On Elementary Extensions in Fuzzy Predicate Logics

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Abstract. Our work is a contribution to the model-theoretic study of equality-free fuzzy predicate logics. We give a characterization of elementary equivalence in fuzzy predicate logics using elementary extensions and introduce an strengthening of this notion, the so-called *strong elementary equivalence*. Using the method of diagrams developed in [5] and elementary extensions we present a counterexample to Conjectures 1 and 2 of [8].

Keywords: Equality-free language, fuzzy predicate logic, model theory, elementary extensions, elementary equivalence.

1 Introduction

This work is a contribution to the model-theoretic study of equality-free fuzzy predicate logics. Model theory is the branch of mathematical logic that studies the construction and classification of structures. Construction means building structures or families of structures, which have some feature that interest us. Classifying a class of structures means grouping the structures into subclasses in a useful way, and then proving that every structure in the collection does belong in just one of the subclasses. The most basic classification in classical model theory is given by the relations of elementary equivalence and isomorphism. Our purpose in the present article is to investigate and characterize the relation of elementary equivalence between two structures in terms of elementary extensions. We introduce also an strengthening of this notion, the so-called *strong elementary equivalence*.

The basic notion of elementary equivalence between models is due to A. Tarski (see [11]) and the fundamental results on elementary extensions and elementary chains were introduced by A. Tarski and R. Vaught in [1]. In the context of fuzzy predicate logics, elementarily equivalent structures were defined in [8] (Definition 10), there the authors presented a characterization of conservative extension theories using the elementary equivalence relation (see Theorems 6 and

11 of [8]). A notion of elementary equivalent models *in a degree d* was presented in [10] (see Definition 4.33).

P. Hájek and P. Cintula proved in Theorem 6 of [8] that, in core fuzzy logics, a theory T_2 is a conservative extension of another theory T_1 if and only if each exhaustive model of T_1 can be elementarily embedded into some model of T_2. Then, they conjectured the same result to be true for arbitrary structures (Conjecture 2 of [8]). In this paper we present a counterexample to Conjecture 2, using the method of diagrams developed in [5] and elementary extensions.

The paper is structured as follows: Section 2 is devoted to preliminaries on fuzzy predicate logics. In Section 3 we introduce some known definitions and basic facts on canonical models (see section 4 and 5 of [8]) and of the method of diagrams for fuzzy predicate logics developed in [5]. Later on we prove some new propositions related to canonical models and diagrams. In Section 4 we present a counterexample to Conjectures 1 and 2 of [8], using the results of Section 3. Finally, in Section 5 we prove a characterization theorem of elementary equivalence in fuzzy predicate logics. We conclude the paper with a section of work in progress and future work.

2 Preliminaries

Our study of the model theory of fuzzy predicate logics is focused on the basic fuzzy predicate logic MTL∀ and stronger t-norm based logics, the so-called *core fuzzy logics*. For a reference on the logic MTL see [6]. We start by introducing the notion of core fuzzy logic in the propositional case.

Definition 1 *A propositional logic L is a* core fuzzy logic *iff L satisfies:*

1. *For all formulas* ϕ, φ, α, $\varphi \leftrightarrow \phi \vdash \alpha(\varphi) \leftrightarrow \alpha(\phi)$.
2. (LDT) Local Deduction Theorem: *for each theory T and formulas* ϕ, φ:

$$T, \varphi \vdash \phi \text{ iff there is a natural number } n \text{ such that } T \vdash \varphi^n \to \phi.$$

3. *L expands* MTL.

For a thorough treatment of core fuzzy logics we refer to [8], [4] and [3]. A *predicate language Γ* is a triple (**P**,**F**,**A**) where **P** is a non-empty set of predicate symbols, **F** is a set of function symbols and **A** is a mapping assigning to each predicate and function symbol a natural number called the *arity of the symbol*. Functions f for which $\mathbf{A}(f) = 0$ are called *object constants*. Formulas of the predicate language Γ are built up from the symbols in (**P**,**F**,**A**), the connectives and constants of L, the logical symbols ∀ and ∃, variables and punctuation. Throughout the paper we consider the equality symbol as a binary predicate symbol not as a logical symbol, we work in equality-free fuzzy predicate logics. That is, the equality symbol is not necessarily present in all the languages and its interpretation is not fixed. Given a propositional core fuzzy logic L we denote by $L\forall$ the corresponding fuzzy predicate logic.

Let L be a fixed propositional core fuzzy logic and $\mathbf{B}$ an L-algebra, we introduce now the semantics for the fuzzy predicate logic $L\forall$. A $\mathbf{B}$-*structure* for predicate language Γ is a tuple $\mathbf{M} = (M, (P_{\mathbf{M}})_{P\in\Gamma}, (F_{\mathbf{M}})_{F\in\Gamma}, (c_{\mathbf{M}})_{c\in\Gamma})$ where:

1. M is a non-empty set.
2. For each n-ary predicate $P \in \Gamma$, $P_{\mathbf{M}}$ is a $\mathbf{B}$-fuzzy relation $P_{\mathbf{M}} : M^n \to \mathbf{B}$.
3. For each n-ary function symbol $F \in \Gamma$, $F_{\mathbf{M}} : M^n \to M$.
4. For each constant symbol $c \in \Gamma$, $c_{\mathbf{M}} \in M$.

Given a $\mathbf{B}$-structure $\mathbf{M}$, we define an $\mathbf{M}$-*evaluation* of the variables as a mapping v which assigns to each variable an element from M. By $\phi(x_1, \ldots, x_k)$ we mean that all the free variables of ϕ are among $x_1, \ldots, x_k$. Let v be an $\mathbf{M}$-evaluation, we denote by $v[x \to d]$ the $\mathbf{M}$-evaluation such that $v[x \to d](x) = d$ and for each variable y different from x, $v[x \to d](y) = v(y)$. Let $\mathbf{M}$ be a $\mathbf{B}$-structure and v an $\mathbf{M}$-evaluation, we define the values of the terms and *truth values* of the formulas as follows:

$$\|c\|^{\mathbf{B}}_{\mathbf{M},v} = c_{\mathbf{M}},\ \|x\|^{\mathbf{B}}_{\mathbf{M},v} = v(x)$$

$$\|F(t_1, \ldots, t_n)\|^{\mathbf{B}}_{\mathbf{M},v} = F_{\mathbf{M}}(\|t_1\|^{\mathbf{B}}_{\mathbf{M},v}, \ldots, \|t_n\|^{\mathbf{B}}_{\mathbf{M},v})$$

for each variable x, each constant symbol $c \in \Gamma$, each n-ary function symbol $F \in \Gamma$ and Γ-terms $t_1, \ldots, t_n$, respectively.

$$\|P(t_1, \ldots, t_n)\|^{\mathbf{B}}_{\mathbf{M},v} = P_{\mathbf{M}}(\|t_1\|^{\mathbf{B}}_{\mathbf{M},v}, \ldots, \|t_n\|^{\mathbf{B}}_{\mathbf{M},v})$$

for each n-ary predicate $P \in \Gamma$,

$$\|\delta(\phi_1, \ldots, \phi_n)\|^{\mathbf{B}}_{\mathbf{M},v} = \delta_{\mathbf{B}}(\|\phi_1\|^{\mathbf{B}}_{\mathbf{M},v}, \ldots, \|\phi_n\|^{\mathbf{B}}_{\mathbf{M},v})$$

for each n-ary connective $\delta \in L$ and Γ-formulas $\phi_1, \ldots, \phi_n$. Finally, for the quantifiers,

$$\|\forall x\phi\|^{\mathbf{B}}_{\mathbf{M},v} = \inf\{\|\phi\|^{\mathbf{B}}_{\mathbf{M},v[x\to d]} : d \in M\}$$

$$\|\exists x\phi\|^{\mathbf{B}}_{\mathbf{M},v} = \sup\{\|\phi\|^{\mathbf{B}}_{\mathbf{M},v[x\to d]} : d \in M\}$$

Remark that, since the L-algebras we work with are not necessarily complete, the above suprema and infima could be not defined in some cases. It is said that a $\mathbf{B}$-structure is *safe* if such suprema and infima are always defined. From now on we assume that all our structures are safe. In particular, throughout the paper we will work only with $\mathbf{B}$-structures such that $\mathbf{B}$ is an L-chain.

If v is an evaluation such that for each $0 < i \leq n$, $v(x_i) = d_i$, and λ is either a Γ-term or a Γ-formula, we abbreviate by $\|\lambda(d_1, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}$ the expression

$\|\lambda(x_1,\ldots,x_n)\|^{\mathbf{B}}_{\mathbf{M},v}$. A Γ-sentence is a Γ-formula without free variables. Let ϕ be a Γ-sentence, given a $\mathbf{B}$-*structure* $\mathbf{M}$, for predicate language Γ, it is said that $\mathbf{M}$ is a *model* of ϕ iff $\|\phi\|^{\mathbf{B}}_{\mathbf{M}} = 1$. And that $\mathbf{M}$ is a model of a set of Γ-sentences Σ iff for all $\phi \in \Sigma$, $\mathbf{M}$ is a model of ϕ.

From now on, given an L-algebra $\mathbf{B}$, we say that $(\mathbf{M},\mathbf{B})$ is a Γ-*structure* instead of saying that $\mathbf{M}$ is a $\mathbf{B}$-structure for predicate language Γ. Let $(\mathbf{M},\mathbf{B})$ be a Γ-structure, by $Alg(\mathbf{M},\mathbf{B})$ we denote the subalgebra of $\mathbf{B}$ whose domain is the set $\{\|\phi\|^{\mathbf{B}}_{\mathbf{M},v} : \phi, v\}$ of truth degrees of all Γ-formulas ϕ under all $\mathbf{M}$-evaluations v of variables. Then, it is said that $(\mathbf{M},\mathbf{B})$ is *exhaustive* iff $Alg(\mathbf{M},\mathbf{B}) = \mathbf{B}$. Now let $(\mathbf{M}_1,\mathbf{B}_1)$ and $(\mathbf{M}_2,\mathbf{B}_2)$ be two Γ-structures, we denote by $(\mathbf{M}_1,\mathbf{B}_1) \equiv (\mathbf{M}_2,\mathbf{B}_2)$ the fact that $(\mathbf{M}_1,\mathbf{B}_1)$ and $(\mathbf{M}_2,\mathbf{B}_2)$ are *elementarily equivalent*, that is, that they are models of exactly the same Γ-sentences.

Finally we recall two notions of preserving mappings: elementary mapping and quantifier-free preserving mapping.

Definition 2 *Let* $(\mathbf{M}_1,\mathbf{B}_1)$ *and* $(\mathbf{M}_2,\mathbf{B}_2)$ *be* Γ*-structures. We say that the pair* (f,g) *is a* quantifier-free preserving mapping *iff*

1. $g : \mathbf{B}_1 \to \mathbf{B}_2$ *is an L-algebra homomorphism of* $\mathbf{B}_1$ *into* $\mathbf{B}_2$.
2. $f : M_1 \to M_2$ *is a mapping of* M_1 *into* M_2.
3. *For each quantifier-free* Γ*-formula* $\phi(x_1,\ldots,x_n)$ *and elements* $d_1,\ldots,d_n \in M_1$, $g(\|\phi(d_1,\ldots,d_n)\|^{\mathbf{B}_1}_{\mathbf{M}_1}) = \|\phi(f(d_1),\ldots,f(d_n))\|^{\mathbf{B}_2}_{\mathbf{M}_2}$

Moreover, if condition 3. holds for every Γ*-formula, it is said that* (f,g) *is an* elementary mapping*. And it is said that* (f,g) *is an* elementary embedding *when both* f *and* g *are one-to-one.*

We have presented so far only a few definitions and basic notation. A detailed introduction to the syntax and semantics of fuzzy predicate logics can be found in [7].

3 Diagrams and Canonical Models

In this section we recall first some definitions and basic facts on canonical models (see section 4 and 5 of [8]) and of the method of diagrams for fuzzy predicate logics developed in [5]. Later on we prove some new propositions related to canonical models and diagrams.

Definition 3 *Let* $(\mathbf{M},\mathbf{B})$ *be a* Γ*-structure, we define:*

1. $Th(\mathbf{M},\mathbf{B})$ *is the set of* Γ*-sentences true in the model* $(\mathbf{M},\mathbf{B})$.
2. $\Gamma_{\mathbf{M}}$ *is the expansion of* Γ *by adding a constant symbol* c_d*, for each* $d \in M$.
3. $(\mathbf{M}',\mathbf{B})$ *is the expansion of* $(\mathbf{M},\mathbf{B})$ *to the language* $\Gamma_{\mathbf{M}}$*, by interpreting for each* $d \in M$*, the constant* c_d *by* d.
4. *The* Elementary Diagram *of* $(\mathbf{M},\mathbf{B})$*, denoted by* $\mathrm{EDIAG}(\mathbf{M},\mathbf{B})$*, is the set of all* $\Gamma_{\mathbf{M}}$*-sentences true in* $(\mathbf{M}',\mathbf{B})$.
5. *The* Complement of the Elementary Diagram *of* $(\mathbf{M},\mathbf{B})$*, denoted by* $\overline{\mathrm{EDIAG}}(\mathbf{M},\mathbf{B})$ *is the set of all* $\Gamma_{\mathbf{M}}$*-sentences* ϕ *such that* $\phi \notin \mathrm{EDIAG}(\mathbf{M},\mathbf{B})$.

Definition 4 *Let* $(\mathbf{M}, \mathbf{B})$ *be a* Γ*-structure, we expand the language further adding new symbols to the predicate language* $\Gamma_{\mathbf{M}}$ *and we define:*

1. $\Gamma_{(\mathbf{M},\mathbf{B})}$ *is the expansion of* $\Gamma_{\mathbf{M}}$ *by adding a nullary predicate symbol* P_b*, for each* $b \in B$.
2. $(\mathbf{M}^*, \mathbf{B})$ *is the expansion of* $(\mathbf{M}', \mathbf{B})$ *to the language* $\Gamma_{(\mathbf{M},\mathbf{B})}$*, by interpreting for each* $b \in B$*, the nullary predicate symbol* P_b *by* b.
3. $\mathrm{EQ}(\mathbf{B})$ *is the set of* $\Gamma_{(\mathbf{B},\mathbf{M})}$*-sentences of the form* $\delta(P_{b_1}, \ldots, P_{b_n}) \leftrightarrow \epsilon(P_{a_1}, \ldots, P_{a_k})$ *such that* $\mathbf{B} \models \delta(b_1, \ldots, b_n) = \epsilon(a_1, \ldots, a_k)$*, where* δ, ϵ *are* L*-terms and* $a_1, \ldots, a_k, b_1, \ldots, b_n \in B$
4. $\mathrm{NEQ}(\mathbf{B})$ *is the set of* $\Gamma_{(\mathbf{B},\mathbf{M})}$*-sentences of the form* $\delta(P_{b_1}, \ldots, P_{b_n}) \leftrightarrow \epsilon(P_{a_1}, \ldots, P_{a_k})$ *such that* $\mathbf{B} \models \delta(b_1, \ldots, b_n) \neq \epsilon(a_1, \ldots, a_k)$*, where* δ, ϵ *are* L*-terms and* $a_1, \ldots, a_k, b_1, \ldots, b_n \in B$
5. The Basic Full Elementary Diagram *of* $(\mathbf{M}, \mathbf{B})$*, denoted by* $\mathrm{FEDIAG}_0(\mathbf{M}, \mathbf{B})$*, is the set*

$$\mathrm{EDIAG}(\mathbf{M}, \mathbf{B}) \cup \mathrm{EQ}(\mathbf{B}) \cup \{\phi \leftrightarrow P_b : \phi \in \Gamma_{\mathbf{M}} \text{ and } \|\phi\|^{\mathbf{B}}_{\mathbf{M}^*} = b\}$$

6. The Full Elementary Diagram *of* $(\mathbf{M}, \mathbf{B})$*, denoted by* $\mathrm{FEDIAG}(\mathbf{M}, \mathbf{B})$*, is the set of all* $\Gamma_{(\mathbf{M},\mathbf{B})}$*-sentences true in* $(\mathbf{M}^*, \mathbf{B})$.

Proposition 5 [Proposition 32 of [5]] *Let* $(\mathbf{M}, \mathbf{B})$ *and* $(\mathbf{N}, \mathbf{A})$ *be two* Γ*-structures. The following are equivalent:*

1. *There is an expansion of* $(\mathbf{N}, \mathbf{A})$ *that is a model of* $\mathrm{FEDIAG}_0(\mathbf{M}, \mathbf{B})$.
2. *There is an elementary mapping* (f, g) *from* $(\mathbf{M}, \mathbf{B})$ *into* $(\mathbf{N}, \mathbf{A})$.

Moreover, g *is one-to-one iff for every sentence* $\psi \in \mathrm{NEQ}(\mathbf{B})$ *the expansion of* $(\mathbf{N}, \mathbf{A})$ *(defined in condition 1.) is not a model of* ψ.

Corollary 6 [Corollary 38 of [5]] *Let* $(\mathbf{M}, \mathbf{B})$ *and* $(\mathbf{N}, \mathbf{A})$ *two* Γ*-structures such that* $(\mathbf{M}, \mathbf{B})$ *is exhaustive. The following are equivalent:*

1. *There is an expansion of* $(\mathbf{N}, \mathbf{A})$ *that is a model of* $\mathrm{EDIAG}(\mathbf{M}, \mathbf{B})$.
2. *There is an elementary mapping* (f, g) *from* $(\mathbf{M}, \mathbf{B})$ *into* $(\mathbf{N}, \mathbf{A})$.

Moreover, g *is one-to-one iff for every sentence of* $\Gamma_{\mathbf{M}}$*,* $\psi \in \overline{\mathrm{EDIAG}}(\mathbf{M}, \mathbf{B})$*, the expansion of* $(\mathbf{N}, \mathbf{A})$ *(defined in condition 1.) is not a model of* ψ.

Remark that, as pointed out in [5], the mapping f of Proposition 5 and of Corollary 6 is not necessarily one-to-one, because we do not work with a crisp equality. Now we will see that, using canonical models, we can improve these results finding elementary expansions of a given model, in which f is one-to-one. We start by recalling some definitions from [4].

Definition 7 *A* Γ*-theory* T *is* linear *iff for each pair of* Γ*-sentences* $\phi, \psi \in \Gamma$*,* $T \vdash \phi \to \psi$ *or* $T \vdash \psi \to \phi$.

Definition 8 *A* Γ*-theory* Ψ *is* directed *iff for each pair of* Γ*-sentences* $\phi, \psi \in \Psi$*, there is a* Γ*-sentence* $\chi \in \Psi$ *such that both* $\phi \to \chi$ *and* $\psi \to \chi$ *are probable.*

Definition 9 *Let Γ and Γ' be predicate languages such that $\Gamma \subseteq \Gamma'$ and let T be a Γ'-theory. We say that T is* Γ-Henkin *if for each formula $\psi(x) \in \Gamma$ such that $T \not\vdash \forall x\psi$, there is a constant $c \in \Gamma'$ such that $T \not\vdash \psi(c)$. And we say that T is* $\exists$-Γ–Henkin *if for each formula $\psi(x) \in \Gamma$ such that $T \vdash \exists x\psi$, there is a constant $c \in \Gamma'$ such that $T \vdash \psi(c)$. Finally, a Γ-theory is called* doubly-Γ-Henkin *if it is both Γ-Henkin and $\exists$-Γ-Henkin. In case that $\Gamma = \Gamma'$, we say that T is* Henkin *($\exists$-Henkin, doubly Henkin, respectively).*

Theorem 10 [Theorem 2.20 of [4]] *Let T_0 be a Γ-theory and Ψ a directed set of Γ-sentences such that $T_0 \not\vdash \Psi$. Then, there is a linear doubly Henkin theory $T \supseteq T_0$ in a predicate language $\Gamma' \supseteq \Gamma$ such that $T \not\vdash \Psi$.*

Definition 11 *Let T be a Γ-theory. The* canonical model *of T, denoted by $(\mathbf{CM}(T), \mathbf{Lind}_T)$, where $\mathbf{Lind}_T$ is the Lindenbaum algebra of T (that is, the L-algebra of classes of T-equivalent Γ–sentences) is defined as follows: the domain of $\mathbf{CM}(T)$ is the set of closed Γ-terms, for every n-ary function symbol $F \in \Gamma$, $F_{(\mathbf{CM}(T),\mathbf{Lind}_T)}(t_1 \dots t_n) = F(t_1 \dots t_n)$ and for each n-ary predicate symbol $P \in \Gamma$, $P_{(\mathbf{CM}(T),\mathbf{Lind}_T)}(t_1 \dots t_n) = [P(t_1 \dots t_n)]_T$.*

From now on we write $\mathbf{CM}(T)$ instead of $(\mathbf{CM}(T), \mathbf{Lind}_T)$.

Lemma 12 [Lemma 2.24 of [4]] *Let T be a Henkin Γ-theory. Then,*

- $\mathbf{Lind}_T$ *is an L-chain iff T is linear*
- *For every sentence $\phi \in \Gamma$, $\|\phi\|^{\mathbf{Lind}_T}_{\mathbf{CM}(T)} = [\phi]_T$*
- *For every sentence $\phi \in \Gamma$, $T \vdash \phi$ iff $\mathbf{CM}(T) \models \phi$*
- $\mathbf{CM}(T)$ *is exhaustive*

Now we prove some new facts on diagrams and elementary extensions, using canonical models.

Proposition 13 *Let $(\mathbf{M}, \mathbf{B})$ be a Σ-structure and $T_0 \supseteq \mathrm{FEDIAG}_0(\mathbf{M}, \mathbf{B})$ a consistent theory in a predicate language $\Gamma \supseteq \Sigma$. If $\Psi \supseteq \mathrm{NEQ}(\mathbf{B})$ is a directed set of Γ-sentences such that $T_0 \not\vdash \Psi$, then there is a linear doubly Henkin theory $T \supseteq T_0$ in a predicate language $\Gamma' \supseteq \Gamma$ such that $T \not\vdash \Psi$ and an elementary mapping (f, g) from $(\mathbf{M}, \mathbf{B})$ into $\mathbf{CM}(T)$, with f and g one-to-one.*

Proof: By Theorem 10, there is a linear doubly Henkin theory $T \supseteq T_0$ in a predicate language $\Gamma' \supseteq \Gamma$ such that $T \not\vdash \Psi$. By Lemma 12, $\mathbf{CM}(T)$ is a model of $\mathrm{FEDIAG}_0(\mathbf{M}, \mathbf{B})$. Then, by Proposition 5 (Proposition 32 of [5]), there is an elementary mapping (f, g) from $(\mathbf{M}, \mathbf{B})$ into $\mathbf{CM}(T)$, defined as follows: for each $d \in M$, $f(d) = c_d$ and for each $b \in B$, $g(b) = [P_b]_T$. Moreover, since $T \not\vdash \mathrm{NEQ}(\mathbf{B})$, for every sentence $\psi \in \mathrm{NEQ}(\mathbf{B})$, $\mathbf{CM}(T)$ is not a model of ψ and thus, g is one-to-one: indeed, if $b \neq b'$, then $P_b \leftrightarrow P_{b'} \in \mathrm{NEQ}(\mathbf{B})$ and, by assumption, it is not true in $\mathbf{CM}(T)$ and consequently, $[P_b]_T \neq [P_{b'}]_T$ and thus $g(b) \neq g(b')$. Finally, by definition of $\mathbf{CM}(T)$, f is also one-to-one. □

Now as a Corollary of Propositions 6 and 13 we obtain the following result for exhaustive structures:

Corollary 14 *Let* $(\mathbf{M}, \mathbf{B})$ *be an exhaustive* Σ*-structure and* $T_0 \supseteq \mathrm{EDIAG}(\mathbf{M}, \mathbf{B})$ *a consistent theory in a predicate language* $\Gamma \supseteq \Sigma$*. If* $\Psi \supseteq \overline{\mathrm{EDIAG}}(\mathbf{M}, \mathbf{B})$ *is a directed set of formulas of* Γ *such that* $T_0 \not\vdash \Psi$*, then there is a linear doubly Henkin theory* $T \supseteq T_0$ *in a predicate language* $\Gamma' \supseteq \Gamma$ *such that* $T \not\vdash \Psi$ *and an elementary mapping* (f, g) *from* $(\mathbf{M}, \mathbf{B})$ *into* $\mathbf{CM}(T)$*, with* f *and* g *one-to-one.*

Now we recall the notion of witnessed model and show a direct application of Proposition 13, giving a generalization of Lemma 5 of [8] for non-exhaustive models. Let $(\mathbf{M}, \mathbf{B})$ be a Γ-structure. We say that $(\mathbf{M}, \mathbf{B})$ is *witnessed* iff for each Γ-formula $\phi(y, x_1, \ldots, x_n)$ and for each $d_1, \ldots, d_n \in M$, there is an element $e \in M$ such that $\|\exists y \phi(d_1, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}} = \|\phi(e, d_1, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}$, and similarly for the universal quantifier. In [8] the following axiom schemes, originally introduced by Baaz, are discussed: (C$\forall$) $\exists x(\phi(x) \to \forall y \phi(y))$ and (C$\exists$) $\exists x(\exists y \phi(y) \to \phi(x))$.

Proposition 15 *Let* T *be a* Γ*-theory and* T' *its extension with axioms* C$\forall$ *and* C$\exists$*. Then every* Γ*-structure model of* T' *can be elementarily embedded into a witnessed model of* T*.*

Proof: Let $(\mathbf{M}, \mathbf{B})$ be a Γ-structure model of T'. We consider the theory $T_0 = \mathrm{FEDIAG}(\mathbf{M}, \mathbf{B})$. Now let Ψ be the closure of $\mathrm{NEQ}(\mathbf{B})$ under disjunctions. Clearly Ψ is a directed set. We show that $T_0 \not\vdash \Psi$: it is enough to prove that for every $\alpha, \beta \in \mathrm{NEQ}(\mathbf{B})$, $T_0 \not\vdash \alpha \vee \beta$. Assume the contrary, since $\mathbf{B}$ is an L-chain, we have that either $\alpha \to \beta \in T_0$ or $\beta \to \alpha \in T_0$. Then, since L is a core fuzzy logic, we will have either that $T_0 \vdash \alpha$ or $T_0 \vdash \beta$, which is absurd, by the same definition of $\mathrm{NEQ}(\mathbf{B})$.

Then, by Proposition 13, since $T_0 \supseteq \mathrm{FEDIAG}_0(\mathbf{M}, \mathbf{B})$ and $\Psi \supseteq \mathrm{NEQ}(\mathbf{B})$, there is a linear doubly Henkin theory $T^* \supseteq T_0$ such that $T^* \not\vdash \Psi$ and $(\mathbf{M}, \mathbf{B})$ is elementarily embedded into $\mathbf{CM}(T^*)$. The rest of the proof follows the same lines that the corresponding part of the proof of Lemma 5 of [8]. $\square$

4 Counterexample to Conjectures 1 and 2 of [8]

Given two theories $T_1 \subseteq T_2$ in the respective predicate languages $\Gamma_1 \subseteq \Gamma_2$, it is said that T_2 is a *conservative extension* of T_1 if and only if each Γ_1-formula provable in T_2 is also provable in T_1. P. Hájek and P. Cintula proved in Theorem 6 of [8] that, in core fuzzy logics, a theory T_2 is a conservative extension of another theory T_1 if and only if each exhaustive model of T_1 can be elementarily embedded into some model of T_2. In Theorem 7 of [8], they conjectured the same result to be true for arbitrary structures, showing that the following two conjectures were equivalent:

Conjecture 1 of [8]: Let P be a nullary predicate symbol and for $i \in \{1, 2\}$, T_i be a Γ_i-theory, and T_i^+ be a $\Gamma_i \cup \{P\}$-theory such that $T_i^+ = T_i$. If T_2 is a conservative extension of T_1, then T_2^+ is a conservative extension of T_1^+.

Conjecture 2 of [8]: A theory T_2 is a conservative extension of another theory T_1 if and only if each model of T_1 can be elementarily embedded into some model of T_2.

We present here a counterexample to Conjecture 2 (and thus to Conjecture 1) using the method of diagrams. Our example is based in one used by F. Montagna in the proof of Theorem 3.11 of [9]. Let L be the logic that has as equivalent algebraic semantics the variety generated by the union of the classes of Łukasiewicz and Product chains, for an axiomatization of this extension of BL we refer to [2] (in this article it is proved that the only chains of the variety are precisely the Łukasiewicz and Product chains). Let now $(\mathbf{M}, \{0,1\})$ be a classical first-order structure in a predicate language Γ, and let $\mathbf{B}_1 = [0,1]_\Pi$ and $\mathbf{B}_2 = [0,1]_{\text{Ł}}$ be the canonical Product and Łukasiewicz chains, respectively.

Remark that the structure $(\mathbf{M}, \{0,1\})$ can also be regarded as a Γ-structure over both $\mathbf{B}_1$ and $\mathbf{B}_2$ chains, since for every two-valued n-ary predicate $P_\mathbf{M} : M^n \to \{0,1\}$, $P_\mathbf{M}$ is also a fuzzy relation $P_\mathbf{M} : M^n \to [0,1]_\Pi$ and $P_\mathbf{M} : M^n \to [0,1]_{\text{Ł}}$. Thus, we have $(\mathbf{M}, \mathbf{B}_1) \equiv (\mathbf{M}, \mathbf{B}_2)$ (in fact we have that $(\mathbf{M}', \mathbf{B}_1) \equiv (\mathbf{M}', \mathbf{B}_2)$, where $\mathbf{M}'$ is the structure of Definition 3).

Let T_1=EDIAG$(\mathbf{M}, \mathbf{B}_1)$ and T_2=FEDIAG$(\mathbf{M}, \mathbf{B}_2)$. We have that T_2 is a conservative extension of T_1: for every $\Gamma_\mathbf{M}$-formula ϕ, if $T_2 \vdash \phi$, then $\|\phi\|^{\mathbf{B_2}}_{\mathbf{M}'}=1$ and since $(\mathbf{M}', \mathbf{B}_1) \equiv (\mathbf{M}', \mathbf{B}_2)$, $\phi \in T_1$. Now we show that there is a model of T_1 that can not be elementarily embedded into some model of T_2, this model is $(\mathbf{M}, [0,1]_\Pi)$. Suppose, contrary to our claim, that there is a model of T_2, say $(\mathbf{N}, \mathbf{A})$, in which $(\mathbf{M}, [0,1]_\Pi)$ is elementarily embedded. By Proposition 5, since $(\mathbf{N}, \mathbf{A})$ is a model of T_2, there is an elementary mapping from $(\mathbf{M}, [0,1]_{\text{Ł}})$ into $(\mathbf{N}, \mathbf{A})$. Consequently, there is an L-embedding k from $[0,1]_\Pi$ into $\mathbf{A}$ and at the same time there is an L-homomorphism h from $[0,1]_{\text{Ł}}$ into $\mathbf{A}$ (not necessarily one-to-one). If $\mathbf{A}$ is an L-chain, it is clear that this is not possible. We show now that, for any arbitrary L-algebra $\mathbf{A}$, this fact leads to a contradiction.

If such embeddings k and h exist, and c and b are the images of $1/2$ under h and k respectively, we have $b = \neg b$ (because h is an L-homomorphism), $c < 1$ and $\neg c = 0$ (because k is an L-embedding and the negation in $[0,1]_\Pi$ is Gödel). If we decompose $\mathbf{A}$ as a subdirect product of an indexed family of subdirectly irreducible BL-chains, say $(\mathbf{A}_i : i \in I)$, every such $\mathbf{A}_i$ is either a Łukasiewicz, or a Product chain (for a reference see [7] and [2]). Therefore, if we take an index i such that the i-component, c_i, satisfies $0 < c_i < 1$, we will have at the same time $\neg c_i = 0$ and for the i-component b_i, $b_i = \neg b_i$, which is absurd, because $\mathbf{A}_i$ can not be, at the same time, a Łukasiewicz and a Product chain.

5 A Characterization Theorem of Elementary Equivalence

In this section we characterize when two exhaustive structures are elementarily equivalent in terms of elementary extensions. We provide an example showing that the result can not be extended to arbitrary models.

Theorem 16 *Let* $(\mathbf{M}_1, \mathbf{B}_1)$ *and* $(\mathbf{M}_2, \mathbf{B}_2)$ *be two exhaustive* Γ*-structures. The following are equivalent:*

1. $(\mathbf{M}_1, \mathbf{B}_1) \equiv (\mathbf{M}_2, \mathbf{B}_2)$.
2. *There is a* Γ*-structure* $(\mathbf{N}, \mathbf{A})$*, such that* $(\mathbf{M}_1, \mathbf{B}_1)$ *and* $(\mathbf{M}_2, \mathbf{B}_2)$ *are elementarily mapped into* $(\mathbf{N}, \mathbf{A})$.

Proof: 2. $\Rightarrow$ 1. is clear. 1. $\Rightarrow$ 2. First we expand the language introducing two disjoint sets of new constants, C_{M_1} and C_{M_2} for the elements of M_1 and M_2, respectively, that are not interpretations of the constant symbols in Γ.

Now consider the theory $T_0 = \text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1) \cup \text{EDIAG}(\mathbf{M}_2, \mathbf{B}_2)$ in the language expanded with the set of constants C_{M_1} and C_{M_2} respectively. Let us show that T_0 is consistent: If $T_0 \vdash \bot$, since $\text{EDIAG}(\mathbf{M}_2, \mathbf{B}_2)$ is closed under conjunction and the proof is finitary, there is $\psi \in \text{EDIAG}(\mathbf{M}_2, \mathbf{B}_2)$ such that $\text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1), \psi \vdash \bot$. Then, by the Local Deduction Theorem (see Definition 1), there is a natural number n such that $\text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1) \vdash (\psi)^n \to \bot$. Let $\widehat{\psi}$ be the formula obtained by replacing each constant $c \in C_{M_2}$ by a new variable x. Thus we have $\text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1) \vdash (\widehat{\psi})^n \to \bot$ and by generalization over the new variables we obtain $\text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1) \vdash (\forall ...)((\widehat{\psi})^n \to \bot)$, thus $(\forall ...)((\widehat{\psi})^n \to \bot) \in Th(\mathbf{M}_1, \mathbf{B}_1) = Th(\mathbf{M}_2, \mathbf{B}_2)$ (because $(\mathbf{M}_1, \mathbf{B}_1) \equiv (\mathbf{M}_2, \mathbf{B}_2)$) and consequently, $\bot \in Th(\mathbf{M}_2, \mathbf{B}_2)$, which is absurd.

Now let $\Psi = \overline{\text{EDIAG}}(\mathbf{M}_1, \mathbf{B}_1)$. It is easy to check that Ψ is a directed set: given $\alpha, \beta \in \Psi$, we show that $\alpha \vee \beta \in \Psi$. If $\alpha \vee \beta \in \text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1)$, using the fact that $\mathbf{B}_1$ is an L-chain, we have that either $\alpha \to \beta \in \text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1)$ or $\beta \to \alpha \in \text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1)$. Then, since L is a core fuzzy logic, we will have either that $\alpha \in \text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1)$ or $\beta \in \text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1)$ which is absurd because $\alpha, \beta \in \Psi$.

We show now that $T_0 \not\vdash \Psi$. Otherwise, if for some $\alpha \in \Psi$, $T_0 \vdash \alpha$, since $\text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1)$ is closed under conjunction and the proof is finitary, there is $\psi \in \text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1)$ such that $\text{EDIAG}(\mathbf{M}_2, \mathbf{B}_2), \psi \vdash \alpha$. Then, by the same kind of argument we have used to show that T_0 is consistent, we would obtain that $\alpha \in \text{EDIAG}(\mathbf{M}_1, \mathbf{B}_1)$, which is absurd.

Then, by Corollary 14, there is a linear doubly Henkin theory $T \supseteq T_0$ in a predicate language $\Gamma' \supseteq \Gamma$ such that $T \not\vdash \Psi$ and an elementary mapping (f, g) from $(\mathbf{M}_1, \mathbf{B}_1)$ into $\mathbf{CM}(T)$, with f and g one-to-one. Moreover, since $\mathbf{CM}(T)$ is also a model of $\text{EDIAG}(\mathbf{M}_2, \mathbf{B}_2)$, by Corollary 6, $(\mathbf{M}_2, \mathbf{B}_2)$ is elementarily mapped into $\mathbf{CM}(T)$. Finally, by Lemma 12, $\mathbf{Lind}_T$ is an L-chain. □

Remark that Theorem 16 can not be generalized to arbitrary structures. If we take the structures of the counterexample to Conjectures 1 and 2 of Section 4, we have $(\mathbf{M}, [0,1]_\Pi) \equiv (\mathbf{M}, [0,1]_{\text{Ł}})$, but there is not a Γ-structure $(\mathbf{N}, \mathbf{A})$ in which both are elementary mapped.

6 Future Work

When working with models over the same L-algebra, we can introduce a stronger notion of elementary equivalence. Given a Γ-structure $(\mathbf{M}, \mathbf{B})$ let $\Gamma_{\mathbf{B}}$ be the expansion of Γ by adding a nullary predicate symbol P_b for each $b \in B$. Let $(\mathbf{M}^\sharp, \mathbf{B})$ be the expansion of $(\mathbf{M}, \mathbf{B})$ to the language $\Gamma_{\mathbf{B}}$, by interpreting for each $b \in B$, the nullary predicate symbol P_b by b. Then we say that two Γ-structures, $(\mathbf{M}_1, \mathbf{B})$ and $(\mathbf{M}_2, \mathbf{B})$, are *strong elementarily equivalent* (denoted by $(\mathbf{M}_1, \mathbf{B}) \equiv_s (\mathbf{M}_2, \mathbf{B})$) if and only if $(\mathbf{M}_1^\sharp, \mathbf{B}) \equiv (\mathbf{M}_2^\sharp, \mathbf{B})$.

By an argument analogue to the one in Theorem 16 (but using Proposition 13 instead of Corollary 14), it is not difficult to check that two strong elementary equivalent structures (not necessarily exhaustive), over the same L-algebra, are elementary embedded in a third structure. Future work will be devoted to the study of the properties of this stronger notion of equivalence.

The work we have done so far can be extended to Δ-core fuzzy logics, by finding analogues to Theorem 10 and Proposition 13 for these logics. Work in progress includes characterizations of elementary equivalence for other expansions of MTL and the study of the relationship between elementarily embeddability and amalgamation properties.

Acknowledgment

Research partially funded by the spanish projects CONSOLIDER (CSD2007-0022), MULOG2 (TIN2007-68005-C04-01) and ARINF (TIN2009-14704-C03-03) by the ESF Eurocores-LogICCC/MICINN project FFI2008-03126- E/FILO and by the Generalitat de Catalunya under the grants 2009-SGR 1433 and 1434.

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