive power to Spence's model by ruling out all but one separating equilibrium, known as the *Riley equilibrium*. The purpose of this section is to show that this does not hold true when beliefs can be updated after the signaling stage. More precisely, we prove that, even though the Riley equilibrium retains its importance as the only separating equilibrium fulfilling the Intuitive Criterion, it is in general not true that all pooling (or mixed) equilibria are ruled out.

A signaling equilibrium is said to fail the Intuitive Criterion if some type could strictly profit by sending a non-equilibrium signal, provided that the sender adopts non-equilibrium beliefs satisfying the following requirement: Assign probability zero to types which could never conceivably profit by sending the considered signal. In our set-up, the Intuitive Criterion amounts to the following. Fix a signaling equilibrium. Say that an unused signal e is equilibrium-dominated for type p if the equilibrium payoff of type p is strictly larger than the payoff that type p would receive with signal e, given any conceivable (non-equilibrium) belief of the industry $\mu'(e)$. The equilibrium is said to fail the Intuitive Criterion if there exists a signal e and a type p such that the equilibrium payoff of type p is strictly smaller than the minimum payoff that this type could get by sending signal e, given any possible (non-equilibrium) beliefs $\mu'(e)$ of the industry which concentrate on the set of types for which signal e is *not* equilibrium dominated.

Separating equilibria. In the absence of employer learning, the *Riley equilibrium* is the only separating equilibrium which survives the Intuitive Criterion: Low types do not acquire any education and high types send the signal $\underline{e_h}$ defined in Proposition 1, implying that low types are indifferent between their equilibrium strategy and acquiring education e_h in order to receive $v(p_l, 1) = p_h/r$. Given that separating equilibria do not depend on the value function, it is not surprising that the result carries over to our set-up.[28]

opposite and potentially countervailing effects on the relationship between education levels across industries and the informativeness of the production process. This finding stands in sharp contrast to predictions based on Informational equilibria. It implies that conclusive evidence cannot be drawn from inter-industry data without disentangling the two components of the signal/noise ratio.

[28]The proof of Proposition 5 is not included in the Appendix because it follows from standard arguments.

Proposition 5. *Consider any value function. The only separating equilibrium which survives the Intuitive Criterion is the Riley equilibrium.*

Survival of pooling equilibria. We now focus on pooling equilibria and use e_p to denote the common level of education. First, notice that no signal $e < e_p$ can be equilibrium dominated for either type. Further, if a deviation to $e > e_p$ is equilibrium-dominated for the high type, it is never profitable for the low type (in the sense of the Intuitive Criterion) to choose e. For such a deviation would induce industry's beliefs $\mu(e) = 0$ and low types would obtain lower payoffs (p_l/r) than at the pooling equilibrium but incur strictly larger educational costs.

Hence, a pooling equilibrium with education level e_p fails the Intuitive Criterion *if and only if* there exists a signal $e > e_p$ such that it is equilibrium dominated for the low types but would result in a better payoff than in equilibrium for the high types when the industry places zero probability on the event that the sender is of the low type given signal e, i.e.

$$\frac{p_h}{r} - c(p_l, e) < v(p_l, \mu_0) - c(p_l, e_p) \text{ (equilibrium dominance for the low type),} \tag{ED}$$

$$\frac{p_h}{r} - c(p_h, e) > v(p_h, \mu_0) - c(p_h, e_p) \text{ (profitable deviation for the high type).} \tag{PD}$$

The equilibrium dominance condition (ED) implies that, even in the best-case scenario where the worker could forever deceive employers, deviating to e is not attractive to low types. The firm can therefore infer by forward induction that any worker with an off-equilibrium signal e has a high productivity. The "profitable deviation" condition (PD) implies in turn that credibly deviating to e is indeed profitable for the high type. Thus such an e_p fails the Intuitive Criterion, or, following the terminology of Kohlberg and Mertens (1986), is not stable.

Let $e^*(e_p)$ denote the minimum education level that does not trigger a profitable deviation for low ability workers, so that (ED) holds with equality at $e^*(e_p)$. Condition (ED) can then be rewritten as $e > e^*(e_p)$. Analogously, let $e^{**}(e_p)$ be such that (PD) holds with equality, then (PD) can be rewritten as $e < e^{**}(e_p)$. These two thresholds always exist because $c(p, e)$ is continuous and strictly increasing in e.[29] There exists an education level satisfying both conditions (ED) and (PD) if and only if $e^*(e_p) < e^{**}(e_p)$. This condition provides us with a straightforward proof that all pooling equilibria fail the Intuitive Criterion in the model without learning. Assume that e_p is stable, so that condition

[29] To see this, observe first that, because $v(p_l, \mu_0) < p_h/r$, we have $p_h/r - c(p_h, e_p) > v(p_h, \mu_0) - c(p_h, e)$ for $e = e_p$. Further, the cost function $c(p, e)$ is continuous, strictly increasing, and convex in e, hence the inequality is reversed for e large enough. This implies that $e^*(e_p)$ and $e^{**}(e_p)$ are well-defined. It also follows that $e^*(e_p) > e_p$ and $e^{**}(e_p) > e_p$.

(PD) is not satisfied at $e^*(e_p)$. This can be true if and only if

$$v(p_h, \mu_0) - v(p_l, \mu_0) \geq c(p_l, e^*(e_p)) - c(p_l, e_p) - [c(p_h, e^*(e_p)) - c(p_h, e_p)] \ . \tag{6}$$

In Spence's model or as $s \to 0$, we have $v(p_l, \mu_0) = v(p_h, \mu_0) = w(\mu_0)/r$. The left hand side of inequality (6) converges to zero while the right-hand side is strictly positive by the single crossing property. The contradiction illustrates that, in the basic signaling model, one can always find a credible and profitable deviation for high types.

When workers' abilities are also revealed on-the-job, the premise leading to a contradiction is no longer true. As stated in property P1, the expectations of high types are higher than those of low types. In other words, $v(p_h, \mu_0|r, s) > v(p_l, \mu_0|r, s)$ for all $s > 0$, and so inequality (6) can hold true for some parameter configurations. We summarize these observations in the following result.

Proposition 6. *The Intuitive Criterion rules out all pooling equilibria in the absence of learning, but this is no longer true for value functions with either weak or strong learning.*

In order to complete the proof of the statement, we need to show that indeed pooling equilibria might survive the Intuitive Criterion. While it is easy to provide numerical examples, we can provide more general results. According to property P2, the more precise correlated information is, the wider the gap in expected income between low and high types. This suggests that a pooling equilibrium is more likely to be stable when signal extraction is efficient. The following proposition substantiates this intuition, showing that *any* pooling equilibrium is stable when precision is high enough. For the particular case of linear costs, we find an even sharper result: if precision is high enough, all pooling equilibria will survive the Intuitive Criterion.

Proposition 7. *Consider a value function with strong learning.*
(a) For any education level e_p, there exists a precision $s^*(e_p)$ such that, for any $s \geq s^*(e_p)$, if e_p can be sustained as a pooling equilibrium then it survives the Intuitive Criterion.
(b) If costs are linear, there exists a precision s^* such that, for any $s \geq s^*$, every pooling equilibrium survives the Intuitive Criterion.

Figure 2 illustrates the mechanism behind Propositions 6 and 7. It displays the indifference curves of high and low types when $s = s^*(e_p)$ and when $s = 0$. The dotted curves correspond to the former case, the undotted curves to the latter one, that is, Spence's model. The level of education $e^*(e_p|s)$ where condition (ED) holds with equality is given by the point where the indifference curve of low types crosses the horizontal line with intercept p_h/r. Similarly, the level of education $e^{**}(e_p|s)$ where condition (PD) holds with equality is given by the point where the indifference curve of high types crosses the same horizontal line. The pooling equilibrium e_p fails the Intuitive Criterion if and only if $e^*(e_p) < e^{**}(e_p)$. We can therefore conclude that e_p is not stable when the indifference curve of low types intersects the horizontal line with intercept p_h/r before the indifference curve of high types .

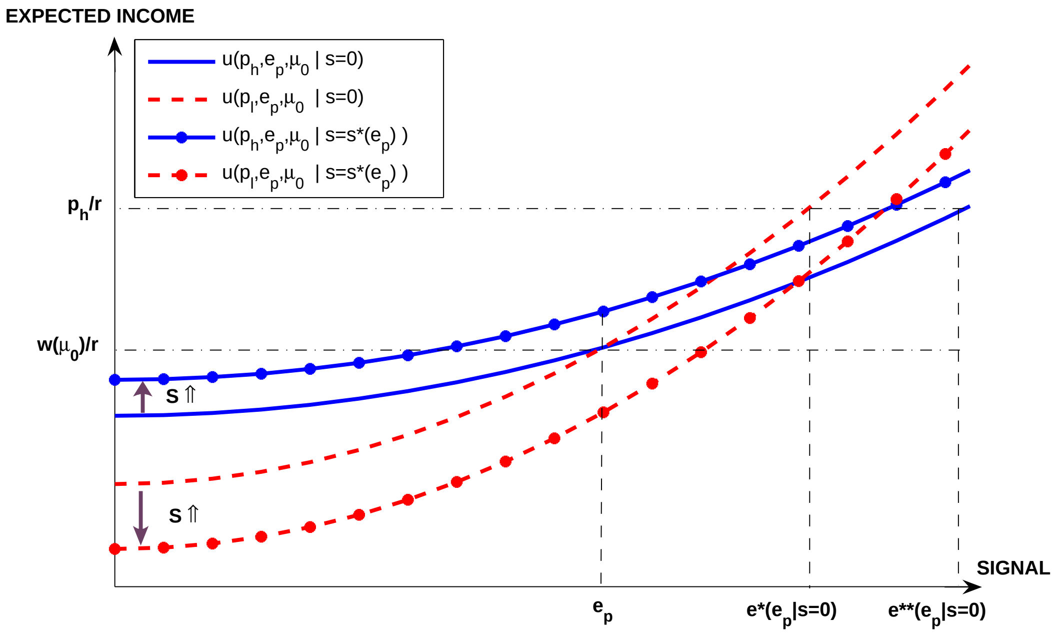

Figure 2: Workers' indifference curves.

Consider first the basic model without learning. At the pooling level of education e_p, the two types enjoy the same asset value $w(\mu_0)/r$. The *single-crossing property* implies that $e^*(e_p)$ lies to the left of $e^{**}(e_p)$, as shown in Figure 2. Thus any pooling equilibrium fails the Intuitive Criterion when there is no learning. Consider now what happens when the precision s increases. As high types are more quickly recognized, their asset value increases and their indifference curve shifts up. Conversely, the indifference curve of low types shifts down. These opposite adjustments shrink the gap between $e^*(e_p|s)$ and $e^{**}(e_p|s)$. The threshold precision $s^*(e_p)$ is identified by the point where the gap vanishes as the two indifference curves concurrently cross the horizontal line with intercept p_h/r. For any value function, property P3 ensures that one can always find such a point for any given e_p because $\lim_{s\to\infty} v(p_h, \mu|r, s) = p_h/r$. Figure 2 also illustrates the fact that $s^*(e_p)$ is unique.

The economics behind Proposition 7 makes intuitive sense. When learning is fast, firms easily infer the actual type of their employees. Then the benefits derived from ex-ante signaling are not important. Conversely, when learning is slow, firms learn little from observed outputs. This leaves fewer opportunities for high types to reveal their ability after the signaling stage and thus raises their incentives to send a message. In the extreme case where precision goes to zero, all the relevant information is collected prior to labor market entry.

According to Proposition 7, when the precision is high enough, the Intuitive Criterion does not rule out any pooling equilibrium. Proposition 2, however, states that the set of pooling equilibria shrinks as precision increases. Taken together, these results imply that there are three possibilities: (i) if learning is slow, there is a large set of pooling equilibria, almost all (or all) of which fail the Intuitive Criterion; (ii) if learning is fast, pooling equilibria would survive the Intuitive Criterion, but the set of such equilibria is small; and (iii) for intermediate values of s, there exists a sizeable set of pooling equilibria which survive the Intuitive Criterion.

This qualitative classification can be made more precise by assuming that the log of the derivative of $c(p,e)$ with respect to e has (strictly) decreasing differences in (p,e) or, in other words, that $c_e(p,e)$ is (strictly) log-submodular:[30]

$$\frac{c_e(p_h,e'')}{c_e(p_h,e')} > \frac{c_e(p_l,e'')}{c_e(p_l,e')}\,, \quad \text{whenever } e'' > e'.$$

Proposition 8. *Let $c_e(\cdot)$ be strictly log-submodular, and consider any value function with strong learning. Let $\overline{e_p}(s)$ be as in Proposition 2. Then, there exists an $s^*(0) > 0$ and an $\overline{s} > s^*(0)$ such that*

(a) For all $s \in [0, s^*(0)[$, all pooling equilibria fail the Intuitive Criterion.

(b) For all $s \in [s^*(0), \overline{s}[$, there exists $\widetilde{e}(s)$, strictly decreasing in s, such that (i) $\widetilde{e}(s) < \overline{e_p}(s)$, (ii) all pooling equilibria with education level $e_p \in [0, \widetilde{e}(s)]$ survive the Intuitive Criterion, and (iii) all pooling equilibria with education level $e_p \in]\widetilde{e}(s), \overline{e_p}(s)]$ fail the Intuitive Criterion.

(c) For all $s \geq \overline{s}$, all pooling equilibria survive the Intuitive Criterion.

If c_e is log-linear, the result holds with $s^*(0) = \overline{s}$, i.e. case (b) cannot occur.

This result is illustrated in Figure 3. Whereas the statement in Proposition 7 is local, assuming log-submodularity allows for a global characterization of the region where the Intuitive Criterion bites. Log-submodularity is more stringent than the single crossing property because it implies that marginal educational costs diverge. If that property were not satisfied, an increase in the level of education could restore the stability of some pooling equilibria. This is why, when $c_e(p,e)$ is not log-submodular, equilibrium stability does not always divide the (s,e) space into two non-overlapping regions.

[30] This assumption might seem restrictive but it is actually satisfied by the functions commonly used to illustrate the single crossing property. Textbook examples of cost functions usually exhibit log-linear marginal costs which, as stated at the end of Proposition 8, yields an even simpler division of the parameter space.

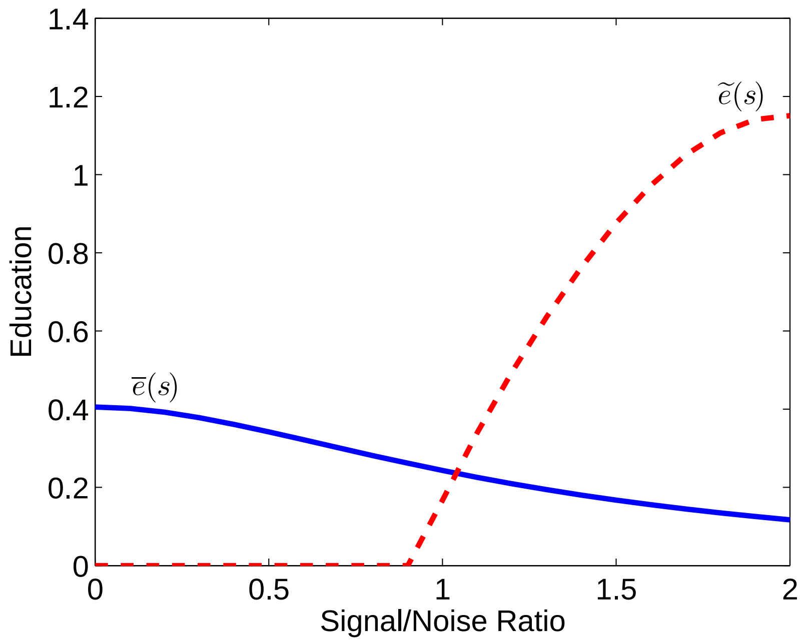

Figure 3: Graphical interpretation of Proposition 8. A pooling equilibrium with education level e exists if and only if $e \leq \overline{e}(s)$. It satisfies the Intuitive Criterion if and only if $e \leq \widetilde{e}(s)$. Parameters: $r = 0.2$, $p_l = 0.5$, $p_h = 1$, $\mu_0 = 0.5$ and $c(p, e) = \exp(e/p) - 1$.

Survival of Mixed-strategy Equilibria. Mixed-strategy equilibria may survive the Intuitive Criterion for the same reason as pooling equilibria. One simply has to verify that conditions (ED) and (PD) are fulfilled when the share μ_0 of high types in the population is replaced by the equilibrium belief $\mu(e)$ resulting from the senders' randomization strategies. Furthermore, one does not need to tediously go through each and every equilibrium signal. As stated in the following proposition, it is sufficient to pick an arbitrary signal that is sent by both types and check its stability.

Proposition 9. *A mixed-strategy equilibrium (with $E_p \neq \emptyset$) fails the Intuitive Criterion if and only if for some $e_p \in E_p$, there exists an education level e such that*

$$\begin{aligned} \frac{p_h}{r} - c(p_l, e) &< v(p_l, \mu(e_p)) - c(p_l, e_p) \, , \\ \frac{p_h}{r} - c(p_h, e) &> v(p_h, \mu(e_p)) - c(p_h, e_p) \, . \end{aligned}$$

As an example, consider the Reports Model with linear costs as in Section 3.2, and focus on the mixed-strategy equilibria with three education levels exhibited in Proposition 4. Note that a failure of the Intuitive Criterion must involve a deviation to a signal $e > e_2$. Then equilibrium dominance for the low types reduces to

$$\frac{p_l}{r} > \frac{p_h}{r} - g(p_l)e \Rightarrow e > \frac{1}{rg(p_l)}\left[p_h - p_l\right] > 0 \ ,$$

which provides a lower bound for the deviation $e > e_2$. That deviation would be strictly profitable for the high types if and only if

$$v(p_h, \mu(e_2)) - g(p_h)e_2 < \frac{p_h}{r} - g(p_h)e \Rightarrow \frac{p_h/r - v(p_h, \mu(e_2))}{g(p_h)} + e_2 > e \ .$$

Given that e is bounded below and $v(p_h, \mu(e_2)) \to \frac{p_h}{r}$ when the signal precision goes to infinity, both conditions are incompatible when signal precision is high enough; hence the equilibrium will survive the Intuitive Criterion.

Further Equilibrium Refinements. We have focused on the Intuitive Criterion because of its relevance for Spence's model. However, once one moves away from the two-type setting, and even without employer learning, the Intuitive Criterion fails to select a unique signaling equilibrium for general signaling games (Cho and Kreps 1987). This problem has been addressed in the literature through the introduction of more sophisticated refinement concepts, and it would be natural to investigate the predictive power of those refinements in our setting. Although such a task is beyond the scope of this paper, we offer here two observations, focusing on two different avenues: Universal Divinity and the Undefeated Criterion.

The first observation is that existing results based on forward-induction refinements do not always help refine the equilibrium set, so that further research is needed. To substantiate this claim, consider the classical uniqueness result of Cho and Sobel (1990). That result concerns three particularly appealing refinement concepts: criterion D1 (Cho and Kreps 1987), Universal Divinity (Banks and Sobel 1987), and Never a Weak Best Response (NWBR; Kohlberg and Mertens 1986). It provides a set of necessary conditions ensuring that those three refinements select a unique signaling equilibrium within a subclass of signaling games known as *monotonic signaling games* (Cho and Sobel 1990, p. 387). A monotonic signaling game is such that, for every fixed signal, all types have the same preferences over the actions of the receiver. This condition is fulfilled by job-market signaling games, with or without employer learning.[31] Proposition 3.1 in Cho and Sobel (1990) then states that the three refinements mentioned above coincide, and their main result establishes uniqueness.[32] The only non-technical condition in their uniqueness theorem is A4, which, in Cho and Sobel's words,

[31]In our setting, this is implied by the fact that $v(p, \mu)$ is strictly increasing in μ for both $p = p_h$ and $p = p_l$

[32]Esö and Schummer's (2008) *Vulnerability to Credible Deviations* provides an alternative interpretation of this selection result within the context of monotonic signaling games.

"is crucial to [the] analysis. It states that if two signal-action pairs yield the same utility to some type of Sender, and one signal is greater (componentwise) than the other, then all higher types prefer to send the greater signal." The formal condition reads as follows: if $p < p'$ and $e < e'$, then $u(e, \mu|p) \leq u(e', \mu'|p) \Rightarrow u(e, \mu|p') < u(e', \mu'|p')$.

In Spence's model, this condition is implied by the single-crossing property, for the wage given an education level is independent of the type. Hence, Universal Divinity (or D1) will always select a unique equilibrium. By contrast, this condition is not always fulfilled in our setting. With employer learning the link fails because lifetime earnings differ across types. Further, it is immediate that, if a signaling game satisfies condition A4, there cannot be equilibria with multiple pooling, for two types cannot be simultaneously indifferent between two signal-action pairs with different signals. In other words, whenever multiple pooling equilibria exist, A4 must be violated.

In our view, Condition A4 would place strong restrictions on a model with employer learning. With two types, it implies that, whenever $v(p_l, \mu') - v(p_l, \mu) \geq c(p_l, e') - c(p_l, e)$, it must also hold that $v(p_h, \mu') - v(p_h, \mu) > c(p_h, e') - c(p_h, e)$. Further, this must hold for all μ, μ', e, e' with $e' > e$. When lifetime incomes are independent of the type, this reduces to a condition on the cost function. With employer learning, it implies a condition on the relation between lifetime incomes (as a function of beliefs) and costs (as a function of education). However, in a model where education does not affect productivity, there is no direct relation between the former and the latter, and so the condition can only hold if the set of possible value and/or cost functions is restricted.[33]

As an illustration, consider the value functions for high and low types depicted in Figure 4 below. Focus on beliefs relatively close to $\mu = 1$. With employer learning, high-productivity workers face a tradeoff between signaling through education and letting their performances on-the-job reveal their type. This is why the marginal return of an increase in the receiver's belief should eventually decrease, which is not true for low-productivity workers. It is therefore natural to expect that, for certain action-signal pairs, low-type workers prefer an education and belief increase while high-type workers do not. This kind of effect is excluded by A4 because this condition prescribes that, whenever an increase in education is (weakly) attractive for the low types, it must be (strictly) attractive for the high types.

Since A4 fails in our model, the analysis and uniqueness result in Cho and Sobel (1990) do not apply. As those authors point out, condition A4 is the crucial one behind their uniqueness result,

[33] Suppose we fix a value function and then require Condition A4 to hold for *any* cost functions fulfilling the single-crossing property. Selecting the education levels in such a way that low types are indifferent and making the cost differences arbitrarily close across types, a limit argument leads to $v(p_h, \mu') - v(p_h, \mu) \geq v(p_l, \mu') - v(p_l, \mu)$ for all pairs μ', μ. Hence, $\partial v(p_h, \mu)/\partial \mu \geq \partial v(p_l, \mu)/\partial \mu$. However, this contradicts P1: for $\mu \in]0, 1[$, $v(p_h, \mu) > v(p_l, \mu)$ and the inequality between the derivatives imply that $p_h/r = v(p_h, 1) > v(p_l, 1) = p_h/r$.

and so there is *a priori* no reason to expect refinements like D1, Universal Divinity, or NWBR to generically restore equilibrium uniqueness in the absence of strong additional restrictions. Indeed, one can build (cumbersome) examples showing that, for general cost and value functions, those criteria do not necessarily refine the set of equilibria. In other words, any attempt to further refine the equilibrium set using standard (stronger) refinement concepts must rely on additional conditions. This has also been observed by Daley and Green (2009), who use the D1 criterion to analyze a signaling game with grades which is isomorphic to our static Reports model. Although they rely on a more demanding notion of equilibrium stability, they find that uniqueness of the equilibrium is not always ensured. They propose a condition under which correlated information is precise enough relative to the cost advantage of high types. They label it *RC-Informativeness* and show that when it is satisfied, a unique equilibrium emerges if D1 is used as a refinement concept. Hence, their analysis also illustrates that forward induction is vulnerable to the introduction of correlated information after the signaling stage and that it crucially depends on the relation between expected income and signaling costs.

Our second observation is related to welfare concerns and the Pareto-efficiency of equilibria. Essentially, we will argue below that pooling equilibria in our model survive the Intuitive Criterion only if the Riley separating equilibrium is Pareto-dominated. This delivers a connection between our results and a different kind of equilibrium refinement criteria, built on a basic message of the signaling literature, namely that signaling might lead to inefficiencies. Mailath, Okuno-Fujiwara, and Postlewaite (1993) postulate the Undefeated Criterion as an alternative to the Intuitive Criterion and argue in favor of pooling equilibria when they Pareto-dominate the Riley equilibrium. The key argument is that, for some parameter constellations, the pooling equilibrium with zero education level might Pareto-dominate the Riley equilibrium: in the latter, high types overeducate to avoid low-type wages even though they would be better off with a pooling wage and minimal education.[34]

Suppose some pooling equilibrium survives the Intuitive Criterion. Then, as our analysis shows, the pooling equilibrium with $e = 0$ will also survive this criterion. By definition of the Riley equilibrium education level $\underline{e_h}$, the low type is indifferent between sending $\underline{e_h}$ in order to obtain the wage p_h and

[34]The Undefeated Criterion is more involved than Pareto-dominance. Mailath, Okuno-Fujiwara, and Postlewaite (1993) restrict attention to pure-strategy signaling equilibria. Within this class, an equilibrium (σ', μ') is defeated if a new equilibrium (σ, μ) can be built where a previously unused signal m is used by some types, in such a way that (i) the new equilibrium is a Pareto-improvement for those types (all of them being weakly better off, and at least one being strictly better off), and (ii) for some type using m in (σ, μ), the receiver's out-of-equilibrium belief in (σ', μ') that the sender is of that type on seeing m can *not* be explained through conditioning on the set of sender types who do use m in (σ, μ), even allowing for the possibility that indifferent types might have randomized. In the context of Spence's model, for certain parameter values the Riley equilibrium is defeated by pooling equilibria with education levels close to zero, while pooling at zero is undefeated. For other parameter values, the Riley equilibrium is undefeated.

receiving the wage p_l at an education level of zero. The payoff at the pooling equilibrium with $e = 0$ is strictly larger than the latter and so the Riley signal $\underline{e_h}$ is always equilibrium-dominated for the low type. If pooling at $e = 0$ survives the Intuitive Criterion, it follows that the payoff of the high types at the pooling equilibrium with $e = 0$ must be weakly better than at the Riley equilibrium. It follows that the pooling equilibrium with $e = 0$ Pareto-dominates the Riley equilibrium. Repeating the argument for pooling equilibria with e close to zero, it can be shown that, if any pooling equilibrium survives the Intuitive Criterion, the Riley equilibrium is defeated in the sense of Mailath, Okuno-Fujiwara, and Postlewaite (1993).[35]

A similar connection between welfare and equilibrium stability is identified in Daley and Green (2009). They show that when correlated reports are RC-Informative, D1 always select equilibria that Pareto dominate the Riley equilibrium. These results suggest that learning restores a link between signaling and efficiency. Recent work by Atkeson et al. (2010) study a related issue in a model where firms may invest in product quality but cannot signal their choice. They find that learning mitigates the 'lemons problem' as it introduces reputation incentives such that some firms find it profitable to invest. They also show that welfare can be improved by the intervention of a regulator. Accordingly, it would be interesting to investigate whether existing labor market regulations, such as income taxes and contractual obligations, can be welfare enhancing in our setup.

5 Bayesian Learning

In this last section, the microfoundation of the learning process are discussed in more detail. We propose two models where beliefs are explicitly derived from output realizations. We show that they deliver value functions satisfying the properties listed in Definition 1.

Note that output realizations are not decisions of the workers, but rather objective signals observed by the industry. Thus the microfoundations developed here concern employer learning but do not correspond to further signaling stages on the side of the workers.[36]

Two period model. We first consider a set-up that maintains the timing conventions of the Reports model, so that two periods are sufficient to characterize a worker's career. The only difference is that

[35] Observe, however, that any pooling equilibrium with $e > 0$ is defeated by the pooling equilibrium with $e = 0$. Hence, in our framework, there will be defeated equilibria which survive the Intuitive Criterion.

[36] Models where signaling can occur over several periods give rise to a different game-theoretic structure. For example, Bar-Isaac (2003) analyzes a signaling game where a monopolist sells a good whose quality is uncertain. Whether the actual quality of the product will be revealed over time or not becomes a question to be answered in equilibrium, rather than a postulated property of the learning process.

output is observed by the industry at the end of the first period. Realizations are not deterministic[37] but randomly drawn from a continuous density $g_i(\cdot)$ with a mean equal to the worker's type p_i for $i \in \{l, h\}$. The sampling distributions share the same support and are common knowledge. The randomness might be inherent to the production process or to the imperfect precision of the monitoring technology.

Lifetime incomes follow, as in the two-period version of the Reports Model, from equation (2). On the other hand, the updating rule differs because second period beliefs now depend on the output density

$$\mu'(x, \mu) = \frac{\mu g_h(x)}{\mu g_h(x) + (1-\mu) g_l(x)} ,$$

whereas wages $w(\mu) = p_l + (p_h - p_l)\mu$ are given by an affine function of beliefs. In spite of these additional features, the two-period model yields predictions that are similar to those of the more stylized Reports model.

Proposition 10. *The value function for the two-period model satisfies property P1 and so exhibits weak learning.*

We have not given a specific definition for s in the discrete time model because it would require further parametric restrictions on the $g(\cdot)$ distributions.[38] In any case, learning is weak in the sense of Definition 1 because P3 does not hold for any conceivable measure of s. To see this, consider the limit case where uncertainty becomes negligible as $g_h(\cdot)$ and $g_l(\cdot)$ converge to Dirac delta functions. Then learning is at its most efficient because types are perfectly revealed at the end of the first period. Nevertheless, the lifetime income of high types converges to $\frac{1}{1+r}\left[w(\mu) + v(p_h, 1)\right] < v(p_h, 1) = p_h/r$.[39] As in the Reports models, the upper limit property in P3 is not fulfilled because first-period earnings solely depend on the prior and thus do not vary with the precision of correlated information.

Continuous time model. In the two period model, the length of time required to elicit information is treated as a primitive parameter. This is an artifice of the discrete time structure as industries where learning is more efficient should also be characterized by shorter periods of information acquisition. This issue can be addressed using a continuous time set-up and establishing that it gives rise to *strong learning* in the sense of Definition 1.[40] The key difference, however, is not continuous vs. discrete time,

[37] Otherwise learning would be perfect by the end of the first period.

[38] For example, it is shown in next subsection that when both $g_i(\cdot)$ distributions are normal with common variance σ^2, a natural measure for s is the signal/noise ratio: $(p_h - p_l)/\sigma^2$.

[39] Conversely, the lifetime income of low types converges to $\frac{1}{1+r}\left[w(\mu) + v(p_l, 0)\right] > v(p_l, 0) = p_l/r$.

[40] We thank an anonymous referee for suggesting an alternative way to obtain strong learning in a continuous time set-up: Let reports of the type described in Subsection 2 arrive at the rate $1/\sigma$. Then strong learning would follow when

but rather the fact that in the continuous-time model there are infinitely many future updates at any given point in time, while in a two-period setting there is exactly one update and the production period before that update never becomes negligible[41]

We assume that output realizations are random draws from a Gaussian distribution with a time invariant average productivity.[42] Thus the *cumulative* output X_t of a match of duration t with a worker of type $i \in \{l, h\}$, follows a Brownian motion with drift

$$dX_t = p_i dt + \sigma dZ_t \ ,$$

where dZ_t is the increment of a standard Brownian motion. The cumulative output $\langle X_t \rangle$ is observed by both parties. The employer uses the filtration $\{\mathcal{F}_t^X\}$ generated by the output sample path to revise his belief about p_i. The variance σ is constant across workers for otherwise firms would be able to infer types with arbitrary precision by observing the quadratic variation of $\langle X_t \rangle$.[43] Starting from a prior μ_0 equal to the fraction of high ability workers in the population, the employer applies Bayes rule to update his belief $\mu_t \triangleq \Pr\left(p = p_h | \, \mathcal{F}_t^X\right)$. His posterior is therefore given by

$$\mu\left(X_t, t | \, \mu_0\right) = \frac{\mu_0 g_h(X_t, t)}{\mu_0 g_h(X_t, t) + (1 - \mu_0) g_l(X_t, t)} \ , \tag{7}$$

where $g_i(X_t, t) \triangleq e^{-\frac{(X_t - p_i t)^2}{2\sigma^2 t}}$ is the rescaled[44] density for a worker of type i.[45] The analysis is simplified by the change of variable $\theta_t \triangleq \mu_t/(1 - \mu_t)$. θ_t is the ratio of "good" to "bad" belief. Given that μ_t is defined over $]0, 1[$, θ_t takes values over the positive real line. It follows from (7) that

$$\theta\left(X_t, t | \, \theta_0\right) = \theta_0 \frac{g_h(X_t, t)}{g_l(X_t, t)} = \theta_0 e^{\frac{s}{\sigma}\left(X_t - \frac{1}{2}(p_h + p_l)t\right)} \ , \tag{8}$$

where $s = (p_h - p_l)/\sigma$ is the signal/noise ratio of output. On the one hand, a larger productivity difference between types increases the informativeness of the observations. On the other hand, a higher variance hinders the industry's ability to identify the mean of the output distribution. Thus

both frequency σ and accuracy s go to infinity.

[41] See Anderson and Smith (2010) for further detail on this point.

[42] One can easily verify that letting workers accumulate general human capital would not substantially modify our conclusions.

[43] See, for instance, Chung and Williams (1990).

[44] The factor $[\sigma\sqrt{2\pi t}]^{-1}$ is omitted because it simplifies in (7).

[45] There are well-known issues with decision problems in continuous time. We refer the reader to Faingold (2008) for a discussion of the significance of information revelation in the limit of discrete-time games, and to Alós-Ferrer and Ritzberger (2008) for a discussion of decision problems defined directly in continuous time. However, in our setting the continuous-time process captures information acquisition only. Since workers take no active decisions, none of those issues apply to our model.

the bigger s, the more efficient learning is. By Ito's lemma, the stochastic differential equation satisfied by the belief ratio reads

$$\begin{aligned} d\theta\left(X_t, t \mid \theta_0\right) &= \frac{\partial \theta\left(X_t, t \mid \theta_0\right)}{\partial X_t} dX_t + \frac{\partial^2 \theta\left(X_t, t \mid \theta_0\right)}{\partial X_t^2} \frac{\sigma^2}{2} dt + \frac{\partial \theta\left(X_t, t \mid \theta_0\right)}{\partial t} dt \\ &= \theta\left(X_t, t \mid \theta_0\right) \left(\frac{s}{\sigma}\right) \left[dX_t - p_l dt\right] . \end{aligned} \tag{9}$$

Replacing in (9) the law of motion of X_t, i.e. $dX_t = p_i dt + \sigma dZ_t$, yields the following stochastic differential equations

$$\begin{aligned} (i) \text{ Low ability worker}: \; d\theta_t &= \theta_t s dZ_t , \\ (ii) \text{ High ability worker}: \; d\theta_t &= \theta_t s \left(s dt + dZ_t\right) . \end{aligned}$$

The belief ratio θ_t increases with time for high types and follows a martingale for low types.[46] In both cases, a higher σ lowers the volatility of beliefs because larger idiosyncratic shocks hamper signal extraction.

We are now in a position to derive expected lifetime incomes as a function of beliefs. Conditional on a given cumulative output X_t, high and low types earn the same wage. Their expected lifetime incomes differ nonetheless because high types are more optimistic about future prospects. Using the laws of motion above, one can derive the Hamilton-Jacobi-Bellman equations

$$\begin{aligned} rv\left(p_l, \theta\right) &= w(\theta) + \frac{1}{2}\left(\theta s\right)^2 v''\left(p_l, \theta\right) , \\ rv\left(p_h, \theta\right) &= w(\theta) + \theta s^2 v'\left(p_h, \theta\right) + \frac{1}{2}\left(\theta s\right)^2 v''\left(p_h, \theta\right) . \end{aligned}$$

Imposing the boundary conditions, $\lim_{\theta \to 0} v\left(p_i, \theta\right) = p_l / r$ and $\lim_{\theta \to \infty} v\left(p_i, \theta\right) = p_h / r$ for $i \in \{l, h\}$, yields the following closed-form solutions for the two ordinary differential equations.

Proposition 11. *For the continuous time model, the expected lifetime incomes of workers as a function of the belief ratio θ are given by*

$$v\left(p_l, \theta\right) = \frac{2\sigma}{s\Delta} \left(\theta^{\alpha^-} \int_0^\theta \frac{1}{(1+x)\, x^{\alpha^-}} dx + \theta^{\alpha^+} \int_\theta^\infty \frac{1}{(1+x)\, x^{\alpha^+}} dx\right) + \frac{p_l}{r}$$

and

$$v\left(p_h, \theta\right) = \frac{2\sigma}{s\Delta} \left(\theta^{\gamma^-} \int_0^\theta \frac{1}{(1+x)\, x^{\gamma^-}} dx + \theta^{\gamma^+} \int_\theta^\infty \frac{1}{(1+x)\, x^{\gamma^+}} dx\right) + \frac{p_l}{r} ,$$

where $\alpha^+ = \frac{1}{2}(1+\Delta)$, $\alpha^- = \frac{1}{2}(1-\Delta)$, $\gamma^+ = \frac{1}{2}(-1+\Delta)$, $\gamma^- = \frac{1}{2}(-1-\Delta)$, *and* $\Delta = \sqrt{1 + 8\left(\frac{r}{s^2}\right)}$.

The value function satisfies properties P1, P2 , and P3, and hence exhibits strong learning.

[46] It may be surprising that the belief ratio θ_t does not drift downward when the worker is of the low type. This is because the belief ratio θ_t is a convex function of μ_t. Reversing the change of variable shows that, as one might expect, the belief μ_t is a strict supermartingale when the worker's ability is low.

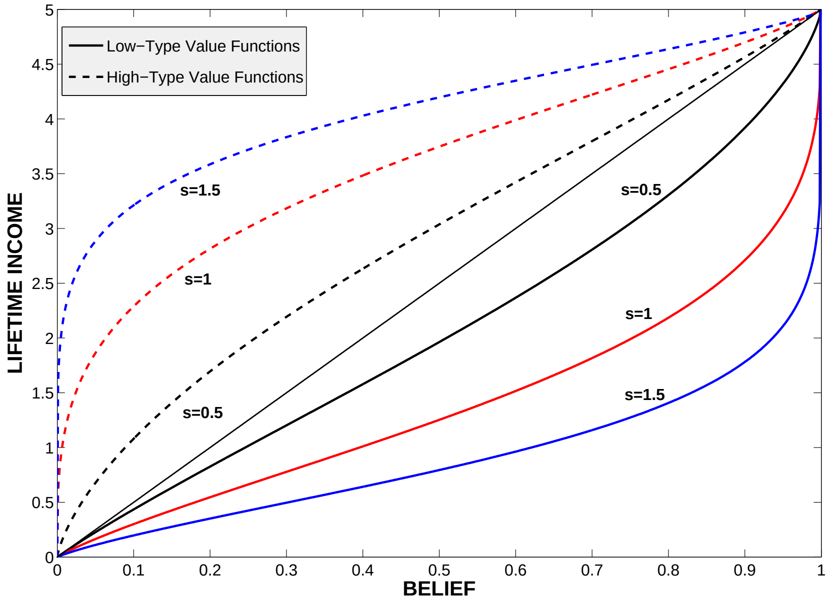

Figure 4: Workers' value functions in the continuous-time model. Parameters: $r = 0.2$, $p_l = 0$, $p_h = 1$.

The closed-form solution is of independent interest, and the fact that it can be derived is an additional contribution of this paper, since continuous-time models often have to be solved numerically.[47] As discussed in the Appendix, the derivation crucially hinges on the change of variable from μ to θ. This technique can be used to simplify a variety of models with continuous time learning, as illustrated by Atkeson et al. (2009) or Daley and Green (2010).

Plots of the value functions for a particular numerical example and several values of s are shown in Figure 4, illustrating the properties listed in Definition 1. First, for any given belief $\mu \in (0, 1)$, the expected incomes of low and high types are respectively smaller and bigger than the discounted value of their current wage, hence P1 is satisfied. Second, as stated in P2, the gap increases when learning becomes more efficient. Finally, the value functions converge to the discounted value of current wages when s goes to zero and to step functions when s goes to infinity, as required by P3.

[47] In the working paper version of this article we also show how to obtain a solution for cases where senders are uncertain about their productivity.

6 Conclusion

We have analyzed a labor market where a worker's ability can be revealed either by his education or by his performance on-the-job. The addition of this realistic feature causes the failure of standard arguments, such as the selection of the Riley equilibrium via the Intuitive Criterion. Available evidence on the speed of employer learning suggests that this observation could be highly relevant for applied work in job market signaling and signaling models in general. Our findings are also relevant for empirical research on signaling theory. Given the existence and stability of multiple pooling outcomes when learning is efficient, tests based on the properties of separating equilibria are likely to be too restrictive.

Our analysis has concentrated on characterizing the set of signaling equilibria and has relied on the Intuitive Criterion as the basic refinement. From a game-theoretic standpoint, a very interesting avenue of research concerns the relevance of more sophisticated equilibrium refinements. As argued above, however, existing forward-looking equilibrium refinements beyond the Intuitive Criterion might not be entirely suitable for signaling models with receiver learning after the signaling stage, which raises a number of theoretical questions on refinements: In which special classes of games could a uniqueness result be restored? How existing refinements can be further adapted to a context with learning on the receiver's side?

Another avenue for future research concerns the learning process itself, whose characteristics can differ from one specific model (value function) to another. The general model presented in this paper abstracts from prevalent features of labor markets. Among other simplifications, it ignores the importance of match-specific uncertainty. This additional source of noise hampers signal extraction and is thus likely to expand the parameter space where the Intuitive Criterion bites. Another implicit premise of our analysis is that wages are a function of current beliefs. This is no longer true when employers can commit to employment contracts. Changing the perspective, it would be interesting to see whether commitment reinforces the informativeness of education signals.

Such extensions would provide a more realistic description of how signaling operates in labor markets. Our basic findings, however, apply to any signaling environment beyond the particular job-market model we have focused on. This suggests plenty of scope for further research on the interactions between learning and signaling.

APPENDIX

Proof of Proposition 2:

We first prove that the given education levels can be sustained as a pooling equilibrium by the particular beliefs $\mu(e_p) = \mu_0$ and $\mu(e) = 0$ for all $e \neq e_p$. These beliefs together with $e_p = 0$ lead to pooling because any deviation yields a strictly larger cost and a strictly lower value (recall that the value function is strictly increasing in μ). Consider $e_p > 0$: Under the stated beliefs, selecting $e = 0$ is always preferred by both types to selecting any other $e \notin \{0, e_p\}$, for both yield the same lifetime income but the cost of $e = 0$ is strictly smaller. Hence it suffices to check that neither of the two types has an incentive to deviate from e_p to $e = 0$. This yields the conditions

$$c(p_l, e_p) - c(p_l, 0) \leq v(p_l, \mu_0) - \frac{p_l}{r} \quad \text{and} \quad c(p_h, e_p) - c(p_h, 0) \leq v(p_h, \mu_0) - \frac{p_l}{r} \ .$$

By the single crossing property, $c(p_h, e_p) - c(p_h, 0) < c(p_l, e_p) - c(p_l, 0)$. Weak learning, however, implies that $v(p_h, \mu_0) \geq v(p_l, \mu_0)$ and so the first condition implies the second one. Given that costs are strictly increasing in education, there exists a unique $\overline{e_p} > 0$ such that the first inequality is fulfilled if and only if $e_p \leq \overline{e_p}$. By continuity, that education level is uniquely determined by the condition given in the statement. It is straightforward to show that an education level can be sustained as a pooling equilibrium under some belief system if and only if it can be sustained under the beliefs specified above. Further, by P1

$$v(p_l, \mu_0) - p_l \leq w(\mu_0) - p_l = \mu_0(p_h - p_l) < p_h - p_l,$$

which, recalling the definition of $\underline{e_h}$ in Proposition 1, implies that $\overline{e_p} < \underline{e_h}$.

Last, assume strong learning. To see that the set of pooling equilibria shrinks as the s increases, simply notice that $v(p_l, \mu_0|r, s)$ is strictly decreasing in s by P2, hence the conclusion follows from the equation defining $\overline{e_p}$. The fact that $\lim_{s\to\infty} \overline{e_p} = 0$ under strong learning follows from P3, i.e. the requirement that $\lim_{s\to\infty} v(p_l, \mu|r, s) = p_l/r$. ■

Proof of Proposition 3:

Suppose that there are at least two education levels in the support of the low types' equilibrium strategy, e and e' with $e < e'$. As both must be optimal, it follows that

$$v(p_l, \mu(e)) - c(p_l, e) = v(p_l, \mu(e')) - c(p_l, e') \ ,$$

which, since $c(p_l, e) < c(p_l, e')$, implies $v(p_l, \mu(e)) < v(p_l, \mu(e'))$. Hence, $\mu(e) < \mu(e')$ (recall that v is strictly increasing in μ), proving that μ needs to be strictly increasing over chosen education levels.

This is only possible if $\mu(e') > 0$, hence in equilibrium e' must also be in the support of the high types' strategy. It follows that there exist at most one education level chosen only by low types, and it must be the lowest one in the support of their strategy. A symmetric argument holds for high types, thereby establishing (i) and (ii). The fact that $e_l = 0$ follows as in Proposition 1.

Suppose that also e is in the support of the high types' strategy. Then

$$v(p_h, \mu(e)) - c(p_h, e) = v(p_h, \mu(e')) - c(p_h, e').$$

By the single crossing property, we obtain

$$v(p_h, \mu(e')) - v(p_h, \mu(e)) = c(p_h, e') - c(p_h, e) < c(p_l, e') - c(p_l, e) = v(p_l, \mu(e')) - v(p_l, \mu(e)).$$

The strict inequality leads to a contradiction in Spence's model because $v(p_h, \mu) = v(p_l, \mu)$. One can then conclude that low types randomize among at most two education levels, where the lower one is only chosen by them. Symmetrically, high types randomize among at most two education levels, where the higher one is only chosen by them.

For other value functions, the last equality does not lead to a contradiction. The analysis of the Reports Model in the main text shows that equilibria where E_p is not a singleton are indeed possible.

It remains to show that $E_p \subseteq [0, \underline{e_h}]$. If $e_p \in E_p$, low-productivity workers must weakly prefer e_p to obtaining wage p_l/r while acquiring education $e = 0$, hence

$$c(p_l, e_p) - c(p_l, 0) \leq v(p_l, \mu(e_p)) - \frac{p_l}{r} \leq \frac{1}{r}(p_h - p_l)$$

(where the last inequality follows from Property P0). Then, $e_p \leq \underline{e_h}$ follows from definition of $\underline{e_h}$. ■

Proof of Proposition 4:

Without loss of generality, we set the discount rate $r = 1$. Denote $g \triangleq g(p_h)/g(p_l)$. Condition (5) holds if and only if

$$d > d(g) \triangleq \frac{1}{2}\left[1 + \sqrt{\frac{1-g}{1+3g}}\right] \tag{10}$$

We require a few preliminary computations. Define $D_\mu \triangleq \mu d + (1-\mu)(1-d) \in [1-d, d]$ and $K_d \triangleq d - g(1-d)$. Observe that, $d \geq \frac{1}{2} > \frac{g(p_h)}{g(p_h)+g(p_l)} = \frac{g}{1+g}$ implies that $K_d > 0$. Straightforward computations show that the function $h(\mu)$ defined in equation (3) satisfies

$$\begin{aligned} r \cdot h(\mu) &= K_d w(\mu, G) + (1 - g - K_d) w(\mu, B) \,, \\ r \cdot h'(\mu) &= d(1-d)(p_h - p_l)\left[\frac{K_d}{D_\mu^2} + \frac{1-g-K_d}{(1-D_\mu)^2}\right] \,, \\ r \cdot h''(\mu) &= -2(2d-1)d(1-d)(p_h - p_l)\left[\frac{K_d}{D_\mu^3} - \frac{1-g-K_d}{(1-D_\mu)^3}\right] \,. \end{aligned}$$

First, we notice that if $d \leq \frac{g(p_l)}{g(p_h)+g(p_l)} = \frac{1}{1+g}$, then there exists no multiple pooling equilibrium. This is because this condition amounts to $1 - g - K_d = 1 - d - gd \geq 0$ and then it follows from the expression above that $h'(\mu) > 0$ for all $\mu \in [0, 1]$, implying that condition (4) cannot hold. Thus we can restrict attention to $d > \frac{1}{1+g}$ from now on.

Under this restriction, or equivalently $1 - g - K_d < 0$, we obtain that $h''(\mu) < 0$ for all $\mu \in [0, 1]$ from its expression above. Hence h is strictly concave, a fact we will rely on below.

The rest of the proof proceeds in three steps.

Step 1. Condition (10) holds if and only if

$$\exists \mu_1, \mu_2 \quad \text{with} \quad \mu_1 < \mu_2 \quad \text{and} \quad h(\mu_1) = h(\mu_2), \tag{11}$$

To see this, note that by differentiability and concavity of h, (11) holds if and only if there exists $\mu \in]0, 1[$ with $h'(\mu) = 0$. Hence, we study the sign of $h'(\cdot)$. If $\mu = 1/2$, then $D_{1/2} = 1 - D_{1/2}$ and from the expression of $h'(\cdot)$ above, one see that $h'(1/2) > 0$. Since $h''(\cdot) < 0$, it follows that $h'(0) > 0$ and $h'(\cdot)$ has a zero in $]0, 1[$ if and only if $h'(1) < 0$. Given that $D_1 = d$,

$$h'(1) < 0 \quad \Longleftrightarrow \quad \frac{K_d}{d^2} + \frac{1 - g - K_d}{(1 - d)^2} < 0 \Longleftrightarrow d(1 - d) < \frac{g}{1 + 3g} .$$

Observe that: (i) $g/[1 + 3g] < 1/4$; (ii) $d(1 - d)$ reaches a maximum value of $1/4$ at $d = 1/2$ and is strictly decreasing for $d \geq 1/2$. Hence the condition is fulfilled if and only if d is above the largest root of the polynomial of second degree above. That root is actually $d(g)$, which completes the proof of this step.

Step 2. If a multiple pooling equilibrium exists, condition (10) is fulfilled and part (a) of the statement holds.

To see this, note that existence of a multiple pooling equilibrium implies (11), so (10) holds by Step 1. Concavity of h implies that, for any given value k, there can be at most two beliefs μ_1, μ_2 with $\mu_1 < \mu_2$ and $h(\mu_1) = h(\mu_2) = k$. It follows that, if a multiple pooling equilibrium exists, it involves exactly two pooling signals, as claimed in (a), i.e. $E_p = \{e_1, e_2\}$ with $e_1 < e_2$.

To complete the proof of (a), it remains only to show that there is no additional separating signal for high types. Suppose there were a third education level, e_3, sent in equilibrium by high types only. High types must be indifferent between e_3 and e_2,

$$v(p_h, \mu_2) - g(p_h)e_2 = \frac{1}{r}p_h - g(p_h)e_3 \quad \Rightarrow \quad e_3 = e_2 + \frac{(p_h/r) - v(p_h, \mu_2)}{g(p_h)}$$

while low types must weakly prefer e_2, so that

$$e_3 > e_2 + \frac{(p_h/r) - v(p_l, \mu_2)}{g(p_l)}.$$

These conditions jointly imply

$$\frac{(p_h/r) - v\left(p_h, \mu_2\right)}{g(p_h)} > \frac{(p_h/r) - v\left(p_l, \mu_2\right)}{g(p_l)} \iff h(\mu_2) < h(1).$$

This leads to a contradiction, because concavity of h and (11) imply $h'(\mu) < 0$ for all $\mu \in [\mu_2, 1]$.

Step 3. If condition (10) is fulfilled, then a multiple pooling equilibrium as in (b) exists. Further, there exists $\mu_{\min} \in]0,1[$ such that a multiple pooling equilibrium as in (c) exists if and only if $\mu_0 > \mu_{\min}$.

By Step 1, Condition (11) holds. Let $\mu_1 < \mu_2$ be such that $h(\mu_1) = h(\mu_2)$. We aim to construct multiple pooling equilibria with $E_p = \{e_1, e_2\}$, $\mu(e_1) = \mu_1$ and $\mu(e_2) = \mu_2$.

First, note that, for given values of $\{e_1, \mu_1, \mu_2\}$, e_2 is uniquely determined since

$$v\left(p_l, \mu_1\right) - g(p_l)e_1 = v\left(p_l, \mu_2\right) - g(p_l)e_2 \iff e_2 = e_1 + \frac{v\left(p_l, \mu_2\right) - v\left(p_l, \mu_1\right)}{g(p_l)} .$$

Second, let us consider e_1. A sufficient condition for low types not to have an incentive to deviate under some out-of-equilibrium belief system is the participation constraint

$$v\left(p_l, \mu_1\right) - g(p_l)e_1 \geq v\left(p_l, 0\right) - g(p_l) \cdot 0 = \frac{p_l}{r}$$

or, equivalently,

$$e_1 \leq \frac{v\left(p_l, \mu_1\right) - (p_l/r)}{g(p_l)} \triangleq \tilde{e}\left(\mu_1\right).$$

Note that $\tilde{e}\left(\mu_1\right) > 0$ for all $\mu_1 > 0$. If low types also send an additional separating signal in equilibrium, it will necessarily be $e_0 = 0$ and then indifference leads to $e_1 = \tilde{e}\left(\mu_1\right)$.

The analogous condition for the high types is redundant, since $v\left(p_h, \mu_1\right) > v\left(p_l, \mu_1\right) > p_l/r + g(p_l)e_1 \geq p_l/r + g(p_h)e_1$ for all $e_1 \in [0, \tilde{e}\left(\mu_1\right)]$. Given that this inequality is strict, $e_1 > 0$, i.e. $0 \notin E_p$ for any multiple pooling equilibrium. In summary, $e_1 \in]0, \tilde{e}\left(\mu_1\right)]$.

Third, let us turn to beliefs. Let $q(e|p)$ denote the equilibrium probability of signal e sent by type p. By Bayes' Rule, weak consistency of beliefs implies, for $j \in \{l, h\}$,

$$\mu_j = \frac{q\left(e_j | p_h\right)\mu_0}{q\left(e_j | p_h\right)\mu_0 + q\left(e_j | p_l\right)(1-\mu_0)} ,$$

where μ_0 is the share of high types in the population. To economize in notation, consider the change of variable $\theta \triangleq \mu/(1-\mu)$ for $0 < \mu < 1$ (the same change of variable considered in Section 5), and denote $\theta_k = \mu_k/(1-\mu_k)$ for $k = 0, 1, 2$. We can then rearrange the last equation as follows

$$q\left(e_j | p_l\right) = q\left(e_j | p_h\right)\frac{\theta_0}{\theta_j} . \tag{12}$$

Since by the Step 1 high types randomize on E_p only, it follows that $q(e_1|p_h)+q(e_2|p_h)=1$. This means that equation (12) is equivalent to

$$1 \geq q(e_1|p_l)+q(e_2|p_l) = q(e_1|p_h)\frac{\theta_0}{\theta_1} + (1-q(e_1|p_h))\,\frac{\theta_0}{\theta_2}\ ,$$

which, rearranging, reduces to

$$q(e_1|p_h)\left[\frac{1}{\theta_1}-\frac{1}{\theta_2}\right] \leq \left[\frac{1}{\theta_0}-\frac{1}{\theta_2}\right]. \tag{13}$$

It follows that an equilibrium with multiple pooling requires $\theta_2 > \theta_0$, or equivalently $\mu_2 > \mu_0$ (else $q(e_1|p_h) > 0$ would yield a contradiction).

We are now ready to show that an equilibrium of the type given in (b) always exist.

Since $h(0) = (1-g)(p_l/r) < (1-g)(p_h/r) = h(1)$, under Condition (10) there always exist pairs (μ_1, μ_2) with $h(\mu_1) = h(\mu_2)$ and μ_2 arbitrarily close to 1. In other words, let $\overline{\mu}$ be such that $h'(\overline{\mu}) = 0$ and let $\Gamma \triangleq]\max\{\mu_0, \overline{\mu}\}, 1[$. For any $\mu_2 \in \Gamma$, one can find a $\mu_1 \in]0, \overline{\mu}[$ such that $h(\mu_1) = h(\mu_2)$. That is, choosing $\mu_2 \in \Gamma$, μ_1 is uniquely determined by $h(\mu_1) = h(\mu_2)$ and this guarantees that both types are indifferent on E_p by definition of $h(\cdot)$.

For the strategy profiles postulated in (b), as argued above indifference with $e_0 = 0$ for the low types leads to $e_1 = \tilde{e}\,(\mu_1)$ and the participation constraints are fulfilled. The only remaining task is to show that randomized strategies $q(e|p)$ can be specified yielding μ_1, μ_2. Since $\mu_2 > \mu_0$, our previous analysis shows that $q(e|p_h)$ can be constructed and then $q(e|p_l)$ are given by equation (12). This proves existence of multiple pooling equilibria as stated in part (b).

We now turn to part (c). For any $e_1 \in]0, \tilde{e}\,(\mu_1)]$, the participation constraints are fulfilled as above. Given that we are focusing on bipolar equilibria, the weak inequality in (13) binds. Then, for $q(e_1|p_h) < 1$ to hold, it must be true that $\mu_1 < \mu_0$. If and only if this requirement is satisfied, consistent $q(e|p_h)$ can be constructed while $q(e|p_l)$ are given by equation (12).

Hence, it remains only to establish under which conditions do there exist pairs (μ_1, μ_2) with $0 < \mu_1 < \mu_0 < \mu_2 < 1$ and $h(\mu_1) = h(\mu_2)$. Given that $h(1) = (1-g)(p_h/r) > (1-g)(p_l/r) = h(0)$, there exists $\overline{\mu}$ with $h'(\overline{\mu}) = 0$, and $h''(\cdot) < 0$. Thus such pairs exist if and only if $\mu_0 > \mu_{\min}$ where $\mu_{\min}$ is defined by $0 < \mu_{\min} < 1$ and $h(\mu_{\min}) = h(1)$. ■

Proof of Proposition 7:

We start with part (a). By continuity, there exists $\varepsilon > 0$ small enough that

$$c(p_l, e) - c(p_l, e_p) < \frac{1}{2}\frac{p_h - p_l}{r} \quad \forall\, e \in (e_p, e_p + \varepsilon)\ .$$

Let $\delta = c(p_h, e_p + \varepsilon) - c(p_h, e_p) > 0$, so that $c(p_h, e) - c(p_h, e_p) > \delta$ for all $e > e_p + \varepsilon$. For a value function with strong learning,

$$\lim_{s\to\infty} v(p_l, \mu|r, s) = \frac{p_l}{r} \quad \text{and} \quad \lim_{s\to\infty} v(p_h, \mu|r, s) = \frac{p_h}{r} \; .$$

Hence there exists s^* such that, for all $s \geq s^*$,

$$\frac{p_h}{r} - v(p_l, \mu|r, s) > \frac{1}{2}\frac{p_h - p_l}{r} \quad \text{and} \quad \frac{p_h}{r} - v(p_h, \mu|r, s) < \frac{1}{2}\delta \; .$$

It follows that, for $s \geq s^*$,

$$c(p_l, e) - c(p_l, e_p) < \frac{p_h}{r} - v(p_l, \mu|r, s) \quad \forall\, e \in (e_p, e_p + \varepsilon) \; ,$$

i.e. (ED) fails, implying that $e^*(e_p) > e_p + \varepsilon$, and

$$c(p_h, e) - c(p_h, e_p) > \frac{p_h}{r} - v(p_h, \mu|r, s) \quad \forall\, e > e_p + \varepsilon \; ,$$

i.e. (PD) fails, implying that $e^{**}(e_p) < e_p + \varepsilon$. We conclude that $e^{**}(e_p) < e^*(e_p)$, or that the considered pooling equilibrium survives the Intuitive Criterion.

To prove part (b), it suffices to notice that, if costs are linear, the quantities ε and δ above are independent of e_p. ■

Proof of Proposition 8:

In this proof, we make the dependence of all involved quantities on s explicit. Recall from Proposition 2 that pooling equilibria correspond to education levels in $[0, \overline{e_p}(s)]$ and that, under strong learning, $\overline{e_p}(s)$ is strictly decreasing in s and $\lim_{s\to\infty} \overline{e_p}(s) = 0$.

Recall also conditions (ED) and (PD). A pooling equilibrium with education level e_p fulfills the Intuitive Criterion if and only if condition (PD) fails at $e = e^*(e_p)$, where the latter education level is defined by taking equality in condition (ED).

Define

$$I_h(e_p|s) = c(p_h, e_p) - c(p_h, e^*(e_p)) + \frac{p_h}{r} - v(p_h, \mu_0|r, s) \; .$$

It follows that the Intuitive Criterion fails at e_p if and only if $I_h(e_p|s) > 0$.

Step 1. I_h is strictly increasing in e_p. Hence, for a given s, if the Intuitive criterion fails at e_p, it also fails at all larger education levels.

To prove this, differentiate the equality defining $e^*(e_p)$ to obtain

$$\frac{\partial e^*(e_p|s)}{\partial e_p} = \frac{c_e(p_l, e_p)}{c_e(p_l, e^*(e_p))} > 0 \; .$$

Now, differentiating I_h with respect to e_p yields

$$\begin{aligned}\frac{\partial I_h(e_p|s)}{\partial e_p} &= c_e(p_h, e_p) - c_e(p_h, e^*(e_p)) \cdot \frac{\partial e^*(e_p|s)}{\partial e_p} \\ &= \frac{c_e(p_h, e_p)c_e(p_l, e^*(e_p)) - c_e(p_h, e^*(e_p))c_e(p_l, e_p)}{c_e(p_l, e^*(e_p))} > 0 \ ,\end{aligned}$$

where the last inequality follows from log-submodularity.

Step 2. There exists $s^*(0)$ such that all pooling equilibria fail the Intuitive Criterion for $s < s^*(0)$, and $e_p = 0$ survives it for $s \geq s^*(0)$.

As in the proof of Proposition 7, we can find a precision $s^*(0)$ such that $e_p = 0$ survives the Intuitive Criterion for $s \geq s^*(0)$ and fails it for $s < s^*(0)$. In the latter case, the conclusion follows from Step 1.

Step 3. Let $s \geq s^*(0)$. There exists a unique education level $\widetilde{e}(s)$ such that $I_h(\widetilde{e}(s)) = 0$ and $I_h(e_p) > 0$ for all $e_p > \widetilde{e}(s)$. Further, $\widetilde{e}(s)$ is strictly increasing in s and $\lim_{s\to\infty} \widetilde{e}(s) = +\infty$.

It follows from Step 1 that either $I_h(e_p) > 0$ for all e_p, and so all pooling equilibria fail the Intuitive Criterion, or there exists a unique education level $\widetilde{e}(s)$ as stated. By Step 2, the former case can only occur if $s < s^*(0)$.

Notice that

$$\frac{\partial I_h(e_p|s)}{\partial s} = -c_e(p_h, e^*(e_p|s)) \cdot \frac{\partial e^*(e_p|s)}{\partial s} - \frac{\partial v(p_h, \mu_0|r, s)}{\partial s} < 0 \ ,$$

where the inequality follows from P2 and the strictly decreasing profile of $e^*(e_p|s)$ in s (see equality in condition (ED)).

Differentiating $I_h(\widetilde{e}(s)) = 0$ with respect to s now yields

$$\frac{\partial I_h(e_p|s)}{\partial e_p}\frac{\partial \widetilde{e}(s)}{\partial s} + \frac{\partial I_h(e_p|s)}{\partial s} = 0$$

which, since $\frac{\partial I_h(e_p|s)}{\partial e_p} > 0$ and $\frac{\partial I_h(e_p|s)}{\partial s} < 0$, implies $\frac{\partial \widetilde{e}(s)}{\partial s} > 0$.

It follows that $\widetilde{e}(s)$ is a strictly increasing function. Thus, either $\lim_{s\to\infty} \widetilde{e}(s) = +\infty$ as claimed, or it has an upper bound and so (by virtue of being increasing) a finite limit L. Suppose the latter case would hold. Recall that $e^*(e_p) > e_p$ for all e_p. As $I_h(\widetilde{e}(s)|s) = 0$, from the definition of I_h we obtain that

$$\lim_{s\to\infty} c(p_h, e^*(\widetilde{s})) - c(p_h, \widetilde{e}(s)) = 0 \ ,$$

because $\lim_{s\to\infty} v(p_h, \mu|r, s) = \frac{p_h}{r}$ in the strong learning case. This is a contradiction with $e^*(L) > L$.

Step 4. For $s \geq s^*(0)$, the rest of the proof follows.

Recapitulating, e_p yields a pooling equilibrium if and only if $e_p \in [0, \overline{e_p}(s)]$, and in that case survives the Intuitive Criterion if and only if $e_p \in [0, \widetilde{e}(s)]$. Given that $\overline{e_p}(s)$ is strictly decreasing and converges to 0 as $s \to \infty$, and $\widetilde{e}(s)$ is strictly increasing and diverges to infinity (and $\widetilde{e}(s^*(0)) = 0 < \overline{e_p}(s^*(0))$), it follows that they must intersect at a unique precision $\overline{s}$ such that part (b) holds below $\overline{s}$ and part (c) holds above it.

It remains to consider the case where c_e is log-linear. Retracing our steps, we can see from the computations in Step 1 that, in this case, $\partial I_h/\partial e_p = 0$, while from Step 2 we still see that $\partial I_h/\partial s < 0$. It follows that, for a fixed s, either all pooling equilibria fail the Intuitive criterion, or all survive it (the locus of $\widetilde{s}$ becomes vertical). ■

Proof of Proposition 9: It is clear that a mixed strategy equilibrium fails the Intuitive Criterion if there exists an e such that the two conditions in **Proposition 9** are fulfilled. To see the *only if* part, note that low and high types are indifferent between signals in E_p, hence $v(p_i, \mu(e)) - c(p_i, \mu(e))$ is constant for all $e \in E_p$, given $i \in \{l, h\}$. It follows that if Equilibrium Dominance for the low types and the Participation Constraint of the high types hold for a given signal in E_p, they also hold for all signals in E_p. ■

Proof of Proposition 10:

The wage is a linear function of current belief given by $w(\mu) = \mu(p_h - p_l) + p_l$. Hence, the value function reads

$$v(p, \mu|r) = \frac{p_l}{r} + \frac{p_h - p_l}{1+r}\left(\mu + \frac{E[\mu'|p, \mu]}{r}\right) .$$

The boundary conditions in P1 follow from $E[\mu'|p, \mu] = \mu$ when $\mu \in \{0, 1\}$. To establish the ranking of the value functions it is sufficient to show that $E[\mu'|p_l, \mu] < \mu < E[\mu'|p_h, \mu]$ for all $\mu \in (0, 1)$. We start by considering the first inequality. Let $\Lambda \triangleq \mu/(1-\mu)$ and X denote the output realization in period 1. Bayes rule implies that $\Lambda'(X) = \Lambda\left[g_h(X)/g_l(X)\right]$ so that

$$E[\Lambda'|p_l, \Lambda] = \Lambda \int \left(\frac{g_h(X)}{g_l(X)}\right) g_l(X) dX = \Lambda .$$

Given that Λ is a convex function of μ, Jensen's inequality yields $E[\mu'|p_l, \mu] < \mu$. A similar reasoning using the converse transformation $\widetilde{\Lambda} \triangleq (1-\mu)/\mu$ yields $E[\mu'|p_h, \mu] > \mu$. ■

Proof of Proposition 11:

The wage does not directly depend on the worker's type, but solely on the current belief ratio θ. It is equal to the expected output $E[p|\theta] = (p_h - p_l)\left(\frac{\theta}{1+\theta}\right) + p_l$.

For a low ability worker, $d\theta_t = \theta_t s dZ_t$ and thus the asset value solves the following Hamilton-Jacobi-Bellman equation

$$rv\left(p_l,\theta\right) - \frac{1}{2}\left(\theta s\right)^2 v''\left(p_l,\theta\right) = \left(p_h - p_l\right)\left(\frac{\theta}{1+\theta}\right) + p_l \ ,$$

which is a second order non-homogeneous ODE with non-constant coefficients. The homogeneous problem satisfies an Euler equation[48] whose solution reads

$$v^H\left(p_l,\theta\right) = C_{1l}\theta^{\alpha^-} + C_{2l}\theta^{\alpha^+} \ ,$$

where α^- and α^+ are the negative and positive roots of the quadratic equation

$$\alpha\left(\alpha-1\right)\frac{s^2}{2} - r = 0 \ .$$

Thus $\alpha^- = \frac{1}{2}(1-\Delta)$ and $\alpha^+ = \frac{1}{2}(1+\Delta)$ with $\Delta = \frac{1}{s}\sqrt{s^2+8r}$. Notice that $\alpha^+ - \alpha^- = \Delta$ and $\alpha^+ + \alpha^- = 1$.

To solve for the non-homogeneous equation we use the method of variations of parameters. The non-homogeneous term is composed of a non-linear function of θ plus a constant term. Thus we can assume that the particular solution is of the form

$$v^{NH}\left(p_l,\theta\right) = \left[y_1\left(\theta\right)\theta^{\alpha^-} + y_2\left(\theta\right)\theta^{\alpha^+}\right] + \frac{p_l}{r} \ .$$

Standard derivations yield the system of equations

$$\begin{pmatrix} \theta^{\alpha^-} & \theta^{\alpha^+} \\ \alpha^-\theta^{\alpha^- -1} & \alpha^+\theta^{\alpha^+ -1} \end{pmatrix}\begin{pmatrix} y_1'\left(\theta\right) \\ y_2'\left(\theta\right) \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2\sigma}{(1+\theta)\theta s} \end{pmatrix} .$$

Given that the Wronskian of the two linearly independent solutions is

$$\theta^{\alpha^-}\alpha^+\theta^{\alpha^+-1} - \theta^{\alpha^+}\alpha^-\theta^{\alpha^- -1} = \alpha^+ - \alpha^- = \Delta \ ,$$

we have

$$y_1\left(\theta\right) = \frac{2\sigma}{s\Delta}\int\frac{1}{\left(1+x\right)x^{\alpha^-}}dx \quad \text{and} \quad y_2\left(\theta\right) = \frac{2\sigma}{s\Delta}\int\frac{1}{\left(1+x\right)x^{\alpha^+}}dx \ .$$

Thus the general form of the particular solution reads

$$v^{NH}\left(p_l,\theta\right) = \frac{2\sigma}{s\Delta}\left(\theta^{\alpha^-}\int\frac{1}{\left(1+x\right)x^{\alpha^-}}dx + \theta^{\alpha^+}\int\frac{1}{\left(1+x\right)x^{\alpha^+}}dx\right) + \frac{p_l}{r} \ . \tag{14}$$

[48] Euler equations are second order homogeneous ODE of the form $\beta v(\theta) + \alpha\theta v'^2 v''(\theta) = 0$, for given constants β and α. They admit a closed form solution as described in e.g. Polyanin and Zaitsev (2003).

The bounds of integration and constants C_{1l} and C_{2l} of the homogeneous solution are pinned down by the boundary conditions

$$v\left(p_l,\theta\right) \xrightarrow[\theta\to 0]{} \frac{p_l}{r} \quad \text{and} \quad v\left(p_l,\theta\right) \xrightarrow[\theta\to\infty]{} \frac{p_h}{r} \; . \tag{15}$$

Let us first consider the homogeneous solution. Given that $\theta^{\alpha^-} \to \infty$ as $\theta \downarrow 0$, the first boundary condition can be satisfied if and only if C_{1l} equals zero. Similarly, because $\theta^{\alpha^+} \to \infty$ as $\theta \uparrow \infty$, the second boundary condition allows us to set C_{2l} equal to zero. All that remains is to determine the integration bounds in equation (14). Consider the following function

$$v\left(p_l,\theta\right) = \frac{2\sigma}{s\Delta}\left(\theta^{\alpha^-}\int_0^\theta \frac{1}{\left(1+x\right)x^{\alpha^-}}dx + \theta^{\alpha^+}\int_\theta^\infty \frac{1}{\left(1+x\right)x^{\alpha^+}}dx\right) + \frac{p_l}{r} \; . \tag{16}$$

Let us examine first the limit when $\theta \downarrow 0$. Given that $\theta^{\alpha^-} \to \infty$ and $\int_0^\theta [\left(1+x\right)x^{\alpha^-}]^{-1}dx \to 0$ as $\theta \downarrow 0$, we can apply l'Hôpital's rule to determine the limit. Straightforward calculations show that $\theta^{\alpha^-}\int_0^\theta [\left(1+x\right)x^{\alpha^-}]^{-1}dx \to -\theta/[(1+\theta)\alpha^-] \to 0$ as $\theta \downarrow 0$. A similar argument yields $\theta^{\alpha^+}\int_\theta^\infty [\left(1+x\right)x^{\alpha^+}]^{-1}dx \to \theta/[(1+\theta)\alpha^+] \to 0$ as $\theta \downarrow 0$.[49] Hence, (16) satisfies the first boundary condition in (15). Now, consider the limit when $\theta \uparrow \infty$. We can again use l'Hôpital's rule because $\theta^{\alpha^-} \to 0$ and $\int_0^\theta [\left(1+x\right)x^{\alpha^-}]^{-1}dx \to \infty$ as $\theta \uparrow \infty$, so that $\theta^{\alpha^-}\int_0^\theta [\left(1+x\right)x^{\alpha^-}]^{-1}dx \to -1/\alpha^-$ as $\theta \uparrow \infty$. Similarly, we obtain $\theta^{\alpha^+}\int_\theta^\infty [\left(1+x\right)x^{\alpha^+}]^{-1}dx \to 1/\alpha^+$ as $\theta \uparrow \infty$. Thus we have

$$\lim_{\theta\to\infty} v\left(p_l,\theta\right) = \frac{2\sigma}{s\Delta}\left(\frac{1}{-\alpha^-} + \frac{1}{\alpha^+}\right) + \frac{p_l}{r} = \frac{2\sigma}{s}\left(\frac{-1}{\alpha^-\alpha^+}\right) + \frac{p_l}{r} = \frac{p_h}{r} \; ,$$

where the last equality follows from $\alpha^-\alpha^+ = -2r/s^2$. We have established that (16) also satisfies the second boundary condition in (15), which completes the derivation of $v(p_l,\theta)$.

The asset value of the high type is derived similarly. For a high ability worker, $d\theta_t = \theta_t s\left(sdt + dZ_t\right)$ and thus the asset value solves

$$rv\left(p_h,\theta\right) - \theta s^2 v'\left(p_h,\theta\right) - \frac{1}{2}\left(\theta s\right)^2 v''\left(p_h,\theta\right) = \left(p_h - p_l\right)\left(\frac{\theta}{1+\theta}\right) + p_l \; .$$

The homogeneous solution reads

$$v^H\left(p_h,\theta\right) = C_{1h}\theta^{\gamma^-} + C_{2h}\theta^{\gamma^+} \; ,$$

where γ^- and γ^+ are the negative and positive roots of the quadratic equation

$$\gamma\left(\gamma+1\right)\frac{s^2}{2} - r = 0 \; ,$$

[49] Notice that $\int_\theta^\infty [\left(1+x\right)x^{\alpha^+}]^{-1}dx < \int_\theta^\infty x^{-\alpha^+-1}dx = \theta^{-\alpha^+}/\alpha^+$. Thus $\int_\theta^\infty [\left(1+x\right)x^{\alpha^+}]^{-1}dx$ is bounded for all $\theta > 0$ and the asset equation is well defined.

so that $\gamma^- = \frac{1}{2}(-1-\Delta)$ and $\gamma^+ = \frac{1}{2}(-1+\Delta)$. The non-homogeneous solution is of the form

$$v^{NH}(p_h,\theta) = \left[z_1(\theta)\,\theta^{\gamma^-} + z_2(\theta)\,\theta^{\gamma^+}\right] + \frac{p_l}{r}\,,$$

where the functions $z_1(\theta)$ and $z_2(\theta)$ satisfy

$$\begin{pmatrix} \theta^{\gamma^-} & \theta^{\gamma^+} \\ \gamma^-\theta^{\gamma^- -1} & \gamma^+\theta^{\gamma^+ -1} \end{pmatrix} \begin{pmatrix} z_1'(\theta) \\ z_2'(\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2\sigma}{(1+\theta)\theta s} \end{pmatrix}.$$

Following the same steps as before yields the solution in Proposition 11.

We now show that the value function obtained above exhibits strong learning. We establish each property in turn.

P1. This property is most easily established reversing the change of variable from θ_t to μ_t

$$(i)\ \text{Low ability worker}: \ d\mu_t = \mu_t(1-\mu_t)\,s\,(-s\mu_t dt + dZ_t)\,, \tag{17}$$

$$(ii)\ \text{High ability worker}: \ d\mu_t = \mu_t(1-\mu_t)\,s\,(s(1-\mu_t)\,dt + dZ_t)\,. \tag{18}$$

By definition

$$\begin{aligned} v(p_i,\mu_t) &= E\left[\int_t^{+\infty} e^{-r(\tau-t)} w(\mu_\tau) d\tau \middle| p_i,\mu_t\right] \\ &= \int_t^{+\infty} e^{-r(\tau-t)} E\left[w(\mu_\tau)|p_i,\mu_t\right] d\tau\ , \text{for all } \mu_t \in (0,1) \text{ and } i \in \{l,h\}\ , \end{aligned} \tag{19}$$

where the second equality follows from Fubini's theorem. When the worker is of the high type ($p_i = p_h$), we know from (18) that μ_t has a positive deterministic trend: $\mu_t(1-\mu_t)^2 s^2$. As $w(\mu_t) = \mu_t(p_h - p_l) + p_l$ is a linear function of μ_t, it follows that $E[w(\mu_\tau)|p_h,\mu_t] > w(\mu_t)$ for all $\tau > t$, and so $v(p_h,\mu_t) > w(\mu_t)/r$. Similarly, condition (17) shows that μ_t has a negative deterministic trend when the worker is of the low type, thus $v(p_l,\mu_t) < w(\mu_t)/r$. Finally, notice that when μ_t goes to zero or one, its stochastic component vanishes which provides us with the two boundary conditions.

P2. Differentiating (19) with respect to s yields

$$\frac{\partial v(p_i,\mu_t)}{\partial s} = (p_h - p_l)\int_t^{+\infty} e^{-r(\tau-t)} \frac{\partial E[\mu_\tau|p_i,\mu_t]}{\partial s} d\tau\ , \text{for all } \mu_t \in (0,1) \text{ and } i \in \{l,h\}\ .$$

It is therefore sufficient to prove that $\partial E[\mu_\tau|p_i,\mu_t]/\partial s$ is positive when $p_i = p_h$ and negative when $p_i = p_l$. This follows from (17) and (18) as beliefs exhibit a negative trend for low types and a positive trend for high types, both of them being increasing in absolute values with respect to s.

P3. The limit condition as $s \to 0$ is satisfied because both deterministic and stochastic terms in (17) and (18) converge to 0. Accordingly, beliefs remain constant, i.e. $\lim_{s\to 0} E\left[\mu_\tau | p_i, \mu_t\right] = \mu_t$ for $i \in \{l, h\}$. To establish the limit condition as $s \to \infty$, let us focus first on high types. Notice that the belief ratio θ_t is a geometric Brownian motion and so $\theta_\tau(Z|p_h, \theta_t) = \theta_t \exp\left(\frac{s^2}{2}(\tau - t) + sZ\right)$, where Z is normally distributed with mean 0 and variance $\sigma^2(\tau - t)$. Given that $\mu(\theta) = \theta/(1+\theta)$, we have: $\mu_\tau(Z|p_h, \mu_t) = \left[1 - \mu_t + \mu_t \exp\left(-\frac{s^2}{2}(\tau - t) - sZ\right)\right]^{-1}$. Hence, for all $\varepsilon > 0$ and Z, there exists a signal/noise ratio $\underline{s}(\varepsilon, Z)$ such that $1 - \mu_\tau(Z|p_h, mu_t) < \varepsilon$ for all $s > \underline{s}(\varepsilon, Z)$. It follows that $\lim_{s\to\infty} E\left[\mu_\tau | p_h, \mu_t\right] = 1$ which in turn implies that $\lim_{s\to\infty} v(\mu|p_h) = p_h/r$. One can establish in a similar fashion that $\lim_{s\to\infty} E\left[\mu_\tau | p_l, \mu_t\right] = 0$ because $\mu_\tau(Z|p_l, \mu_t) = \left[1 - \mu_t + \mu_t \exp\left(\frac{s^2}{2}(\tau - t) - sZ\right)\right]^{-1}$. ■

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